# Koiter Estimate Revisited

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**Abstract:** We prove a universal energy estimate between the solution of the three-dimensional Lamé system on a thin *clamped* shell and a displacement reconstructed from the solution of the classical Koiter model. The mid-surface of the shell is an arbitrary smooth manifold with boundary. The bound of our energy estimate only involves the thickness parameter, constants attached to the midsurface, the loading, the two-dimensional energy of the solution of the Koiter model and "wave-lengths" associated with this solution. This result is in the same spirit as Koiter's who gave a heuristic estimate in 1970. Taking boundary layers into account, we obtain rigorous estimates, which prove to be sharp in the cases of plates and elliptic shells.

**Key-words:** Mechanics of solids, Shell theory, Multiscale expansions, Boundary layers

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# Retour sur l'estimation de Koiter

**Résumé :** Nous démontrons l'existence d'une estimation en énergie entre la solution du problème de l'élasticité tridimensionnelle posé sur une coque mince et un déplacement reconstruit à partir de la solution du modèle bidimensionnel classique de Koiter. La surface moyenne de la coque est une variété régulière de dimension deux à bord régulier. L'estimation fait intervenir l'épaisseur de la coque, des constantes géométriques liées à la surface moyenne de la coque, le chargement, l'énergie bidimensionnelle de la solution du modèle de Koiter, ainsi que des "longueurs d'ondes" associées à cette solution. Ce résultat est dans le même esprit que l'estimation heuristique donnée par Koiter en 1970. La prise en compte de termes de couches limites au voisinage du bord de la coque permet d'obtenir une estimation rigoureuse, et optimale dans le cas des plaques et des coques elliptiques.

**Mots-clés :** Mécanique du solide, Théorie des coques, Développements multiéchelles, Couches limites

## 1 Introduction

This paper deals with *shell theory* whose aim is the approximation of the three-dimensional linear elastic shell problem by a two-dimensional problem posed on the mid-surface. This is an old and difficult question. As written by Koiter & Simmonds in 1972 [21] *"Shell theory attempts the impossible: to provide a two-dimensional representation of an intrinsically three-dimensional phenomenon."*

Let us recall that a shell is a three-dimensional object characterized by its mid-surface $S$ and its (half-)thickness $\varepsilon$. The mid-surface is a two-dimensional manifold embedded in $\mathbb{R}^3$. We assume that $S$ if a $\mathscr{C}^\infty$ smooth compact orientable manifold with boundary. Let $S \ni P \mapsto \boldsymbol{n}(P) \in \mathbb{R}^3$ be a continuous unit normal field on $S$. We denote the shell by $\Omega^\varepsilon$ in order to remind the value $\varepsilon$ of the thickness parameter which is small enough, $0 < \varepsilon \leq \varepsilon_0$, so that the representation

$$S \times (-\varepsilon, \varepsilon) \ni (P, x_3) \mapsto P + x_3\, \boldsymbol{n}(P) \in \mathbb{R}^3, \tag{1.1}$$

is a $\mathscr{C}^\infty$ diffeomorphism onto $\Omega^\varepsilon$. In simpler words, $\Omega^\varepsilon$ is the surface $S$ thickened in its normal direction by the thickness $\varepsilon$. Of course, if $S$ a plane domain, $\Omega^\varepsilon$ is a plate.

As material law for the body $\Omega^\varepsilon$, the most standard assumption is to consider the case of an homogeneous and isotropic material like in the literature quoted below. Such a material is characterized by its Lamé constants $\lambda$ and $\mu$, or, alternatively by its Young modulus $E$ and its Poisson coefficient $\nu$. We are interested by the displacement $\boldsymbol{u}$ solution of the problem $(P_{\mathsf{3D}})$ consisting of the three-dimensional Lamé system on $\Omega^\varepsilon$ with clamped boundary conditions on its lateral boundary. We consider this $\boldsymbol{u}$ as the "exact" solution and adress the question of the approximation of $\boldsymbol{u}$ via the solution $\boldsymbol{z}$ of a problem $(P_{\mathsf{2D}})$ posed on the mid-surface $S$.

Many papers deal with this question. Concerning the classical aspects of the derivation of shell models, let us quote Koiter [18, 19, 20], John [16], Naghdi [23], Novozhilov [25]. Concerning plates the derivation of the first two-dimensional model is much earlier, see Kirchhoff [17].

Most of shell models rely on a $3 \times 3$ system of equations on $S$ depending on $\varepsilon$, which can be written in the form

$$\mathsf{K}(\varepsilon) := \mathsf{M} + \varepsilon^2 \mathsf{B} \tag{1.2}$$

where $\mathsf{M}$ is the *membrane* operator on $S$ and $\mathsf{B}$ a *bending* operator. The above authors all agree about the definition of the membrane operator $\mathsf{M}$. On the contrary, different expression for $\mathsf{B}$ can be found in the literature. The most natural in a geometrical and mechanical point of view, is the one given by W. T. Koiter (see [19]) but the question of determinating the *best* model was very controversial (see in particular [2] and the discussion in [20, 23]). Without special mention, *we always take as* $\mathsf{K}(\varepsilon)$ *the Koiter operator.*

When considering laterally clamped shells, the equation inside $S$ has to be complemented by the Dirichlet boundary condition and define problem $(P_{\mathsf{2D}})$. The unique solvability of this problem was proved by Bernadou & Ciarlet [1]. Let $\boldsymbol{z}$ be the solution of problem $(P_{\mathsf{2D}})$. Natural questions arise:

- Q1 Is $\boldsymbol{z}$ itself a "valid" approximation of $\boldsymbol{u}$ ? In what sense ?
- Q2 Is it possible to reconstruct with $\boldsymbol{z}$ only, a three-dimensional displacement $\mathsf{U} = \mathsf{U}(\boldsymbol{z})$ which would be an approximation of $\boldsymbol{u}$ in (relative) energy norm?

To the authors' knowledge, the first question to be addressed was Q2, by Koiter himself. Indeed, the energy norm seems the most natural one and the easiest to deal with. But in general $\boldsymbol{z}$ *is not an approximation of* $\boldsymbol{u}$ *in energy norm* but in weaker norms, as stated and proved by Sanchez-Palencia [26] and Ciarlet, Lods, Miara [3, 6, 4] who gave answers to question Q1. Let us go back to Q2, which is our main point of interest.

KOITER proposed for $\mathsf{U}(\boldsymbol{z})$ (which we will also denote by $\mathsf{U}\boldsymbol{z}$) a *modified Kirchhoff-Love* three-dimensional displacement, which we may write as

$$\mathsf{U}(\boldsymbol{z}) := \mathsf{U}^{\mathsf{KL}}(\boldsymbol{z}) + \mathsf{U}^{\mathsf{cmp}}(\boldsymbol{z}), \tag{1.3}$$

where $\mathsf{U}^{\mathsf{KL}}(\boldsymbol{z})$ is the Kirchhoff-Love displacement associated with $\boldsymbol{z}$ and the complementary term $\mathsf{U}^{\mathsf{cmp}}(\boldsymbol{z})$ is a transverse displacement quadratic in the normal variable $x_3$. It is easy to provide the formulas for $\mathsf{U}^{\mathsf{KL}}$ and $\mathsf{U}^{\mathsf{cmp}}$ in the case of plates: in the situation with zero curvature, we choose as system of coordinates the Cartesian coordinates $(x_1, x_2, x_3)$ with $x_1$, $x_2$ (also denoted $(x_\alpha)$) coordinates in the plane containing $S$ and $x_3$ the coordinate in normal direction. The corresponding components of $\mathsf{U}$ are $\mathsf{U}_\alpha$, $\alpha = 1, 2$ and $\mathsf{U}_3$ and similarly for the components $z_\alpha$ and $z_3$ of $\boldsymbol{z}$. We have

$$\begin{aligned} &\mathsf{U}^{\mathsf{KL}}_\alpha(\boldsymbol{z}) = z_\alpha - x_3\partial_\alpha z_3, && \mathsf{U}^{\mathsf{cmp}}_\alpha(\boldsymbol{z}) = 0,\\ &\mathsf{U}^{\mathsf{KL}}_3(\boldsymbol{z}) = z_3, && \mathsf{U}^{\mathsf{cmp}}_3(\boldsymbol{z}) = -x_3 p(\partial_1 z_1 + \partial_2 z_2) + \tfrac{x_3^2}{2} p(\partial_1^2 + \partial_2^2) z_3, \end{aligned} \tag{1.4}$$

where $p = \lambda(\lambda + 2\mu)^{-1}$. Formulas for general shells are a natural geometrical extension of these formulas, see (2.11) later.

In his main papers [19, 20], KOITER obtained the following tentative energy estimate:

$$\mathsf{E}^{\varepsilon}_{\mathsf{3D}}\big[\boldsymbol{u} - \mathsf{U}(\boldsymbol{z})\big] \le C_S \left(\frac{\varepsilon^2}{L^2} + \frac{\varepsilon}{R}\right) \mathsf{E}^{\varepsilon}_{\mathsf{2D}}[\boldsymbol{z}], \tag{1.5}$$

where $\mathsf{E}^{\varepsilon}_{\mathsf{3D}}$ is the quadratic energy functional associated with the problem $(P_{\mathsf{3D}})$ and $\mathsf{E}^{\varepsilon}_{\mathsf{2D}}$ is the quadratic "physical" energy associated with problem $(P_{\mathsf{2D}})$. Moreover $1/R$ denotes the maximum principal curvature of $S$ and $L$ a "wave length" associated with the solution $\boldsymbol{z}$. Indeed $L$ is a constant appearing in *inverse estimates* concerning the membrane and bending tensors of $\boldsymbol{z}$, see §2.E later.

Let us stress that $\boldsymbol{z}$ depends on $\varepsilon$, and that the wave length $L$ may also depend on $\varepsilon$. But in the situation of plates, $L$ *does not depend on* $\varepsilon$ and, of course, $\frac{1}{R} = 0$. Two years after the publication of [19, 20], it was already known that estimate (1.5) does not hold as $\varepsilon \to 0$ for plates. We read in [21] *"The somewhat depressing conclusion for most shell problems is, similar to the earlier conclusions of* GOL'DENWEIZER*, that no better accuracy of the solutions can be expected than of order $\frac{\varepsilon}{L} + \frac{\varepsilon}{R}$, even if the equations of first-approximation shell theory would permit, in principle, an accuracy of order $\frac{\varepsilon^2}{L^2} + \frac{\varepsilon}{R}$."*

The reason for this is also explained by JOHN [16] in these terms *"Concentrating on the interior we sidestep all kinds of delicate questions, with an attendant gain in certainty and generality. The information about the interior behavior can be obtained much more cheaply (in the mathematical sense) than that required for the discussion of boundary value problems, which form a more "transcendental" stage."*.

The presence of boundary layer terms for thin plates in the vicinity of the lateral part of the boundary was already pointed out by GOL'DENWEIZER [14] but a multi-scale asymptotic expansion combining (for plates) inner (boundary layer) and outer (regular) parts was only available later, see [22, Ch. 15, 16] and its bibliographical comments. A more specific form adapted for clamped thin plates is provided by NAZAROV & ZORIN in [24] and DAUGE & GRUAIS in [8]. From these results we can deduce the sharp estimates for plates, valid for a "standard" load, see [10, §12]

$$\mathsf{E}^{\varepsilon}_{\mathsf{3D}}\big[\boldsymbol{u} - \mathsf{U}(\boldsymbol{z})\big] \le b_S\, \varepsilon\, \mathsf{E}^{\varepsilon}_{\mathsf{2D}}[\boldsymbol{z}], \quad \text{as} \quad \varepsilon \to 0. \tag{1.6}$$

In (1.6), the factor $\varepsilon$ in the bound comes from the contribution of the three-dimensional boundary layer term along the lateral part of the boundary, and $b_S^{-1}$ has the dimension of a length.

For shells, the complexity of a multi-scale analysis (if possible) is much higher. There is at least one situation where such an analysis was succesfully performed: the case of clamped elliptic shells. In [12, 13], Faou proved that

1. The solution $\boldsymbol{z} = \boldsymbol{z}^\varepsilon$ of the Koiter problem $(P_{\mathsf{2D}})$ has a boundary layer in the vicinity of $\partial S$ with length-scale $\sqrt{\varepsilon}$, which yields that the wave length $L$ is also a $\mathcal{O}(\sqrt{\varepsilon})$,
2. The solution $\boldsymbol{u} = \boldsymbol{u}^\varepsilon$ of the Lamé problem $(P_{\mathsf{3D}})$ has a three scale complete asymptotics, which yields estimate (1.6) again, and it is also sharp. But now, both terms in the sum $\frac{\varepsilon^2}{L^2} + \frac{\varepsilon}{R}$ are a $\mathcal{O}(\varepsilon)$ and this proves that *the first Koiter estimate* (1.5) *is asymptotically valid for clamped elliptic shells.*

Thus, we may provide a final answer to the validity of an energy estimate if we know complete asymptotic expansions of the displacement $\boldsymbol{u}$ and of the two-dimensional solution $\boldsymbol{z}$ including boundary layer terms. In both previous situations it is remarkable that the reconstruction operator $\boldsymbol{z} \mapsto \mathsf{U}\boldsymbol{z}$ is the exactly the same as Koiter's.

In this paper, our aim is to prove an universal estimate in the spirit of (1.5) without a priori knowledge of multi-scale expansions for $\boldsymbol{u}$ and $\boldsymbol{z}$. Our estimate is:

$$\mathsf{E}^\varepsilon_{\mathsf{3D}}\big[\boldsymbol{u} - \mathsf{U}(\boldsymbol{z})\big] \le a_S\,\big(B_S(\varepsilon;\boldsymbol{z})\,\mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}] + \mathsf{d}^2 E^{-1}\|\boldsymbol{f}^{\mathsf{rem}}\|^2_{\mathrm{L}^2(\Omega^\varepsilon)}\big)$$
$$\text{with}\quad B_S(\varepsilon;\boldsymbol{z}) = \frac{\varepsilon}{L_\flat}\left(1 + \frac{\varepsilon^2}{\ell^2} + \frac{\varepsilon^2}{\mathsf{r}^2} + \frac{\varepsilon r_0}{\mathsf{r}^2}\right) + \frac{\varepsilon^2}{L^2} + \frac{\varepsilon^2}{\mathsf{r}^2} + \frac{\varepsilon^4\mathsf{d}^2}{L^6} + \sum_F \frac{\varepsilon^{2i}\mathsf{d}^{2j}}{L^{2k}\mathsf{r}^{2l}} \tag{1.7}$$

where $E$ is the Young modulus, and where, see §2.E for the precise definitions,

*a)* $L$ is a global wave length for $\boldsymbol{z}$ similar to the one which Koiter used,

*b)* $\ell$ is a lateral wave length for $\boldsymbol{z}$,

*c)* $L_\flat$ is third constant, involved in trace liftings for $\boldsymbol{z}$,

*d)* $\mathsf{r}$ is a constant depending on the curvature of $S$,

*e)* $r_0 = \inf(\mathsf{r}, 1)$ is a constant having the dimension of a length, and defining the size of a geodesic tubular neighborhood in the vicinity of $\partial S$.

*f)* $\mathsf{d}$ is a constant appearing in the 3D Korn inequalities.

Note that all these constant have the dimension of a length, making the term $B_S(\varepsilon;\boldsymbol{z})$ *adimensional.* The term $\boldsymbol{f}^{\mathsf{rem}}$ is the remaining part of the load $\boldsymbol{f}$ when the mean value of $\boldsymbol{f}$ across each fiber is subtracted from the total load $\boldsymbol{f}$. The set $F$ is the finite set

$$F = \{(i,j,k,l) \in \mathbb{N}^4 \quad | \quad i + j = k + l \le 4\,, \quad i \ge k \quad \text{and} \quad i \ge 1\}. \tag{1.8}$$

Note that the constant $a_S$ is adimensional, as the term depending on the load. In [8, 9, 13], the following is proved:

- For *plates*, the three wave-lengths $L$, $\ell$, and $L_\flat$ are $\mathcal{O}(1)$.
- For *elliptic shells*, $\ell$ and $L_\flat$ are $\mathcal{O}(1)$, whereas $L$ is $\mathcal{O}(\sqrt{\varepsilon R_\partial})$ where $R_\partial$ is the curvature radius along the boundary of $S$.

In both cases our general estimate (1.7) gives back the optimal estimate (1.6) in the case of standard loads (where $\boldsymbol{f} \ne \boldsymbol{f}^{\mathsf{rem}}$). If $\boldsymbol{f}$ is constant along each fiber (which was Koiter's hypothesis), $\boldsymbol{f}^{\mathsf{rem}}$ is 0: Thus the bound of $\mathsf{E}^\varepsilon_{\mathsf{3D}}\big[\boldsymbol{u} - \mathsf{U}\boldsymbol{z}\big]$ depends only on two-dimensional objects. Moreover, we find the following bound for the difference between the energies of $\boldsymbol{z}$ and $\mathsf{U}\boldsymbol{z}$:

$$\big|\mathsf{E}^\varepsilon_{\mathsf{3D}}[\mathsf{U}\boldsymbol{z}] - \mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}]\big| \le a_S\left(\frac{\varepsilon}{R} + \frac{\varepsilon^2}{L^2}\right)\mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}], \tag{1.9}$$

where $1/R$ is the maximum principal curvature of $S$. Therefore, if for $\varepsilon$ small enough $a_S(\varepsilon R^{-1} + \varepsilon^2 L^{-2})$ is less than $\frac{1}{2}$, estimate (1.7) combined with (1.9) yields the *relative energy estimate*:

$$\frac{\mathsf{E}^\varepsilon_{\mathsf{3D}}\big[\boldsymbol{u} - \mathsf{U}\boldsymbol{z}\big]}{\mathsf{E}^\varepsilon_{\mathsf{3D}}[\mathsf{U}\boldsymbol{z}]} \leq 2a_S\, B(\varepsilon; \boldsymbol{z}). \tag{1.10}$$

The plan of the paper is the following. We introduce the three- and two-dimensional problems in §2, their solutions $\boldsymbol{u}$ and $\boldsymbol{z}$, the different wave-lengths associated with $\boldsymbol{z}$, and finally the reconstruction operator $\boldsymbol{z} \mapsto \mathsf{U}\boldsymbol{z}$. In §3, we prove a priori estimates for Sobolev norms of $\boldsymbol{z}$ by Sobolev norms of its membrane and bending strain tensors $\boldsymbol{\gamma}$ and $\boldsymbol{\rho}$. This will serve to convert any norm of $\boldsymbol{z}$ appearing in our estimates in norms of $\boldsymbol{\gamma}$ and $\boldsymbol{\rho}$. After using the wave-lengths, these norms can be compared to the energy of $\boldsymbol{z}$. In §5, in a preliminary step, we prove the estimate (1.9) between the three- and two-dimensional energies. In §6, we describe the strategy of proof of the main estimate (1.7): We take advantage of the framework developed in [12, 11] combining operator formal series and inner-outer expansions, and we split our estimates in 2 parts, investigated in §8 and 9.

## 2 Setting of the problems

In this section, we now give precisely our assumptions, the definitions of problems $(P_{\mathsf{3D}})$ and $(P_{\mathsf{2D}})$ and of the different wave lengths occuring in estimates (1.7) and (1.9). We use everywhere the convention of repeated indices for the contraction of tensors.

### 2.a The three-dimensional problem

In all this work $\{\Omega^\varepsilon\}_{\varepsilon \leq \varepsilon_0}$ denotes a family of elastic shells defined for $\varepsilon_0$ sufficiently small, made with an isotropic and homogeneous material characterized by its two Lamé coefficients $\lambda$ and $\mu$. The mid-surface of the shell is represented by a smooth 2-manifold $S$ embedded in $\mathbb{R}^3$, compact with non-empty boundary $\partial S$. We stress that no other assumption is made on the geometry of the surface $S$. In particular, its main curvatures may have different signs, or even be zero, in which case the shell is a plate. The domain $\Omega^\varepsilon$ is then the image of the manifold $S \times (-\varepsilon, \varepsilon)$ by the application :

$$S \times (-\varepsilon, \varepsilon) \ni (P, x_3) \mapsto P + x_3\, \boldsymbol{n}(P) \in \mathbb{R}^3, \tag{2.1}$$

where $\boldsymbol{n}$ is a continuous unit normal field on $S$. The shell has two faces $\Gamma^\varepsilon_\pm$ images by the previous application of $S \times \{\pm\varepsilon\}$ and a lateral boundary $\Gamma^\varepsilon_0$ image of $\partial S \times (-\varepsilon, \varepsilon)$. The boundary conditions applied to the shell are the free traction conditions on the two faces $\Gamma^\varepsilon_\pm$ and the clamped conditions on $\Gamma^\varepsilon_0$. The space of admissible displacements is then

$$V(\Omega^\varepsilon) = \big\{ \boldsymbol{u} \in \mathrm{H}^1(\Omega^\varepsilon)^3 \quad | \quad \boldsymbol{u} = 0 \quad \text{on} \quad \Gamma^\varepsilon_0 \big\}.$$

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are two displacements on $\Omega^\varepsilon$, we define the energy scalar product

$$a^\varepsilon_{\mathsf{3D}}(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Omega^\varepsilon} A^{ijk\ell}\, e_{ij}(\boldsymbol{u})\, e_{k\ell}(\boldsymbol{v})\, \mathrm{d}V,$$

where $\mathrm{d}V = \mathrm{d}t^1 \mathrm{d}t^2 \mathrm{d}t^3$ with $\{t^i\}$ a system of Cartesian coordinates in $\mathbb{R}^3$ and where

$$A^{ijk\ell} = \lambda \delta^{ij} \delta^{k\ell} + \mu(\delta^{ik}\delta^{j\ell} + \delta^{i\ell}\delta^{jk})$$

is the rigidity tensor of the material. The tensor $e_{ij}(\boldsymbol{v}) = \frac{1}{2}(\partial_i v_j + \partial_j v_i)$ is the deformation tensor in Cartesian coordinates, where $\partial_i$ denotes the derivative with respect to $t^i$. The associated quadratic three-dimensional energy of a displacement $\boldsymbol{v}$ is then:

$$\mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{v}] \;:=\; a^\varepsilon_{\mathsf{3D}}(\boldsymbol{v}, \boldsymbol{v}). \tag{2.2}$$

Our "exact solution" $\boldsymbol{u}$ is the displacement solution of the variational problem :

$$(P_{\mathsf{3D}}) \qquad \text{Find } \boldsymbol{u} \in V(\Omega^\varepsilon) \text{ such that } \quad \forall\, \boldsymbol{v} \in V(\Omega^\varepsilon), \quad a^\varepsilon_{\mathsf{3D}}(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Omega^\varepsilon} \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}V,$$

where $\boldsymbol{f}$ represents the loading force. Of course, $\boldsymbol{u}$ depends on $\varepsilon$, but we leave this dependence implied.

### 2.b Normal coordinates and tensors

The shell $\Omega^\varepsilon$ is diffeomorphic to the manifold $S \times (-\varepsilon, \varepsilon)$ via the application (2.1). Any local coordinate system $(x_\sigma)$ on $S$ yields a coordinate system $(x_\sigma, x_3)$ on $S \times (-\varepsilon, \varepsilon)$ and thus an atlas on $S$ provides an atlas on $\Omega^\varepsilon$ whose local maps are $U \times (-\varepsilon, \varepsilon)$ where $U$ are the maps of the atlas on $S$. Such a coordinate system is called *normal coordinate system*, and induces a basis for tensor fields on $\Omega^\varepsilon$. Moreover we can show that the jacobians of the change of coordinates leave the transverse component unchanged.

This implies that by fixing indices to the transverse index 3 and by letting other indices vary as Greek indices, we obtain functions of $x_3$ with values in surfacic tensors. Thus any displacement (i.e. a 1-form on $\Omega^\varepsilon$) $\boldsymbol{v}$ splits into

*(i)* a surfacic displacement $(v_\sigma)$, which means that $x_3 \mapsto (v_\sigma(x_3))$ takes its values in 1-forms on $S$.

*(ii)* a function $v_3$, in other words $x_3 \mapsto v_3(x_3)$ takes its values in functions on $S$.

On the same way, consider the strain tensor $e_{ij}$. In normal coordinates system, for each fixed $x_3$ this tensor field splits in $e_{33}$, which is a function on $S$, $(e_{\sigma 3})$ which is a covariant tensor of order 1 on $S$, and $(e_{\alpha\beta})$ which is a covariant tensor of order 2 on $S$. These three surfacic tensors depend smoothly on $x_3$.

### 2.c The two-dimensional problem

The Koiter operator on $S$ is defined as $\mathsf{K}(\varepsilon) = \mathsf{M} + \varepsilon^2 \mathsf{B}$ where $\mathsf{M}$ is the *membrane* operator and $\mathsf{B}$ the *bending* operator. Both of them involve the rigidity tensor $M^{\alpha\beta\sigma\delta}$ corresponding to the modified Lamé constants $\tilde{\lambda} = 2\lambda\mu/(\lambda + 2\mu)$ and $\mu$:

$$M^{\alpha\beta\sigma\delta} = \frac{2\lambda\mu}{\lambda + 2\mu} a^{\alpha\beta} a^{\sigma\delta} + \mu (a^{\alpha\sigma} a^{\delta\beta} + a^{\alpha\delta} a^{\beta\sigma})$$

with $a_{\alpha\beta}$ the metric tensor on $S$.

Both operators $\mathsf{M}$ and $\mathsf{B}$ act on spaces of $\boldsymbol{z} = (z_\sigma, z_3)$ where $(z_\sigma)$ is a 1-form on $S$ and $z_3$ a function on $S$. The target space is a space of $\boldsymbol{g} = (g_\sigma, g_3)$ where $(g_\sigma)$ is a 1-form on $S$ and $g_3$ a function on $S$.

We denote by $\mathbf{H}^k(S)$ in a general way the space of 1-forms whose both components belong to the Sobolev space $\mathrm{H}^k(S)$. We keep the notation $\mathrm{H}^k(S)$ for functions. Typical spaces for the $\boldsymbol{z}$ are $\mathbf{H}^1 \times \mathrm{L}^2(S)$ and $\mathbf{H}^1 \times \mathrm{H}^2(S)$.

The operator $\mathsf{M}$ is the operator associated with the bilinear form defined for any $\boldsymbol{z} = (z_\sigma, z_3)$ and $\boldsymbol{\eta} = (\eta_\sigma, \eta_3)$ in $\mathbf{H}^1 \times \mathrm{L}^2(S)$ by

$$(\boldsymbol{z}, \boldsymbol{\eta}) \mapsto a_{\mathsf{M}}(\boldsymbol{z}, \boldsymbol{\eta}) = \int_S M^{\alpha\beta\sigma\delta}\, \gamma_{\alpha\beta}(\boldsymbol{z})\, \gamma_{\sigma\delta}(\boldsymbol{\eta})\, \mathrm{d}S,$$

where the membrane strain tensor field

$$\gamma_{\alpha\beta}(\boldsymbol{z}) = \tfrac{1}{2}(\mathrm{D}_\alpha z_\beta + \mathrm{D}_\beta z_\alpha) - b_{\alpha\beta} z_3$$

is the change of metric tensor. Here $b_{\alpha\beta}$ is the curvature on $S$ and $\mathrm{D}_\alpha$ is the covariant derivative on $S$.

The operator $\mathsf{B}$ is associated with the bilinear form defined for any $\boldsymbol{z}$ and $\boldsymbol{\eta}$ in $\mathbf{H}^1 \times \mathrm{H}^2(S)$ by

$$(\boldsymbol{z},\boldsymbol{\eta}) \mapsto a_{\mathsf{B}}(\boldsymbol{z},\boldsymbol{\eta}) = \frac{1}{3}\int_S M^{\alpha\beta\sigma\delta}\,\rho_{\alpha\beta}(\boldsymbol{z})\,\rho_{\sigma\delta}(\boldsymbol{\eta})\,\mathrm{d}S$$

where

$$\rho_{\alpha\beta}(\boldsymbol{z}) = \mathrm{D}_\alpha\mathrm{D}_\beta z_3 - b_\alpha^\sigma b_{\sigma\beta} z_3 + b_\alpha^\sigma \mathrm{D}_\beta z_\sigma + \mathrm{D}_\alpha b_\beta^\sigma z_\sigma$$

is the change of curvature tensor.

The two-dimensional energy scalar product is defined for $\boldsymbol{z}$, $\boldsymbol{\eta} \in \mathbf{H}^1 \times \mathrm{H}^2(S)$ by

$$a^\varepsilon_{\mathsf{2D}}(\boldsymbol{z},\boldsymbol{\eta}) = a_{\mathsf{M}}(\boldsymbol{z},\boldsymbol{\eta}) + \varepsilon^2 a_{\mathsf{B}}(\boldsymbol{z},\boldsymbol{\eta}). \tag{2.3}$$

This bilinear form is associated with the Koiter operator $\mathsf{K}(\varepsilon) = \mathsf{M} + \varepsilon^2\mathsf{B}$. The physical quadratic energy associated with a displacement $\boldsymbol{z}$ is defined as:

$$\mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}] \;:=\; 2\varepsilon\, a^\varepsilon_{\mathsf{2D}}(\boldsymbol{z},\boldsymbol{z}). \tag{2.4}$$

The right-hand side $\boldsymbol{g} = (g_\sigma, g_3)$ of the two-dimensional problem $(P_{\mathsf{2D}})$ is defined on $S$ as

$$\boldsymbol{g} = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \boldsymbol{f}(x_3)\,\mathrm{d}x_3.$$

The admissible two-dimensional displacement space is $\mathbf{H}^1_0 \times \mathrm{H}^2_0(S)$. The two-dimensional problem then writes:

$$(P_{\mathsf{2D}})\qquad \begin{array}{l} \text{Find } \boldsymbol{z} \in \mathbf{H}^1_0 \times \mathrm{H}^2_0(S) \quad \text{such that} \\ \qquad \forall\,\boldsymbol{\eta} \in \mathbf{H}^1_0 \times \mathrm{H}^2_0(S), \quad a^\varepsilon_{\mathsf{2D}}(\boldsymbol{z},\boldsymbol{\eta}) = \displaystyle\int_S (a^{\alpha\beta} g_\alpha \eta_\beta + g_3\eta_3)\,\mathrm{d}S. \end{array}$$

The residual load is defined as

$$\boldsymbol{f}^{\mathsf{rem}} := \boldsymbol{f} - \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \boldsymbol{f}(x_3)\,\mathrm{d}x_3$$

In the sequel, we also use the notation $\Sigma(S)$ for all couple $(z_\alpha, z_3)$ of $\mathscr{C}^\infty$ 1-form field $(z_\alpha)$ and $\mathscr{C}^\infty$ function $z_3$ defined on $S$. In other words, $\Sigma(S) = \Gamma(T_1 S) \times \mathscr{C}^\infty(S)$ where $\Gamma(T_1 S)$ denotes the space of smooth 1-form fields on $S$ (see [12] for details).

## 2.d Physical dimensions

We recall here the physical dimensions of the different objects present in the problem. We first give the dimensions of the 3D objects:

| Physical object | Notation | Dimension |
|---|---|---|
| Displacement | $\boldsymbol{u}$ | $m$ |
| Volumic force | $\boldsymbol{f}$ | $N.m^{-3}$ |
| Energy | $\mathsf{E}_{\mathsf{3D}}[\boldsymbol{u}]$ | $N.m$ (Joule) |
| Deformation | $e_{ij}(\boldsymbol{u})$ | Adimensional |
| Material coefficients | $\lambda$, $\mu$, $E$ | $N.m^{-2}$ (Pascal) |

**Table 1.** *Physical dimensions of the* 3D *objects*

In this table, $E$ denotes the Young modulus of the material. We recall that

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$$

and

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1+\nu)}$$

where $\nu$ is the adimensional Poisson coefficient

$$\nu = \frac{\lambda}{2(\lambda + \mu)}.$$

With these relations we have

$$\widetilde{\lambda} = \frac{\mathsf{E}\nu}{1-\nu^2}.$$

For the surfacic objects, we have similarly

| Physical object | Notation | Dimension |
|---|---|---|
| Displacement | $\boldsymbol{z}$ | $m$ |
| Volumic force | $\boldsymbol{g}$ | $N.m^{-3}$ |
| Energy | $\mathsf{E}_{\mathsf{2D}}[\boldsymbol{z}]$ | $N.m$ (Joule) |
| Curvature | $b_\alpha^\beta$ | $m^{-1}$ |
| Covariant derivative | $\mathrm{D}_\alpha$ | $m^{-1}$ |
| Change of metric tensor | $\gamma_{\alpha\beta}(\boldsymbol{z})$ | Adimensional |
| Change of curvature tensor | $\rho_{\alpha\beta}(\boldsymbol{z})$ | $m^{-1}$ |

**Table 2.** *Physical dimensions of the* 2D *objects*

Note that in a local coordinate system on $S$, a partial derivative has the dimension of the inverse of a length. The expression of the Christoffel symbols

$$\Gamma_{\alpha\beta}^\sigma = a^{\sigma\delta}(\partial_\alpha a_{\beta\delta} + \partial_\beta a_{\alpha\delta} - \partial_\delta a_{\alpha\beta})$$

then shows that they also have the dimension of the inverse of a length. Thus the dimension of the covariant derivative is coherent.

### 2.e Sobolev norms and wave lengths

Let us first recall the definition of the Sobolev norm of a tensor on a manifold. Consider a tensor field $\boldsymbol{\tau}$ of order $p$ on $S$ written $\tau_{\alpha_1\alpha_2\cdots\alpha_p}$ in any coordinate system. The norm of $\tau$ at a fixed point $P \in S$ is defined as

$$|\boldsymbol{\tau}|^2 = \tau^{\alpha_1\alpha_2\cdots\alpha_p}\tau_{\alpha_1\alpha_2\cdots\alpha_p}.$$

The $\mathrm{L}^2$ norm of $\boldsymbol{\tau}$ is defined as

$$|\boldsymbol{\tau}|_{0\,;\,S}^2 := \int_S |\boldsymbol{\tau}|^2 \mathrm{d}S.$$

Similarly we can consider the tensor $\mathrm{D}^{[\ell]}\boldsymbol{\tau}$ of order $p+\ell$ written $\mathrm{D}_{\beta_1}\cdots\mathrm{D}_{\beta_\ell}\tau_{\alpha_1\alpha_2\cdots\alpha_p}$ in any coordinate system. Thus semi norm of order $\ell$ of $\tau$ is thus the norm

$$|\boldsymbol{\tau}|_{\ell\,;\,S} = |\mathrm{D}^{[\ell]}\boldsymbol{\tau}|_{0\,;\,S}\,.$$

We define similarly the norms $|\boldsymbol{\tau}|_{\ell\,;\,\partial S}$ on the lateral boundary $\partial S$.

Before defining the wave length attached to the solution $\boldsymbol{z}$ of $(P_{\mathsf{2D}})$, we introduce a characteristic wave length depending on $S$.

**Definition 2.1** *We define the constant* $\mathsf{r}$ *as the largest constant such that*

$$\forall\, j = 1,\ldots,7 \qquad \mathsf{r}^{-j} \geq \max_{P\in(\overline{S})} |\mathrm{D}^{[j-1]}\boldsymbol{b}| \tag{2.5}$$

*where* $\boldsymbol{b}$ *is the curvature tensor. We then define the norm* $\|\boldsymbol{\tau}\|^{(\mathsf{r})}_{\ell\,;\,S}$ *by the expression*

$$\Big(\|\boldsymbol{\tau}\|^{(\mathsf{r})}_{\ell\,;\,S}\Big)^2 = \sum_{j=0}^{\ell} \mathsf{r}^{2j-2\ell}|\boldsymbol{\tau}|^2_{j\,;\,S} \tag{2.6}$$

*for any tensor field on* $S$*.*

Note that with the definition (2.5) the constant $\mathsf{r}$ has the dimension of a length, and that we have $\mathsf{r} \leq R$ where $1/R$ is the maximum principal curvature of $S$, as we have $|b_\alpha^\beta| \leq 1/R$. As the covariant derivative has the dimension of the inverse of a length, we see that all the terms in the sum of the right-hand side of (2.6) have the same dimension.

**Definition 2.2** *Let* $\mathsf{L}$ *be an operator acting on tensor spaces on* $S$*. We say that* $\mathsf{L}$ *is* $\mathsf{r}$*-homogeneous of degree* $\ell$ *if it is a linear combination with adimensional coefficients of terms of the form* $\boldsymbol{b}^{[i]}\mathrm{D}^{[k]}\boldsymbol{b}^{[n]}\mathrm{D}^{[j]}$ *with* $i+k+n+j=\ell$*, where* $\boldsymbol{b}^{[i]}$ *denote the product of* $i$ *times the curvature tensor.*

With this definition, we have the following estimate: if $\mathsf{L}$ is a $\mathsf{r}$-homogeneous operator of degree $\ell$ acting on some tensor field $\boldsymbol{\tau}$, we have

$$|\mathsf{L}\boldsymbol{\tau}|_{0\,;\,S} \leq A\|\boldsymbol{\tau}\|^{(\mathsf{r})}_{\ell\,;\,S}$$

where $A$ is an adimensional constant.

Note that the previous definition extend naturally to operators acting on $\Sigma(S)$. We easily see that the operator $\gamma_{\alpha\beta}$ and $\rho_{\alpha\beta}$ are $\mathsf{r}$-homogeneous operators of orders 1 and 2 respectively. Similarly the membrane is $\mathsf{r}$-homogeneous of orders 2, which means that both surfacic and transverse components are $\mathsf{r}$-homogeneous of orders 2, and the bending operator is $\mathsf{r}$-homogeneous of orders 4.

Let $\boldsymbol{\gamma}$ and $\boldsymbol{\rho}$ denote the membrane and bending strain tensors $\gamma_{\alpha\beta}(\boldsymbol{z})$ and $\rho_{\alpha\beta}(\boldsymbol{z})$ of the solution $\boldsymbol{z}$ of problem $(P_{\mathsf{2D}})$. With our notations, we can reformulate Koiter's definition of the quantity $L$ in [18, 19] as *"the wave length of the deformation pattern of shell theory, defined by the order of magnitude relations* $\mathrm{D}^{[1]}\boldsymbol{\gamma} = \mathcal{O}(\boldsymbol{\gamma}/L)$ *and* $\mathrm{D}^{[1]}\boldsymbol{\rho} = \mathcal{O}(\boldsymbol{\rho}/L)$*."*

Without being exactly the same, our definitions retain the idea of *inverse inequalities* for the membrane and bending strain tensors $\boldsymbol{\gamma}$ and $\boldsymbol{\rho}$. We introduce the three constants $L$, $\ell$ and $L_\flat$ in the next definitions.

**Definition 2.3** *For $\boldsymbol{z}$ the solution of problem $(P_{\mathsf{2D}})$ we denote by $\boldsymbol{\gamma} = \gamma_{\alpha\beta}(\boldsymbol{z})$ and $\boldsymbol{\rho} = \rho_{\alpha\beta}(\boldsymbol{z})$ the membrane and bending strain tensors associated with $\boldsymbol{z}$. Then we define $L$ as the* largest *constant such that there holds, for $k \leq 4$*

$$|\boldsymbol{\gamma}|_{k\,;\,S} \leq L^{-k}|\boldsymbol{\gamma}|_{0\,;\,S} \quad \mathit{and} \quad |\boldsymbol{\rho}|_{k\,;\,S} \leq L^{-k}|\boldsymbol{\rho}|_{0\,;\,S}\,. \tag{2.7}$$

Note that $L$ has the dimension of a length. This global wave length $L$ will be sufficient for an estimate like (1.9) between the 2D energy of $\boldsymbol{z}$ and its associated reconstructed 3D displacement $\mathsf{U}(\boldsymbol{z})$. Note that we have the estimate for $\ell \geq 0$

$$\|\boldsymbol{\gamma}\|^{(\mathsf{r})}_{\ell\,;\,S} \leq A\left(\sum_{j=0}^{\ell} L^{-j}\mathsf{r}^{j-\ell}\right)|\boldsymbol{\gamma}|_{0\,;\,S}\,,$$

and a similar estimate for $\boldsymbol{\rho}$, where $A$ is an adimensional constant. We now define a similar wave length but for the norms on the boundary $\partial S$.

**Definition 2.4** *With $\boldsymbol{\gamma}$ and $\boldsymbol{\rho}$ as in Definition* 2.3*, we define $\ell$ as the* largest *constant such that there holds*

$$|\boldsymbol{\gamma}|_{1\,;\,\partial S} \leq \ell^{-1}|\boldsymbol{\gamma}|_{0\,;\,\partial S} \quad \mathit{and} \quad |\boldsymbol{\rho}|_{1\,;\,\partial S} \leq \ell^{-1}|\boldsymbol{\rho}|_{0\,;\,\partial S}\,. \tag{2.8}$$

The third constant $L_\flat$ is involved in inverse estimates bounding norms on $\partial S$ by norms on $S$.

**Definition 2.5** *With $\boldsymbol{\gamma}$ and $\boldsymbol{\rho}$ as in Definition* 2.3*, we define $L_\flat$ as the* largest *constant such that there holds*

$$|\boldsymbol{\gamma}|_{0\,;\,\partial S} \leq L_\flat^{-1/2}|\boldsymbol{\gamma}|_{0\,;\,S} \quad \mathit{and} \quad |\boldsymbol{\rho}|_{0\,;\,\partial S} \leq L_\flat^{-1/2}|\boldsymbol{\rho}|_{0\,;\,S}\,. \tag{2.9}$$

The wave length $\ell$ and $\mathsf{L}$ have the dimension of a length. It is clear for $\ell$, and for $\mathsf{L}$ it is due to the fact that the boundary $\partial S$ is one dimensional while the $S$ is two-dimensional.

### 2.f Shifted displacement and reconstructed displacement

Let $\boldsymbol{u}$ the displacement solution of $(P_{\mathsf{3D}})$. We can express this displacement in *shifted* normal coordinates introduced by Naghdi (see [23]) and currently used in shell theory. It appears that computations are easier when considering the shifted components. That is why we briefly recall their definition.

Let $\boldsymbol{Y}$ be a vector field on $\Omega^\varepsilon$ (i.e. a contravariant tensor of order 1) and let $(x_\alpha, x_3)$ be a local normal coordinate system on the manifold $S \times (-\varepsilon, \varepsilon)$. Let $\mathbf{x}_i = (\mathbf{x}_\sigma, \mathbf{x}_3)(x_\alpha, x_3)$ be a local basis of the space of vector fields induced by the diffeomorphism between $\Omega^\varepsilon$ and $S \times (-\varepsilon, \varepsilon)$. In this basis, $\boldsymbol{Y}$ writes :

$$\boldsymbol{Y} = Y^\sigma(x_\alpha, x_3)\mathbf{x}_\sigma(x_\alpha, x_3) + Y^3(x_\alpha, x_3)\mathbf{x}_3(x_\alpha, x_3).$$

For $x_3 = 0$, the basis $(\mathbf{x}_i(x_\alpha, 0))$ consists simply of a local basis $\mathbf{x}_\sigma(x_\alpha, 0)$ on $U$ and of the normal vector field $\mathbf{x}_3(x_\alpha, 0) = \boldsymbol{n}(x_\alpha)$. However, as $U \times (-\varepsilon, \varepsilon)$ is embedded in $\mathbb{R}^3$, this basis extends by translation over the domain corresponding to $U \times (-\varepsilon, \varepsilon)$ in $\Omega^\varepsilon$. Hence, we can decompose $\boldsymbol{Y}$ in this basis as

$$\boldsymbol{Y} = \tilde{Y}^\sigma(x_\alpha, x_3)\mathbf{x}_\sigma(x_\alpha, 0) + \tilde{Y}^3(x_\alpha, x_3)\mathbf{x}_3(x_\alpha, 0).$$

Using the diffeomorphism (2.1), we can see that we have for all $x_3$

$$\mathbf{x}_3(x_\alpha, x_3) = \mathbf{x}_3(x_\alpha, 0) \quad \text{and} \quad \mathbf{x}_\sigma(x_\alpha, x_3) = \mathbf{x}_\sigma(x_\alpha, 0) - x_3 b_\sigma^\beta(x_\alpha)\mathbf{x}_\beta(x_\alpha, 0),$$

and this implies the relations

$$\tilde{Y}^3(x_\alpha, x_3) = Y^3(x_\alpha, x_3) \quad \text{and} \quad \tilde{Y}^\sigma(x_\alpha, x_3) = \mu^\sigma_\beta(x_\alpha, x_3) Y^\beta(x_\alpha, x_3),$$

where $\mu^\beta_\sigma$ is the *shifter* (see [23]) defined by

$$\mu^\beta_\sigma(x_\alpha, x_3) = \delta^\beta_\sigma - x_3 b^\beta_\sigma(x_\alpha).$$

Hence a vector field $\boldsymbol{Y}$ can be represented by its components $(Y^i)$ or $(\tilde{Y}^i)$ and the shifter appears as the jacobian of a change of coordinates. Similarly, a displacement (which is a covariant tensor of order 1) can be represented by its coordinates $(v_\sigma, v_3)$ along the basis induced by the diffeomorphism (2.1) or by its coordinates $(\tilde{v}_\sigma, \tilde{v}_3)$ along the coordinates associated with the fact that $S$ is included in $\mathbb{R}^3$, and we have the relations

$$\tilde{v}_3 = v_3 \quad \text{and} \quad \tilde{v}_\sigma = (\mu^{-1})^\beta_\sigma v_\beta,$$

where $(\mu^{-1})^\beta_\sigma$ is the inverse of the shifter.

As they will be of constant use, we will denote by $(w_\sigma, w_3)$ the shifted components of the displacement $\boldsymbol{u}$ solution of $(P_{\mathsf{3D}})$, instead of $(\tilde{u}_\sigma, \tilde{u}_3)$. We denote by $\boldsymbol{w}$ the corresponding shifted displacement.

Let $\boldsymbol{z} = (z_\sigma, z_3)$ be the solution of $(P_{\mathsf{2D}})$. To $\boldsymbol{z}$, we associate the three-dimensional shifted displacement $\mathsf{W}\boldsymbol{z}$ defined by the formula

$$\mathsf{W}\boldsymbol{z} = \begin{cases} z_\sigma - x_3\theta_\sigma(\boldsymbol{z}), \\ z_3 - x_3 p\gamma^\alpha_\alpha(\boldsymbol{z}) + \frac{x_3^2}{2} p\rho^\alpha_\alpha(\boldsymbol{z}), \end{cases} \tag{2.10}$$

where $\theta_\sigma(\boldsymbol{z}) = \mathrm{D}_\sigma z_3 + b^\alpha_\sigma z_\alpha$ and $p = \lambda(\lambda + 2\mu)^{-1}$. To this displacement $\mathsf{W}\boldsymbol{z}$ corresponds the displacement $\mathsf{U}\boldsymbol{z}$ in "unshifted" normal coordinates:

$$\mathsf{U}\boldsymbol{z} = \begin{cases} z_\sigma - x_3(\mathrm{D}_\sigma z_3 + 2b^\alpha_\sigma z_\alpha) + x_3^2 b^\alpha_\sigma \theta_\alpha(\boldsymbol{z}), \\ z_3 - x_3 p\gamma^\alpha_\alpha(\boldsymbol{z}) + \frac{x_3^2}{2} p\rho^\alpha_\alpha(\boldsymbol{z}), \end{cases} \tag{2.11}$$

## 3 A PRIORI ESTIMATES

We prove in this section estimates for the Sobolev norms (standard and anisotropic) of any $\boldsymbol{z} = (z_\sigma, z_3)$ where $(z_\sigma)$ is a 1-form on $S$ and $z_3$ a function on $S$, first by Sobolev norms of its strain tensors $\boldsymbol{\gamma} := \boldsymbol{\gamma}(\boldsymbol{z})$ and $\boldsymbol{\rho} := \boldsymbol{\rho}(\boldsymbol{z})$ and then, with the help of its wave length $L$, by its quadratic energy $\mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}]$, *cf* (2.4).

**Lemma 3.1** *There exists a positive adimensional constant $A$ such that*

$$\forall \boldsymbol{z} \in \mathbf{H}^1 \times \mathrm{L}^2(S), \qquad \|z_\sigma\|^{(\mathsf{r})}_{1\,;\,S} \;\leq\; A\big(|\boldsymbol{\gamma}|_{0\,;\,S} + \mathsf{r}^{-1}|\boldsymbol{z}|_{0\,;\,S}\big), \tag{3.1}$$

$$\forall \boldsymbol{z} \in \mathbf{H}^1 \times \mathrm{H}^2(S), \qquad \|z_3\|^{(\mathsf{r})}_{2\,;\,S} \;\leq\; A\big(|\boldsymbol{\rho}|_{0\,;\,S} + \mathsf{r}^{-1}\|\boldsymbol{z}\|^{(\mathsf{r})}_{1\,;\,S}\big). \tag{3.2}$$

**Proof.** In the following, $A$ always denotes an adimensional constant independent of $\boldsymbol{z}$. We clearly have

$$|\tfrac{1}{2}(\mathrm{D}_\alpha z_\beta + \mathrm{D}_\beta z_\alpha)|_{0\,;\,S} \leq A\big(|\boldsymbol{\gamma}|_{0\,;\,S} + \mathsf{r}^{-1}|\boldsymbol{z}|_{0\,;\,S}\big).$$

But, by using the standard Korn inequalities on each map of an atlas over $S$, and patching them together, we obtain the estimate for $(z_\sigma) \in \mathbf{H}^1(S)$:

$$|z_\sigma|_{1\,;\,S} \le A\big(|\tfrac{1}{2}(\mathrm{D}_\alpha z_\beta + \mathrm{D}_\beta z_\alpha)|_{0\,;\,S} + \mathsf{r}^{-1}|z_\sigma|_{0\,;\,S}\big),$$

provided we have taken the maps of the atlas sufficiently small so that the Christoffel symbols are smaller than $\mathsf{r}^{-1}$. The two previous inequalities give (3.1).

For (3.2), this estimate is an easy consequence of the inequality

$$|\mathrm{D}_\alpha \mathrm{D}_\beta z_3|_{0\,;\,S} \le A\big(|\boldsymbol{\rho}|_{0\,;\,S} + \mathsf{r}^{-1}\|\boldsymbol{z}\|^{(\mathsf{r})}_{1\,;\,S}\big).$$

■

From the previous estimates, we are going to deduce bounds for $\|z_\sigma\|^{(\mathsf{r})}_{\ell\,;\,S}$ and $\|z_3\|^{(\mathsf{r})}_{\ell\,;\,S}$ by induction over $\ell$.

If $\boldsymbol{z} \in \mathbf{H}^2 \times \mathrm{H}^1(S)$, we apply (3.1) to the first order derivatives $\mathrm{D}_\delta z_\sigma$ of $z_\sigma$ and obtain in particular

$$|\mathrm{D}^{[1]}\mathrm{D}_\delta z_\sigma|_{0\,;\,S} \le A\big(|\boldsymbol{\gamma}(\mathrm{D}_\delta \boldsymbol{z})|_{0\,;\,S} + \mathsf{r}^{-1}|\mathrm{D}_\delta \boldsymbol{z}|_{0\,;\,S}\big).$$

The commutator of $D_\delta$ and $\boldsymbol{\gamma}$ is a partial differential operator of order 1 which is $\mathsf{r}$-homogeneous of order 2: this is due to the fact that the commutator of two covariant derivative $\mathrm{D}_\alpha\mathrm{D}_\beta - \mathrm{D}_\beta\mathrm{D}_\alpha$ depends only on the Gaussian curvature on $S$, which is of order less than $\mathsf{r}^{-2}$. We deduce that

$$|\mathrm{D}^{[1]}\mathrm{D}_\delta z_\sigma|_{0\,;\,S} \le A\big(|\mathrm{D}_\delta \boldsymbol{\gamma}|_{0\,;\,S} + \mathsf{r}^{-1}\|\boldsymbol{z}\|^{(\mathsf{r})}_{1\,;\,S}\big).$$

Whence the estimate

$$\|z_\sigma\|^{(\mathsf{r})}_{2\,;\,S} \le A\big(|\boldsymbol{\gamma}|_{1\,;\,S} + \mathsf{r}^{-1}\|\boldsymbol{z}\|^{(\mathsf{r})}_{1\,;\,S}\big). \tag{3.3}$$

Then, applying the estimates (3.2) and (3.3) to $\mathrm{D}_\delta \boldsymbol{z}$ and using induction, we find the estimate for any $j \ge 0$:

$$\|z_\sigma\|^{(\mathsf{r})}_{j+3\,;\,S} \le A\big(|\boldsymbol{\gamma}|_{j+2\,;\,S} + \mathsf{r}^{-1}|\boldsymbol{\rho}|_{j\,;\,S} + \mathsf{r}^{-1}\|\boldsymbol{z}\|^{(\mathsf{r})}_{j+2\,;\,S}\big) \tag{3.4}$$

and

$$\|z_3\|^{(\mathsf{r})}_{j+2\,;\,S} \le A\big(|\boldsymbol{\rho}|_{j\,;\,S} + \mathsf{r}^{-1}|\boldsymbol{\gamma}|_{j\,;\,S} + \mathsf{r}^{-1}\|\boldsymbol{z}\|^{(\mathsf{r})}_{j+1\,;\,S}\big) \tag{3.5}$$

Combining (3.4) and (3.5) for $j = k,\ \ell-1, \ldots, 0$ we obtain for all $\boldsymbol{z} \in \mathbf{H}^{\ell+3} \times \mathrm{H}^{\ell+2}(S)^3$

$$\|z_\sigma\|^{(\mathsf{r})}_{\ell+3\,;\,S} \le A\big(\|\boldsymbol{\gamma}\|^{(\mathsf{r})}_{\ell+2\,;\,S} + \mathsf{r}^{-1}\|\boldsymbol{\rho}\|^{(\mathsf{r})}_{\ell\,;\,S} + \mathsf{r}^{-\ell-2}\|\boldsymbol{z}\|^{(\mathsf{r})}_{1\,;\,S}\big). \tag{3.6}$$

and

$$\|z_3\|^{(\mathsf{r})}_{\ell+2\,;\,S} \le A\big(\|\boldsymbol{\rho}\|^{(\mathsf{r})}_{\ell\,;\,S} + \mathsf{r}^{-1}\|\boldsymbol{\gamma}\|^{(\mathsf{r})}_{\ell\,;\,S} + \mathsf{r}^{-\ell-1}\|\boldsymbol{z}\|^{(\mathsf{r})}_{1\,;\,S}\big). \tag{3.7}$$

We can eliminate the norms $\|\boldsymbol{z}\|^{(\mathsf{r})}_{1\,;\,S}$ with the coercivity estimate, see [1]. Indeed we can prove that

$$\forall \boldsymbol{z} \in \mathbf{H}^1_0 \times \mathrm{H}^2_0(S), \quad \mathsf{r}^{-1}\|z_\sigma\|^{(\mathsf{r})}_{1\,;\,S} + \|z_3\|^{(\mathsf{r})}_{2\,;\,S} \le A\big(\mathsf{r}^{-1}|\boldsymbol{\gamma}|_{0\,;\,S} + |\boldsymbol{\rho}|_{0\,;\,S}\big) \tag{3.8}$$

and finally obtain the following result

**Proposition 3.2** *For any $\ell \ge 0$, there exists a positive adimensional constant $A$ such that for any $\boldsymbol{z} \in \mathbf{H}^1_0 \times \mathrm{H}^2_0(S)$ with sufficient regularity there holds:*

$$\|z_\sigma\|^{(\mathsf{r})}_{\ell+3\,;\,S} \le A\big(\|\boldsymbol{\gamma}\|^{(\mathsf{r})}_{\ell+2\,;\,S} + \mathsf{r}^{-1}\|\boldsymbol{\rho}\|^{(\mathsf{r})}_{\ell\,;\,S}\big). \tag{3.9}$$

*and*

$$\|z_3\|^{(\mathsf{r})}_{\ell+2\,;\,S} \le A\big(\|\boldsymbol{\rho}\|^{(\mathsf{r})}_{\ell\,;\,S} + \mathsf{r}^{-1}\|\boldsymbol{\gamma}\|^{(\mathsf{r})}_{\ell\,;\,S}\big). \tag{3.10}$$

Using the relation between $\lambda$, $\mu$ and the Young modulus $E$ we see that there exist adimensional constants $a$ and $A$ such that

$$aE \le \lambda \le AE \quad \text{and} \quad aE \le \mu \le AE.$$

As $E > 0$, the definition (2.4) then yields that

$$|\boldsymbol{\gamma}|^2_{0\,;\,S} \le AE^{-1}\varepsilon^{-1}\mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}] \quad \text{and} \quad |\boldsymbol{\rho}|^2_{0\,;\,S} \le AE^{-1}\varepsilon^{-3}\mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}]. \tag{3.11}$$

Proposition 3.2 combined with (3.11) yields the following energy estimates for the solution $\boldsymbol{z}$ of problem $(P_{\mathsf{2D}})$:

**Theorem 3.3** *Let $\boldsymbol{z}$ be the solution of problem $(P_{\mathsf{2D}})$. For any $j \le 2$ and $\ell \le 5$, there exists a positive adimensional constant $A$ such that there holds for any $\varepsilon > 0$*

$$\big(\|z_\sigma\|^{(\mathsf{r})}_{\ell+3\,;\,S}\big)^2 \le AE^{-1}\varepsilon^{-3}L^{-2\ell}\Big(\mathsf{r}^{-2} + \frac{\varepsilon^2}{L^4}\Big)\Big(\sum_{j=0}^{\ell} L^{2j}\mathsf{r}^{-2j}\Big)\mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}]\,, \tag{3.12}$$

*and*

$$\big(\|z_3\|^{(\mathsf{r})}_{\ell+2\,;\,S}\big)^2 \le AE^{-1}\varepsilon^{-3}L^{-2\ell}\Big(1 + \frac{\varepsilon^2}{L^2}\Big)\Big(\sum_{j=0}^{\ell} L^{2j}\mathsf{r}^{-2j}\Big)\mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}]\,, \tag{3.13}$$

*where $E$ is the Young modulus, $L$ is the global wave length of $\boldsymbol{z}$ and $\mathsf{r}$ the constant estimating the curvature,* cf *Definitions* 2.3 *and* 2.1.

## 4 Korn inequalities

We give now Korn inequalities on $\Omega^\varepsilon$ with physical constants. We introduce the constant $\mathsf{d}$ having the dimension of a lentgh and appearing in the Korn inequalities.

Recall that if $\boldsymbol{v}$ is a 1-form field in $\Omega^\varepsilon$, it can be split into two parts: a surfacic 1-form field $v_\alpha$ and a function $v_3$ on $S$, both depending on $x_3$. Thus the derivative $\partial_3$ with respect to $x_3$ and the covariant derivative $\mathrm{D}_\alpha$ act on $\boldsymbol{v}$.

**Proposition 4.1** *There exists a constant $\mathsf{d}$ independent on $\varepsilon$, having the dimension of a length, such that for all $\boldsymbol{v} \in \mathbf{H}^1(\Omega^\varepsilon)$ satisfying the boundary condition $\boldsymbol{v}\big|_{\Gamma_0^\varepsilon} = 0$, we have*

$$\begin{aligned} \|\boldsymbol{v}\|^2_{\mathbf{L}^2(\Omega^\varepsilon)} &\le \mathsf{d}^4E^{-1}\varepsilon^{-2}\mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{v}] \\ \|\mathrm{D}_\alpha\boldsymbol{v}\|^2_{\mathbf{L}^2(\Omega^\varepsilon)} &\le \mathsf{d}^2E^{-1}\varepsilon^{-2}\mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{v}] \\ \|\partial_3\boldsymbol{v}\|^2_{\mathbf{L}^2(\Omega^\varepsilon)} &\le \mathsf{d}^2E^{-1}\varepsilon^{-2}\mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{v}]. \end{aligned} \tag{4.1}$$

**Proof.** We make the scaling $X_3 = \varepsilon^{-1}x_3$ mapping the manifold $S \times (-\varepsilon, \varepsilon)$ to the manifold $\Omega := S \times (-1, 1)$. In this variable, we write $e_{ij}(\varepsilon)(\boldsymbol{v})$ the deformation tensor image of $e_{ij}(\boldsymbol{v})$ by the scaling. We have for example $e_{33}(\varepsilon)(\boldsymbol{v}) = \varepsilon^{-1}\partial_{X_3}v_3$. Note that the variable $X_3$ is adimensional, and that the tensor $e_{ij}(\varepsilon)(\boldsymbol{v})$ is also adimensional.

The lateral boundary $\Gamma_0 = \partial S \times (-1, 1)$ is the image of $\Gamma_0^\varepsilon$ by the scaling. We then define the space $V(\Omega) = \{\boldsymbol{v} \in \mathbf{H}^1(\Omega) \quad | \quad \boldsymbol{v}\big|_{\Gamma_0} = 0\}$. 0n the manifold $\Omega$, we have the following inequalities (see [5]): for all $\boldsymbol{v} \in V(\Omega)$ we have

$$\begin{aligned} \|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)} &\leq C_1\varepsilon^{-1}\|e_{ij}(\varepsilon)(\boldsymbol{v})\|_{\mathbf{L}^2(\Omega)} \\ \|\mathrm{D}_\alpha\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)} &\leq C_2\varepsilon^{-1}\|e_{ij}(\varepsilon)(\boldsymbol{v})\|_{\mathbf{L}^2(\Omega)} \\ \|\partial_{X_3}\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)} &\leq C_3\varepsilon^{-1}\|e_{ij}(\varepsilon)(\boldsymbol{v})\|_{\mathbf{L}^2(\Omega)}. \end{aligned}$$

Using the fact that

$$\partial_{X_3}v_i = \varepsilon e_{i3}(\varepsilon)(\boldsymbol{v}) + \mathcal{O}(\varepsilon\|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)}) + \mathcal{O}(\varepsilon\|\mathrm{D}_\alpha\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)}),$$

we see that the last equation can be replaced by

$$\|\partial_{X_3}\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)} \leq C_4\|e_{ij}(\varepsilon)(\boldsymbol{v})\|_{\mathbf{L}^2(\Omega)}.$$

We get the result after a change of coordinate, taking into account the dimensions of the norms in the estimates. ■

## 5 Energy of the reconstructed displacement

Let $\boldsymbol{z} = (z_\sigma, z_3) \in \mathbf{H}^2 \times \mathrm{H}^3(S)$ where $(z_\sigma)$ is a 1-form on $S$ and $z_3$ a function on $S$. The aim of this section is the comparison between the two-dimensional energy $\mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}]$ of $\boldsymbol{z}$ and the three-dimensional energy $\mathsf{E}^\varepsilon_{\mathsf{3D}}[\mathsf{U}\boldsymbol{z}]$ of the associated reconstructed displacement $\mathsf{U}\boldsymbol{z}$ given by (2.11):

$$\mathsf{U}\boldsymbol{z} = \begin{cases} z_\sigma - x_3(\mathrm{D}_\sigma z_3 + 2b^\alpha_\sigma z_\alpha) + x_3^2 b^\alpha_\sigma \theta_\alpha(\boldsymbol{z}), \\ z_3 - x_3 p\gamma^\alpha_\alpha(\boldsymbol{z}) + \frac{x_3^2}{2} p\rho^\alpha_\alpha(\boldsymbol{z}), \end{cases}$$

In the following, $\frac{1}{R}$ denotes the maximum principal curvature of $S$, which means that we have $|b^\beta_\alpha| \leq \frac{1}{R}$ everywhere on $S$. Of course, if $S$ is a plane domain (when the shell is a plate) we have $\frac{1}{R} = 0$.

The main result of this section is:

**Theorem 5.1** *For all $\boldsymbol{z} \in \mathbf{H}^2 \times \mathrm{H}^3(S)$, we have the following estimate*

$$\big|\mathsf{E}^\varepsilon_{\mathsf{3D}}[\mathsf{U}\boldsymbol{z}] - \mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}]\big| \leq A\Big(\frac{\varepsilon}{R} + \frac{\varepsilon^2}{L^2}\Big)\,\mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}], \tag{5.1}$$

*for an adimensional constant $A$, where $L$ is the wave length for $\boldsymbol{z}$ defined according to Definition 2.3.*

**Proof.** The proof is organized in three steps.

Step 1. The proof is easier when using the shifted displacement $\mathsf{W}\boldsymbol{z}$, *see* (2.10), corresponding to the reconstructed displacement $\mathsf{U}\boldsymbol{z}$. For any three-dimensional displacement $\boldsymbol{u}$, we recall that $\mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{u}]$ denotes its quadratic energy, *cf* (2.2). If $\boldsymbol{w}$ is the shifted displacement associated with $\boldsymbol{u}$ we denote the corresponding energy by $\widetilde{\mathsf{E}}^\varepsilon_{\mathsf{3D}}[\boldsymbol{w}]$ which is defined so that $\widetilde{\mathsf{E}}^\varepsilon_{\mathsf{3D}}[\boldsymbol{w}] = \mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{u}]$. Hence we have

$$\widetilde{\mathsf{E}}^\varepsilon_{\mathsf{3D}}[\boldsymbol{w}] = \int_{\Omega^\varepsilon} A^{ijk\ell}\tilde{e}_{ij}(\boldsymbol{w})\tilde{e}_{k\ell}(\boldsymbol{w})\,\mathrm{d}V,$$

where the modified strain $\tilde{e}_{ij}(\boldsymbol{w})$ is defined so that $\tilde{e}_{ij}(\boldsymbol{w}) = e_{ij}(\boldsymbol{u})$. In normal coordinates we have the following expressions for the tensor $\tilde{e}_{ij}(\boldsymbol{w})$, see [12]:

$$\tilde{e}^3_3(\boldsymbol{w}) = \partial_{x_3} w_3, \tag{5.2}$$

$$\tilde{e}^3_\beta(\boldsymbol{w}) = \tfrac{1}{2}\big(\partial_{x_3} w_\beta - x_3 b^\alpha_\beta \partial_{x_3} w_\alpha + \theta_\beta(\boldsymbol{w})\big), \tag{5.3}$$

$$\tilde{e}^\alpha_\beta(\boldsymbol{w}) = \gamma^\alpha_\beta(\boldsymbol{w}) + \sum_{n=1}^{\infty} x_3^n (b^n)^\alpha_\delta \gamma^\delta_\beta(\boldsymbol{w}) + \sum_{n=1}^{\infty} n x_3^n (b^{n-1})^\alpha_\delta \Lambda^{\delta}{}_{\cdot\beta}(\boldsymbol{w}), \tag{5.4}$$

where $\theta_\beta(\boldsymbol{z}) = \mathrm{D}_\beta z_3 + b^\alpha_\beta z_\alpha$ and $\Lambda_{\alpha\beta}(\boldsymbol{z}) = \frac{1}{2}(b^\sigma_\alpha \mathrm{D}_\sigma z_\beta - b^\sigma_\beta \mathrm{D}_\alpha z_\sigma)$.

Using the definition of the rigidity matrix, we obtain

$$\widetilde{\mathsf{E}}^\varepsilon_{3\mathsf{D}}[\boldsymbol{w}] = \int_{\Omega^\varepsilon} \Big[(\lambda + 2\mu)\tilde{e}^3_3(\boldsymbol{w})\tilde{e}^3_3(\boldsymbol{w}) + 2\lambda \tilde{e}^3_3(\boldsymbol{w})\tilde{e}^\alpha_\alpha(\boldsymbol{w}) + \lambda \tilde{e}^\alpha_\alpha(\boldsymbol{w})\tilde{e}^\beta_\beta(\boldsymbol{w}) + 4\mu a^{\alpha\beta}(x_3)\tilde{e}^3_\alpha(\boldsymbol{w})\tilde{e}^3_\beta(\boldsymbol{w}) + 2\mu \tilde{e}^\beta_\alpha(\boldsymbol{w})\tilde{e}^\alpha_\beta(\boldsymbol{w})\Big]\mathrm{d}V, \tag{5.5}$$

where $a^{\alpha\beta}(x_3)$ is the inverse of the metric tensor on the surface at the level $x_3$ in the shell.

Thus, we reduce the proof to showing $\big|\tilde{\mathsf{E}}^\varepsilon_{3\mathsf{D}}[\mathsf{W}\boldsymbol{z}] - \mathsf{E}^\varepsilon_{2\mathsf{D}}[\boldsymbol{z}]\big|$ is less than the right hand side (5.1). We note that $\mathsf{E}^\varepsilon_{2\mathsf{D}}[\boldsymbol{z}]$ is associated with the material law of Lamé coefficients $2\mu p$ and $\mu$ and writes

$$\mathsf{E}^\varepsilon_{2\mathsf{D}}[\boldsymbol{z}] = 2\varepsilon \int_S \Big[2\mu p \gamma^\alpha_\alpha(\boldsymbol{z})\gamma^\beta_\beta(\boldsymbol{z}) + 2\mu \gamma^\beta_\alpha(\boldsymbol{z})\gamma^\alpha_\beta(\boldsymbol{z})\Big]\mathrm{d}S + \frac{2}{3}\varepsilon^3 \int_S \Big[2\mu p \rho^\alpha_\alpha(\boldsymbol{z})\rho^\beta_\beta(\boldsymbol{z}) + 2\mu \rho^\beta_\alpha(\boldsymbol{z})\rho^\alpha_\beta(\boldsymbol{z})\Big]\mathrm{d}S, \tag{5.6}$$

STEP 2. We are going to evaluate $\mathsf{E}^\varepsilon_{3\mathsf{D}}[\mathsf{W}\boldsymbol{z}]$ with the help of the following splitting of $\mathsf{W}\boldsymbol{z}$ into the sum of a displacement of Kirchhoff-Love type $\mathsf{W}^{\mathsf{KL}}\boldsymbol{z}$ and of a complementary term $\mathsf{W}^{\mathsf{cmp}}\boldsymbol{z}$ which is a transverse quadratic displacement:

$$\mathsf{W}^{\mathsf{KL}}\boldsymbol{z} = \begin{cases} z_\sigma - x_3\theta_\sigma(\boldsymbol{z}), \\ z_3, \end{cases} \quad \text{and} \quad \mathsf{W}^{\mathsf{cmp}}\boldsymbol{z} = \begin{cases} 0, \\ -x_3 p \gamma^\alpha_\alpha(\boldsymbol{z}) + \frac{x_3^2}{2} p \rho^\alpha_\alpha(\boldsymbol{z}). \end{cases}$$

The following Lemma 5.2 justifies the denomination of the first part $\mathsf{W}^{\mathsf{KL}}\boldsymbol{z}$:

**Lemma 5.2** *For all $\boldsymbol{z} \in \mathbf{H}^2 \times \mathrm{H}^3(S)$, we have:*

$$\tilde{e}^3_i(\mathsf{W}^{\mathsf{KL}}\boldsymbol{z}) = 0.$$

**Proof.** It is clear that $\tilde{e}^3_3(\mathsf{W}^{\mathsf{KL}}\boldsymbol{z}) = \partial_{x_3} z_3 = 0$. Using equality (5.3), we calculate

$$2\tilde{e}^3_\sigma(\mathsf{W}^{\mathsf{KL}}\boldsymbol{z}) = -\theta_\sigma(\boldsymbol{z}) + x_3 b^\alpha_\sigma \theta_\alpha(\boldsymbol{z}) + \theta_\sigma(\boldsymbol{z}) - x_3 b^\alpha_\sigma \theta_\alpha(\boldsymbol{z}) = 0.$$

Hence we have obtained the lemma. ■

**Lemma 5.3** *We have the following relations:*

$$\tilde{e}^3_3(\mathsf{W}^{\mathsf{cmp}}\boldsymbol{z}) = -p\gamma^\alpha_\alpha(\boldsymbol{z}) + p x_3 \rho^\alpha_\alpha(\boldsymbol{z}) \tag{5.7}$$

$$2\tilde{e}^3_\sigma(\mathsf{W}^{\mathsf{cmp}}\boldsymbol{z}) = -x_3 p \mathrm{D}_\sigma \gamma^\alpha_\alpha(\boldsymbol{z}) + \frac{x_3^2}{2} p \mathrm{D}_\sigma \rho^\alpha_\alpha(\boldsymbol{z}). \tag{5.8}$$

**Proof.** The equation (5.7) is clear. The expression (5.3) of the operator $\tilde{e}^3_\sigma(\boldsymbol{w})$ yields (5.8). ■

**Lemma 5.4** *We have*

$$\tilde{e}^\alpha_\sigma(\mathsf{W}^{\mathsf{KL}}\boldsymbol{z}) = \gamma^\alpha_\sigma(\boldsymbol{z}) - x_3\big(\rho^\alpha_\sigma(\boldsymbol{z}) - 2b^\alpha_\delta\gamma^\delta_\sigma(\boldsymbol{z})\big) + \sum_{n=1}^\infty x_3^{1+n}(P_n^{\mathsf{KL}})^\alpha_\sigma(\boldsymbol{z}),$$

*and*

$$\tilde{e}^\alpha_\sigma(\mathsf{W}^{\mathsf{cmp}}\boldsymbol{z}) = x_3 p b^\alpha_\sigma\gamma^\delta_\delta(\boldsymbol{z}) + \sum_{n=1}^\infty x_3^{1+n}(P_n^{\mathsf{cmp}})^\alpha_\sigma(\boldsymbol{z})$$

*where the tensors $(P_n^{\mathsf{KL}})(\boldsymbol{z})$ and $(P_n^{\mathsf{cmp}})(\boldsymbol{z})$ satisfy the estimates for all $n \geq 1$*

$$|(P_n^{\mathsf{KL}})(\boldsymbol{z})|_{0\,;\,S} + |(P_n^{\mathsf{cmp}})(\boldsymbol{z})|_{0\,;\,S} \leq \frac{An}{R^n}\big(|\boldsymbol{\rho}(\boldsymbol{z})|_{0\,;\,S} + \frac{1}{R}|\boldsymbol{\gamma}(\boldsymbol{z})|_{0\,;\,S}\big), \tag{5.9}$$

*for an adimensional constant $A$.*

**Proof.** With expression (5.4) we compute that

$$\tilde{e}^\alpha_\sigma(\mathsf{W}^{\mathsf{KL}}\boldsymbol{z}) = \gamma^\alpha_\sigma(\boldsymbol{z}) + x_3 b^\alpha_\delta\gamma^\delta_\sigma(\boldsymbol{z}) + x_3\Lambda^{\alpha\,\cdot}_{\cdot\,\sigma}(\boldsymbol{z}) - x_3\overline{\rho}^\alpha_\sigma + \sum_{n=1}^\infty x_3^{1+n}(P_n^{\mathsf{KL}})^\alpha_\sigma(\boldsymbol{z}),$$

where the tensors $(P_n^{\mathsf{KL}})(\boldsymbol{z})$ satisfy estimate (5.9) and where $\overline{\rho}_{\alpha\beta} = \frac{1}{2}(\mathrm{D}_\alpha\theta_\beta + \mathrm{D}_\beta\theta_\alpha)$. But we have

$$\overline{\rho}^\alpha_\beta - \Lambda^{\alpha\,\cdot}_{\cdot\,\beta} = \rho^\alpha_\beta - b^\alpha_\sigma\gamma^\sigma_\beta.$$

Thus we obtain the estimate

$$|(P_n^{\mathsf{KL}})(\boldsymbol{z})|_{0\,;\,S} + |(P_n^{\mathsf{cmp}})(\boldsymbol{z})|_{0\,;\,S} \leq \frac{An}{R^n}\big(|z_3|_{2\,;\,S} + \frac{1}{R}|\boldsymbol{z}|_{1\,;\,S} + \frac{1}{R^2}|\boldsymbol{z}|_{0\,;\,S}\big).$$

But as for (3.8) we can prove that

$$|z_3|_{2\,;\,S} + \frac{1}{R}|\boldsymbol{z}|_{1\,;\,S} + \frac{1}{R^2}|\boldsymbol{z}|_{0\,;\,S} \leq A\big(|\boldsymbol{\rho}(\boldsymbol{z})|_{0\,;\,S} + \frac{1}{R}|\boldsymbol{\gamma}(\boldsymbol{z})|_{0\,;\,S}\big)$$

and thus we get the first formula. The proof of the second formula is similar. ∎

Step 3. Gathering the previous results and setting $(P_n)(\boldsymbol{z}) = (P_n^{\mathsf{KL}})(\boldsymbol{z}) + (P_n^{\mathsf{cmp}})(\boldsymbol{z})$, we find that

$$\begin{aligned}
\tilde{e}^3_3(\mathsf{W}\boldsymbol{z}) &= -p\gamma^\alpha_\alpha(\boldsymbol{z}) + px_3\rho^\alpha_\alpha(\boldsymbol{z}),\\
\tilde{e}^3_\sigma(\mathsf{W}\boldsymbol{z}) &= -\tfrac{x_3}{2}p\mathrm{D}_\sigma\gamma^\alpha_\alpha(\boldsymbol{z}) + \tfrac{x_3^2}{4}p\mathrm{D}_\sigma\rho^\alpha_\alpha(\boldsymbol{z}),\\
\tilde{e}^\alpha_\sigma(\mathsf{W}\boldsymbol{z}) &= \gamma^\alpha_\sigma(\boldsymbol{z}) - x_3\big(\rho^\alpha_\sigma(\boldsymbol{z}) - pb^\alpha_\sigma\gamma^\delta_\delta(\boldsymbol{z}) - 2b^\alpha_\delta\gamma^\delta_\sigma(\boldsymbol{z})\big) + \textstyle\sum_{n=1}^\infty x_3^{1+n}(P_n)^\alpha_\sigma(\boldsymbol{z})
\end{aligned}$$

where $(P_n)(\boldsymbol{z})$ satisfies the estimate (5.9).

We compute now the different contributions in the integral (5.5). The previous computations yield a convergent series expansion of each term in powers of $x_3$. Therefore each contribution in the integral (5.5) has also a convergent series expansion in powers of $x_3$. When integrating with respect to $x_3$ from $-\varepsilon$ to $\varepsilon$, the odd powers of $x_3$ have no contribution. Based on this remark we immediately obtain, first:

$$\int_{\Omega^\varepsilon}(\lambda+2\mu)\tilde{e}^3_3(\mathsf{W}\boldsymbol{z})\tilde{e}^3_3(\mathsf{W}\boldsymbol{z})\,\mathrm{d}V = 2\varepsilon(\lambda+2\mu)p^2\int_S\gamma^\alpha_\alpha(\boldsymbol{z})\gamma^\beta_\beta(\boldsymbol{z})\,\mathrm{d}S$$
$$+\,\varepsilon^3\tfrac{2}{3}(\lambda+2\mu)p^2\int_S\rho^\alpha_\alpha(\boldsymbol{z})\rho^\beta_\beta(\boldsymbol{z})\,\mathrm{d}S.$$

And then :

$$\int_{\Omega^\varepsilon} 2\lambda \tilde{e}^3_3(\mathsf{W}\boldsymbol{z})\tilde{e}^\alpha_\alpha(\mathsf{W}\boldsymbol{z})\,\mathrm{d}V = -4\varepsilon\lambda p \int_S \gamma^\alpha_\alpha(\boldsymbol{z})\gamma^\beta_\beta(\boldsymbol{z})\,\mathrm{d}S \\ - \varepsilon^3 \tfrac{4}{3}\lambda p \int_S \rho^\alpha_\alpha(\boldsymbol{z})\rho^\beta_\beta(\boldsymbol{z})\,\mathrm{d}S + Q_1(\varepsilon,\boldsymbol{z})$$

where

$$Q_1(\varepsilon,\boldsymbol{z}) = \int_{\Omega^\varepsilon} \Big(2x_3^2\lambda p \rho^\alpha_\alpha(\boldsymbol{z})\big(pb^\nu_\nu\gamma^\delta_\delta(\boldsymbol{z}) + 2b^\nu_\delta\gamma^\delta_\nu(\boldsymbol{z})\big) \\ - 2\lambda x_3^2 p\gamma^\alpha_\alpha(\boldsymbol{z})(P_1)^\nu_\nu(\boldsymbol{z}) + 2\lambda x_3^4 p\rho^\alpha_\alpha(\boldsymbol{z})(P_2)^\nu_\nu(\boldsymbol{z}) + \text{h.o.t.}\Big)\,\mathrm{d}V.$$

Hence using (5.9) we see that $Q_1(\varepsilon,\boldsymbol{z})$ satisfies:

$$|Q_1(\varepsilon,\boldsymbol{z})| \leq AE\Big(\frac{\varepsilon^3}{R^2}|\boldsymbol{\gamma}|^2_{0\,;\,S} + \frac{\varepsilon^3}{R}|\boldsymbol{\gamma}|_{0\,;\,S}\,|\boldsymbol{\rho}|_{0\,;\,S} + \frac{\varepsilon^5}{R^2}|\boldsymbol{\rho}|^2_{0\,;\,S} + \frac{\varepsilon^5}{R^4}|\boldsymbol{\gamma}|^2_{0\,;\,S}\Big),$$

where we used the fact that $\varepsilon R^{-1} < 1$. As we have

$$\frac{\varepsilon^3}{R}|\boldsymbol{\gamma}|_{0\,;\,S}\,|\boldsymbol{\rho}|_{0\,;\,S} \leq A\Big(\frac{\varepsilon^2}{R}|\boldsymbol{\gamma}|^2_{0\,;\,S} + \frac{\varepsilon^4}{R}|\boldsymbol{\rho}|^2_{0\,;\,S}\Big)$$

we get using (3.11)

$$|Q_1(\varepsilon,\boldsymbol{z})| \leq A\frac{\varepsilon}{R}\mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}].$$

Similarly we compute that :

$$\int_{\Omega^\varepsilon} \lambda \tilde{e}^\alpha_\alpha(\mathsf{W}\boldsymbol{z})\tilde{e}^\beta_\beta(\mathsf{W}\boldsymbol{z})\,\mathrm{d}V = 2\varepsilon\lambda \int_S \gamma^\alpha_\alpha(\boldsymbol{z})\gamma^\beta_\beta(\boldsymbol{z})\,\mathrm{d}S \\ + \varepsilon^3 \tfrac{2}{3}\lambda \int_S \rho^\alpha_\alpha(\boldsymbol{z})\rho^\beta_\beta(\boldsymbol{z})\,\mathrm{d}S + Q_2(\varepsilon,\boldsymbol{z})$$

where again we have

$$|Q_2(\varepsilon,\boldsymbol{z})| \leq A\frac{\varepsilon}{R}\mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}].$$

We also have

$$\int_{\Omega^\varepsilon} 4\mu a^{\alpha\beta}(x_3)\tilde{e}^3_\alpha(\mathsf{W}\boldsymbol{z})\tilde{e}^3_\beta(\mathsf{W}\boldsymbol{z})\,\mathrm{d}V = Q_3(\varepsilon,\boldsymbol{z}),$$

with :

$$|Q_3(\varepsilon,\boldsymbol{z})| \leq AE\Big(\varepsilon^3|\boldsymbol{\gamma}|^2_{1\,;\,S} + \varepsilon^5|\boldsymbol{\rho}|^2_{1\,;\,S}\Big)$$

and thus using the definition (2.7) of $L$ and the estimates (3.11),

$$|Q_3(\varepsilon,\boldsymbol{z})| \leq AE\frac{\varepsilon^2}{L^2}\mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}].$$

Finally, we have :

$$\int_{\Omega^\varepsilon} 2\mu \tilde{e}^\alpha_\beta(\mathsf{W}\boldsymbol{z})\tilde{e}^\beta_\alpha(\mathsf{W}\boldsymbol{z})\,\mathrm{d}V \\ = 4\mu\varepsilon \int_S \gamma^\alpha_\beta(\boldsymbol{z})\gamma^\beta_\alpha(\boldsymbol{z})\,\mathrm{d}S + \frac{4\mu}{3}\,\varepsilon^3 \int_S \rho^\alpha_\beta(\boldsymbol{z})\rho^\beta_\alpha(\boldsymbol{z})\,\mathrm{d}S + Q_4(\varepsilon,\boldsymbol{z})$$

where again we have the estimate :

$$|Q_4(\varepsilon,\boldsymbol{z})| \leq A\frac{\varepsilon}{R}\mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}].$$

Finally, using the relation :

$$\lambda - 2\lambda p + p^2(\lambda + 2\mu) = 2\mu p,$$

we find that

$$\tilde{E}^{\varepsilon}_{\mathsf{3D}}[\mathsf{W}\boldsymbol{z}] = 2\varepsilon \int_S \Big[2\mu p \gamma^\alpha_\alpha(\boldsymbol{z})\gamma^\beta_\beta(\boldsymbol{z}) + 2\mu\gamma^\beta_\alpha(\boldsymbol{z})\gamma^\alpha_\beta(\boldsymbol{z})\Big]\,\mathrm{d}S$$

$$+ \frac{2}{3}\,\varepsilon^3 \int_S \Big[2\mu p \rho^\alpha_\alpha(\boldsymbol{z})\rho^\beta_\beta(\boldsymbol{z}) + 2\mu\rho^\beta_\alpha(\boldsymbol{z})\rho^\alpha_\beta(\boldsymbol{z})\Big]\,\mathrm{d}S + Q_0(\varepsilon,\boldsymbol{z}) \quad (5.10)$$

where $Q_0(\varepsilon,\boldsymbol{z})$ is the sum $\sum_{\ell=1}^4 Q_\ell(\varepsilon,\boldsymbol{z})$, and thus

$$|Q_0(\varepsilon,\boldsymbol{z})| \le A\Big(\frac{\varepsilon}{R} + \frac{\varepsilon^2}{L^2}\Big)\mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}].$$

But, compared with (5.6), the right-hand side of (5.10) writes

$$\mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}] + Q_0(\varepsilon,\boldsymbol{z}).$$

Hence we have

$$\tilde{\mathsf{E}}^\varepsilon_{\mathsf{3D}}[\mathsf{W}\boldsymbol{z}] - \mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}] = Q_0(\varepsilon,\boldsymbol{z}),$$

and this yields the result. ■

**Remark 5.5** The part $\mathsf{U}^{\mathsf{cmp}}\boldsymbol{z}$ has a significative energy. If we evaluate the energy of $\mathsf{U}^{\mathsf{KL}}\boldsymbol{z}$ instead of the full $\mathsf{U}\boldsymbol{z}$, we obtain the plain strain energy $2\varepsilon b^\varepsilon_{\mathsf{2D}}(\boldsymbol{z},\boldsymbol{z})$ of $\boldsymbol{z}$ instead of the plain stress energy $2\varepsilon a^\varepsilon_{\mathsf{2D}}(\boldsymbol{z},\boldsymbol{z})$: let us recall that *cf* (2.3)-(5.6)

$$a^\varepsilon_{\mathsf{2D}}(\boldsymbol{z},\boldsymbol{\eta}) = \int_S \tilde{\lambda}\gamma^\alpha_\alpha(\boldsymbol{z})\gamma^\beta_\beta(\boldsymbol{\eta}) + 2\mu\gamma^\beta_\alpha(\boldsymbol{z})\gamma^\alpha_\beta(\boldsymbol{\eta})\,\mathrm{d}S$$

$$+ \frac{\varepsilon^2}{3}\int_S \tilde{\lambda}\rho^\alpha_\alpha(\boldsymbol{z})\rho^\beta_\beta(\boldsymbol{\eta}) + 2\mu\rho^\beta_\alpha(\boldsymbol{z})\rho^\alpha_\beta(\boldsymbol{\eta})\,\mathrm{d}S,$$

and let us define

$$b^\varepsilon_{\mathsf{2D}}(\boldsymbol{z},\boldsymbol{\eta}) = \int_S \lambda\gamma^\alpha_\alpha(\boldsymbol{z})\gamma^\beta_\beta(\boldsymbol{\eta}) + 2\mu\gamma^\beta_\alpha(\boldsymbol{z})\gamma^\alpha_\beta(\boldsymbol{\eta})\,\mathrm{d}S$$

$$+ \frac{\varepsilon^2}{3}\int_S \lambda\rho^\alpha_\alpha(\boldsymbol{z})\rho^\beta_\beta(\boldsymbol{\eta}) + 2\mu\rho^\beta_\alpha(\boldsymbol{z})\rho^\alpha_\beta(\boldsymbol{\eta})\,\mathrm{d}S.$$

Using the previous computations, we can show that

$$\big|\mathsf{E}^\varepsilon_{\mathsf{3D}}[\mathsf{U}^{\mathsf{KL}}\boldsymbol{z}] - 2\varepsilon b^\varepsilon_{\mathsf{2D}}(\boldsymbol{z},\boldsymbol{z})\big| \le AE\Big(\frac{\varepsilon^2}{R}|\boldsymbol{\gamma}|^2_{0\,;\,S} + \frac{\varepsilon^4}{R}|\boldsymbol{\rho}|^2_{0\,;\,S}\Big).$$

■

## 6 Outline of the proof of the main estimate

To prove (1.7), we had rather to work with the shifted displacements, i.e. prove that

$$\widetilde{\mathsf{E}}^\varepsilon_{\mathsf{3D}}\big[\boldsymbol{w} - \mathsf{W}\boldsymbol{z}\big] \le A\,\big(B(\varepsilon;\boldsymbol{z})\,\mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}] + \mathsf{d}E^{-1}\|\boldsymbol{f}^{\mathsf{rem}}\|^2_{\mathrm{L}^2(\Omega^\varepsilon)}\big), \quad (6.1)$$

where $A$ is an adimensional constant. Problem $(P_{\mathsf{3D}})$ written with shifted displacements is equivalent to the boundary value problem

$$\left\{\begin{array}{rcll}\mathsf{L}\boldsymbol{w} &=& \boldsymbol{f} & \text{in } \Omega^\varepsilon\\ \mathsf{T}\boldsymbol{w} &=& 0 & \text{on } \Gamma^\varepsilon_\pm\\ \boldsymbol{w} &=& 0 & \text{on } \Gamma^\varepsilon_0,\end{array}\right. \tag{6.2}$$

the solution of which is our $\boldsymbol{w}$ in (6.1).

As $\mathsf{W}\boldsymbol{z}$ does not satisfy the lateral boundary conditions in general, we will add a correction term $\boldsymbol{w}^{\mathsf{cor}}$ to it so that $\mathsf{W}\boldsymbol{z}+\boldsymbol{w}^{\mathsf{cor}}$ is zero on $\Gamma^\varepsilon_0$.

Let us recall that $\widetilde{\mathsf{E}}^\varepsilon_{\mathsf{3D}}[\boldsymbol{w}]$ denotes the energy of the shifted displacement. In a similar way, let us denote by $\tilde{a}^\varepsilon_{\mathsf{3D}}$ the energy bilinear form acting on shifted displacements. The plan of the proof of (6.1) originates from the following

**Theorem 6.1** *Let $\boldsymbol{w}$ be solution of problem* (6.2)*, $\boldsymbol{z}$ the solution of problem* $(P_{\mathsf{2D}})$ *and $\boldsymbol{w}^{\mathsf{cor}}$ constructed so that $\mathsf{W}\boldsymbol{z}+\boldsymbol{w}^{\mathsf{cor}}\in V(\Omega^\varepsilon)$. If we have the estimates*

$$\forall \boldsymbol{v}\in V(\Omega^\varepsilon)\qquad \tilde{a}^\varepsilon_{\mathsf{3D}}(\boldsymbol{w}-\mathsf{W}\boldsymbol{z},\boldsymbol{v})\le B_1^{1/2}\widetilde{\mathsf{E}}^\varepsilon_{\mathsf{3D}}[\boldsymbol{v}]^{1/2}, \tag{6.3}$$

*and*

$$\widetilde{\mathsf{E}}^\varepsilon_{\mathsf{3D}}[\boldsymbol{w}^{\mathsf{cor}}]\le B_2, \tag{6.4}$$

*then there holds*

$$\widetilde{\mathsf{E}}^\varepsilon_{\mathsf{3D}}[\boldsymbol{w}-\mathsf{W}\boldsymbol{z}]\le \big(B_1^{1/2}+2B_2^{1/2}\big)^2. \tag{6.5}$$

**Proof.** Let $\boldsymbol{w}^{\mathsf{new}}=\mathsf{W}\boldsymbol{z}+\boldsymbol{w}^{\mathsf{cor}}\in V(\Omega^\varepsilon)$. Since $\boldsymbol{w}-\mathsf{W}\boldsymbol{z}=(\boldsymbol{w}-\boldsymbol{w}^{\mathsf{new}})+\boldsymbol{w}^{\mathsf{cor}}$, we start from the triangle inequality

$$\widetilde{\mathsf{E}}^\varepsilon_{\mathsf{3D}}[\boldsymbol{w}-\mathsf{W}\boldsymbol{z}]^{1/2}\le \widetilde{\mathsf{E}}^\varepsilon_{\mathsf{3D}}[\boldsymbol{w}-\boldsymbol{w}^{\mathsf{new}}]^{1/2}+\widetilde{\mathsf{E}}^\varepsilon_{\mathsf{3D}}[\boldsymbol{w}^{\mathsf{cor}}]^{1/2}. \tag{6.6}$$

The last term of the rhs is bounded by $B_2^{1/2}$. As for the first one we write

$$\begin{array}{rcl}\widetilde{\mathsf{E}}^\varepsilon_{\mathsf{3D}}[\boldsymbol{w}-\boldsymbol{w}^{\mathsf{new}}] &=& \tilde{a}^\varepsilon_{\mathsf{3D}}(\boldsymbol{w}-\boldsymbol{w}^{\mathsf{new}},\boldsymbol{w}-\boldsymbol{w}^{\mathsf{new}})\\ &=& \tilde{a}^\varepsilon_{\mathsf{3D}}(\boldsymbol{w}-\mathsf{W}\boldsymbol{z},\boldsymbol{w}-\boldsymbol{w}^{\mathsf{new}})+\tilde{a}^\varepsilon_{\mathsf{3D}}(\boldsymbol{w}^{\mathsf{cor}},\boldsymbol{w}-\boldsymbol{w}^{\mathsf{new}}).\end{array}$$

Since $\boldsymbol{w}-\boldsymbol{w}^{\mathsf{new}}$ belongs to $V(\Omega^\varepsilon)$, we may use (6.3) and obtain:

$$\widetilde{\mathsf{E}}^\varepsilon_{\mathsf{3D}}[\boldsymbol{w}-\boldsymbol{w}^{\mathsf{new}}]\quad\le\quad \big(B_1^{1/2}+\widetilde{\mathsf{E}}^\varepsilon_{\mathsf{3D}}[\boldsymbol{w}^{\mathsf{cor}}]^{1/2}\big)\widetilde{\mathsf{E}}^\varepsilon_{\mathsf{3D}}[\boldsymbol{w}-\boldsymbol{w}^{\mathsf{new}}]^{1/2},$$

whence, using (6.4) again

$$\widetilde{\mathsf{E}}^\varepsilon_{\mathsf{3D}}[\boldsymbol{w}-\boldsymbol{w}^{\mathsf{new}}]^{1/2}\quad\le\quad \big(B_1^{1/2}+B_2^{1/2}\big).$$

With (6.6) this gives the estimate (6.5). ■

Thus, to obtain (6.1), is suffices to prove estimates (6.3)-(6.4) with $B_1,\ B_2\lesssim A_S(\varepsilon,\boldsymbol{z},\boldsymbol{f}^{\mathsf{rem}})$ with

$$A_S(\varepsilon,\boldsymbol{z},\boldsymbol{f}^{\mathsf{rem}})=B_S(\varepsilon;\boldsymbol{z})\,\mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}]+\mathsf{d}^2E^{-1}\|\boldsymbol{f}^{\mathsf{rem}}\|^2_{\mathbf{L}^2(\Omega^\varepsilon)}, \tag{6.7}$$

where the symbol $f\lesssim g$ means that there exist a numerical adimensional constant $A$ such that $f\le Ag$ and $B_S(\varepsilon;\boldsymbol{z})$ is defined in (1.7). In §8, we do this for $B_1$ and in §9 we construct the correction term $\boldsymbol{w}^{\mathsf{cor}}$ and prove that $B_2\lesssim A_S(\varepsilon,\boldsymbol{z},\boldsymbol{f}^{\mathsf{rem}})$.

## 7 Formal series

In this section we recall results given in [12, 11] concerning the formal series reduction of the 3D problem. The operators constructed are then used to obtain the estimate.

We consider the 3D operators $\mathsf{L}$ and $\mathsf{T}$ (see Equation (6.2)) acting on the shifted displacement $\boldsymbol{w}$. We make the scaling $X_3 = \varepsilon^{-1}x_3$ in order to write these operators on the manifold $\Omega := S \times (-1,1)$. Thus, if $\mathsf{L}(x_\alpha, x_3; \mathrm{D}_\alpha, \partial_3)$ denotes the 3D interior operator on $\Omega^\varepsilon$, we define the operator $\mathsf{L}(\varepsilon)$ by

$$\mathsf{L}(\varepsilon)(x_\alpha, X_3; \mathrm{D}_\alpha, \partial_{X_3}) := \mathsf{L}(x_\alpha, \varepsilon X_3; \mathrm{D}_\alpha, \varepsilon^{-1}\partial_{X_3}),$$

and similarly for $\mathsf{T}(\varepsilon)$.

In [12] we show that the operators $\mathsf{L}(\varepsilon)$ and $\mathsf{T}(\varepsilon)$ expand in power series of $\varepsilon$:

$$\mathsf{L}(\varepsilon) = \varepsilon^{-2}\sum_{k=0}^{\infty} \varepsilon^k \mathsf{L}^k \quad \text{and} \quad \mathsf{T}(\varepsilon) = \varepsilon^{-1}\sum_{k=0}^{\infty} \varepsilon^k \mathsf{T}^k,$$

where $\mathsf{L}^k$ and $\mathsf{T}^k$ are intrinsic operators in $\Omega$. Note that on the manifold $\Omega$, the variable $X_3$ and the partial derivative $\partial_{X_3}$ are adimensional. Moreover, the expressions of the operators $\mathsf{L}^k$ and $\mathsf{T}^k$ show that these operators are polynomials in $X_3$ and of $\partial_3$ with coefficients $\mathsf{r}$-homogeneous operators of degree $k$.

Now with these expansions, we associate the *formal series*

$$\mathsf{L}[\varepsilon] = \varepsilon^{-2}\sum_{k\geq 0} \varepsilon^k \mathsf{L}^k \quad \text{and} \quad \mathsf{T}[\varepsilon] = \varepsilon^{-1}\sum_{k\geq 0} \varepsilon^k \mathsf{T}^k,$$

and we consider the formal series problem

$$\begin{aligned} \mathsf{L}[\varepsilon]\boldsymbol{w}[\varepsilon] &= \boldsymbol{f}[\varepsilon] \\ \mathsf{T}[\varepsilon]\boldsymbol{w}[\varepsilon] &= 0, \end{aligned} \tag{7.1}$$

where $\boldsymbol{f}[\varepsilon] = \sum_{k\geq 0} \varepsilon^k \boldsymbol{f}^k$ is a given formal series with 1-form field coefficients and where $\boldsymbol{w}[\varepsilon]$ is seek as a formal series with 1-form field coefficients. The product between two formal series is the standard Cauchy product. In the following, we denote by $\Gamma(T_1 S)$ the space of 1-form field on $S$. Hence, a displacement independent of $X_3$ as our $\boldsymbol{z}$ belongs to the space $\Sigma(S) := \Gamma(T_1 S) \times \mathscr{C}^\infty(S)$.

In [12] we show how the solution of (7.1) reduces to the solution of a two-dimensional problem posed on the mean surface: There exists formal series operators $\mathsf{V}[\varepsilon]$, $\mathsf{Q}[\varepsilon]$, $\mathsf{A}[\varepsilon]$ and $\mathsf{G}[\varepsilon]$ such that if $\boldsymbol{z}[\varepsilon] = \sum_{k\geq 0} \varepsilon^k \boldsymbol{z}^k$ is a formal with coefficients $\boldsymbol{z}^k \in \Sigma(S)$ satisfying the equation

$$\mathsf{A}[\varepsilon]\boldsymbol{z}[\varepsilon] = \mathsf{G}[\varepsilon]\boldsymbol{f}[\varepsilon] \tag{7.2}$$

then we can construct a formal series

$$\boldsymbol{w}[\varepsilon] = \mathsf{V}[\varepsilon]\boldsymbol{z}[\varepsilon] + \mathsf{Q}[\varepsilon]\boldsymbol{f}[\varepsilon] \tag{7.3}$$

solution of (7.1). Here, the coefficients of the formal series $\mathsf{V}[\varepsilon] = \sum_{k\geq 0} \varepsilon^k \mathsf{V}^k$ are operators acting on $\boldsymbol{z}$ and polynomials in $X_3$. Actually, the reconstruction operator $\mathsf{W}$: $\boldsymbol{z} \to \mathsf{W}\boldsymbol{z}$ coincides with $\mathsf{V}^0 + \varepsilon \mathsf{V}^1 + \varepsilon^2 \overline{\mathsf{V}}^2$ where $\overline{\mathsf{V}}^2$ is a part of the full operator $\mathsf{V}^2$. Following the proof in [12] we can show that the operators $\mathsf{V}^k = (\mathsf{V}^k_\alpha, \mathsf{V}^k_3)$ are $\mathsf{r}$-homogeneous operators of degree $k$. Thus the displacement $\mathsf{W}\boldsymbol{z}$ has indeed the dimension of a length.

The formal series $\mathsf{A}[\varepsilon]$ writes $\mathsf{A}[\varepsilon] = \mathsf{M} + \varepsilon^2 \mathsf{A}^2 + \cdots$, where $\mathsf{M}$ is the membrane operator, and as before we can show that the operators $\mathsf{A}^k$ are $\mathsf{r}$-homogeneous operators of degree $k+2$. Hence the formal series equation (7.2) involves an operator similar to the Koiter operator. Moreover, we can estimate the difference $\mathsf{A}^2 - \mathsf{B}$ where $\mathsf{B}$ is the bending operator (see [12]). We can in particular adapt the proof in [12] and show the following estimate: Let $\boldsymbol{z}$ and $\boldsymbol{\eta} \in \Sigma(S)$. If $\boldsymbol{\eta}$ satisfies the boundary condition $\boldsymbol{\eta}\big|_{\partial S} = 0$, then we have

$$\Big|\big\langle(\mathsf{A}^2 - \mathsf{B})\boldsymbol{z}, \boldsymbol{\eta}\big\rangle_{\mathrm{L}^2(S)}\Big| \lesssim E\Big(\|\boldsymbol{\gamma}(\boldsymbol{z})\|^{(\mathsf{r})}_{2\,;\,S}\,|\boldsymbol{\gamma}(\boldsymbol{\eta})|_{0\,;\,S} + \mathsf{r}^{-2}\|\boldsymbol{z}\|^{(\mathsf{r})}_{1\,;\,S}\,|\boldsymbol{\gamma}(\boldsymbol{\eta})|_{0\,;\,S} + \mathsf{r}^{-1}|z_3|_{2\,;\,S}\,|\boldsymbol{\gamma}(\boldsymbol{\eta})|_{0\,;\,S} + \mathsf{r}^{-1}\|\boldsymbol{\gamma}(\boldsymbol{z})\|^{(\mathsf{r})}_{1\,;\,S}\,\|\boldsymbol{\eta}\|^{(\mathsf{r})}_{1\,;\,S}\Big), \quad (7.4)$$

where $\mathsf{B}$ is the bending Koiter operator. By definition, the formal series $\mathsf{A}[\varepsilon]$ and $\mathsf{V}[\varepsilon]$ satisy the formal series equation:

$$\begin{aligned} \mathsf{L}[\varepsilon]\mathsf{V}[\varepsilon] &= -\mathscr{I} \circ \mathsf{A}[\varepsilon], \\ \mathsf{T}[\varepsilon]\mathsf{V}[\varepsilon] &= 0, \end{aligned} \quad (7.5)$$

in the space of formal series operators acting on $\Sigma(S)$. Here, the operator $\mathscr{I}$ is the natural embedding operator from $\Sigma(S)$ to $\mathscr{C}^\infty(I, \Sigma(S))$, the space of smooth 1-form field on $\Omega$.

## 8 Inner estimate

In this section, we prove the following result: (6.3).

**Proposition 8.1** *For $\boldsymbol{v} \in V(\Omega^\varepsilon)$, we have the estimate*

$$\tilde{a}^\varepsilon_{\mathsf{3D}}(\boldsymbol{w} - \mathsf{W}\boldsymbol{z}, \tilde{\boldsymbol{v}}) \lesssim B_1^{1/2}\widetilde{\mathsf{E}}^\varepsilon_{\mathsf{3D}}[\tilde{\boldsymbol{v}}]^{1/2}$$

*where*

$$B_1 = \mathsf{d}^2 E^{-1}\|\boldsymbol{f}^{\mathsf{rem}}\|^2_{\mathbf{L}^2(\Omega^\varepsilon)} + B^1_S(\varepsilon; \boldsymbol{z})\mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}], \quad (8.1)$$

*where*

$$B^1_S(\varepsilon; \boldsymbol{z}) = \frac{\varepsilon^2}{L^2} + \frac{\varepsilon^2}{\mathsf{r}^2} + \frac{\varepsilon^4\mathsf{d}^2}{L^6} + \sum_F \frac{\varepsilon^{2i}\mathsf{d}^{2j}}{L^{2k}\mathsf{r}^{2l}}, \quad (8.2)$$

*where the set $F$ is given by* (1.8).

Before starting the proof of the proposition, let us see that $B_1$ in (8.1) satisfies $B_1 \lesssim A_S(\varepsilon, \boldsymbol{z}, \boldsymbol{f}^{\mathsf{rem}})$. This is clear with the definitions (6.7) of $A_S(\varepsilon, \boldsymbol{z}, \boldsymbol{f}^{\mathsf{rem}})$ and (1.7) of $B_S(\varepsilon; \boldsymbol{z})$.

The leading idea of the proof of Proposition 8.1 is to replace $\mathsf{W}\boldsymbol{z}$ with a reconstructed displacement $\mathsf{W}^{\mathsf{asy}}\boldsymbol{z}$ which contains more terms out of the formal series expansions constructed in [12, 11], that is

$$\mathsf{W}^{\mathsf{asy}}\boldsymbol{z} = \sum_{n=0}^{4} \varepsilon^n \mathsf{V}^n \boldsymbol{z} \quad (8.3)$$

where $\mathsf{V}^n$ are the coefficients of formal series operators $\mathsf{V}[\varepsilon]$ defined in (7.3).

In the following, we denote by $\Gamma_\pm$ the upper and lower faces $S \times \{\pm 1\}$ in the manifold $\Omega$ and $\Gamma_0$ the lateral boundary $\partial S \times (-1, 1)$. Note that the scaling $X_3 = \varepsilon^{-1}x_3$ induces a diffeomorphism between $\Omega^\varepsilon$ and $\Omega$. In the following, we do not make differences between 1-form fields on $\Omega^\varepsilon$ or on $\Omega$.

By definition we have that

$\forall \boldsymbol{w} \in \mathrm{H}^1(\Omega^\varepsilon)^3, \forall \boldsymbol{v} \in V(\Omega^\varepsilon)$,

$$\tilde{a}^\varepsilon_{\mathsf{3D}}(\boldsymbol{w}, \tilde{\boldsymbol{v}}) = -\varepsilon\big\langle \mathsf{L}(\varepsilon)\boldsymbol{w}, \boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)} - \varepsilon\big\langle \mathsf{T}(\varepsilon)\boldsymbol{w}, \boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Gamma_\pm)}, \quad (8.4)$$

where $\tilde{\boldsymbol{v}}$ is the shifted displacement associated with $\boldsymbol{v}$.

We first prove the following lemma:

**Lemma 8.2** *For $\boldsymbol{v} \in V(\Omega^\varepsilon)$, we have the estimate*

$$\tilde{a}^\varepsilon_{\mathsf{3D}}(\boldsymbol{w} - \mathsf{W}^{\mathsf{asy}}\boldsymbol{z}, \tilde{\boldsymbol{v}}) \lesssim B_1^{1/2} \widetilde{\mathsf{E}}^\varepsilon_{\mathsf{3D}}[\tilde{\boldsymbol{v}}]^{1/2}$$

*where $B_1$ is given by* (8.1).

**Proof of Lemma 8.2.** By definition of the operator $\mathsf{W}^{\mathsf{asy}}$, we have that

$$-\mathsf{L}(\varepsilon)\mathsf{W}^{\mathsf{asy}} = \mathsf{M} + \varepsilon^2\mathsf{A}^2$$
$$+\varepsilon^3(\mathsf{L}^1\mathsf{V}^4 + \mathsf{L}^2\mathsf{V}^3 + \mathsf{L}^3\mathsf{V}^2 + \mathsf{L}^4\mathsf{V}^1 + \mathsf{L}^5\mathsf{V}^0) + \varepsilon^4 \sum_{i=0}^{4} \overline{\mathsf{L}}^i(\varepsilon)\mathsf{V}^i, \quad (8.5)$$

where the operators $\overline{\mathsf{L}}^i(\varepsilon)$ are of order of derivative 2 in $\mathrm{D}_\alpha$ and are of order 0 in $\varepsilon$ on the manifold $\Omega$.

Similarly, we have

$$-\mathsf{T}(\varepsilon)\mathsf{W}^{\mathsf{asy}} = \varepsilon^4 \sum_{i=0}^{4} \overline{\mathsf{T}}^i(\varepsilon)\mathsf{V}^i$$

where $\overline{\mathsf{T}}^i(\varepsilon)$ are operator of order 1 on $\mathrm{D}_\sigma$ and 0 in $\varepsilon$. Using the definition of $\boldsymbol{z}$, we have

$$\mathsf{M}\boldsymbol{z} + \varepsilon^2\mathsf{A}^2\boldsymbol{z} = \boldsymbol{g} + \varepsilon^2(\mathsf{A}^2 - \mathsf{B})\boldsymbol{z}.$$

Let $\boldsymbol{v} \in V(\Omega^\varepsilon)$. By definition, we have

$$\tilde{a}^\varepsilon_{\mathsf{3D}}(\boldsymbol{w}, \tilde{\boldsymbol{v}}) = \langle \boldsymbol{f}, \boldsymbol{v}\rangle_{\mathrm{L}^2(\Omega^\varepsilon)}.$$

Thus, we have

$$\begin{aligned} \tilde{a}^\varepsilon_{\mathsf{3D}}(\boldsymbol{w} - \mathsf{W}^{\mathsf{asy}}\boldsymbol{z}, \tilde{\boldsymbol{v}}) = \; & \varepsilon\langle \boldsymbol{f} - \boldsymbol{g}, \boldsymbol{v}\rangle_{\mathrm{L}^2(\Omega)} + \varepsilon^3\big\langle(\mathsf{A}^2 - \mathsf{B})\boldsymbol{z}, \boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)} \\ & + \varepsilon^4\langle \mathsf{L}^1\mathsf{V}^4\boldsymbol{z} + \mathsf{L}^2\mathsf{V}^3\boldsymbol{z} + \mathsf{L}^3\mathsf{V}^2\boldsymbol{z} + \mathsf{L}^4\mathsf{V}^1\boldsymbol{z} + \mathsf{L}^5\mathsf{V}^0\boldsymbol{z}, \boldsymbol{v}\rangle_{\mathrm{L}^2(\Omega)} \\ & + \varepsilon^5 \textstyle\sum_{i=1}^4 \big\langle \overline{\mathsf{L}}^i(\varepsilon)\mathsf{V}^i\boldsymbol{z}, \boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)} + \varepsilon^5 \sum_{i=1}^4 \big\langle \overline{\mathsf{T}}^i(\varepsilon)\mathsf{V}^i\boldsymbol{z}, \boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Gamma_\pm)}. \end{aligned} \quad (8.6)$$

The proof of Lemma 8.2 consists in estimating each terms in the previous equation with respect to $\widetilde{\mathsf{E}}^\varepsilon_{\mathsf{3D}}[\tilde{\boldsymbol{v}}]$.

We first give an estimation of the first term in the right-hand side of (8.6):

**Sublemma 8.2.1** *For $\boldsymbol{v} \in V(\Omega^\varepsilon)$, we have the estimate:*

$$\varepsilon\langle \boldsymbol{f} - \boldsymbol{g}, \boldsymbol{v}\rangle_{\mathrm{L}^2(\Omega)} \lesssim \mathsf{d}E^{-1/2}\|\boldsymbol{f}^{\mathsf{rem}}\|_{\mathbf{L}^2(\Omega^\varepsilon)} \widetilde{\mathsf{E}}^\varepsilon_{\mathsf{3D}}[\tilde{\boldsymbol{v}}]^{1/2}. \quad (8.7)$$

**Proof of sublemma 8.2.2.** Recall that $\mathsf{G}^0$ is the operator defined by (see [12])

$$\mathsf{G}^0\boldsymbol{f} = \frac{1}{2}\int_{-1}^{1} \boldsymbol{f}(X_3)\mathrm{d}X_3$$

and thus after the change of variable, we have on $\Omega$ that $\boldsymbol{g} = \mathsf{G}^0\boldsymbol{f}$. But as $\mathsf{G}^0\boldsymbol{v}$ is independent of $X_3$, we compute that we have

$$\left|\varepsilon\langle \boldsymbol{f}-\boldsymbol{g},\boldsymbol{v}\rangle_{\mathrm{L}^2(\Omega)}\right| = \left|\varepsilon\langle \boldsymbol{f}-\mathsf{G}^0\boldsymbol{f},\boldsymbol{v}-\mathsf{G}^0\boldsymbol{v}\rangle_{\mathrm{L}^2(\Omega)}\right| \le \varepsilon\|\boldsymbol{f}-\boldsymbol{g}\|_{\mathbf{L}^2(\Omega)}\,\|\boldsymbol{v}-\mathsf{G}^0\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)}\,.$$

Now using the classical Bramble-Hilbert Lemma on $(-1,1)$, using the fact that $X_3$ is an adimensional variable, we get

$$\|\boldsymbol{v}-\mathsf{G}^0\boldsymbol{v}\|^2_{\mathbf{L}^2(\Omega)} \lesssim \|\partial_{X_3}\boldsymbol{v}\|^2_{\mathbf{L}^2(\Omega)} = \varepsilon\|\partial_3\boldsymbol{v}\|^2_{\mathbf{L}^2(\Omega^\varepsilon)}\,.$$

But using Korn inequality (4.1) we have

$$\|\partial_3\boldsymbol{v}\|^2_{\mathbf{L}^2(\Omega^\varepsilon)} \lesssim \mathsf{d}^2E^{-1}\varepsilon^{-2}\mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{v}],$$

and finally we find

$$\|\boldsymbol{v}-\mathsf{G}^0\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)} \lesssim \mathsf{d}E^{-1/2}\varepsilon^{-1/2}\mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{v}]^{1/2}.$$

We conclude using $\varepsilon^{1/2}\|\boldsymbol{f}-\boldsymbol{g}\|_{\mathbf{L}^2(\Omega)} = \|\boldsymbol{f}^{\mathsf{rem}}\|_{\mathbf{L}^2(\Omega^\varepsilon)}$. ■

The following sublemma give an estimation for the second term in (8.6):

**Sublemma 8.2.2** *For $\boldsymbol{v}\in V(\Omega^\varepsilon)$, we have the estimate:*

$$\left|\varepsilon^3\big\langle(\mathsf{A}^2-\mathsf{B})\boldsymbol{z},\boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)}\right| \lesssim B^1_S(\varepsilon;\boldsymbol{z})^{1/2}\mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}]^{1/2}\mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{v}]^{1/2}, \tag{8.8}$$

*where $B^1_S(\varepsilon;\boldsymbol{z})$ is given by* (8.2).

**Proof of sublemma 8.2.2.** Using (7.4), we have for a 3D displacement $\boldsymbol{v}$ satifying the homogeneous lateral boundary condition

$$\left|\big\langle(\mathsf{A}^2-\mathsf{B})\boldsymbol{z},\boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)}\right| \lesssim E\left(\|\boldsymbol{\gamma}(\boldsymbol{z})\|^{(\mathsf{r})}_{2\,;\,S}+\mathsf{r}^{-2}\|\boldsymbol{z}\|^{(\mathsf{r})}_{1\,;\,S}+\mathsf{r}^{-1}|z_3|_{2\,;\,S}\right)\|\boldsymbol{\gamma}(\boldsymbol{v})\|_{\mathbf{L}^2(\Omega)}$$
$$+\,E\mathsf{r}^{-1}\|\boldsymbol{\gamma}(\boldsymbol{z})\|^{(\mathsf{r})}_{1\,;\,S}\left(\|\mathrm{D}_\alpha\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)}+\mathsf{r}^{-1}\|\boldsymbol{v}\|_{\mathbf{L}^2(\Omega)}\right), \tag{8.9}$$

But recall that for any $\boldsymbol{v}$ we have

$$\gamma_{\alpha\beta}(\boldsymbol{v}) = \tilde{e}_{\alpha\beta}(\boldsymbol{v}) - x_3\big(c_{\alpha\beta}v_3 - \tfrac12 b^\sigma_\alpha\mathrm{D}_\beta v_\sigma - \tfrac12 b^\sigma_\beta\mathrm{D}_\alpha v_\sigma\big).$$

Thus we have

$$\|\boldsymbol{\gamma}(\boldsymbol{v})\|_{\mathbf{L}^2(\Omega)} \lesssim \|\tilde{e}_{\alpha\beta}(\boldsymbol{v})\|_{\mathrm{L}^2(\Omega)} + \varepsilon\mathsf{r}^{-2}\|v_3\|_{\mathrm{L}^2(\Omega)} + \varepsilon\mathsf{r}^{-1}\|\mathrm{D}_\alpha v_\beta\|_{\mathrm{L}^2(\Omega)}$$

and hence using Korn inequalities (4.1) we find

$$\|\boldsymbol{\gamma}(\boldsymbol{v})\|_{\mathbf{L}^2(\Omega)} \lesssim \varepsilon^{-1/2}E^{-1/2}(1+\mathsf{d}^1\mathsf{r}^{-1}+\mathsf{d}^2\mathsf{r}^{-2})\mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{v}]^{1/2}. \tag{8.10}$$

Using again Korn inequalities in (8.9) we find

$$\left|\big\langle(\mathsf{A}^2-\mathsf{B})\boldsymbol{z},\boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)}\right| \lesssim E^{1/2}\mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{v}]^{1/2}\Big(\varepsilon^{-3/2}\mathsf{r}^{-1}(\mathsf{d}+\mathsf{r}^{-1}\mathsf{d}^2)\|\boldsymbol{\gamma}(\boldsymbol{z})\|^{(\mathsf{r})}_{1\,;\,S}$$
$$+\,\varepsilon^{-1/2}(1+\mathsf{d}^1\mathsf{r}^{-1}+\mathsf{d}^2\mathsf{r}^{-2})\big(\|\boldsymbol{\gamma}(\boldsymbol{z})\|^{(\mathsf{r})}_{2\,;\,S}+\mathsf{r}^{-2}\|z_\alpha\|^{(\mathsf{r})}_{1\,;\,S}+\mathsf{r}^{-1}\|z_3\|^{(\mathsf{r})}_{2\,;\,S}\big)\Big)$$

Using (3.8) and the definition of $L$ we find

$$\left|\varepsilon^3\big\langle(\mathsf{A}^2-\mathsf{B})\boldsymbol{z},\boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)}\right| \lesssim E^{1/2}\mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{v}]^{1/2}\Big(\varepsilon^{3/2}\mathsf{r}^{-1}(\mathsf{d}+\mathsf{r}^{-1}\mathsf{d}^2)(L^{-1}+\mathsf{r}^{-1})|\boldsymbol{\gamma}(\boldsymbol{z})|_{0\,;\,S}$$
$$+\,\varepsilon^{5/2}(1+\mathsf{d}^1\mathsf{r}^{-1}+\mathsf{d}^2\mathsf{r}^{-2})\big((L^{-2}+\mathsf{r}^{-1}L^{-1}+\mathsf{r}^{-2})|\boldsymbol{\gamma}(\boldsymbol{z})|_{0\,;\,S}+\mathsf{r}^{-1}|\boldsymbol{\rho}(\boldsymbol{z})|_{0\,;\,S}\big)\Big)$$

and using (3.11) we find

$$\left|\varepsilon^3\big\langle(\mathsf{A}^2-\mathsf{B})\boldsymbol{z},\boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)}\right| \lesssim \mathsf{E}^\varepsilon_{2\mathsf{D}}[\boldsymbol{z}]^{1/2}\mathsf{E}^\varepsilon_{3\mathsf{D}}[\boldsymbol{v}]^{1/2}\Big(\varepsilon\mathsf{r}^{-1}(\mathsf{d}+\mathsf{r}^{-1}\mathsf{d}^2)(L^{-1}+\mathsf{r}^{-1}) + \varepsilon^2(1+\mathsf{d}^1\mathsf{r}^{-1}+\mathsf{d}^2\mathsf{r}^{-2})\big(L^{-2}+\mathsf{r}^{-1}L^{-1}+\mathsf{r}^{-2}+\mathsf{r}^{-1}\varepsilon^{-1}\big)\Big).$$

Computing the right-hand side we find

$$\left|\varepsilon^3\big\langle(\mathsf{A}^2-\mathsf{B})\boldsymbol{z},\boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)}\right| \lesssim a_S(\varepsilon,\boldsymbol{z})\mathsf{E}^\varepsilon_{2\mathsf{D}}[\boldsymbol{z}]^{1/2}\mathsf{E}^\varepsilon_{3\mathsf{D}}[\boldsymbol{v}]^{1/2}$$

where

$$a_S(\varepsilon,\boldsymbol{z}) = \varepsilon\mathsf{r}^{-1}(1+\mathsf{d}L^{-1}+\mathsf{d}\mathsf{r}^{-1}+\mathsf{d}^2\mathsf{r}^{-1}L^{-1}+\mathsf{d}^2\mathsf{r}^{-2}) + \varepsilon^2(L^{-2}+\mathsf{r}^{-1}L^{-1}+\mathsf{r}^{-2}+\mathsf{d}\mathsf{r}^{-1}L^{-2}+\mathsf{d}L^{-1}\mathsf{r}^{-2}+\mathsf{d}\mathsf{r}^{-3}+\mathsf{d}^2L^{-2}\mathsf{r}^{-2}+\mathsf{d}^2L^{-1}\mathsf{r}^{-3}+\mathsf{d}^2\mathsf{r}^{-4}),$$

which can be written

$$a_S(\varepsilon,\boldsymbol{z}) = \Big(1+\frac{\mathsf{d}^2}{\mathsf{r}^2}\Big)\Big(\frac{\varepsilon}{L}+\frac{\varepsilon^2}{L^2}+\frac{\varepsilon}{\mathsf{r}}+\frac{\varepsilon^2}{\mathsf{r}^2}\Big)$$

and we get the result. ■

The following result give estimates for the contribution of the terms $\mathsf{L}^1\mathsf{V}^4\boldsymbol{z}$ and $\mathsf{L}^2\mathsf{V}^3\boldsymbol{z}$.

**Sublemma 8.2.3** *For $\boldsymbol{v}\in V(\Omega^\varepsilon)$, we have the estimates*

$$\left|\varepsilon^4\big\langle\mathsf{L}^1\mathsf{V}^4\boldsymbol{z},\boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)}\right| + \left|\varepsilon^4\big\langle\mathsf{L}^2\mathsf{V}^3\boldsymbol{z},\boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)}\right| \lesssim B^1_S(\varepsilon;\boldsymbol{z})\mathsf{E}^\varepsilon_{2\mathsf{D}}[\boldsymbol{z}]^{1/2}\mathsf{E}^\varepsilon_{3\mathsf{D}}[\boldsymbol{v}]^{1/2}.$$

*where $B^1_S(\varepsilon;\boldsymbol{z})$ is given by* (8.2).

**Proof of sublemma 8.2.3.** (i) The operators $\mathsf{V}^3$ and $\mathsf{V}^4$ are polynomials in $X_3$ with 2D operator coefficients. These operators are $\mathsf{r}$-homogeneous operators of degree 3 and 4 respectively. If $A$ is a 2D operator acting on $\Sigma(S)$, the equation

$$\deg A \le \begin{pmatrix} a_\sigma(\alpha) & a_3(\alpha)\\ a_\sigma(3) & a_3(3)\end{pmatrix}$$

means that the operator $A_\sigma$ acting on $\boldsymbol{z}\in\Sigma(S)$ is of order of derivative less than $a_\sigma(\alpha)$ in $z_\alpha$ and less that $a_\sigma(3)$ in $z_3$, and similarly, $A_3$ is of order less than $a_3(\alpha)$ in $z_\alpha$ and less that $a_3(3)$ in $z_3$. With these notations, we have (see [12]) that

$$\deg\mathsf{V}^4 \le \begin{pmatrix} 4 & 3\\ 3 & 4\end{pmatrix}.$$

Using the expression of the operator $\mathsf{L}^1$ (see [12]),

$$\mathsf{L}^1_\sigma(\boldsymbol{w}) = -\mu b^\alpha_\alpha\partial_{X_3}w_\sigma + (\lambda+\mu)\mathrm{D}_\sigma\partial_{X_3}w_3 - X_3\mu b^\alpha_\sigma\partial^2_{X_3}w_\alpha,$$

$$\mathsf{L}^1_3(\boldsymbol{w}) = -\mu b^\alpha_\alpha\partial_{X_3}w_3 + (\lambda+\mu)\gamma^\alpha_\alpha(\partial_{X_3}\boldsymbol{w}),$$

we have

$$\big\langle\mathsf{L}^1\mathsf{V}^4\boldsymbol{z},\boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)} = \int_\Omega(L_\sigma(\mathsf{V}^4\boldsymbol{z})v^\sigma + L_3(\mathsf{V}^4\boldsymbol{z})v^3)\mathrm{d}V$$

and thus

$$\big\langle\mathsf{L}^1\mathsf{V}^4\boldsymbol{z},\boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)} = \int_\Omega(-\mu b^\alpha_\alpha\partial_{X_3}\mathsf{V}^4_\sigma\boldsymbol{z} + (\lambda+\mu)\mathrm{D}_\sigma\partial_{X_3}\mathsf{V}^4_3\boldsymbol{z} - X_3\mu b^\alpha_\sigma\partial^2_{X_3}\mathsf{V}^4_\alpha\boldsymbol{z})v^\sigma\mathrm{d}V + \int_\Omega(-(\lambda+2\mu)b^\alpha_\alpha\partial_{X_3}\mathsf{V}^4_3\boldsymbol{z} + (\lambda+\mu)\mathrm{D}^\alpha\partial_{X_3}\mathsf{V}^4_\alpha\boldsymbol{z})v_3\mathrm{d}V.$$

Here we used the expression $\gamma^\alpha_\alpha(\boldsymbol{v}) = \mathrm{D}^\alpha v_\alpha - b^\alpha_\alpha v_3$ to transform the expression of $L^1_3\boldsymbol{v}$. In the following, we set $\Gamma_0$ the lateral boundary $\partial S\times(-1,1)$ of $\Omega$. Using the fact that $\boldsymbol{v}\big|_{\Gamma_0}=0$ we can integrate by part with respect to the surfacic derivative $\mathrm{D}_\sigma$, and we obtain (we do not write the $\mathrm{d}V$) :

$$\big\langle \mathsf{L}^1\mathsf{V}^4\boldsymbol{z},\boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)} = -\int_\Omega (\mu b^\alpha_\alpha \partial_{X_3}\mathsf{V}^4_\sigma\boldsymbol{z} + X_3\mu b^\alpha_\sigma\partial^2_{X_3}\mathsf{V}^4_\alpha\boldsymbol{z})v^\sigma - \int_\Omega(\lambda+\mu)(\partial_{X_3}\mathsf{V}^4_3\boldsymbol{z})\mathrm{D}_\sigma v^\sigma$$
$$- \int_\Omega(\lambda+2\mu)(\partial_{X_3}\mathsf{V}^4_3\boldsymbol{z})b^\alpha_\alpha v_3 - \int_\Omega(\lambda+\mu)(\partial_{X_3}\mathsf{V}^4_\alpha\boldsymbol{z})\mathrm{D}^\alpha v_3$$

and hence

$$\big\langle \mathsf{L}^1\mathsf{V}^4\boldsymbol{z},\boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)} = -\int_\Omega (\mu b^\alpha_\alpha \partial_{X_3}\mathsf{V}^4_\sigma\boldsymbol{z} + x_3\mu b^\alpha_\sigma\partial^2_{X_3}\mathsf{V}^4_\alpha\boldsymbol{z})v^\sigma - \int_\Omega\mu(\partial_{X_3}\mathsf{V}^4_3\boldsymbol{z})b^\alpha_\alpha v_3$$
$$- \int_\Omega(\lambda+\mu)(\partial_{X_3}\mathsf{V}^4_3\boldsymbol{z})\gamma^\sigma_\sigma(\boldsymbol{v}) - \int_\Omega(\lambda+\mu)(\partial_{X_3}\mathsf{V}^4_\alpha\boldsymbol{z})\mathrm{D}^\alpha v_3.$$

But we have after an integration by parts that

$$\Big|\int_\Omega \mu(\partial_{X_3}\mathsf{V}^4_3\boldsymbol{z})b^\alpha_\alpha v_3\Big| \lesssim E\mathsf{r}^{-1}\|\boldsymbol{z}\|^{(\mathsf{r})}_{3\,;\,S}\,\big(\|\mathrm{D}_\alpha v_3\|_{\mathrm{L}^2(\Omega)} + \mathsf{r}^{-1}\|v_3\|_{\mathrm{L}^2(\Omega)}\big).$$

Thus we obtain finally

$$\Big|\big\langle \mathsf{L}^1\mathsf{V}^4\boldsymbol{z},\boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)}\Big| \lesssim E\big(\|z_\alpha\|^{(\mathsf{r})}_{4\,;\,S} + \mathsf{r}^{-1}\|z_3\|^{(\mathsf{r})}_{3\,;\,S}\big)\big(\|\mathrm{D}_\alpha\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)} + \mathsf{r}^{-1}\|\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)}\big)$$
$$+ E\big(\|z_3\|^{(\mathsf{r})}_{4\,;\,S} + \mathsf{r}^{-1}\|z_\alpha\|^{(\mathsf{r})}_{3\,;\,S}\big)\|\gamma_{\alpha\beta}(\boldsymbol{v})\|_{\mathrm{L}^2(\Omega)}. \quad (8.11)$$

Using Korn inequalities (4.1) and (8.10) we find

$$\Big|\big\langle \mathsf{L}^1\mathsf{V}^4\boldsymbol{z},\boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)}\Big| \lesssim \varepsilon^{-3/2}E^{1/2}\big(\|z_\alpha\|^{(\mathsf{r})}_{4\,;\,S} + \mathsf{r}^{-1}\|z_3\|^{(\mathsf{r})}_{3\,;\,S}\big)(\mathsf{d}+\mathsf{d}^2\mathsf{r}^{-1})\mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{v}]^{1/2}$$
$$+ \varepsilon^{-1/2}E^{1/2}\big(\|z_3\|^{(\mathsf{r})}_{4\,;\,S} + \mathsf{r}^{-1}\|z_\alpha\|^{(\mathsf{r})}_{3\,;\,S}\big)(1+\mathsf{d}^2\mathsf{r}^{-2})\mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{v}]^{1/2}.$$

Using the estimates (3.12) and (3.13) we get

$$\Big|\big\langle \mathsf{L}^1\mathsf{V}^4\boldsymbol{z},\boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)}\Big| \lesssim \mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}]^{1/2}\mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{v}]^{1/2}\times$$
$$\Big(\varepsilon^{-3}(\mathsf{r}^{-1}+L^{-1})(\mathsf{r}^{-1}+\varepsilon\mathsf{r}^{-1}L^{-1}+\varepsilon L^{-2})(\mathsf{d}+\mathsf{d}^2\mathsf{r}^{-1})$$
$$+ \varepsilon^{-2}(1+\mathsf{d}^2\mathsf{r}^{-2})(L^{-2}+\mathsf{r}^{-2}+\varepsilon L^{-3}+\varepsilon L^{-1}\mathsf{r}^{-2}+\varepsilon\mathsf{r}^{-1}L^{-2})\Big),$$

and this yields the result.

(ii) Similarly, using the fact that

$$\deg\mathsf{V}^3 \le \begin{pmatrix}2 & 3\\ 3 & 2\end{pmatrix},$$

and the expression of $\mathsf{L}^2$ (see [12]),

$$\mathsf{L}^2_\sigma(\boldsymbol{w}) = -\mu X_3 c^\alpha_\alpha\partial_{X_3}w_\sigma + \mu X_3 b^\alpha_\alpha b^\beta_\sigma\partial_{X_3}w_\beta - \mu b^\alpha_\alpha\mathrm{D}_\sigma w_3 - \mu b^\beta_\beta b^\alpha_\sigma w_\alpha + \lambda\mathrm{D}_\sigma\gamma^\alpha_\alpha(\boldsymbol{w})$$
$$+ 2\mu\mathrm{D}_\alpha\gamma^\alpha_\sigma(\boldsymbol{w}),$$

$$\mathsf{L}^2_3(\boldsymbol{w}) = -\mu X_3 c^\alpha_\alpha\partial_{X_3}w_3 + (\lambda+\mu)b^\beta_\alpha\gamma^\alpha_\beta\big(\partial_{X_3}(X_3\boldsymbol{w})\big) + \mu b^\beta_\alpha\gamma^\alpha_\beta(\boldsymbol{w}) + \mu\mathrm{D}^\alpha\theta_\alpha(\boldsymbol{w}),$$

we have after integration by parts that

$$\big\langle \mathsf{L}^2\mathsf{V}^3\boldsymbol{z},\boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)} = -\int_\Omega (\mu X_3 c^\alpha_\alpha \partial_{X_3}\mathsf{V}^3_\sigma\boldsymbol{z} - \mu X_3 b^\alpha_\alpha b^\beta_\sigma \partial_{X_3}\mathsf{V}^3_\beta\boldsymbol{z} + \mu b^\beta_\beta b^\alpha_\sigma \mathsf{V}^3_\alpha\boldsymbol{z})v^\sigma\,\mathrm{d}V$$

$$+\int_\Omega (\mu b^\alpha_\alpha \mathsf{V}^3_3\boldsymbol{z} - \lambda\gamma^\alpha_\alpha(\mathsf{V}^3\boldsymbol{z}))\mathrm{D}_\sigma v^\sigma\,\mathrm{d}V - \int_\Omega 2\mu\gamma^\alpha_\sigma(\mathsf{V}^3\boldsymbol{z})\mathrm{D}_\alpha v^\sigma\,\mathrm{d}V$$

$$-\int_\Omega (\lambda+2\mu)(c^\alpha_\alpha\partial_3(X_3\mathsf{V}^3_3\boldsymbol{z})v^3\,\mathrm{d}V - \int_\Omega \big((\lambda+\mu)\partial_{X_3}(X_3\mathsf{V}^3_\alpha\boldsymbol{z})\big)(\mathrm{D}^\beta b^\alpha_\beta \boldsymbol{v}^3)\,\mathrm{d}V$$

$$-\int_\Omega (\mu\mathsf{V}^3_\alpha\boldsymbol{z})(\mathrm{D}^\beta b^\alpha_\beta\boldsymbol{v}^3)\,\mathrm{d}V - \int_\Omega \mu\theta_\alpha(\mathsf{V}^3\boldsymbol{z})(\mathrm{D}^\alpha v^3)\,\mathrm{d}V.$$

Using the relation

$$\tau^{\alpha\beta}\mathrm{D}_\alpha w_\beta = \tau^\beta_\alpha\gamma^\alpha_\beta(\boldsymbol{w}) + \tau^\beta_\alpha b^\alpha_\beta w_3$$

valid for any symmetric tensor $\tau_{\alpha\beta}$, we find the same estimate as in (8.11), and this yields the result. ■

**End of proof of lemma 8.2.** We now prove that the remaining terms in equation (8.6) can be estimated by terms of the form $B^1_S(\varepsilon;\boldsymbol{z})^{1/2}\mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{z}]^{1/2}$ where the expression of the bound is given by (8.2). We conclude using the fact that

$$\widetilde{\mathsf{E}}^\varepsilon_{\mathsf{3D}}[\tilde{\boldsymbol{v}}] = \mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{v}].$$

Recall that for all $k$ the operators $\mathsf{V}^k$ are $\mathsf{r}$-homogeneous operators of degree $k$. The operators $\mathsf{L}^k$ and $\mathsf{T}^k$ act on the surfacic displacements as $\mathsf{r}$-homogeneous operators of degree $k$ but their degrees of derivative are equal to 2 for $k \geq 2$. Moreover, they are homogeneous of degree $k-2$ in the transverse variable $X_3$. This means after integration by parts that for any $\boldsymbol{v}\in\mathrm{H}^1(\Omega)$ and $\boldsymbol{w}$ a homogeneous polynomial in $X_3$, we have the estimate for $k\geq 2$:

$$\big|\langle \mathsf{L}^k\boldsymbol{w},\boldsymbol{v}\rangle_{\mathrm{L}^2(\Omega)}\big| \lesssim E\mathsf{r}^{2-k}\big(\|\mathrm{D}_\alpha\boldsymbol{w}\|_{\mathrm{L}^2(\Omega)} + \mathsf{r}^{-1}\|\boldsymbol{w}\|_{\mathrm{L}^2(\Omega)}\big)\big(\|\mathrm{D}_\alpha\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)} + \mathsf{r}^{-1}\|\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)}\big).$$

Using this estimate we see that for all $k\geq 2$ and $i\geq 0$

$$\begin{aligned}\big|\langle \mathsf{L}^k\mathsf{V}^i\boldsymbol{z},\boldsymbol{v}\rangle_{\mathrm{L}^2(\Omega)}\big| &\lesssim E\mathsf{r}^{2-k}\|\boldsymbol{z}\|^{(\mathsf{r})}_{i+1\,;\,S}\big(\|\mathrm{D}_\alpha\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)} + \mathsf{r}^{-1}\|\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)}\big)\\ &\lesssim E^{1/2}\varepsilon^{-3/2}\mathsf{r}^{2-k}\mathsf{d}(1+\mathsf{d}\mathsf{r}^{-1})\|\boldsymbol{z}\|^{(\mathsf{r})}_{i+1\,;\,S}\,\mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{v}]^{1/2},\end{aligned}$$

using Korn inequalities (4.1).

This estimate yields immediatly that

$$\Big|\varepsilon^4\big\langle \mathsf{L}^3\mathsf{V}^2\boldsymbol{z} + \mathsf{L}^4\mathsf{V}^1\boldsymbol{z} + \mathsf{L}^5\mathsf{V}^0\boldsymbol{z},\boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)}\Big| \lesssim \varepsilon^{5/2}E^{1/2}\mathsf{r}^{-1}\mathsf{d}(1+\mathsf{d}\mathsf{r}^{-1})\|\boldsymbol{z}\|^{(\mathsf{r})}_{3\,;\,S}\,\mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{v}]^{1/2},$$

and using the a priori estimates (3.12) and (3.13) we get

$$\begin{aligned}&\Big|\varepsilon^4\big\langle \mathsf{L}^3\mathsf{V}^2\boldsymbol{z} + \mathsf{L}^4\mathsf{V}^1\boldsymbol{z} + \mathsf{L}^5\mathsf{V}^0\boldsymbol{z},\boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)}\Big|\\ &\qquad\lesssim \varepsilon\mathsf{r}^{-1}\mathsf{d}(1+\mathsf{d}\mathsf{r}^{-1})\big(\mathsf{r}^{-1} + \varepsilon L^{-2} + L^{-1}(1+\varepsilon L^{-1})(1+L\mathsf{r}^{-1})\big)\mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}]^{1/2}\mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{v}]^{1/2},\end{aligned}$$

and we see easily that the right-hand side is less than

$$B^1_S(\varepsilon;\boldsymbol{z})^{1/2}\mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}]^{1/2}\mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{v}]^{1/2},$$

which yields the result.

The operators $\overline{\mathsf{L}}^i$ of the equation (8.6) are power series of the operators $\mathsf{L}^k$. As before, we thus can show that we have

$$\begin{aligned}\Big|\varepsilon^5\big\langle \textstyle\sum_{i=0}^4 \overline{\mathsf{L}}^i(\varepsilon)\mathsf{V}^i\boldsymbol{z},\boldsymbol{v}\big\rangle_{\mathrm{L}^2(\Omega)}\Big| &\lesssim \varepsilon^5 E\|\boldsymbol{z}\|^{(\mathsf{r})}_{5\,;\,S}\big(\|\mathrm{D}_\alpha\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)}+\mathsf{r}^{-1}\|\boldsymbol{v}\|_{\mathrm{L}^2(\Omega)}\big)\\ &\lesssim \varepsilon^{7/2}E^{1/2}d(1+d\mathsf{r}^{-1})\|\boldsymbol{z}\|^{(\mathsf{r})}_{5\,;\,S}\,\mathsf{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{v}]^{1/2}.\end{aligned}$$

But using (3.12) and (3.13) we see that

$$\|\boldsymbol{z}\|^{(\mathsf{r})}_{5\,;\,S}\lesssim E^{-1/2}\varepsilon^{-3/2}\Big(1+\frac{\varepsilon}{L}\Big)\Big(\frac{1}{L^3}+\frac{1}{L\mathsf{r}^2}+\frac{1}{\mathsf{r}^3}\Big),$$

and we conclude as before. The estimate for the traction terms involving the operators $\overline{\mathsf{T}}^i(\varepsilon)$ can be done similarly, and this proves the lemma. ∎

We can now prove the main result of this section.

**Proof of Proposition 8.1.** Using Lemma 8.2 we have

$$\begin{aligned}\tilde a^\varepsilon_{\mathsf{3D}}(\boldsymbol{w}-\mathsf{W}\boldsymbol{z},\tilde{\boldsymbol{v}}) &= \tilde a^\varepsilon_{\mathsf{3D}}(\boldsymbol{w}-\mathsf{W}^{\mathsf{asy}}\boldsymbol{z},\tilde{\boldsymbol{v}})+\tilde a^\varepsilon_{\mathsf{3D}}(\mathsf{W}\boldsymbol{z}-\mathsf{W}^{\mathsf{asy}}\boldsymbol{z},\tilde{\boldsymbol{v}})\\ &\lesssim \big(B_1^{1/2}+\widetilde{\mathsf{E}}^\varepsilon_{\mathsf{3D}}[\mathsf{W}\boldsymbol{z}-\mathsf{W}^{\mathsf{asy}}\boldsymbol{z}]^{1/2}\big)\widetilde{\mathsf{E}}^\varepsilon_{\mathsf{3D}}[\tilde{\boldsymbol{v}}]^{1/2}.\end{aligned}$$

Thus the proposition is proved provided we show

$$\widetilde{\mathsf{E}}^\varepsilon_{\mathsf{3D}}[\mathsf{W}\boldsymbol{z}-\mathsf{W}^{\mathsf{asy}}\boldsymbol{z}]\lesssim B_1.$$

By definition, we have

$$\mathsf{W}^{\mathsf{asy}}\boldsymbol{z}=\mathsf{W}\boldsymbol{z}+\mathsf{v}^2\boldsymbol{z}+\mathsf{V}^3\boldsymbol{z}+\mathsf{V}^4\boldsymbol{z} \tag{8.12}$$

where

$$\mathsf{v}^2\boldsymbol{z}=\begin{cases}\frac{X_3^2}{2}p\mathrm{D}_\sigma\gamma^\alpha_\alpha(\boldsymbol{z}),\\ \frac{X_3^2}{2}\big(-p^2b^\alpha_\alpha\gamma^\beta_\beta(\boldsymbol{z})-2pb^\beta_\alpha\gamma^\alpha_\beta(\boldsymbol{z})\big),\end{cases}$$

We now succesively estimate the energy of the three terms $\mathsf{v}^2\boldsymbol{z}$, $\mathsf{V}^3\boldsymbol{z}$ and $\mathsf{V}^4\boldsymbol{z}$.

(i) We have $\widetilde{\mathsf{E}}^\varepsilon_{\mathsf{3D}}[\mathsf{v}^2\boldsymbol{z}]\lesssim\varepsilon E\|\tilde e_{ij}(\varepsilon)(\mathsf{v}^2\boldsymbol{z})\|^2_{\mathrm{L}^2(\Omega)}$, where $\tilde e_{ij}(\varepsilon)$ is the deformation tensor $\tilde e_{ij}$ after the change of variable $x_3\mapsto X_3$. Recall that $p$ is adimensional. It is then clear that we have

$$\begin{aligned}\|\tilde e_{33}(\varepsilon)(\mathsf{v}^2\boldsymbol{z})\|^2_{\mathrm{L}^2(\Omega)} &\lesssim \varepsilon^{-2}\mathsf{r}^{-2}|\boldsymbol{\gamma}|^2_{0\,;\,S},\\ \|\tilde e_{\sigma3}(\varepsilon)(\mathsf{v}^2\boldsymbol{z})\|^2_{\mathrm{L}^2(\Omega)} &\lesssim \varepsilon^{-2}(1+\varepsilon^2\mathsf{r}^{-2})\big(\|\boldsymbol{\gamma}\|^{(\mathsf{r})}_{1\,;\,S}\big)^2\quad\text{and}\\ \|\tilde e_{\alpha\beta}(\varepsilon)(\mathsf{v}^2\boldsymbol{z})\|^2_{\mathrm{L}^2(\Omega)} &\lesssim (1+\varepsilon^2\mathsf{r}^{-2})\big(\|\boldsymbol{\gamma}\|^{(\mathsf{r})}_{2\,;\,S}\big)^2.\end{aligned}$$

Hence we have

$$\widetilde{\mathsf{E}}^\varepsilon_{\mathsf{3D}}[\mathsf{v}^2\boldsymbol{z}]\lesssim\varepsilon^{-1}E(1+\varepsilon^2\mathsf{r}^{-2})\Big(\big(\|\boldsymbol{\gamma}\|^{(\mathsf{r})}_{1\,;\,S}\big)^2+\varepsilon^2\big(\|\boldsymbol{\gamma}\|^{(\mathsf{r})}_{2\,;\,S}\big)^2\Big)$$

and hence using the definition of $L$ we have (8.13) after multiplying by $\varepsilon^4$.

$$\widetilde{\mathsf{E}}^\varepsilon_{\mathsf{3D}}[\varepsilon^2\mathsf{v}^2\boldsymbol{z}]\lesssim\varepsilon^3E(1+\varepsilon^2\mathsf{r}^{-2})\Big((\mathsf{r}^{-1}+L^{-1})^2+\varepsilon^2((\mathsf{r}^{-2}+\mathsf{r}^{-1}L^{-1}+L^{-2})^2\Big)|\boldsymbol{\gamma}|^2_{0\,;\,S} \tag{8.13}$$

and thus we have

$$\widetilde{\mathsf{E}}^\varepsilon_{\mathsf{3D}}[\varepsilon^2\mathsf{v}^2\boldsymbol{z}]\lesssim B^1_S(\varepsilon;\boldsymbol{z})\mathsf{E}^\varepsilon_{\mathsf{2D}}[\boldsymbol{z}]$$

where $B^1_S(\varepsilon;\boldsymbol{z})$ is given in (8.2).

(ii) Again, we have $\widetilde{\mathsf{E}}^{\varepsilon}_{\mathsf{3D}}[\mathsf{V}^3\boldsymbol{z}] \lesssim \varepsilon\|\tilde{e}_{ij}(\varepsilon)(\mathsf{V}^3\boldsymbol{z})\|^2_{\mathrm{L}^2(\Omega)}$. But recall that we have the following estimate for the orders of the derivatives of $\mathsf{V}^3$:

$$\deg \mathsf{V}^3 \le \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix},$$

and that $\mathsf{V}^3$ is a $\mathsf{r}$-homogeneous operator of order 3. We deduce that

$$\|\tilde{e}_{33}(\varepsilon)(\mathsf{V}^3\boldsymbol{z})\|^2_{\mathrm{L}^2(\Omega)} \lesssim \varepsilon^{-2}\Big(|z_\alpha|_{3\,;\,S} + \mathsf{r}^{-1}\|\boldsymbol{z}\|^{(\mathsf{r})}_{2\,;\,S}\Big)^2$$

and

$$\|\tilde{e}_{\alpha 3}(\varepsilon)(\mathsf{V}^3\boldsymbol{z})\|^2_{\mathrm{L}^2(\Omega)} \lesssim \varepsilon^{-2}\Big(|z_3|_{3\,;\,S} + \mathsf{r}^{-1}\|\boldsymbol{z}\|^{(\mathsf{r})}_{2\,;\,S}\Big)^2 + \Big(|z_\alpha|_{4\,;\,S} + \mathsf{r}^{-1}\|\boldsymbol{z}\|^{(\mathsf{r})}_{3\,;\,S}\Big)^2$$

and

$$\|\tilde{e}_{\alpha\beta}(\varepsilon)(\mathsf{V}^3\boldsymbol{z})\|^2_{\mathrm{L}^2(\Omega)} \lesssim \Big(|z_3|_{4\,;\,S} + \mathsf{r}^{-1}\|\boldsymbol{z}\|^{(\mathsf{r})}_{3\,;\,S}\Big)^2$$

Hence we have

$$\begin{aligned}\widetilde{\mathsf{E}}^{\varepsilon}_{\mathsf{3D}}[\mathsf{V}^3\boldsymbol{z}] &\lesssim \varepsilon^{-1}E\Big(|z_3|_{3\,;\,S} + \mathsf{r}^{-1}\|\boldsymbol{z}\|^{(\mathsf{r})}_{2\,;\,S}\Big)^2 + \varepsilon\Big(|z_\alpha|_{4\,;\,S} + \mathsf{r}^{-1}\|\boldsymbol{z}\|^{(\mathsf{r})}_{3\,;\,S}\Big)^2 \\ &\lesssim \varepsilon^{-1}E(1+\varepsilon^2\mathsf{r}^{-2})\Big(\|\boldsymbol{z}\|^{(\mathsf{r})}_{3\,;\,S}\Big)^2 + \varepsilon|z_\alpha|^2_{4\,;\,S}\end{aligned}$$

whence the result after multiplying by $\varepsilon^6$ and using (3.12) and (3.13):

$$\widetilde{\mathsf{E}}^{\varepsilon}_{\mathsf{3D}}[\varepsilon^3\mathsf{V}^3\boldsymbol{z}] \lesssim B^1_S(\varepsilon;\boldsymbol{z})\mathsf{E}^{\varepsilon}_{\mathsf{2D}}[\boldsymbol{z}]$$

where $B^1_S(\varepsilon;\boldsymbol{z})$ is given in (8.2).

(iii) On the same way, we easily find :

$$\|\tilde{e}_{ij}(\varepsilon)(\mathsf{V}^4\boldsymbol{z})\|^2_{\mathrm{L}^2(\Omega)} \lesssim \varepsilon^{-2}\Big(\|\boldsymbol{z}\|^{(\mathsf{r})}_{4\,;\,S}\Big)^2 + \Big(\|\boldsymbol{z}\|^{(\mathsf{r})}_{5\,;\,S}\Big)^2$$

and thus

$$\widetilde{\mathsf{E}}^{\varepsilon}_{\mathsf{3D}}[\mathsf{V}^4\boldsymbol{z}] \lesssim \varepsilon^{-1}\Big(\|\boldsymbol{z}\|^{(\mathsf{r})}_{4\,;\,S}\Big)^2 + \varepsilon\Big(\|\boldsymbol{z}\|^{(\mathsf{r})}_{5\,;\,S}\Big)^2$$

whence the result after multiplying by $\varepsilon^8$ and using (3.12) and (3.13).

## 9 Estimate for the corrector term

The goal of this section is to construct a displacement $\boldsymbol{w}^{\mathsf{cor}}$ satisfying the equation (6.4) for $B_2 \lesssim B_S(\varepsilon,\boldsymbol{z},\boldsymbol{f}^{\mathsf{rem}})$, and such that $\mathsf{W}\boldsymbol{z} + \boldsymbol{w}^{\mathsf{cor}} \in V(\Omega^\varepsilon)$.

Let us first recall properties of the boundary layer formal series operators introduced in [12, 7]. In the following, we denote by $r$ the geodesic distance in $S$ to the boundary $\partial S$, and by $s$ the arc-length along $\partial S$. We denote by $r_0$ the size of the tubular neighborhood in which the coordinate system $(r,s)$ is defined. It is clear that $r_0$ has the dimension of a length and that $r_0$ is proportional to the maximum radius of curvature near the boundary. Thus, we can always assume that $r_0 = \inf(\mathsf{r},1)$.

In [12], we construct a formal series operator acting on $\Sigma(S)$, and taking value in a space of boundary layer exponentially decreasing with respect to $R = r/\varepsilon$. Let us recall this construction. Consider the coordinate system $(r,s,X_3)$ in a neighborhood of $\Gamma_0$ in $\Omega$. We set $R = \varepsilon^{-1}r$. The coordinate system $(R,X_3,s)$ is defined on the manifold $\Sigma^+ \times \partial S$ where $\Sigma^+ := \mathbb{R}^+ \times I \ni (R,X_3)$ is a semi-strip. The boundary of $\Sigma^+$ decomposes into a lateral boundary $\gamma_0 := \{R=0\}\times I$ and the two half-lines $\gamma_\pm := \mathbb{R}^+ \times \{X_3 = \pm 1\}$. In coordinates $(r,s,x_3)$, we write $(\mathsf{L},\mathsf{T})(r,s,x_3;\partial_r,\partial_s,\partial_3)$ the 3D operators. For $\varepsilon \le \varepsilon_0$, we define the operators $\big(\mathscr{L}(\varepsilon),\mathscr{T}(\varepsilon)\big)$ on $\Sigma^+\times\partial S$ by the formulas

$$\begin{cases} \mathscr{L}(\varepsilon)(R,s,X_3;\partial_R,\partial_s,\partial_{X_3}) := \mathsf{L}(\varepsilon R, s, \varepsilon X_3; \varepsilon^{-1}\partial_R, \partial_s, \varepsilon^{-1}\partial_{X_3}) \quad \text{and} \\ \mathscr{T}(\varepsilon)(R,s,X_3;\partial_R,\partial_s,\partial_{X_3}) := \mathsf{T}(\varepsilon R, s, \varepsilon X_3; \varepsilon^{-1}\partial_R, \partial_s, \varepsilon^{-1}\partial_{X_3}). \end{cases} \tag{9.1}$$

The formal series $\big(\mathscr{L}[\varepsilon],\mathscr{T}[\varepsilon]\big)$ are then the formal series associated with these operators using the Taylor expansion in $R=0$ and $X_3=0$ of the coefficients.

We then write

$$\mathscr{L}[\varepsilon] = \varepsilon^{-2}\sum_{k\ge 0}\varepsilon^k\mathscr{L}^k \quad \text{and} \quad \mathscr{T}[\varepsilon] = \varepsilon^{-1}\sum_{k\ge 0}\varepsilon^k\mathscr{T}^k,$$

where $\mathscr{L}^k : \mathscr{C}^\infty\big(\Sigma^+\times\partial S\big)^3 \to \mathscr{C}^\infty\big(\Sigma^+\times\partial S\big)^3$ and $\mathscr{T}^k : \mathscr{C}^\infty\big(\Sigma^+\times\partial S\big)^3 \to \mathscr{C}^\infty\big(\gamma_\pm\times\partial S\big)^3$ are operators of degree 2 polynomials in $R$ and $X_3$. The computations of the first terms $\mathscr{L}^0$ and $\mathscr{T}^0$ is given in [12].

As in [7], we introduce the following spaces: Let $\mathfrak{H}(\Sigma^+)$ be the space of *adimensional* $\mathscr{C}^\infty$ functions $\varphi$ on the semi-strip $\Sigma^+$ except in the non regular points $(R=0, X_3=\pm1)$, and such that $\varphi$ is exponentially decreasing with $R$ in the following sense:

$$\forall\, i,j,k\in\mathbb{N}, \qquad e^{\delta R}\, R^k\, \partial_R^i\partial_{X_3}^j\varphi \in \mathrm{L}^2(\Sigma^+), \tag{9.2}$$

where $\delta > 0$ is a real strictly less than the smallest Papkovich-Fadle exponent (see [15]). Notice that as the variables $R$ and $X_3$ are adimensional, the functions in (9.2) are all adimensional. In the neighborhood of the two corners of the semi-strip, we impose the following: if $\rho$ denote the distance in $\Sigma^+$ to a point $(R=0, X_3=\pm1)$, we suppose that each $\varphi$ in $\mathfrak{H}(\Sigma^+)$ satisfies

$$\forall\, i,j\in\mathbb{N},\ i+j\ne 0, \qquad \rho^{i+j-1}\,\partial_R^i\partial_{X_3}^j\varphi\in\mathrm{L}^2(\Sigma^+).$$

We then define the corresponding displacement space

$$\boldsymbol{\mathfrak{H}}(\Sigma^+) := \big\{\boldsymbol{\varphi} = (\varphi_R,\varphi_s,\varphi_3)\in\mathfrak{H}(\Sigma^+)^3\big\}.$$

As the arc-length appears as a parameter, the natural space in which the equations will be posed is hence $\mathscr{C}^\infty\big(\partial S, \boldsymbol{\mathfrak{H}}(\Sigma^+)\big)$.

We now define the associated range spaces: We set $\mathfrak{K}(\Sigma^+)$ the space of *adimensional* functions $\psi\in\mathscr{C}^\infty(\Sigma^+)$ such that

$$\forall\, i,j,k\in\mathbb{N},\, e^{\delta R}\,R^k\,\partial_R^i\partial_{X_3}^j\psi\in\mathrm{L}^2(\Sigma^+) \quad\text{and}\quad \forall\, i,j\in\mathbb{N},\, \rho^{i+j+1}\,\partial_R^i\partial_{X_3}^j\psi\in\mathrm{L}^2(\Sigma^+)$$

with the same notations. Similarly, we introduce the same space corresponding to the trace operators on $\gamma_\pm$: let $\mathfrak{K}(\gamma_\pm)$ the space of couple of functions $\psi^\pm\in\mathscr{C}^\infty(\gamma_\pm)$ such that

$$\forall\, i,k\in\mathbb{N}, \quad e^{\delta R}\,R^k\,\partial_R^i\psi^\pm\in\mathrm{L}^2(\gamma_\pm) \quad\text{and}\quad \forall\, i,j\in\mathbb{N}, \quad \rho^{i+j+1/2}\,\partial_R^i\psi^\pm\in\mathrm{L}^2(\gamma_\pm).$$

We then define the spaces

$$\boldsymbol{\mathfrak{K}}(\Sigma^+) := \big\{\boldsymbol{\psi} = (\psi_R,\psi_s,\psi_3)\in\mathfrak{K}(\Sigma^+)^3\big\},$$

and

$$\boldsymbol{\mathfrak{K}}(\gamma_\pm) := \big\{\boldsymbol{\psi}^\pm = (\psi_R^\pm,\psi_s^\pm,\psi_3^\pm)\in\mathfrak{K}(\gamma_\pm)^3\big\}.$$

Thus the operators $\mathscr{L}^0$ et $\mathscr{T}^0$ act on the space $\mathscr{C}^\infty\big(\partial S, \mathfrak{H}(\Sigma^+)\big)$ and take values in $\mathscr{C}^\infty\big(\partial S, \mathfrak{K}(\Sigma^+)\big)$ and $\mathscr{C}^\infty\big(\partial S, \mathfrak{K}(\gamma_\pm)\big)$ respectively.

The properties of the operators $\mathscr{L}^0$ and $\mathcal{T}^0$ involve the rigid displacement space $\mathfrak{Z}$ spanned by the four following displacements, written in coordinates $(R, s, X_3)$ (see [10]) :

$$\boldsymbol{\mathcal{Z}}^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \boldsymbol{\mathcal{Z}}^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \boldsymbol{\mathcal{Z}}^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \boldsymbol{\mathcal{Z}}^4 = \begin{pmatrix} -X_3 \\ 0 \\ R \end{pmatrix}. \tag{9.3}$$

These displacements are in the kernel of the operator $(\mathscr{L}^0, \mathscr{T}^0)$ without boundary condition on the lateral boundary. The operators $\mathscr{L}^0$ and $\mathscr{T}^0$ have the following property (see for example [10, section 5]):

**Proposition 9.1** *Let $\boldsymbol{\psi} \in \mathfrak{K}(\Sigma^+)$, $\boldsymbol{\psi}^\pm \in \mathfrak{K}(\gamma_\pm)$ and $\boldsymbol{v} \in \mathscr{C}^\infty(\overline{\gamma}_0)^3$. There exist a unique $\boldsymbol{\varphi} \in \mathfrak{H}(\Sigma^+)$ and a unique $\boldsymbol{\mathcal{Z}} \in \mathfrak{Z}$ such that*

$$\begin{cases} \mathscr{L}^0(\boldsymbol{\varphi} - \boldsymbol{\mathcal{Z}}) &= \boldsymbol{\psi} \quad \textit{in} \quad \Sigma^\pm, \\ \mathscr{T}^0(\boldsymbol{\varphi} - \boldsymbol{\mathcal{Z}}) &= \boldsymbol{\psi}^\pm \quad \textit{on} \quad \gamma_+ \times \gamma_-, \\ (\boldsymbol{\varphi} - \boldsymbol{\mathcal{Z}})\big|_{R=0} + \boldsymbol{v}\big|_{\gamma_0} &= 0. \end{cases} \tag{9.4}$$

Remark that as $\boldsymbol{\mathcal{Z}} \in \mathfrak{Z}$ the left-hand sides of the two first equations of (9.4) are equals to $\mathscr{L}^0(\boldsymbol{\varphi})$ et $\mathscr{T}^0(\boldsymbol{\varphi})$. The following corollary is clear using the fact that the operator $(\mathscr{L}^0, \mathscr{T}^0)$ does not depend on $s$:

**Corollary 9.2** *If in the previous proposition we have $\boldsymbol{\psi} \in \mathscr{C}^\infty\big(\partial S, \mathfrak{K}(\Sigma^+)\big)$, $\boldsymbol{\psi}^\pm \in \mathscr{C}^\infty\big(\partial S, \mathfrak{K}(\gamma_\pm)\big)$ and $\boldsymbol{v} \in \mathscr{C}^\infty(\overline{\Gamma}_0)^3$ then the functions solution of* (9.4) *are in the spaces $\boldsymbol{\varphi} \in \mathscr{C}^\infty\big(\partial S, \mathfrak{H}(\Sigma^+)\big)$ and $\boldsymbol{\mathcal{Z}} \in \mathscr{C}^\infty\big(\partial S, \mathfrak{Z}\big)$.*

In [12] this corollary is used to construct two formal series $\Psi[\varepsilon] = \sum_{k\geq 0} \varepsilon^k \Psi^k$ and $\mathfrak{d}[\varepsilon] = \sum_{k\geq 0} \varepsilon^k \mathfrak{d}^k$ with coefficients operators

$$\Psi^k : \Sigma(S) \to \mathscr{C}^\infty\big(\partial S, \mathfrak{H}(\Sigma^+)\big) \quad \text{and} \quad \mathfrak{d}^k : \Sigma(S) \to \mathscr{C}^\infty\big(\partial S, \mathfrak{Z}\big)$$

satisfying the formal series equations:

$$\begin{cases} \mathscr{L}[\varepsilon]\Psi[\varepsilon] &= 0, \\ \mathscr{T}[\varepsilon]\Psi[\varepsilon] &= 0, \\ \big(\Psi[\varepsilon] - \mathfrak{d}[\varepsilon]\big)\big|_{R=0} + \mathsf{V}[\varepsilon]\big|_{\Gamma_0} &= 0, \end{cases} \tag{9.5}$$

in the space of formal series operator acting on $\Sigma(S)$. These formal series are introduced in order to compensate the traces of $\mathsf{V}[\varepsilon]$ on the lateral boundary $\Gamma_0$. Indeed, we can show that in general, if $\boldsymbol{z}[\varepsilon]$ is a formal series solution of (7.2), the formal series $\boldsymbol{w}[\varepsilon]$ constructed in (7.3) does not satisfy the boundary condition $\boldsymbol{w}[\varepsilon]\big|_{\Gamma_0} = 0$. In [12] we show that if $\boldsymbol{z}[\varepsilon]$ satisfies a boundary condition of the form

$$\mathfrak{d}[\varepsilon]\boldsymbol{z}[\varepsilon] = \mathfrak{h}[\varepsilon]\boldsymbol{f}[\varepsilon]$$

we can construct a boundary layer formal series

$$\boldsymbol{\varphi}[\varepsilon] := \Psi[\varepsilon]\boldsymbol{z}[\varepsilon] + \Theta[\varepsilon]\boldsymbol{f}[\varepsilon]$$

such that $\boldsymbol{\varphi}[\varepsilon]\big|_{R=0}+\boldsymbol{w}[\varepsilon]\big|_{\Gamma_0}=0$ and $\big(\mathscr{L}[\varepsilon],\mathscr{T}[\varepsilon]\big)\boldsymbol{\varphi}[\varepsilon]=0$. Here, the formal series $\mathfrak{h}[\varepsilon]$ and $\Theta[\varepsilon]$ are constructed in a same way as $\Psi[\varepsilon]$ and $\mathfrak{d}[\varepsilon]$.

The first terms of the formal series $\mathfrak{d}[\varepsilon]$ are

$$\mathfrak{d}^0\boldsymbol{z}=\big(z_r\big|_{\partial S}\big)\boldsymbol{\mathcal{Z}}^1+\big(z_s\big|_{\partial S}\big)\boldsymbol{\mathcal{Z}}^2+\big(z_3\big|_{\partial S}\big)\boldsymbol{\mathcal{Z}}^3,$$

and

$$\mathfrak{d}^1\boldsymbol{z}=\big(c_1\gamma^\alpha_\alpha(\boldsymbol{z})\big|_{\partial S}\big)\boldsymbol{\mathcal{Z}}^1+\big(\theta_r(\boldsymbol{z})\big|_{\partial S}\big)\boldsymbol{\mathcal{Z}}^4$$

where $c_1$ is an adimensional constant. The first terms of the formal series $\Psi[\varepsilon]$ are $\Psi^0=0$ and

$$\Psi^1_R\boldsymbol{z}=\big(p\gamma^\alpha_\alpha(\boldsymbol{z})\big|_{\partial S}\big)\overline{\varphi}^1_R,\quad \Psi^1_s\boldsymbol{z}=\big(\theta_s(\boldsymbol{z})\big|_{\partial S}\big)\overline{\varphi}^1_s\quad\text{and}\quad \Psi^1_3\boldsymbol{z}=\big(p\gamma^\alpha_\alpha(\boldsymbol{z})\big|_{\partial S}\big)\overline{\varphi}^1_3, \tag{9.6}$$

where $\overline{\boldsymbol{\varphi}}^1=(\overline{\varphi}^1_R,\overline{\varphi}^1_s,\overline{\varphi}^1_3)$ is an element of $\boldsymbol{\mathfrak{H}}(\Sigma^+)$ independent on $\varepsilon$. Moreover, following the result of Proposition 5.4 in [12], we can show that for all $k$, we have decompositions:

$$\Psi^k\boldsymbol{z}=\sum_{j\in F_k}(P^k_j\boldsymbol{z})\big|_{\partial S}\boldsymbol{\varphi}^{k,j}\quad\text{and}\quad \mathfrak{d}^k\boldsymbol{z}=\sum_{i=1}^4(\mathfrak{D}^k_i\boldsymbol{z})\big|_{\partial S}\boldsymbol{\mathcal{Z}}^i,$$

where for all $k$, $F_k$ is a finite set. In this decompositions, the functions $\boldsymbol{\varphi}^{k,j}$ are adimensional elements of $\mathscr{C}^\infty\big(\partial S,\boldsymbol{\mathfrak{H}}(\Sigma^+)\big)$ independent of $\boldsymbol{z}$ and the operators $P^k_j$ and $\mathfrak{D}^k_i$ are $\mathsf{r}$-homogeneous operators of order $k$.

In our situation, we will consider formal series $\hat{\Psi}[\varepsilon]$ and $\hat{\mathfrak{d}}[\varepsilon]$ defined by the equation (9.5) with the truncated formal series $\mathsf{W}[\varepsilon]=\mathsf{V}^0+\varepsilon\mathsf{V}^1+\varepsilon^2\overline{\mathsf{V}}^2$ associated with the operator $\mathsf{W}$ instead of $\mathsf{V}[\varepsilon]$.

In this case, we can show that $\hat{\Psi}^0=0$ and $\hat{\Psi}^1=\Psi^1$ given in (9.6). Moreover, we can easily show that $\hat{\Psi}^2_3\boldsymbol{z}=\big(p\rho^\alpha_\alpha(\boldsymbol{z})\big|_{\partial S}\big)\overline{\varphi}^2_3$ where $\overline{\varphi}^2_3\in\mathfrak{H}(\Sigma^+)$ and that

$$\hat{\mathfrak{d}}^2_3\boldsymbol{z}\big|_{R=0}=c_2\rho^\alpha_\alpha(\boldsymbol{z})\big|_{\partial S}$$

where $c_2$ is a numerical adimensional constant depending only on $\lambda$ and $\mu$.

Consider now the displacement $\mathsf{W}\boldsymbol{z}$. As $\boldsymbol{z}$ satisfies the boundary conditions $\boldsymbol{z}\big|_{\partial S}=0$ and $\partial_r z_3\big|_{\partial S}=0$ we have that $\mathsf{W}_\alpha\boldsymbol{z}\big|_{\Gamma_0^\varepsilon}=0$. Moreover we have the formula:

$$\mathsf{W}_3\boldsymbol{z}\big|_{\Gamma_0}+\varepsilon\hat{\Psi}^1_3\boldsymbol{z}\big|_{R=0}+\varepsilon^2\hat{\Psi}^2_3\boldsymbol{z}\big|_{R=0}=\varepsilon^2c_2\rho^\alpha_\alpha(\boldsymbol{z})\big|_{\partial S}.$$

As corrector, we will thus consider the displacement defined as follows: let $\chi(r)$ be a cut-off function satisfying $\chi(0)=1$ and $\chi(r)=0$ for $r$ sufficiently large ($r\geq r_0$). We can always suppose that $|\partial_r\chi(r)|\leq r_0^{-1}$. We set

$$w^{\mathsf{cor}}_\alpha=0\quad\text{and}\quad w^{\mathsf{cor}}_3=\chi(r)\big(\varepsilon\hat{\Psi}^1_3\boldsymbol{z}+\varepsilon^2\hat{\Psi}^2_3\boldsymbol{z}-\varepsilon^2c_2\rho^\alpha_\alpha\big|_{\partial S}\big). \tag{9.7}$$

Here, the boundary layer terms $\hat{\Psi}^1_3\boldsymbol{z}$ and $\hat{\Psi}^2_3\boldsymbol{z}$ are functions of $(\varepsilon^{-1}r,s,\varepsilon^{-1}x_3)$ and exponentially decreasing in $R=\varepsilon^{-1}r$. Notice that using the expressions of $\hat{\Psi}^1_3$ and $\hat{\Psi}^2_3$, we verify that $\boldsymbol{w}^{\mathsf{cor}}$ has the dimension of a lenght. By definition, we have that $\boldsymbol{w}+\boldsymbol{w}^{\mathsf{cor}}\in V(\Omega^\varepsilon)$. It remains to estimate the energy of $\boldsymbol{w}^{\mathsf{cor}}$.

**Proposition 9.3** - *Let* $\boldsymbol{w}^{\mathsf{cor}}$ *defined by the equation* (9.7)*, then we have the estimate*

$$\widetilde{E}^\varepsilon_{\mathsf{3D}}[\boldsymbol{w}^{\mathsf{cor}}]\lesssim E\varepsilon^2\Big(|\boldsymbol{\gamma}|^2_{0\,;\,\partial S}+\varepsilon^2|\boldsymbol{\rho}|^2_{0\,;\,\partial S}\Big)\Big(1+\varepsilon^2\mathsf{r}^{-2}+\varepsilon^4\mathsf{r}^{-4}\Big)+E\varepsilon^4\Big(|\boldsymbol{\gamma}|^2_{1\,;\,\partial S}+\varepsilon^2|\boldsymbol{\rho}|^2_{1\,;\,\partial S}\Big)$$
$$+E|\boldsymbol{\rho}|^2_{0\,;\,\partial S}\Big(\varepsilon^5r_0\mathsf{r}^{-2}(1+\varepsilon^2\mathsf{r}^{-2})\Big)+E\varepsilon^5r_0|\boldsymbol{\rho}|^2_{1\,;\,\partial S} \tag{9.8}$$

Using the definitions of $L_\flat$ and $\ell$, this estimate proves that

$$\widetilde{E}^{\varepsilon}_{\mathsf{3D}}[\boldsymbol{w}^{\mathsf{cor}}] \lesssim \breve{B}_S(\varepsilon;\boldsymbol{z})\mathsf{E}^{\varepsilon}_{\mathsf{2D}}[\boldsymbol{z}]$$

where $\breve{B}_S(\varepsilon;\boldsymbol{z})$ is defined by (1.7).

**Proof of Proposition 9.3.** Using the fact that only the transverse component of $\boldsymbol{w}^{\mathsf{cor}}$ is non zero, we have that

$$\widetilde{E}^{\varepsilon}_{\mathsf{3D}}[\boldsymbol{w}^{\mathsf{cor}}] \lesssim E\|\partial_i w_3^{\mathsf{cor}}\|^2_{\mathrm{L}^2(\Omega^\varepsilon)} + E\mathsf{r}^{-2}(1+\varepsilon^2\mathsf{r}^{-2})\|w_3^{\mathsf{cor}}\|^2_{\mathrm{L}^2(\Omega^\varepsilon)}\,.$$

We easily see that

$$\|\chi(r)\varepsilon^2 c_2\rho^\alpha_\alpha\big|_{\partial S}\|^2_{\mathrm{L}^2(\Omega^\varepsilon)} \lesssim \varepsilon^5 r_0|\boldsymbol{\rho}|^2_{0\,;\,\partial S}$$

and using the fact that $|\partial_r\chi(r)| \leq r_0^{-1} \leq \mathsf{r}^{-1}$, we have

$$\|\partial_\alpha\chi(r)\varepsilon^2 c_2\rho^\alpha_\alpha\big|_{\partial S}\|^2_{\mathrm{L}^2(\Omega^\varepsilon)} \lesssim \varepsilon^5\Big(r_0\mathsf{r}^{-2}|\boldsymbol{\rho}|^2_{0\,;\,\partial S} + r_0|\boldsymbol{\rho}|^2_{1\,;\,\partial S}\Big).$$

Moreover, by doing the change of coordinates $(r,s,x_3)\mapsto(R,s,X_3)$ and using the fact that $\overline{\varphi}{}^1_3(R,X_3)$ is exponentially decaying with respect to $R$ and adimensional, we have that

$$\|\chi(r)\varepsilon\hat{\Psi}^1_3\boldsymbol{z}\|^2_{\mathrm{L}^2(\Omega^\varepsilon)} = \varepsilon^4\int_{\partial S}\int_{\Sigma^+}\chi(\varepsilon R)^2\Big(p\gamma^\alpha_\alpha(\boldsymbol{z})\big|_{\partial S}\Big)^2\overline{\varphi}{}^1_3(R,X_3)^2\,\mathrm{d}s\,\mathrm{d}R\,\mathrm{d}X_3 \lesssim \varepsilon^4|\boldsymbol{\gamma}|^2_{0\,;\,\partial S}\,.$$

Moreover, using again the fact that $|\partial_r\chi(r)| \leq r_0^{-1} \leq \mathsf{r}^{-1}$, we have

$$\|\partial_\alpha\chi(r)\varepsilon\hat{\Psi}^1_3\boldsymbol{z}\|^2_{\mathrm{L}^2(\Omega^\varepsilon)} \lesssim \varepsilon^2(1+\varepsilon^2\mathsf{r}^{-2})|\boldsymbol{\gamma}|^2_{0\,;\,\partial S} + \varepsilon^4|\boldsymbol{\gamma}|^2_{1\,;\,\partial S}\,.$$

But similar computations for the terms $\chi(r)\varepsilon^2\hat{\Psi}^2_3\boldsymbol{z}$ show that

$$\|\chi(r)\varepsilon^2\hat{\Psi}^2_3\boldsymbol{z}\|^2_{\mathrm{L}^2(\Omega^\varepsilon)} \lesssim \varepsilon^6|\boldsymbol{\rho}|^2_{0\,;\,\partial S}$$

and

$$\|\partial_\alpha\chi(r)\varepsilon^2\hat{\Psi}^2_3\boldsymbol{z}\|^2_{\mathrm{L}^2(\Omega^\varepsilon)} \lesssim \varepsilon^4(1+\varepsilon^2\mathsf{r}^{-2})|\boldsymbol{\rho}|^2_{0\,;\,\partial S} + \varepsilon^6|\boldsymbol{\rho}|^2_{1\,;\,\partial S}$$

and this shows the result. ∎

Using the definition of the wave length $L_\flat$ and $\ell$, we deduce the bound (1.7) from the previous Proposition.

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