Reconciling landmarks and level sets: geometric shape warping and matching using generalized gradients and correspondence-augmented implicit representations

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Abstract: Shape warping is a key problem in statistical shape analysis. This paper proposes a framework for geometric shape warping based on both shape distances and landmarks. Taking advantage of the recently proposed spatially coherent flows, our method is mathematically well-posed and uses only intrinsic shape information, namely some similarity measure between shapes and the correspondence of landmarks provided on the shape surface. No extrinsic quantity is considered, neither a diffeomorphism of the embedding space nor point correspondences in this space. Thanks to a recent extension of the level set method allowing point tracking and tangential velocities, our method is compatible with implicit representations. Moreover, a matching between shape surfaces is provided at no additional cost. Although some recent work deals with implicit representations and landmarks, it is, to our knowledge, the first time that landmarks and shape distances are reconciled in a pure geometric level set framework. The feasibility of the method is demonstrated with two- and three-dimensional examples. Combining shape distance and landmarks, our approach reveals to need only a small number of landmarks to obtain improvements on both warping and matching.

Key-words: Shape metrics, distance fonctions, level set, shape warping, shape matching, landmarks

Points d'amer et méthode des ensembles de niveaux: Déformation de formes et mise en correspondance

Résumé : La déformation de formes est un problème important de l'analyse statistique des formes. Nous proposons dans cet article un cadre permettant de déformer une forme en une autre. Notre méthode utilise à la fois des distances entre formes et des points d'amer (ou landmarks). En utilisant la technique récemment proposée des gradients avec cohérence spatiale, notre travail est mathématiquement bien posé et utilise seulement l'information intrinsèque des formes: une mesure de la similarité entre formes et la correspondance entre points d'amer fournie sur la surface de la forme. Aucune quantité extrinsèque n'est considérée: nul besoin d'un difféomorphisme de l'espace dans lequel la forme est plongée, ni de correspondances entre les points de cet espace. Grâce à la récente extension de la méthode des ensembles de niveaux permettant le suivi de point ainsi que l'introduction de vitesses tangentielles, notre méthode est compatible avec une représentation implicite des formes. De plus un appariement entre les formes est fourni sans coût supplémentaire. Bien que certains récents travaux traitent du sujet de représentation implicite et de points d'amer, il s'agit, à notre connaissance, de la première fois que les points d'amer et les distances entre formes sont réunis dans un cadre purement géométrique et implicite. La faisabilité de cette méthode est démontrée par des exemples en deux et trois dimensions. Associant distance entre forme et points d'amer, notre approche ne nécessite qu'un petit nombre de points d'amer pour obtenir une amélioration à la fois de la déformation de la forme mais aussi de la mise en correspondance.

Mots-clés : Métriques de formes, fonctions distances, ensembles de niveaux, déformation de formes, appariement de formes, points d'amer

1 Introduction

Understanding shapes and their basic empirical statistics is a fascinating problem that has attracted the attention of many scientists for many years[32, 23, 24, 25]. Warping one shape into another is one of the keys leading to statistical shape analysis [4, 14]. It offers a way to compare shapes, to compute their mean, to analyse their variability, and eventually to obtain correspondence between them. Roughly speaking, the warping problem consists in transforming an initial shape into a target one: the result is the family of the intermediate shapes. Slightly different, though closely related, is the matching problem, where a correspondence (sometimes one-to-one, but not always) between two given shapes has to be established, regardless to some path from one of the shape into the other one. Indeed, if a warping process keeps track of the motion of every point of the initial shape all along the transformation, it induces a matching.

The diffeomorphic matching problem [33, 42] is often cited in that context. Indeed, as far as applications like brain warping [41, 40] are concerned, shapes (surfaces) and images are closely coupled. In that particular case, considering the images (e.g. MR scans) from which the shapes (e.g. the cortical surface) are extracted is natural and matching the images themselves is justified. Depending on the particular method [44, 16, 18, 17], the shapes are more or less involved in the image matching process, usually as two or three-dimensional curves or points. They are referred to as landmarks. However, one should not be confused: here, the recovered matching is a diffeomorphism of the spaces in which the shapes are embedded.

Rather different is the original problem of shape warping, motivated by recognition, tracking or segmentation tasks. Since the beginning, it has been formulated in geometric term of shape distance[12]. Landmarks have most of the time been part of the proposed methods, sometimes as guides, given by the user (e.g. anatomical landmarks) or geometrically determined (e.g. high curvature points), sometimes to define the shapes themselves (pseudo-landmarks between the landmarks). Among successful methods, let us cite the Procrustes analysis [22, 24], the thin-plate splines [5, 3] (still inspiring recent work like [43]), the active shape models [9] (later extended into the active appearance models [10], in which the underlying images are again considered).

Introduced as a way to cope with interface evolution simulation, the level set method [13, 34] is based on an implicit representation of surfaces. In the original version of this method, the signed distance function was only a convenient choice to get an implicit representation of the initial surface. Then, some methods were designed [19, 36] to preserve this distance during the whole evolution: the evolving implicit function remains the distance to its zero level set. Consequently, the emergence of shape statistics in the implicit framework is not surprising. The pioneering piece of work considered the distance function as the only object of analysis: warping, matching, or statistical analysis were directly performed on the distance functions [29, 35, 39]. Nevertheless, mainly because the combination of two distance functions is not a distance function anymore, these methods could not be considered as definitive answers.

More recently, in a large collection of papers [31, 30, 27, 26, 28], the authors proposed to mix implicit representations and landmarks. Yet, their work belongs to the diffeomorphic matching family: a space diffeomorphism is recovered thanks to the implicit representation of the landmarks (closed curves, but also open curves or points). It does not consider any surface evolution, thus, to our opinion, cannot be really seen as a true level set method application.

Back to pure geometric shape warping, the Hausdorff distance has always been one of the most considered shape similarity measure[12]. Hopelessly, trying to make a shape evolve toward another one minimizing their Hausdorff distance yields an irregular motion. Yet, the authors of [6, 7] introduced a family of smoothing approximations[1] of the Hausdorff distance that makes this warping possible. Although mathematically well justified, the resulting warping does not seem to be completely satisfying, in the sense that it does not reveal to be the one a human observer would have chosen. To correct this behavior, the same authors proposed in [8] a generalized gradient definition, yielding what they called spatially coherent minimizing flows.

Though, in the case of complex shapes, providing corresponding landmarks on both the initial and the target shapes reveals to be inevitable[15, 20]. In [1], the authors proposed a first try to modify a curve evolution in the implicit framework in a way that encourages the motion of the landmarks toward their respective targets. Turning back to simple ideas, the natural way to guide an evolution with landmarks is to try to minimize the distance between the landmarks on the evolving shape and the corresponding ones on the target shape. Again, this yields an irregular motion. In this paper, we present a novel usage of the *generalized gradients* introduced in [8] that turns this motion into a regular and well posed one.

Remarkably, recents advances in the level set method make our shape evolution compatible with it. To do this, two techniques are required. First, a way to simulate a Partial Differential Equation (PDE) embedded on a surface. This has been affordable for some times[2]. Second, a way to deal with *surface evolutions involving non normal velocities* (and, but this is related, to *track the surface points along time*). This last need is not usual in the level set framework, but a recent work[37] proposed a solution to that problem. As a result, we present what is, to our knowledge, the first usage of landmarks for true geometrical shape warping in the level set framework. Moreover, tracking points during the evolution gives us a matching (again, not necessarily one-to-one) between the shapes at no additional cost.

As a benchmark, we test the effect of adding our landmark-guided force to the distance-based spatially coherent evolution given by [8]. Our method proves to be robust and gives the expected improvements: the obtained warping in more satisfying. Although not required, we track points in the original method too, and show that adding landmarks greatly improves the induced matching, even in cases where the original warping seems to be good enough. Again, our landmark-based evolution could be added to any other shape warping scheme. Yet, we have found the tested scheme, namely shape distance plus landmarks (plus eventually

[1] Apparently rediscovered and mentioned in [26]

spatially coherent flows), an interesting combination that reveals to need only a small number of landmarks to obtain improvements on both warping and matching.

The rest of the paper is organized as follows. We first review some shape distances and their usage for the shape warping problem. Then, after introducing the generalized gradient proposed by [8], we present our landmark-guided warping. The next section discusses the level set implementation of our method. The final section shows two- and three-dimensional results and comparisons.

2 Shapes and Shape Metrics

Fully defining the notion of shape is beyond the scope of this article in which we use a limited, i.e purely *geometric*, definition. In our context we define a shape Γ to be the boundary of a regular and bounded subset of $\mathbb{R}^n$. Since we are driven by image applications we also assume that all our shapes are contained in a hold-all regular open bounded subset of $\mathbb{R}^n$ which we denote by Ω. We suppose Γ to be a smooth codimension-one manifold of $\mathbb{R}^n$, and denote by S the set of shapes. We refer the reader to [12] for a more rigorous and complete analysis.

Since we want to be able to compare shapes, a way to quantify the similarity between them must be defined. One of the broadly used distance between shapes is the *Hausdorff distance*:

$$d_H(\Gamma_1, \Gamma_2) = \max \left\{ \sup_{\mathbf{x} \in \Gamma_1} d_{\Gamma_2}(\mathbf{x}), \sup_{\mathbf{x} \in \Gamma_2} d_{\Gamma_1}(\mathbf{x}) \right\}$$

where d_Γ is the *distance function* to the shape Γ:

$$d_\Gamma(\mathbf{x}) = \inf_{\mathbf{y} \in \Gamma} d(\mathbf{x}, \mathbf{y})$$

Some other distances are based on signed distance functions. The *signed distance function* to a shape Γ, denoted by $\tilde{d}_\Gamma$, is equal to d_Γ outside Γ and equal to $-d_\Gamma$ inside Γ:

$$\begin{array}{cccc} \tilde{d}_\Gamma : & \mathbb{R}^n & \to & \mathbb{R} \\ & \mathbf{x} & \mapsto & \varepsilon_\Gamma(\mathbf{x})\ d_\Gamma(\mathbf{x}) \end{array}$$

where

$$\varepsilon_\Gamma(\mathbf{x}) = \left\{ \begin{array}{ll} -1 & \text{if } \mathbf{x} \text{ is inside } \Gamma \\ 1 & \text{otherwise} \end{array} \right.$$

A possible shape distance is then the norm of the Sobolev space, $W^{1,2}(\Omega)$, of square integrable functions with square integrable derivatives:

$$d_{W^{1,2}}(\Gamma_1, \Gamma_2)^2 = \left\| \tilde{d}_{\Gamma_1} - \tilde{d}_{\Gamma_2} \right\|^2_{L^2(\Omega, \mathbb{R})} + \left\| \nabla \tilde{d}_{\Gamma_1} - \nabla \tilde{d}_{\Gamma_2} \right\|^2_{L^2(\Omega, \mathbb{R}^n)}$$

Although d_H and $d_{W^{1,2}}$ can be shown to be equivalent (see [7]), the choice of one or the other is not neutral with respect to warping and to computational complexity. We will not discuss this choice here, our landmark-based warping being independent from it.

3 Variational Shape Warping

In this section, we review the initial work of [7] and its extension [8]. Indeed, we will use it as a benchmark to test our own scheme. Moreover, it will be the occasion to introduce useful concepts and notations.

We assume that we are given a function $E : S \times S \to \mathbb{R}^+$, the energy. This energy can be thought of as a measure of dissimilarity between two shapes. Warping a shape Γ_1 into another one Γ_2 can be stated as the minimization of the energy $E(.,\Gamma_2)$ starting from Γ_1, i.e. finding a family of shapes $\{\Gamma(t),\ t \geq 0\}$ with $\Gamma(0) = \Gamma_1$ and $\Gamma(t)$ following some gradient descent toward Γ_2.

3.1 Shape gradient

In order to define the gradient of the energy functional, the first step is to compute its Gâteaux derivatives in all directions, i.e. for all admissible velocity fields $\mathbf{v} : \Gamma \to \mathbb{R}^n$. Let us denote by $\mathcal{G}_\Gamma\big(E(\Gamma,\Gamma_2),\mathbf{v}\big)$ the Gâteaux derivatives of the energy function $E(\Gamma,\Gamma_2)$ with respect to the shape Γ and in the direction $\mathbf{v}$:

$$\mathcal{G}_\Gamma\big(E(\Gamma,\Gamma_2),\mathbf{v}\big) = \lim_{\varepsilon\to 0} \frac{E(\Gamma + \varepsilon\,\mathbf{v},\Gamma_2) - E(\Gamma,\Gamma_2)}{\varepsilon}$$

We would like to pick the gradient as the direction of steepest descent of the energy. But, to be able to assess the steepness of the energy, the deformation space needs to be equipped with an inner product. We model the space of admissible deformations as an inner product space $\big(F, \langle,\rangle_F\big)$. Under certain regularity conditions, there exists a vector $\mathbf{w} \in F$ such that:

$$\forall\ \mathbf{v} \in F,\ \ \mathcal{G}_\Gamma\big(E(\Gamma,\Gamma_2),\mathbf{v}\big) = \langle \mathbf{w}|\ \mathbf{v}\rangle_F$$

We call it the *shape gradient* of E relative to the inner product $\langle,\rangle_F$ and we note it $\mathbf{w} = \mathbf{D}_\Gamma^{(F,\langle,\rangle_F)} E(\Gamma,\Gamma_2)$. Usually F is taken as the set $L^2(\Gamma,\mathbb{R}^n)$ of the square integrable velocity fields on Γ, and $\langle,\rangle_F$ its associated inner product:

$$\langle f|g\rangle_{L^2} = \int_\Gamma f(\mathbf{x}) \cdot g(\mathbf{x})\, d\Gamma(\mathbf{x})$$

In that case, we will only denote the gradient by $\mathbf{D}_\Gamma E(\Gamma,\Gamma_2)$.

Equipped some shape gradient, we can define the warping of a shape Γ_1 into another one Γ_2 as finding the family $\Gamma(t)$ solution of the following Partial Differential Equation:

$$\begin{array}{rcl} \Gamma(0) & = & \Gamma_1 \\ \dfrac{\partial \Gamma}{\partial t} & = & -\mathbf{D}_\Gamma^{(F,\langle,\rangle_F)} E(\Gamma,\Gamma_2) \end{array} \qquad (1)$$

Natural candidates for the energy function E are the distances presented in the previous section. The Hausdorff distance is not Gâteaux differentiable. Yet, this problem can be solved using an smooth approximation of this distance, denoted by $\tilde{d}_H(\Gamma_1, \Gamma_2)$, which presents the advantage of being differentiable (see [7] for more details). Regarding $d_{W^{1,2}}$, we mention that, although it is based on distance functions, the correct minimization does not consist in directly making $\tilde{d}_{\Gamma(t)}$ evolve toward a minimum of $d_{W^{1,2}}(\Gamma(t), \Gamma_2)$. One has to deform $\Gamma(t)$ so that its distance to Γ_2 vanishes (indirectly via a change of $\tilde{d}_{\Gamma(t)}$). More precisely, the gradient is not taken with respect to the distance function but with respect to the shape itself[2].

3.2 Generalized gradient and spatially coherent flows

Although mathematically well justified, the warpings induced by $E = \tilde{d}_H$ or $E = d_{W^{1,2}}$ do not reveal to be completely satisfying: the obtained deformations do not seem to be the one a human observer would have chosen. Having implemented these warpings, we confirm this behavior (see figure 2). To cope with this, the same authors introduced in [8] a way to favor rigid (translations and rotations) and scaling motions. Their approach consists in recalling that the gradient depends on the choice of $\big(F, \langle,\rangle_F\big)$. They simply *change the inner product used in the definition of the gradient.* They decompose the deformation space L^2 into a sum of several mutually orthogonal linear subspaces: the subspace T of translations, the subspace R of rotations around the centroid, the subspace S of scaling motions centered on the centroid and the orthogonal complement of these subspaces G:

$$L^2 = T \overset{\perp}{\oplus} R \overset{\perp}{\oplus} S \overset{\perp}{\oplus} G$$

Applying different penalty factors to the different types of motions, they define a new inner product related to this decomposition and these penalty factors. The gradient of the Energy function relative to this new inner product can be easily deduced from the standard L^2-gradient $\mathbf{D}_\Gamma E$ (see [8] for more details). The resulting warping promotes spatially coherent motions, keeping the warping of small details for the end of the evolution.

Actually, only *global* coherent motions are promoted by this new gradient. We will see that the symptom of "unnatural" warping persists in case of complex shapes or shapes related by an articulated motion (see the hands example on figure 2).

4 Landmarks-guided warping

Landmarks are then necessary in many cases. Provided by the user (anatomical landmarks), or automatically extracted (geometric landmarks), we assume that we are given p pairs of corresponding points on the initial and on the target shapes, $\{(\mathbf{x}_{1i}, \mathbf{x}_{2i}) \in \Gamma_1 \times \Gamma_2, 1 \le i \le p\}$. We would like to use the information given by theses correspondences to guide the evolution.

[2] A important point generally not mentioned in previous literature.

4.1 A naive definition

As an example, let us suppose we want to provide landmark guidance to the previous evolution given by equation (1). We do this by adding a landmark term $E_{\mathscr{L}}$ to the energy:

$$E_{tot}(\Gamma,\Gamma_2) = E(\Gamma,\Gamma_2) + E_{\mathscr{L}}(\Gamma,\Gamma_2)$$

A first idea would be to track the evolution of the landmarks with points $\mathbf{x}_i(t)$.

$$\begin{aligned} \mathbf{x}_i(0) &= \mathbf{x}_{1i} \\ \frac{\partial \mathbf{x}_i(t)}{\partial t} &= -\mathbf{D}_\Gamma E_{tot}(\Gamma,\Gamma_2)(\mathbf{x}_i(t)) \end{aligned} \tag{2}$$

and to simply choose:

$$E_{\mathscr{L}}(\Gamma,\Gamma_2) = \sum_{i=1}^{p} d(\mathbf{x}_i(t),\mathbf{x}_{2i})^2 \tag{3}$$

Hopelessly, this is not well posed: the existence of point $\mathbf{x}_i(t)$ is not even guaranteed! The framework of viscosity solutions adapted to PDEs like equation (1) allows some points to disappear (see [11, 38]).

4.2 A correct definition

In fact, forward correspondences may not exist if the interface evolution forms shocks; the interface may even collapse and merely disappear. On the contrary, backward correspondences are guaranteed: each point of the evolving interface comes from one point at time 0 (see [37]).

We note $\psi_t : \Gamma(t) \to \Gamma_1$ the family of functions giving for each point $\mathbf{x}$ of $\Gamma(t)$ the point $\psi_t(\mathbf{x})$ on Γ_1 from which $\mathbf{x}$ comes. Let $\gamma_i(t) = \psi_t^{-1}(\{\mathbf{x}_{1i}\})$ be the subset of $\Gamma(t)$ coming from $\mathbf{x}_{1i}$. Equipped with this correspondence, we are now able to define a correct landmark-based energy as the sum, for each landmark of Γ_2, of the squared distance between this point and the corresponding set $\gamma_i(t)$:

$$E_{\mathscr{L}} = \sum_i d(\mathbf{x}_{2i},\gamma_i(t))^2 \tag{4}$$

with the convention that the distance to an empty set is zero. Note that some landmarks might disappear (shock) or become a continuous infinity of points (rarefaction). Actually, we conjecture that, for reasonable choices of the landmarks, rarefaction does not happen with smooth curves. Yet, depending on the initial energy E, there might be some shocks, even with smooth curves. Such considerations are beyond the scope of this paper and might still be open questions. So far, we can only mention that this is what we have observed in our experiments.

In the sequel, we will suppose that either an initial landmark $\mathbf{x}_{1i}$ remains one point $\mathbf{x}_i(t)$ ($\gamma_i(t) = \{\mathbf{x}_i(t)\}$), or it disappears ($\gamma_i(t) = \emptyset$). Under these hypothesis, the difference between our energy (equation (4)) and the "naive" one (equation (3) boils down to the possible disappearance of some landmarks:

$$E_{\mathscr{L}} = \sum_{\{i,\gamma_i(t)\neq\emptyset\}} d(\mathbf{x}_i(t),\mathbf{x}_{2i})^2 \tag{5}$$

keeping in mind that point $\mathbf{x}_i(t)$ are not given by evolution (2), but come from the backward correspondences ψ_t.

4.3 Adapted gradient

Formally, the energy given by equation (5) yields Dirac peaks in the expression of the gradient of the energy:

$$\mathbf{D}_\Gamma^{L^2} E_{tot}(\mathbf{x}) = \mathbf{D}_\Gamma^{L^2} E(\mathbf{x}) + \sum_{\{i,\gamma_i(t)\neq\emptyset\}} \delta_{\mathbf{x}_i(t)}(\mathbf{x})(\mathbf{x}_i(t)-\mathbf{x}_{2i}) \tag{6}$$

where $\delta_{\mathbf{x}}$ denotes the Dirac function centered at point $\mathbf{x}$. This is indeed not a good candidate for a gradient descent.

The solution here is inspired by [8] where the authors mention a way of smoothing the shape gradient. However, we go further and use this smoothing to "get rid" of the Dirac peaks. Let us detail what is actually a standard "trick" in numerical analysis. Again, we change the inner product which appears in the definition of the gradient. Let $H^1(\Gamma,\mathbb{R}^n)$ be the Sobolev space of square integrable velocity fields with square integrable derivatives. We consider the canonical inner product of $H^1(\Gamma,\mathbb{R}^n)$:

$$\langle f|g\rangle_{H^1} = \int_\Gamma f(\mathbf{x})\cdot g(\mathbf{x})\, d\Gamma(\mathbf{x}) + \int_\Gamma \nabla_\Gamma f(\mathbf{x})\cdot\nabla_\Gamma g(\mathbf{x})\, d\Gamma(\mathbf{x})$$

where $\nabla_\Gamma f$ and $\nabla_\Gamma g$ are respectively the intrinsic derivatives[3] on Γ. Recalling the definitions of the gradient given section 3.1, we get:

$$\begin{aligned}\forall\, \mathbf{v}, \quad \left\langle \mathbf{D}_\Gamma^{L^2} E_{tot} \,\middle|\, \mathbf{v}\right\rangle_{L^2} &= \mathcal{G}_\Gamma\big(E(\Gamma,\Gamma_2),\mathbf{v}\big) = \left\langle \mathbf{D}_\Gamma^{H^1} E_{tot} \,\middle|\, \mathbf{v}\right\rangle_{H^1} \\ &= \left\langle \mathbf{D}_\Gamma^{H^1} E_{tot} \,\middle|\, \mathbf{v}\right\rangle_{L^2} + \left\langle \nabla_\Gamma\, \mathbf{D}_\Gamma^{H^1} E_{tot} \,\middle|\, \nabla_\Gamma\, \mathbf{v}\right\rangle_{L^2}\end{aligned}$$

We get that the H^1 gradient can be obtained from the L^2 gradient by solving an intrinsic heat equation with a data attachment term: $\mathbf{D}_\Gamma^{H^1} E_{tot}$ is solution of

$$\Delta_\Gamma\, u \;=\; u \;-\; \mathbf{D}_\Gamma^{L^2} E_{tot} \tag{7}$$

[3]Note that, as soon as $n > 2$, these quantities are not elements of $\mathbb{R}^n$ anymore. However, their correct definition is straightforward and we do not detail it for sake of simplicity.

where Δ_Γ denotes the intrinsic Laplacian operator on the surface, often called the Laplace-Beltrami operator. The solution of this equation coincides with:

$$\arg\min_u \int_\Gamma |\, u(\mathbf{x}) \;-\; \mathbf{D}_\Gamma^{L^2} E_{tot}(\mathbf{x})\,|^2 \, d\Gamma(\mathbf{x}) \;+\; \int_\Gamma |\, \nabla_\Gamma u(\mathbf{x})\,|^2 \, d\Gamma(\mathbf{x}) \tag{8}$$

and the H^1 gradient is finally a smoothed version of the L^2 gradient, given by an image restoration process on a manifold Γ, a problem familiar to the image processing community. However, let us make two remarks:

- Introducing this smoothing via a modification of the gradient rather than directly from equation (8) warrants that the gradient descent will decrease the energy (see [8]).
- We have knowingly omit to mention the space of admissible velocities. All the computations have been written formally. A rigorous demonstration would have involved the theory of distributions. Sketchily, in an appropriate distribution space, we should have considered the PDE obtained from (7) when replacing the formal symbol $\mathbf{D}_\Gamma^{L^2} E_{tot}$ by the second term of equation (6). The solution of this PDE can be shown to be a member of H^1 and exactly the desired smooth gradient $\mathbf{D}_\Gamma^{H^1} E_{tot}$.

In summary, starting from the irregular gradient $\mathbf{D}_\Gamma^{L^2} E_{tot}$ given by equation (6), we obtain a smooth gradient $\mathbf{D}_\Gamma^{H^1} E_{tot}$, given by the PDE (7) and mathematically justified by an adapted choice of inner product that guarantees a decrease of the energy.

4.4 Matching

Let us suppose that the warping process of Γ_1 into Γ_2 has converged. More precisely, we suppose there exists some time T such that $\Gamma(T)$ is very close to Γ_2 (e.g. $d_H(\Gamma(T), \Gamma_2) < \epsilon$ or $E_{tot}(\Gamma(T), \Gamma_2) < \epsilon'$), and a way to assimilate points of Γ_2 to points of $\Gamma(T)$ (e.g. taking the closest point[4]). Then, the backward correspondence ψ_T supplies a natural matching from Γ_2 to Γ_1. This matching is not one to one if some points of Γ_1 have disappeared during the evolution (shocks).

5 Level set implementation

There is no need to introduce the broadly known level set method [13, 34]. However, implementing our scheme in that framework requires two adaptations of the original method: implementing a PDE on an implicit surface and being able to track points during the evolution.

[4]This could be a problem if the evolution gets stuck into some local minimum. Yet, we have never experienced this case.

5.1 H^1 gradient

The H^1 gradient, solution of equation (7), is obtained from an iterative minimization induced by (8). Since the work introduced in [2], implementing a PDE on a surface is affordable in the implicit framework. The only hard point in our case could be the Dirac peaks in the data term. We indeed use a smooth approximation of them.

It should also be mentioned that, in the two dimensional case, the explicit solution of the equation $\Delta_\Gamma\ u = u - v$ is known and given by:

$$\begin{aligned} u(x) &= \tfrac{e^{\sigma(x)}}{2}\left(A - \int_0^{\sigma(x)} e^{-\sigma}\ v(\sigma)\ d\sigma\right) + \tfrac{e^{-\sigma(x)}}{2}\left(B + \int_0^{\sigma(x)} e^{\sigma}\ v(\sigma)\ d\sigma\right) \\ &\text{with } A = \tfrac{e^{|\Gamma|}}{e^{|\Gamma|}-1}\ \oint_\Gamma e^{-\sigma}\ v(\sigma)\ d\sigma \\ &\text{and } B = \tfrac{1}{e^{|\Gamma|}-1}\ \oint_\Gamma e^{\sigma}\ v(\sigma)\ d\sigma \end{aligned}$$

where σ is the arc length and $|\Gamma|$ the length of the curve. Using this explicit solution might be attractive to avoid the iterative minimization giving u. Yet, it requires the extraction of the zero level set Γ of the implicit function, a process generally considered awkward in the level set community.

5.2 Point Correspondences

Because it codes interfaces with implicit representations, the original level set method can not follow the evolution of each point of the initial interface. Only the geometric location of the whole interface is recovered. For the same reason, considered velocities are usually normal to the interface (or projected onto the normal for simplification reasons because it does not modify the location of the interfaces). In our case, we need to follow the landmark points through the backward correspondences ψ_t and to cope with the non normal velocity $-\mathbf{D}_\Gamma^{H^1} E_{tot}$. In [37], the authors propose an approach to maintain an explicit backward correspondence from the evolving surface to the initial one. Let the curves $\Gamma(t)$ be represented by a level set function $\phi : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$, and $\mathbf{v}$ be the (non necessarily normal) velocity fields. ϕ evolves according to:

$$\begin{aligned} \phi(\mathbf{x}, 0) &= \tilde{d}_{\Gamma_1}(\mathbf{x}) \\ \frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \nabla \phi &= 0 \end{aligned}$$

Let us consider a function $\Psi : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ such as:

$$\begin{aligned} \Psi(\mathbf{x}, 0) &= \mathbf{x} \\ \frac{\partial \Psi}{\partial t} + D\Psi\ \mathbf{v} &= 0 \end{aligned} \tag{9}$$

where $D\Psi$ stands for the Jacobian matrix of Ψ. It is shown in [37] that $\Psi(\mathbf{x}, t)$ holds the position that this point was occupying at time $t = 0$. Our needed backward correspondence is then straightforward: $\psi_t(\mathbf{x}) = \Psi(\mathbf{x}, t)$.

6 Experiments

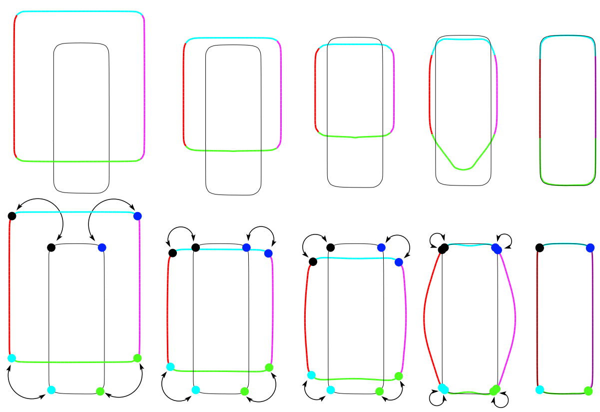

Figure 1: Warping of a rectangle shape into another one. Top row: evolution with $E = d_{W^{1,2}}$. Bottom row: evolution with the same energy, augmented with four provided landmarks, marked by color spots. The colors on the evolving curve shows the evolution of different parts of it. See text for comments.

As a benchmark, we warp some artificial two dimensional curves with the original energy $E = d_{W^{1,2}}$ and test how our landmark-guided force modifies the warping and the final matching. Figure 1 shows the warping of a rectangle into another one. The different parts of the curves are shown with different colors, so that their respective evolution can be followed. Although the initial warping without any landmark seems natural, it fails discovering the matching between the edges of the rectangles, a matching indeed recovered when providing landmarks. Figure 2 shows the warping between two hand shapes. The energy $E = d_{W^{1,2}}$ yields an unnatural warping. Adding spatially coherent flows makes the warping a bit better but still fails in some parts, mainly because the difference between the two shapes can not be summed up to a global motion. With three landmarks only, both a satisfying warping and a good matching are recovered. Figure 3 shows the warping of a teddy bear into a cartoon character. Without any landmarks, the top row evolution fails matching the ears and arms of the characters. The bottom row shows the evolution with four landmarks. Red spots allow to check a good matching between landmarks.

7 Conclusion

We propose a framework for shape warping based on both shape distances and landmarks. Our method is purely geometric and no extrinsic quantity like a space diffeomorphism has to

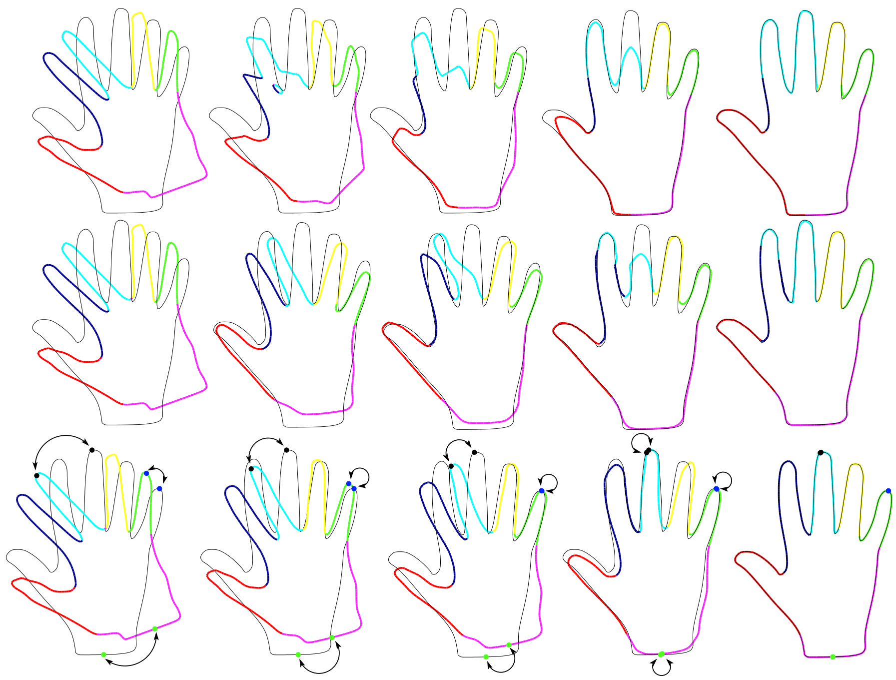

Figure 2: Warping of a hand shape into another one. Top row: evolution with $E = d_{W^{1,2}}$. Middle row: evolution with the same energy plus spatially coherent flows. Bottom row: evolution with the same energy plus coherent flows plus three provided landmarks. See text for comments.

be considered. Thanks to recent advances in the level set techniques, a level set implementation is possible, reconciling landmarks and the level set methods. Moreover, a matching between shapes is provided at no additional cost. Two- and three-dimensional examples, combining shape distance and landmarks, demonstrate the improvement brought by our approach on both warping and matching, even with a small number of landmarks. Further work includes investigating for a one-to-one matching between shapes, and a way to cope with other landmarks, such as curves on surfaces in $\mathbb{R}^3$.

References

[1] R. Abu-Gharbieh, C. Kaminski, T. Gustavsson, and G. Hamarheh. Flame fromt matching and tracking in plif images using geodesic paths and level sets. In *IEEE Workshop*

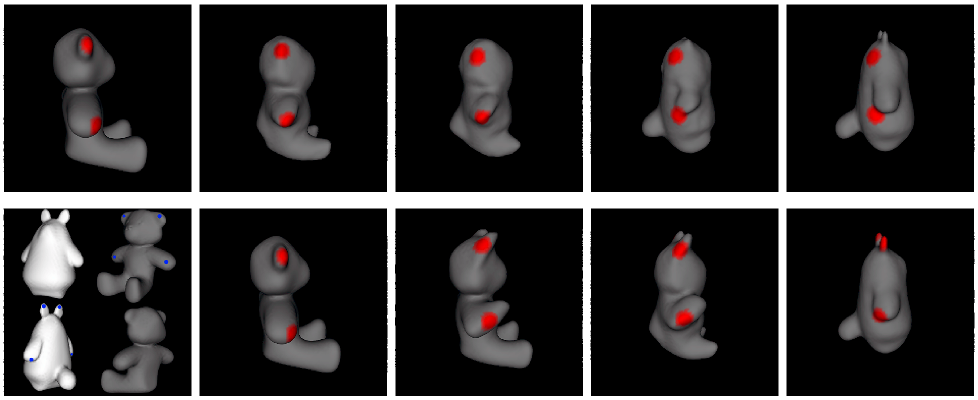

Figure 3: Warping of a teddy bear into a cartoon character. Top row: evolution with $E = d_{W^{1,2}}$. Bottom row, first image: four landmarks provided on the two shapes, indicated by blue spots. Bottom row, remaining images: evolution with $E = d_{W^{1,2}}$ plus the provided landmarks. In red, some parts of the shapes are tracked. See text for comments.

on Variational and Level Set Methods, Vancouver, Canada, July 2001.

[2] M. Bertalmio, G. Sapiro, L.T Cheng, and S. Osher. Variational problems and PDE's on implicit surfaces. In IEEE, editor, *IEEE Workshop on Variational and Level Set Methods*, pages 186–193, Vancouver, Canada, July 2001.

[3] F-L. Bookstein. Landmark methods for forms without landmarks: localizing group differences in outline shapes. *Medical Image Analysis*, 1(3):225–244, 1997.

[4] F.L. Bookstein. Size and shape spaces for landmark data in two dimensions. *Statistical Science*, 1:181–242, 1986.

[5] F.L. Bookstein. Principal warps: Thin-plate splines and the decomposition of deformations. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 11:567–585, 1989.

[6] G. Charpiat, O. Faugeras, and R. Keriven. Shape metrics, warping and statistics. In *Proceedings of the International Conference on Image Processing*. IEEE Signal Processing Society, 2003.

[7] G. Charpiat, O. Faugeras, and R. Keriven. Approximations of shape metrics and application to shape warping and empirical shape statistics. *Foundations of Computational Mathematics*, 5(1):1–58, February 2005.

[8] G. Charpiat, R. Keriven, J.P. Pons, and O. Faugeras. Designing spatially coherent minimizing flows for variational problems based on active contours. In *10th International Conference on Computer Vision*, Beijing, China, 2005.

[9] T. Cootes, C. Taylor, D. Cooper, and J. Graham. Active shape models-their training and application. *Computer Vision and Image Understanding*, 61(1):38–59, 1995.

[10] Timothy F. Cootes, Gareth J. Edwards, and Christopher J. Taylor. Active appearance models. *IEEE Trans. Pattern Anal. Mach. Intell.*, 23(6):681–685, 2001.

[11] M.G. Crandall, L.C. Evans, and P.L. Lions. Some properties of viscosity solutions of Hamilton–Jacobi equations. *Trans. AMS*, 282:487–502, 1984.

[12] M.C. Delfour and J.-P. Zolésio. Shape analysis via distance functions: Local theory. In *Boundaries, interfaces and transitions*, volume 13 of *CRM Proc. Lecture Notes*, pages 91–123. AMS, Providence, RI, 1998.

[13] A. Dervieux and F. Thomasset. A finite element method for the simulation of Rayleigh-Taylor instability. *Lecture Notes in Mathematics*, 771:145–159, 1979.

[14] I.L. Dryden and K.V. Mardia. *Statistical Shape Analysis*. John Wiley & Son, 1998.

[15] M. Ferrant, A. Nabavi, B. Macq, R. Kikinis, and S. Warfield. Registration of 3d intraoperative mr images of the brain using a finite element biomechanical model. *IEEE Transactions on Medical Imaging*, 20(12):1384–1387, 2001.

[16] L. Garcin, A. Rangarajan, and L. Younes. Non rigid registration of shapes via diffeomorphic point matching and clustering. In *International Conference on Image Processing*, Singapore, 2004.

[17] J. Glaunes, M. Vaillant, and M. Miller. Landmark matching via large deformation diffeomorphisms on the sphere. *Journal of Mathematical Imaging and Vision*, 20:179–200, 2004.

[18] J. Glaunes, L. Younes, and A. Trouvé. iffeomorphic matching of distributions: A new approach for unlabelled point-sets and sub-manifolds matching. In *International Conference on Pattern Recognition*, Cambridge, 2004.

[19] J. Gomes and O. Faugeras. Reconciling distance functions and level sets. *Journal of Visual Communication and Image Representation*, 11(2):209–223, 2000.

[20] S. Haker, S. Warfield, and C. Tempany. Landmark-guided surface matching and volumetric warping for improved prostate biopsy targeting and guidance. In *Medical Image Computing and Computer-Assisted Intervention-MICCAI2004*, Lecture Notes in Computer Science. Springer-Verlag, 2004.

[21] A. Heyden, G. Sparr, M. Nielsen, and P. Johansen, editors. *Proceedings of the 7th European Conference on Computer Vision*, Copenhagen, Denmark, May 2002. Springer–Verlag.

[22] Hurley and Cattell. The procrustes program: producing a direct rotation to test an hypothesized factor structure. *Behav. Sci.*, 7:258–262, 1962.

[23] D. G. Kendall. *Stochastic Geometry*, chapter Foundation of a theory of random sets, pages 322–376. John Wiley Sons, New-York, 1973.

[24] D.G. Kendall. Shape manifolds, procrustean metrics and complex projective spaces. *Bulletin of London Mathematical Society*, 16:81–121, 1984.

[25] D.G. Kendall. A survey of the statistical theory of shape. *Statist. Sci.*, 4(2):87–120, 1989.

[26] A. Leow, M-C. Chiang, H. Protas, P. Thompson, L. Vese, and S-C. Huang. Linear and non-linear geometric object matching with implicit representation. In *International Conference on Pattern Recognition*, Cambridge, 2004.

[27] A. Leow, A. Wang, H. Protas, and P. Thompson. Brain warping via landmark points and curves with a level set representation. Technical Report 4-14, UCLA CAM Report, 2004.

[28] A. Leow, A. Wang, H. Protas, A. Toga, and P. Thompson. Brain warping via landmark points and curves with a level set representation. Technical report, submitted to MICCAI'04, 2004.

[29] M. Leventon, O. Faugeras, E. Grimson, and W. Wells. Level Set Based Segmentation with Intensity and Curvature Priors. In *Mathematical Method in Biomedical Image Analysis*, 2000.

[30] W-H. Liao, A. Khuu, M. Bergsneider, L. Vese, S-C. Huang, and S. Osher. From landmark matching to shape and open curve matching: a level set approach. In *IEEE Conference on Vision and Pattern Recognition*, Madison, 2003.

[31] W-H. Liao, H. Protas, M. Bergsneider, L. Vese, S-C. Huang, and S. Osher. A new framework for object warping: a semi-lagrangian level set approach. In *International Conference on Image Processing*, Barcelona, 2003.

[32] G. Matheron. *Random Sets and Integral Geometry.* John Wiley & Sons, 1975.

[33] M. Miller and L. Younes. Group actions, homeomorphisms, and matching : A general framework. *International Journal of Computer Vision*, 41(1/2):61–84, 2001.

[34] S. Osher and J.A. Sethian. Fronts propagating with curvature-dependent speed: Algorithms based on Hamilton–Jacobi formulations. *Journal of Computational Physics*, 79(1):12–49, 1988.

[35] Nikos Paragios, Mikael Rousson, and Visvanathan Ramesh. Matching distance functions: A shape-to-area variational approach for global-to-local registration. In Heyden et al. [21], pages 775–789.

[36] D. Peng, B. Merriman, S. Osher, H.-K. Zhao, and M. Kang. A PDE-based fast local level set method. *Journal of Computational Physics*, 155(2):410–438, 1999.

[37] J.-P. Pons, G. Hermosillo, R. Keriven, and O. Faugeras. How to deal with point correspondences and tangential velocities in the level set framework. In *International Conference on Computer Vision*, volume 2, pages 894–899, 2003.

[38] J.A. Sethian. *Level Set Methods and Fast Marching Methods: Evolving Interfaces in Computational Geometry, Fluid Mechanics, Computer Vision, and Materials Sciences*. Cambridge Monograph on Applied and Computational Mathematics. Cambridge University Press, 1999.

[39] S. Soatto and A.J. Yezzi. DEFORMOTION, deforming motion, shape average and the joint registration and segmentation of images. In Heyden et al. [21], pages 32–47.

[40] A. Toga and P. Thompson. The role of image registration in brain mapping. *Image and Vision Computing*, 19(1-2):3–24, 2001.

[41] Arthur Toga, editor. *Brain Warping*. Academic Press, 1998.

[42] A. Trouvé and L. Younes. Diffeomorphic matching problems in one dimension: designing and minimizing matching functionals. In *Proceedings of the 6th European Conference on Computer Vision*, pages 573–587, Dublin, Ireland, June 2000.

[43] S. Wang, J. Xiuquan Ji, and Z-P. Liang. Landmark-based shape deformation with topology preserving constraints. In *9th International Conference on Computer Vision*, Nice, France, 2003.

[44] L. Younes. Computable elastic distances between shapes. *SIAM Journal of Applied Mathematics*, 58(2):565–586, 1998.

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