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Abstract: An $M/M/1$ queue whose server rate depends on the state of an independent Ornstein-Uhlenbeck diffusion process is studied in this paper by means of a regular perturbation analysis. Specifically, if $(X(t))$ denotes the modulating Ornstein-Uhlenbeck process, then the server rate at time t is $\phi(X(t))$, where ϕ is some given function. After establishing the Fokker-Planck equation characterizing the joint distribution of the occupation process of the $M/M/1$ queue and the state of the modulating Ornstein-Uhlenbeck process, we show, under the assumption that the server rate is weakly perturbed by the diffusion process, that the problem can be solved via a perturbation analysis of a self-adjoint operator defined in an adequate Hilbert space. We then perform a detailed analysis when the perturbation function is linear, namely of the form $\phi(x) = 1 - \varepsilon x$. We compute in particular the different terms of the expansion of the solution in power series of ε and we determine the radius of convergence of the solution. The results are finally applied to study of the integration of elastic and streaming flows in telecommunication network and we show that at the first order the reduced service rate approximation is valid.

Key-words: $M/M/1$ queue, Self-Adjoint Operators, Perturbation Analysis, Power Series Expansion, Reduced Service Rate

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Étude de perturbation d'une file $M/M/1$ dirigée par une diffusion

Résumé : On étudie le processus du nombre de clients d'une file d'attente $M/M/1$ dont le taux de service est fonction d'un processus d'Ornstein-Uhlenbeck. Une analyse de perturbation est conduite pour ce modèle.

Mots-clés : File $M/M/1$. Opérateur auto-adjoint.

1 Introduction

We study in this paper an $M/M/1$ queue whose server rate is time varying. We specifically assume that the server rate depends upon a random process $(X(t))$ so that the server rate at time t is $\phi(X(t))$ for some function ϕ. The study of this model is motivated by the following problem related to bandwidth sharing in telecommunication networks. Consider a link carrying elastic traffic corresponding to long file transfers together with a small proportion of traffic, which does not adapt to the level of congestion of the network, referred to as unresponsive traffic. On the one hand, long flows are usually controlled by TCP, which adapts the transmission rate of long flows according to the level of congestion of the network. If we consider the bottleneck link, then a reasonable assumption consists of using the processor sharing discipline for modeling how bandwidth is shared among long file transfers. Moreover, long flows are assumed to arrive according to a Poisson process. Under these classical modeling assumptions, we thus have an $M/G/1$ processor sharing queue (see for instance Massoulié and Roberts [12]).

On the other hand, unresponsive traffic is usually due to short file transfers, which are too small to achieve some bandwidth sharing within the network. It is however worth noting that with the emergence of multimedia applications in the Internet, unresponsive traffic may also be generated by streaming applications. This type of traffic is carried by the uncontrolled UDP protocol, which is not able to adapt to network conditions. As far as responsive flows are concerned, everything happens as if the transmission capacity seen by responsive long flows were reduced up to the aggregated bit rate of short flows.

Queueing systems with time varying server rate have been studied in the literature in many different situations. In Núñez-Queija and Boxma [16], the authors consider a queueing system where priority is given to some flows driven by Markov Modulated Poisson Processes (MMPP) with finite state spaces and the low priority flows share the remaining server capacity according to the processor sharing discipline. By assuming that arrivals are Poisson and service times are exponentially distributed, the authors solve the system via a matrix analysis. Similar models have been investigated in Núñez-Queija [14, 15] by still using the quasi-birth and death process associated with the system and a matrix analysis. The integration of elastic and streaming flows has been studied by Delcoigne *et al* [7], where stochastic bounds for the mean number of active flows have been established. More recently, priority queueing systems with fast dynamics, which can be described by means of quasi birth and death processes, have been studied via a perturbation analysis of a Markov chain by Altman *et al* [2].

In this paper, we assume that the process modulating the server rate is a diffusion process and more precisely an Ornstein-Uhlenbeck (OU) process. This assumption is motivated by the following facts.

1. An OU process reasonably represents the aggregated bit rate of the superposition of a large number of short flows. In particular, when those flows have exponentially distributed duration, then exact heavy traffic results show that the aggregated bit

properly rescaled converges in distribution to an OU process, see Iglehart [10] for example.

2. An OU process has only two parameters (namely the mean and the variance), which can be empirically identified in practical situations. Furthermore, the impact of these parameters on the performance of the system will be much easier to understand, when compared with the case of MMPP environment where these variables are somewhat hidden in the numerous parameters of the MMPP environment.

In a first step, we establish the Fokker-Planck equations of the system and we show that these equations can be seen as an eigenvalue problem for a self-adjoint operator defined in some adequate Hilbert space. This last property is closely related to the time-reversibility properties of the $M/M/1$ occupation process and of the Ornstein-Uhlenbeck process. We then show that when the interaction between the OU process and the $M/M/1$ queue is weak and depends upon a small parameter ε, the problem of computing the generating function of the number of customers in the $M/M/1$ queue can be solved by means of a regular perturbation analysis. It is possible to completely compute the coefficients of the expansion in power series of ε of the solution. Also, the radius of convergence is determined. By taking into account the first order only, the above analysis shows that a reduced service rate pertains.

This paper is organized as follows: In Section 3, the Fokker-Planck equation is established. This equation can be interpreted as an eigenvalue problem for a self-adjoint operator is given in Section 4. When the interaction between the $M/M/1$ queue and the OU process is weak, the formulation as a perturbation problem is presented in Section 5, where the case of a linear perturbation function is completely solved.

2 Problem formulation

2.1 Model description

We consider a single link with transmission capacity equal to unity. We suppose that two classes of customers are multiplexed on this link and that the first class has priority over the second class. More precisely, if there are $N(t)$ customers of the first class in the system at time t, we assume that the service rate for the customers of the second class is equal to $\phi(N(t))$ for some function $\phi(x)$ which is decreasing and such that $\phi(0) = 1$. Moreover, we assume that the number of class 1 customers is sufficiently large so that the process $N(t)$ properly rescaled converges in distribution to an OU process (X_t) satisfying the stochastic differential equation

$$dX(t) = -\alpha(X(t) - m)dt + \sigma dB(t), \tag{1}$$

where $(B(t))$ is a standard Brownian motion and α and σ are positive constants.

This situation typically occurs when class 1 customers arrive according to a Poisson process with rate u, require exponential service times with unit mean and have a peak bit rate which is negligible with respect to the link transmission capacity. Since class 1 customers have priority over class 2 customers and contention for those customers can be neglected, the process describing the number of class 1 customers then corresponds to the occupation process of an $M/M/\infty$ queue. When u tends to infinity, classical heavy traffic results (see Borovkov [4] or Iglehart [10]) then yield

$$\left(\frac{N(t)-u}{\sqrt{u}}, t \geq 0\right) \xrightarrow{d} (X(t), t \geq 0),$$

where the OU process $(X(t))$ satisfies Equation (1) with $\alpha = -1$ and $\sigma = \sqrt{2}$.

With the above assumptions, the server rate for class 2 customers is a function of $X(t)$, which is denoted by $\phi(X(t))$ (with $\phi(0) = 1$). We now assume that class 2 customers arrive according to a Poisson process with intensity λ and require exponential service times with mean $1/\mu$. If $L(t) = l$ denotes the number of class 2 customers in the system and $X(t) = x$ at time t, then the transitions of $(L(t))$ are given by

$$l \to \begin{cases} l+1 & \text{with rate } \lambda, \\ l-1 & \text{with rate } \mu\phi(x). \end{cases}$$

The process describing the number of class 2 customers is thus equal to the occupation process of an $M/M/1$ queue, which server rate depends upon a diffusion process. In the following, the function $\phi(x)$ will be referred to as perturbation function.

Throughout this paper, we assume that the diffusion process $(X(t))$ is in stationary regime. Its stationary distribution is a normal distribution with mean m and variance $\sigma^2/(2\alpha)$; its density function on $\mathbb{R}$ is therefore given by

$$p(x) \stackrel{\text{def.}}{=} \frac{1}{\sigma}\sqrt{\frac{\alpha}{\pi}} \exp\left(-\frac{\alpha(x-m)^2}{\sigma^2}\right). \tag{2}$$

Let us note that the stability condition for the system reads

$$\rho \stackrel{\text{def.}}{=} \frac{\lambda}{\mu} < \mathbb{E}[\phi(X(0))],$$

and will be assumed to hold throughout the paper. Under this assumption, it is straightforward to show the existence of a stationary probability distribution for the Markov process $(X(t), L(t))$. See Meyn and Tweedie [13] for example.

2.2 The independent case and its perturbation

When $\phi(x) \equiv 1$, the processes $(X(t))$ and $(L(t))$ are clearly independent one of each other. In this case, at equilibrium, for $t \geq 0$, the variable $L(t)$ has a geometric distribution with parameter ρ and we consequently have the relation

$$\mathbb{E}\left(u^{L(t)} \mathbf{1}_{[x,x+dx]}(X(t))\right) = \frac{(1-\rho)}{(1-\rho u)} p(x)\, dx. \tag{3}$$

As it will be seen in Section 3, when ϕ is not constant, it is extremely difficult to get some explicit results on the equilibrium distribution of $(L(t))$. For this reason, this paper addresses the case when the queue is almost independent with respect to the OU process. More precisely, it is assumed that the function $\phi(x)$ is given by $1 - \varepsilon x$ for some small $\varepsilon \geq 0$. The goal of this paper is to derive an expansion of the distribution of the stationary distribution of $(L(t))$ with respect to ε. In particular, the following theorem will be proved.

Theorem 1. *For ε sufficiently small, the first order expansion of the generating function of the stationary distribution of $(L(t))$ is given by*

$$\mathbb{E}\left(u^L\right) = \frac{1-\rho}{1-\rho u} - \frac{\rho(1-u)}{(1-\rho u)^2} m\varepsilon + o(\varepsilon).$$

Therefore, $\mathbb{E}[u^{L(t)}] \sim \mathbb{E}[u^{L_\varepsilon}]$, where L_ε has the stationary distribution of the number of customers in an $M/M/1$ queue when the server rate is $1 - \varepsilon m$. This shows a principle of reduced service rate approximation.

3 Fokker-Planck equations

The goal of this section is to establish the Fokker-Planck equation for the process $(X(t), L(t))$ in the stationary regime, i.e., the evolution equation for the probability density function $p(x, \ell)$ for $x \in \mathbb{R}$ and $\ell \in \mathbb{N}$. By construction, it is easily checked that the process $(X(t), L(t))$ is a Markov process taking values in $\mathbb{R} \times \mathbb{N}$. The following result gives its infinitesimal generator.

Lemma 1. *The process $(X(t), L(t))$ is a Markov process in $\mathbb{R}\times\mathbb{N}$ with infinitesimal generator $\mathcal{G}$ defined by*

$$\mathcal{G}f(x,\ell) = \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}(x,\ell) - \alpha(x-m)\frac{\partial f}{\partial x}(x,\ell) \\ + \lambda[f(x,\ell+1) - f(x,\ell)] + \mu\phi(x)f(x)\mathbf{1}_{\{\ell>0\}}[f(x,\ell-1) - f(x,\ell)], \quad (4)$$

for every function $f(x, \ell)$ from $\mathbb{R} \times \mathbb{N}$ in $\mathbb{R}$, twice differentiable with respect to the first variable.

Proof. According to Equation (1), the infinitesimal generator of an Ornstein-Uhlenbeck process applied to some twice differentiable function g on $\mathbb{R}$ is given by

$$\frac{\sigma^2}{2}\frac{\partial^2 g}{\partial x^2}(x) - \alpha(x-m)\frac{\partial g}{\partial x}(x).$$

The second part of Equation (4) corresponds to the infinitesimal generator of the number of customers of a classical $M/M/1$ queue with arrival rate λ and service rate $\mu\phi(x)$, when the OU process is in state x. ☐

Let P denote the stationary probability distribution of the couple $(X(t), L(t))$,

$$P(x,\ell) = \mathbb{P}(X \leq x, L = \ell),$$

the probability density function $p(x,\ell)$ is

$$p(x,\ell) = \frac{\partial P}{\partial x}(x,\ell)$$

and the generating function is given by

$$g_u(x) = \sum_{\ell=0}^{\infty} p(x,\ell) u^\ell \tag{5}$$

for $u \in (0,1)$ and $x \in \mathbb{R}$.

The equation of invariant measure for the Markov process $(L(t), X(t))$ is given by

$$\sum_{\ell \geq 0} \int_{\mathbb{R}} \mathcal{G} f(x,\ell) P(dx,\ell) = 0,$$

for a twice differentiable function f with respect to the first variable. By choosing convenient test functions, one readily gets the Fokker-Planck equations.

Lemma 2 (Fokker-Planck equations). *The function $p(x,\ell)$ satisfies the relation*

$$\begin{aligned}\frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2} + \alpha(x-m)\frac{\partial p}{\partial x} + \alpha p(x,\ell) + \lambda \mathbf{1}_{\{\ell>0\}} p(x,\ell-1) \\ - \left(\lambda + \mu\phi(x)\mathbf{1}_{\{\ell>0\}}\right) p(x,\ell) + \mu\phi(x)\mathbf{1}_{\{\ell>0\}} p(x,\ell+1) = 0. \end{aligned} \tag{6}$$

An easy consequence of the above Fokker-Planck equation is the following equation for the function g_u.

Proposition 1. *The generating function $g_u(x)$ is such that*

$$\begin{aligned}\frac{\sigma^2}{2}\frac{\partial^2 g_u}{\partial x^2} + \alpha(x-m)\frac{\partial g_u}{\partial x} + \left(\lambda(u-1) + \alpha + \mu\left(\frac{1}{u}-1\right)\phi(x)\right) g_u(x) \\ = \mu\left(\frac{1}{u}-1\right)\phi(x) g_0(x). \end{aligned} \tag{7}$$

4 Operator theoretic analysis of the Fokker Planck equation

4.1 Notation

By definition, the function $g_u(x)$ is twice weakly differentiable with respect to the variable x and is analytic in variable u in the open unit disk and continuous in the closed unit disk.

Hence, the function $g_u(x)$ can be seen as an element of the tensor product $H^2(\mathbb{R}) \otimes S(U)$, where $H^2(\mathbb{R})$ is the Sobolev space of functions which admit a second order weak derivative and $S(U)$ is the set of functions which are analytic in the unit disk and continuous in the closed unit disk.

From the previous section, the function $g : (u,x) \to g_u(x)$ defined by Equation (5) satisfies the relation

$$\Omega g(u,x) = 0$$

where Ω is the operator defined as follows: for a function $f \in H^2(\mathbb{R}) \otimes S(U)$

$$\Omega f(u,x) = \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} + \alpha(x-m)\frac{\partial f}{\partial x} + \left(\lambda(u-1) + \alpha + \mu\left(\frac{1}{u} - 1\right)\phi(x)\right) f(u,x) - \mu\left(\frac{1}{u}-1\right)\phi(x) f(0,x). \quad (8)$$

In the following, we refine the domain of definition of Ω so as to obtain a self-adjoint operator defined in an appropriate Hilbert space.

By construction, the function $g_u(x)$ is given by

$$g_u(x) = \frac{1}{\sigma}\sqrt{\frac{\alpha}{\pi}} \exp\left(-\frac{\alpha(x-m)^2}{\sigma^2}\right) \mathbb{E}[u^L \mid X = x]$$

The function $g_u(x)$ of the variable u is analytic in the open unit disk and is continuous in the closed unit disk. Moreover, the function $g_u(x)$ is such that for all $|u| \le 1$,

$$\frac{1}{\sigma}\sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{\infty} g_u(x)^2 \exp\left(\frac{\alpha(x-m)^2}{\sigma^2}\right) dx \le 1.$$

This clearly implies that for fixed u with $|u| \le 1$, the function $g_u(x)$ is in the Hilbert space H defined by

$$H = \left\{ f : \mathbb{R} \to \mathbb{C} : f \exp\left(\frac{\alpha(x-m)^2}{2\sigma^2}\right) \in H^2(\mathbb{R}) \right\}, \quad (9)$$

This Hilbert space is equipped with the scalar product defined by: for all $f, g \in H$,

$$<f, g> = \int_{-\infty}^{\infty} f(x)\overline{g(x)} \exp\left(\frac{\alpha(x-m)^2}{\sigma^2}\right) dx,$$

where $\overline{g(x)}$ is the complex conjugate of $g(x)$; the norm of an element $f \in H$ is

$$\|f\| = \sqrt{\int_{-\infty}^{\infty} |f(x)|^2 \exp\left(\frac{\alpha(x-m)^2}{\sigma^2}\right) dx}.$$

Now, for fixed x, we identify $g_u(x)$ with the sequence $(p(x,\ell), l \ge 0)$. By definition,

$$p(x,\ell) = \frac{1}{\sigma}\sqrt{\frac{\alpha}{\pi}} \exp\left(-\frac{\alpha(x-m)^2}{\sigma^2}\right) \mathbb{P}(L = \ell \mid X = x),$$

and clearly

$$\sum_{\ell=0}^{\infty} |p(x,\ell)|^2 < \frac{\alpha}{\pi\sigma^2} \exp\left(-\frac{2\alpha(x-m)^2}{\sigma^2}\right),$$

it follows that the sequence $(p(x,\ell), \ell \geq 0)$ is in the Hilbert space $L^2(\mathbb{N})$ composed of square summable sequences in $\mathbb{C}$, that is,

$$L^2(\mathbb{N}) = \left\{ f = (f_n) \in \mathbb{C}^{\mathbb{N}} : \sum_{n=0}^{\infty} |f_n|^2 < \infty \right\},$$

equipped with the scalar product defined by

$$<f, g> = \sum_{n=0}^{\infty} f_n \overline{g_n}, \tag{10}$$

for two elements $f = (f_n)$ and $g = (g_n)$. The norm of an element f of $L^2(\mathbb{N})$ is given by

$$\|f\| = \sqrt{\sum_{n=0}^{\infty} |f_n|^2}. \tag{11}$$

The operator Ω can be seen as an operator defined in the tensor product $H \otimes L^2(\mathbb{N})$, that we still denote by Ω and given by

$$\Omega = A \otimes \mathbb{I} + \mathbb{I} \otimes B + V,$$

where

- The symbol $\mathbb{I}$ denotes the identity operator in the appropriate Hilbert space.
- The operator A is defined by

$$Af = \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} + \alpha(x-m)\frac{\partial f}{\partial x} + \alpha f;$$

  the domain of definition of A is denoted by $D(A)$ and is given by

$$D(A) = \left\{ f \in H : x^2 f \exp\left(\frac{\alpha(x-m)^2}{2\sigma^2}\right) \in H^2(\mathbb{R}) \right\}.$$

- The operator B is defined by the infinite matrix

$$\begin{pmatrix} -\lambda & \mu & 0 & . & . & . \\ \lambda & -(\lambda+\mu) & \mu & 0 & . & . \\ 0 & \lambda & -(\lambda+\mu) & \mu & 0 & . \\ 0 & 0 & \lambda & -(\lambda+\mu) & \mu & . \\ . & . & . & . & . & . \end{pmatrix} \tag{12}$$

  the domain of B will be determined in the following.

- The operator V is equal to $\Phi \otimes C$, where Φ is the operator defined on $L^2(\mathbb{R})$ by the multiplication by $\phi - 1$, i.e.

$$\Phi(f)(x) = (\phi(x) - 1)f(x),$$

and C is the pure death operator, defined by the infinite matrix

$$\begin{pmatrix} 0 & \mu & 0 & . & . & . & . \\ 0 & -\mu & \mu & 0 & . & . & . \\ 0 & 0 & -\mu & \mu & 0 & . & . \\ 0 & 0 & 0 & -\mu & \mu & 0 & . \\ . & . & . & . & . & . & . \end{pmatrix} \tag{13}$$

4.2 Self-adjointness properties

In this section, we examine the properties of the operators A and B. We specifically determine under which conditions these operators are self-adjoint. The self-adjointness property will be crucial in subsequent sections to carry out a perturbation analysis. To prove self-adjointness, we use the classical tools of spectral analysis (see Dautray and Lions [6], Dunford and Schwartz [8], Reed and Simon [17] or Rudin [18] for basic elements of spectral theory).

The operator B defined in $L^2(\mathbb{N})$ is not symmetric. However, by reducing the underlying Hilbert space, we can obtain a symmetric operator as follows. Let us consider the Hilbert space $L^2_\rho(\mathbb{N})$ defined by

$$L^2_\rho(\mathbb{N}) = \left\{ f = (f_n) \in \mathbb{C}^{\mathbb{N}} : \sum_{n=0}^{\infty} |f_n|^2 \rho^{-n} < \infty \right\},$$

where $\rho = \lambda/\mu < 1$. The scalar product in $L^2_\rho(\mathbb{N})$ is defined by

$$<f, g>_\rho = \sum_{n=0}^{\infty} f_n \overline{g_n} \rho^{-n}$$

and the norm by

$$\|f\|_\rho = \sqrt{\sum_{n=0}^{\infty} |f_n|^2 \rho^{-n}}.$$

Since $\rho < 1$, the space $L^2_\rho(\mathbb{N})$ is clearly a subspace of $L^2(\mathbb{N})$. The operator B induces in $L^2_\rho(\mathbb{N})$ an operator, that we still denote by B and that is defined by the infinite matrix given by Equation (12). In the following, the symbol B refers to that operator defined in $L^2_\rho(\mathbb{N})$ by the infinite matrix (12).

Let $D(B)$ denote the domain of the operator B, i.e., the subset of $L^2_\rho(\mathbb{N})$ composed of those elements $f \in L^2_\rho(\mathbb{N})$ such that $Bf \in L^2_\rho(\mathbb{N})$. The adjoint of the operator B is denoted by B^* and is defined by: for $f \in D(B)$ and $g \in H$ by $<Bf, g>_\rho = <f, B^* g>_\rho$ and $D(B^*)$ is the domain of B^*.

In the following, the operator B is shown to be self-adjoint, i.e. $B = B^*$, which requires in particular that $D(B) = D(B^*)$. To get this property it is sufficient to prove that the operator B is

- symmetric: for all $f, g \in D(B)$, $<Bf, g>_\rho = <f, Bg>_\rho$;
- bounded: the quantity

$$\|B\|_\rho = \inf\left\{|<Bf, f>_\rho| : f \in L^2_\rho(\mathbb{N}), \|f\|^2_\rho = 1\right\}$$

is finite.

Lemma 3. *The operator B is symmetric and bounded with*

$$\|B\|_\rho \leq (\sqrt{\lambda} + \sqrt{\mu})^2, \tag{14}$$

the operator B is consequently self-adjoint.

Proof. The symmetry of the operator B is straightforward. For $f \in L^2_\rho(\mathbb{N})$,

$$<Bf, f>_\rho = \sum_{n=0}^{\infty} \left((\lambda + \mu \mathbf{1}_{\{n \geq 0\}}) f_n - \lambda f_{n-1} - \mu f_{n+1}\right) \overline{f_n} \rho^{-n},$$

consequently

$$|<Bf, f>_\rho| \leq (\lambda + \mu)||f||^2_\rho + \left|\sum_{n=0}^{\infty} \mu f_{n+1} \overline{f_n} \rho^{-n}\right| + \left|\sum_{n=0}^{\infty} \lambda f_{n-1} \overline{g_n} \rho^{-n}\right|,$$

by using Schwarz inequality, we get

$$\left|\sum_{n=0}^{\infty} \mu f_{n+1} \overline{f_n} \rho^{-n}\right| \leq \sqrt{\lambda\mu} ||f||^2_\rho,$$
$$\left|\sum_{n=0}^{\infty} \lambda f_{n-1} \overline{f_n} \rho^{-n}\right| \leq \sqrt{\lambda\mu} ||f||^2_\rho,$$

and Equation (14) follows. □

The spectrum $\sigma(B)$ of the operator B is defined by

$$\sigma(B) = \{z \in \mathbb{C} : (B - z\mathbb{I}) \text{ is not invertible}\},$$

since B is self-adjoint, $\sigma(B) \subset \mathbb{R}$. Moreover, Equation (14) implies the relation $\sigma(B) \subset [-(\sqrt{\lambda} + \sqrt{\mu})^2, \infty]$. Standard spectral theory shows that the spectrum $\sigma(B)$ can be decomposed as follows:

$$\sigma(B) = \overline{\sigma_p(B)} \cup \sigma_c(B),$$

where $\overline{\sigma_p(B)}$ is the closure of the set composed of the eigenvalues of B, referred to as point spectrum, and $\sigma_c(B)$ is the continuous spectrum. The point spectrum is purely discrete and $z \in \sigma_p(B)$ if and only if there exists some $f \in L^2_\rho(\mathbb{N})$ such that $Bf = zf$. From spectral theory, the following proposition holds.

Proposition 2. *There exists a measure $d\psi(z)$, referred to as spectral measure, whose support is $\sigma(B)$, and a family of spaces $\{\mathcal{H}_z\}$, $z \in \sigma(B)$, such that*

- *the Hilbert space $L^2_\rho(\mathbb{N})$ is equal to the direct sum of the spaces $\mathcal{H}_z$, that is,*

$$H = \int^{\oplus} \mathcal{H}_z d\psi(z), \tag{15}$$

  i.e. every $f \in L^2_\rho(\mathbb{N})$ can be decomposed into a family $(f_z, z \in \sigma(B))$, where $f_z \in \mathcal{H}_z$ and $\int \|f_z\|^2_\rho \psi(z) < \infty$. Moreover,

$$<f, g>_\rho = \int <f_z, g_z>_\rho d\psi(z).$$

- *The operator B is such $(Bf)_z = zf_z$ for $z \in \sigma(B)$, where $(Bf)_z$ is the projection of (Bf) on the space $\mathcal{H}_z$.*

Note that z is an eigenvalue of the operator B if and only if $\psi(\{z\}) > 0$ and that the space $\mathcal{H}_z$ is a subset of $L^2_\rho(\mathbb{N})$ if and only if z is an eigenvalue. The next result gives the explicit representation of the spectral measure and the spaces $\mathcal{H}_z$ appearing in Decomposition (15).

Proposition 3. *The spectral measure $d\psi(x)$ is given by*

$$\int f(x) d\psi(x) = (1-\rho) f(0) \\ - \frac{\sqrt{\rho}}{\pi} \int_{-(\sqrt{\lambda}+\sqrt{\mu})^2}^{-(\sqrt{\lambda}-\sqrt{\mu})^2} \frac{f(x)}{x} \sqrt{1 - \left(\frac{x+\lambda+\mu}{2\sqrt{\lambda\mu}}\right)^2}\, dx, \tag{16}$$

for any non-negative Borelian function f.

The operator B has a unique eigenvalue at 0, and the associated eigenvector is up to a multiplicative constant the sequence $e(\rho)$ whose nth element is equal to ρ^n. The space $\mathcal{H}_0$ is the space spanned by the vector $e(\rho)$.

For $z \in (-(\sqrt{\lambda}+\sqrt{\mu})^2, -(\sqrt{\lambda}-\sqrt{\mu})^2)$, the space $\mathcal{H}_z$ is the vector space spanned by the sequence $(Q_n(z))$ defined by the following recursion:

$$Q_0(z) = 1, Q_1(z) = (z+\lambda)/\mu$$
$$\mu Q_{n+1}(z) - (z+\lambda+\mu) Q_n(z) + \mu Q_{n-1}(z) = 0, \quad n \geq 2.$$

The sequence $(Q_n(z))$ for $z \in (-(\sqrt{\lambda}+\sqrt{\mu})^2, -(\sqrt{\mu}-\sqrt{\lambda})^2)$ forms an orthogonal family.

Proof. To determine the spectrum of the operator B, we consider the equation $Bf(z) = zf(z)$, where $f(z) = (f_n(z)) \in \mathbb{C}^{\mathbb{N}}$. By assuming that $f_0(z) = 1$, the sequence $f(z)$ satisfies the recurrence relation:

$$\begin{aligned} & f_0(z) = 1, f_1(z) = (z+\lambda)/\mu, \\ & \mu f_{n+1}(z) - (z+\lambda+\mu) f_n(z) + \lambda f_{n-1}(z) = 0, \quad n \geq 2. \end{aligned} \tag{17}$$

The above three-term recurrence relation implies that $f_n(z)$ is a polynomial in variable z with degree n and that the polynomials $(f_n(z))$ form an orthogonal polynomial system since Favard's condition is obviously satisfied (see Askey and Ismail [3] for details). The orthogonality measure of these polynomials is precisely the spectral measure of the operator B since B is self-adjoint.

To determine $d\psi(x)$, we compute the limiting value as n tends to infinity of the ratio $f_n^*(z)/f_n(z)$, where $f_n^*(z)$, $n = 0, 1, 2, \ldots$ are the associate polynomials, which satisfy recurrence relation (17) with the initial conditions

$$f_0(z) = 0 \text{ and } f_1(z) = 1/\mu.$$

Straightforward computations yield that, for $z \notin [-(\sqrt{\lambda}+\sqrt{\mu})^2, -(\sqrt{\lambda}-\sqrt{\mu})^2]$,

$$f_n(z) = \frac{1}{Z_+ - Z_-} \left[\left(\frac{\lambda + z - \mu + \sqrt{\delta(z)}}{2\mu} \right) Z_+^n + \left(\frac{-\lambda - z + \mu + \sqrt{\delta(z)}}{2\mu} \right) Z_-^n \right],$$

where

$$Z_\pm = \frac{z + \lambda + \mu \pm \sqrt{\delta(z)}}{2\mu}$$

with $\delta(z) = (z+\lambda+\mu)^2 - 4\lambda\mu$. Moreover, the associated polynomials $f_n^*(z)$ are given by: for $z \notin [-(\sqrt{\lambda}+\sqrt{\mu})^2, -(\sqrt{\lambda}-\sqrt{\mu})^2]$

$$f_n^*(z) = \frac{1}{\mu(Z_+ - Z_-)} \left[Z_+^n - Z_-^n \right].$$

Stieltjes theory [9] states that the measure $d\psi(x)$ has a support included in $(-\infty, 0]$ and that

$$\int_{-\infty}^0 \frac{d\psi(x)}{z - x} = \chi(z),$$

where $\chi(z)$ is the continued fraction whose nth approximant is $f_n^*(z)/f_n(z)$. The function $\chi(z)$ for $z \notin (-\infty, 0]$ is given by

$$\chi(z) = \lim_{n\to\infty} \frac{f_n^*(z)}{f_n(z)}.$$

It is easily checked that for $z > 0$, $Z_+ > Z_- > 0$ and then for $z > 0$,

$$\chi(z) = \frac{2}{\lambda + z - \mu + \sqrt{\delta(z)}}. \tag{18}$$

The function on the right hand side of Equation (18) can be analytically continued to the complex plane deprived of the segment $[-(\sqrt{\lambda}+\sqrt{\mu})^2, -(\sqrt{\lambda}-\sqrt{\mu})^2]$ and the origin. More precisely, the function $\chi(z)$ has a unique pole at $z=0$ and its residue is equal to $(1-\rho)$. The eigenvector associated with the eigenvalue 0 is the vector which nth component is ρ^n. (This vector clearly belongs to $L^2_\rho(\mathbb{N})$.)

From Perron-Stieltjes inversion formula, see Askey and Ismail [3] and Henrici [9], the continuous spectrum of the measure $d\psi(x)$ is given by

$$\frac{d\psi(x)}{dx} = \lim_{\varepsilon\to 0}\frac{1}{2i\pi}(\chi(x-i\varepsilon)-\chi(x+i\varepsilon)).$$

It is easily checked that the above limit is non null only for x in the interval $(-(\sqrt{\lambda}+\sqrt{\mu})^2, -(\sqrt{\lambda}-\sqrt{\mu})^2)$ and, in that case,

$$d\psi(x) = -\frac{\sqrt{\rho}}{\pi x}\sqrt{1-\frac{(x+\lambda+\mu)^2}{4\lambda\mu}}dx.$$

It is worth noting that $d\psi(x)$ is very close to that the corresponding spectral measure associated with Chebyshev polynomials. In fact, the polynomials under consideration here differ from Chebyshev polynomials only through the initial conditions (see Chihara [5] for an exhaustive treatment of classical orthogonal polynomials).

It is easily checked that

$$\begin{aligned}\int_{-(\sqrt{\lambda}+\sqrt{\mu})^2}^{-(\sqrt{\lambda}-\sqrt{\mu})^2} d\psi(x) &= \frac{2\sqrt{\lambda\mu\rho}}{\pi}\int_{-1}^{1}\frac{\sqrt{1-x^2}}{\lambda-2\sqrt{\lambda\mu}x+\mu}\,dx\\ &= \frac{2\rho}{\pi}\sum_{n=0}^{\infty}\rho^{n/2}\int_{-1}^{1}U_n(x)\sqrt{1-x^2}\,dx = \rho,\end{aligned}$$

where $U_n(x)$, $n=0,1,2,\dots$ are the Chebyshev polynomials of the second kind, which are orthonormal with respect to the weight measure $w(x)dx$ with

$$w(x) = \sqrt{1-x^2}\,\mathbf{1}_{(-1,1)}(x).$$

It follows that the total mass of $d\psi(x)$ is

$$\int_{-\infty}^{0} d\psi(x) = 1.$$

The orthogonality of the sequences $(Q_n(z))$ for $z\in[-(\sqrt{\lambda}+\sqrt{\mu})^2, -(\sqrt{\lambda}-\sqrt{\mu})^2]$ and Equation (16) are therefore established. □

It is worth noting that the point spectrum of the operator B contains only one point and that the continuous spectrum is the interval $(-(\sqrt{\lambda}+\sqrt{\mu})^2, -(\sqrt{\lambda}-\sqrt{\mu})^2)$.

Let us now examine the properties of the operator A. This operator is closely related to the harmonic oscillator in quantum mechanics (see Reed and Simon [17] for details). Indeed, for $f \in D(A)$ and $h(x) = f(x)\exp[\alpha(x-m)^2/(2\sigma^2)]$ we have

$$Af = \alpha\left(\frac{\sigma^2}{2\alpha}\frac{\partial^2 h}{\partial x^2} + \left(\frac{1}{2} - \frac{\alpha(x-m)^2}{2\sigma^2}\right)h\right)\exp\left(-\frac{\alpha(x-m)^2}{2\sigma^2}\right).$$

We then have the following result, where we use the Hermite functions $H_\nu(x)$, the Hermite polynomials $H_n(x)$, as well as the parabolic cylinder functions $D_n(x)$, also referred to as Whittaker functions, see Abramowitz and Stegun [1] or Lebedev [11].

Proposition 4. *The operator A is self-adjoint in H. Its spectrum is purely discrete, composed of the numbers of the form $-2\alpha n$ for $n \geq 0$. The eigenvector associated with the eigenvalue $-\alpha n$ is the function*

$$\varphi_n(x) = \gamma_n \exp\left(-\frac{\alpha(x-m)^2}{\sigma^2}\right) H_n\left(\frac{\sqrt{\alpha}(x-m)}{\sigma}\right), \tag{19}$$

where $H_n(x)$ is the nth Hermite polynomial and

$$\gamma_n^2 = \frac{\sqrt{\alpha}}{2^n n! \sigma\sqrt{\pi}}.$$

The sequence (φ_n) forms an orthonormal basis of H.

Proof. The Hilbert space H defined by Equation (9) and $H^2(\mathbb{R})$ are obviously isomorphic.

Let κ denote canonical isomorphism from H into $H^2(\mathbb{R})$. By construction, this isomorphism preserves the scalar product. The image of the operator A by the isomorphism κ is the operator $\alpha(-\mathcal{A} + \frac{1}{2}\mathbb{I})$ where the operator $\mathcal{A}$ is the harmonic oscillator operator defined by

$$\mathcal{A}h = -\frac{\sigma^2}{2\alpha}\frac{\partial^2 h}{\partial x^2} + \frac{\alpha(x-m)^2}{2\sigma^2}h.$$

The domain of definition of the operator $\mathcal{A}$ is

$$D(\mathcal{A}) = \left\{h \in H^2(\mathbb{R}) : x^2 h \in L^2(\mathbb{R})\right\}.$$

It is well known that the operator $\mathcal{A}$ is self-adjoint. The functions $h_n(x) = D_n(\sqrt{2\alpha}(x - m)/\sigma)$, $n \geq 0$, where the functions D_n are Whittaker parabolic cylinder functions [11], satisfy

$$\mathcal{A}h_n = \left(n + \frac{1}{2}\right)h_n$$

Since the functions D_n for $n \geq 0$ form an orthogonal basis of $H^2(\mathbb{R})$, the spectrum of $-\mathcal{A}$ is purely discrete and composed of the numbers $n + 1/2$ for $n \geq 0$. It follows that the operator

A is self-adjoint in H. Its eigenvalues are the numbers $-\alpha n$, $n \geq 0$ and the eigenvectors associated with the eigenvalue $-\alpha n$ is

$$\tilde{\varphi}_n(x) = \exp\left(-\frac{\alpha(x-m)^2}{2\sigma^2}\right) D_n(\sqrt{2\alpha}(x-m)/\sigma).$$

By using the relation between Whittaker and Hermite functions [11, p. 284] and by normalizing, Equation (19) follows. □

The main Hilbert space $\mathcal{H}$ used in this paper is defined as the tensor product of the spaces H and $L^2_\rho(\mathbb{N})$, that is, $\mathcal{H} = H \otimes L^2_\rho(\mathbb{N})$. In view of the above results, an element of this Hilbert space is defined by a sequence $(c_{n,k})$ and can be written as

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} c_{n,k}\varphi_n \otimes e_k,$$

where e_k is the sequence with all elements equal to 0 except the kth one equal to 1 and φ_n is defined by Equation (19).

The Hilbert space $\mathcal{H}$ is equipped with the scalar product $<\cdot,\cdot>$ defined by: for $f = (f_{n,k})$ and $g = (g_{n,k})$ in $\mathcal{H}$

$$<f,g> = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} f_{n,k}\overline{g}_{n,k}\rho^{-n};$$

the norm is defined as

$$\|f\|^2 = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} |f_{n,k}|^2\rho^{-n}.$$

Note also that, as a consequence of Proposition 3, an element $f \in \mathcal{H}$ can also be represented as

$$f = \sum_{n\geq 0}\int_{-(\sqrt{\lambda}+\sqrt{\mu})^2}^{-(\sqrt{\lambda}-\sqrt{\mu})^2} c_n(y)\varphi_n \otimes \mathcal{Q}(y)\, d\psi(y) + \sum_{n\geq 0} c_n(0)\varphi_n \otimes e(\rho),$$

where the measure $d\psi(y)$ and the sequence $\mathcal{Q}(y)$ are defined by Proposition 3 and the function $c_n : \mathbb{R} \to \mathbb{C}$ is such that

$$\sum_{n\geq 0}\int_{-(\sqrt{\lambda}+\sqrt{\mu})^2}^{-(\sqrt{\lambda}-\sqrt{\mu})^2} |c_n(y)|^2\, d\psi(y) < \infty.$$

If we consider the equation $(A\otimes\mathbb{I}+\mathbb{I}\otimes B)f = 0$ for a non-zero function f of $\mathcal{H}$ represented by

$$f = \sum_{n\geq 0} c_n\varphi_n \otimes g,$$

for some $g = (g_n) \in L^2_\rho(\mathbb{N})$ and (c_n) a sequence of $\mathbb{C}$, then we have for all $n \geq 0$,

$$c_n(-\alpha n g + Bg) = 0.$$

Since the single eigenvalue of B is 0, we have $c_n = 0$ for $n > 1$. Now, if we want that $c_0 \neq 0$, then $g = \kappa e(\rho)$ for some $\kappa > 0$. If we impose that the sum of the g_n is 1, then $\kappa = 1 - \rho$. This shows that, for $\phi \equiv 1$, the unique non-null solution to the equation $(A\otimes\mathbb{I}+\mathbb{I}\otimes B)f = 0$ is proportional to $\varphi_0 \otimes e(\rho)$.

To conclude this section, let us examine the properties of the operator V.

Proposition 5. *If the function $|\phi - 1|$ is upper bounded by a constant K, then the operator V is bounded and its norm $||V|| \leq K\mu$.*

Proof. For $g \in L^2_\rho(\mathbb{N})$, $<Dg, g>_\rho \leq \mu ||g||^2$. Hence, for any element $f = (f_n(x)) \in \mathcal{H}$,

$$(Vf, f) \leq \int_{-\infty}^{\infty} |\phi(x) - 1| \mu \sum_{n\geq 0} ||f_n(x)||^2 \exp\left(\frac{\alpha(x-m)^2}{\sigma^2}\right) dx \leq K\mu ||f||^2,$$

and the result follows. □

The above result indicates that when the function $\phi(x) - 1$ is bounded, then the operator V appears as a nice self-adjoint perturbation of the operator $A\otimes\mathbb{I}+\mathbb{I}\otimes B$. In the following, we have to deal with a more complex perturbation function of the form $\phi(x) = 1 - \varepsilon x$. The multiplication by x is clearly not bounded in H and the above result can not be applied.

In the remainder of this paper, an element $f = (f_{n,k})$ is identified with the function in $\mathcal{H}$ defined by

$$f_u(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} c_{n,k} u^k \varphi_n(x).$$

5 Perturbation analysis

In this section, we assume that the perturbation function ϕ is of the form $1 - \varepsilon x$ for some $\varepsilon \ll 1$. The operator V thus appears as a small perturbation of the self-adjoint operator $A\otimes\mathbb{I}+\mathbb{I}\otimes B$. We then perform a classical perturbation analysis by studying the modification of the function $g_u(x)$ given by Equation (3) due to the perturbation.

In the following, we search for a solution to the Fokker-Planck Equation (7), which belongs to the reference Hilbert space $\mathcal{H}$. We specifically assume that the solution can be expanded as

$$g_{u,\varepsilon}(x) = g_u^{(0)}(x) + \varepsilon g_u^{(1)}(x) + \varepsilon^2 g_u^{(2)}(x) + \cdots, \tag{20}$$

where the functions $g_u^{(n)}(x)$ for $n \geq 0$ belongs to the Hilbert space $\mathcal{H}$. The function $g_u^{(0)}(x)$ corresponds to the case $\varepsilon = 0$ and is given by Equation (3).

The ultimate goal of this section is to prove that the elements $g^{(n)}$ has to satisfy a recurrence relation of the form $g^{(n)} = \Theta(x g^{(n-1)})$ for $n \geq 1$ and for some linear operator

Θ whose norm is finite. This shows that the expansion (20) actually defines for sufficiently small ε an element which is in $\mathcal{H}$ and that by construction, this is the unique solution to the perturbed Fokker-Planck equation.

In the following, we assume that the expansion (20) is valid and we investigate the conditions which have to be satisfied by the elements $g^{(n)}$. In a first step, we prove the following property satisfied by the functions $(g_u^{(n)}(x))$.

Lemma 4. *For $n \geq 0$, the function $g_u^{(n)}(x)$ in Expansion (20) can be expressed as a linear combination of $\varphi_0, \ldots, \varphi_n$. In particular, for $N > n$,*

$$\lim_{x \to \pm\infty} \frac{1}{x^N} g_u^{(n)}(x) \exp\left(\frac{\alpha(x-m)^2}{\sigma^2}\right) = 0. \tag{21}$$

Proof. The proof is by induction. The result is true for $n = 0$ since $g_u^{(0)}(x)$ is given by Equation (3).

If the result is true for n, then for fixed u, $g_u(x)$ belongs to the vector space spanned by the functions φ_i for $i = 0, \ldots, n$, denoted by $\text{span}(\varphi_0, \ldots, \varphi_n)$. From Fokker-Planck Equation (7), we have

$$(A \otimes \mathbb{I} + \mathbb{I} \otimes B) g^{(n+1)} = \Psi \otimes D g^{(n)},$$

where the operator Ψ is the multiplication by x in the Hilbert space H. By using the recurrence relation satisfied by Hermite polynomials, it is easily checked that the image by the operator Ψ of the vector space $\text{span}(\varphi_0, \ldots, \varphi_n)$ is the vector space $\text{span}(\varphi_0, \ldots, \varphi_{n+1})$. Therefore, since by assumption $g^{(n)}$ belongs to $\text{span}(\varphi_0, \ldots, \varphi_n) \otimes L^2_\rho(\mathbb{N})$, we immediately deduce from the uniqueness of the decomposition on the basis (φ_n) that $g^{(n+1)}$ is in $\text{span}(\varphi_0, \ldots, \varphi_{n+1}) \otimes L^2_\rho(\mathbb{N})$ and the result follows. □

5.1 First order term

In a first step, we pay special attention to the derivation of the first order term because it gives the basic arguments to derive higher order terms. Moreover, the explicit form of the first order term will be used to examine the validity of the reduced service rate approximation (see Theorem 1).

On the basis of the domination property given by Lemma 4, we explicitly compute the function $g_u^{(1)}(x)$. From Equation (7), it is easily checked that the function $g_u^{(1)}(x)$ satisfies the equation

$$\begin{aligned} &\frac{\sigma^2}{2}\frac{\partial^2 g_u^{(1)}}{\partial x^2} + \alpha(x-m)\frac{\partial g_u^{(1)}}{\partial x} + \alpha(\nu(u)+1) g_u^{(1)}(x) \\ &\quad = \mu\left(\frac{1}{u}-1\right)\left(g_0^{(1)}(x) - x(g_0^{(0)}(x) - g_u^{(0)}(x))\right) \\ &\quad = \mu\left(\frac{1}{u}-1\right)\left(g_0^{(1)}(x) + \frac{x}{\sigma}\sqrt{\frac{\alpha}{\pi}}\frac{\rho u(1-\rho)}{(1-\rho u)}\exp\left(-\frac{\alpha(x-m)^2}{\sigma^2}\right)\right) \end{aligned} \tag{22}$$

where the constant $\nu(u)$ is given by

$$\nu(u) = \frac{\mu(1-u)(1-\rho u)}{\alpha u}.$$

In a first step, we search for a particular solution to the ordinary differential equation

$$\frac{\sigma^2}{2}\frac{\partial^2 \xi_u}{\partial x^2} + \alpha(x-m)\frac{\partial \xi_u}{\partial x} + \alpha(\nu(u)+1)\xi_u(x) = \frac{x}{\sigma}\sqrt{\frac{\alpha}{\pi}}\frac{\rho\mu(1-u)(1-\rho)}{(1-\rho u)}\exp\left(-\frac{\alpha(x-m)^2}{\sigma^2}\right) \quad (23)$$

of the form

$$\xi_u(x) = (a(u)+b(u)x)\exp\left(-\frac{\alpha(x-m)^2}{\sigma^2}\right).$$

Straightforward manipulations show that

$$b(u) = \frac{1}{\sigma}\sqrt{\frac{1}{\alpha\pi}}\frac{\rho\mu(1-u)(1-\rho)}{(\nu(u)-1)(1-\rho u)}\exp\left(-\frac{\alpha(x-m)^2}{\sigma^2}\right) \quad \text{and} \quad a(u) = -\frac{m}{\nu(u)}b(u).$$

Noting that $\xi_0(x) \equiv 0$, it follows that if we write $g_u^{(1)}(x) = \xi_u(x) + \psi_u(x)$, then the function $\psi_u(x)$ is solution to the equation

$$\frac{\sigma^2}{2}\frac{\partial^2 \psi_u}{\partial x^2} + \alpha(x-m)\frac{\partial \psi_u}{\partial x} + \alpha(\nu(u)+1)\psi_u(x) = \mu\left(\frac{1}{u}-1\right)\psi_0(x). \quad (24)$$

By using the domination property of Lemma 4, we can determine the form of the function $\psi_0(x)$.

Lemma 5. *The function $\psi_0(x)$ is given by*

$$\psi_u(x) = \left(c_0 + c_1\frac{\sqrt{\alpha}(x-m)}{\sigma}\right)\exp\left(-\frac{\alpha(x-m)^2}{\sigma^2}\right)$$

for some constants c_0 and c_1.

Proof. By introducing the function $k_u(x)$ defined by

$$k_u(x) = \exp\left(\frac{\alpha(x-m)^2}{2\sigma^2}\right)\psi_u(x) \quad (25)$$

and then the change of variable

$$z = \frac{\sqrt{\alpha}(x-m)}{\sigma}, \quad (26)$$

Equation (24) becomes

$$\frac{\partial^2 k_u}{\partial z^2} + (2\nu(u) + 1 - z^2)k_u(z) = \frac{2\mu}{\alpha}\left(\frac{1}{u} - 1\right) k_0(z). \tag{27}$$

The homogeneous equation reads

$$\frac{\partial^2 k_u}{\partial z^2} + (2\nu(u) + 1 - z^2)k_u = 0,$$

which solutions are parabolic cylinder functions (see Lebedev [11] for details). Two independent solutions $v_1(u;z)$ and $v_2(u;z)$ of this homogeneous equation are given in terms of Hermite functions as

$$v_1(u;z) = e^{-z^2/2}H_{\nu(u)}(z) \quad \text{and} \quad v_2(u;z) = e^{z^2/2}H_{-\nu(u)-1}(iz). \tag{28}$$

The Wronskian $\mathcal{W}$ of these two functions is given by

$$\mathcal{W}(z) = e^{-(\nu+1)\pi i/2}.$$

By using the method of variation of parameters, the solution to Equation (27) is given by

$$\begin{aligned} k_u(z) &= \gamma_1(u)v_1(u;z) + \gamma_2(u)v_2(u;z) \\ &\quad - \frac{2\mu}{\alpha}\left(\frac{1}{u} - 1\right) e^{(\nu+1)\pi i/2} \int_0^z \left[v_1(u;y)v_2(u;z) - v_1(u;z)v_2(u;y)\right] k_0(y)\, dy, \end{aligned}$$

where $\gamma_1(u)$ and $\gamma_2(u)$ are constants, which depend upon u.

The function $\psi_u(x)$ enjoys the same domination property as function $g_u^{(1)}(x)$, given by Lemma 4. Hence, for $N > 1$

$$\lim_{z\to\pm\infty} \frac{1}{z^N} e^{z^2/2} k_u(z) = 0. \tag{29}$$

From Lebedev [11], we have the following asymptotic estimates

$$H_\nu(z) \sim (2z)^\nu \tag{30}$$

when $|z| \to \infty$ and $|\arg z| \leq 3\pi/4 - \delta$ for some $\delta > 0$. Moreover, when $z \to -\infty$

$$H_\nu(z) \sim \begin{cases} \dfrac{\sqrt{\pi}}{\Gamma(-\nu)}|z|^{-\nu-1}e^{z^2}, & \nu \notin \mathbb{N} \\[2ex] (2z)^\nu, & \nu \in \mathbb{N}. \end{cases}$$

From the above asymptotic estimates and Lemma (29), we deduce that for $u \in (0,1)$ such that $\nu(u) \in \mathbb{N}$ with $\nu > 1$, we have

$$
\begin{aligned}
\gamma_1(u) &= -\frac{2\mu}{\alpha}\left(\frac{1}{u}-1\right)e^{(\nu+1)\pi i/2}\int_0^{\infty} v_2(u;y)k_0(y)\,dy \\
&= -\frac{2\mu}{\alpha}\left(\frac{1}{u}-1\right)e^{(\nu+1)\pi i/2}\int_0^{-\infty} v_2(u;y)k_0(y)\,dy
\end{aligned}
$$

and

$$
\begin{aligned}
\gamma_2(u) &= \frac{2\mu}{\alpha}\left(\frac{1}{u}-1\right)e^{(\nu+1)\pi i/2}\int_0^{\infty} v_1(u;y)k_0(y)\,dy \\
&= \frac{2\mu}{\alpha}\left(\frac{1}{u}-1\right)e^{(\nu+1)\pi i/2}\int_0^{-\infty} v_1(u;y)k_0(y)\,dy.
\end{aligned}
$$

The latter equation implies that for all $n > 1$

$$
\int_{-\infty}^{\infty} e^{-y^2/2}k_0(y)H_n(y)\,dy = 0, \tag{31}
$$

where $H_n(x)$ is the nth Hermite polynomial. Property (29) implies that the function $y \to \exp(y^2/2)k_0(y)$ is in $L^2(\mathbb{R}, \exp(-y^2)\,dy)$, which is the Hilbert space of the functions square integrable with respect to the measure $\exp(-y^2)\,dy$, i.e.,

$$
L^2(\mathbb{R}, \exp(-y^2)\,dy) = \left\{f : \int_{-\infty}^{\infty} |f(y)|^2 e^{-y^2} dy < \infty\right\},
$$

equipped with the scalar product

$$
<f,g>_2 = \int_{-\infty}^{\infty} f(y)\overline{g(y)}e^{-y^2}\,dy.
$$

Since Hermite polynomials form an orthogonal basis in this Hilbert space, equation (31) entails that the function $y \to \exp(y^2/2)k_0(y)$ is orthogonal to all Hermite polynomials H_n with $n > 1$ and then that this function belongs to the vector space spanned by H_0 and H_1. Hence, function $k_0(z)$ should be of the form

$$
k_0(z) = (c_0 + c_1 z)e^{-z^2/2}
$$

for some constants c_0 and c_1 and the result follows. □

By using the above lemma, we are now able to establish the expression of $g_u^{(1)}(x)$.

Proposition 6. *The function $g_u^{(1)}(x)$ is given by*

$$g_u^{(1)}(x) = \left(\frac{(u_1-1)(\tilde{u}_1-\rho u)(u-1)}{u_1\tilde{u}_1(u-\tilde{u}_1)(1-\rho u_1)(1-\rho u)^2} + \frac{(1-\rho)(1-u)}{(u-\tilde{u}_1)(1-\rho u_1)(1-\rho u)} x \right) \frac{1}{\sigma}\sqrt{\frac{\alpha}{\pi}} e^{\alpha(x-m)^2/\sigma^2}, \quad (32)$$

where u_1 and $\tilde{u}_1$ are the two real solutions to the quadratic equation

$$\rho u^2 - \left(1+\rho+\frac{\alpha}{\mu}\right) u + 1 = 0$$

with $0 < u_1 < 1 < \tilde{u}_1$

Proof. By taking into account Lemma 5, the function $K_u(z)$ defined by

$$K_u(x) = g_u^{(1)}(x) \exp\left(\frac{\alpha(x-m)^2}{2\sigma^2}\right)$$

and the change of variable (26), satisfies the equation

$$\frac{\partial^2 K_u}{\partial z^2} + (2\nu(u)+1-z^2)K_u(z) = \frac{2\mu}{\alpha}\left(\frac{1}{u}-1\right)\left(c_0 + c_1 z + \frac{1}{\sigma}\sqrt{\frac{\alpha}{\pi}}\frac{\rho u(1-\rho)}{1-\rho u}\left(\frac{\sigma z}{\sqrt{\alpha}}+m\right)\right)e^{-z^2/2}. \quad (33)$$

We search for a particular solution of the form

$$K_u(z) = (a(u)+b(u)z)\, e^{-z^2/2}.$$

Straightforward computations yield

$$a(u) = \frac{1}{(1-\rho u)}\left(c_0 + \frac{1}{\sigma}\sqrt{\frac{\alpha}{\pi}}\frac{\rho u(1-\rho)}{1-\rho u} m\right),$$

$$b(u) = \frac{(1-u)}{\rho(u-u_1)(u-\tilde{u}_1)}\left(c_1 + \frac{\rho u(1-\rho)}{\sqrt{\pi}(1-\rho u)}\right).$$

It follows that the general solution to the above equation can be written as

$$K_u(z) = (a(u)+b(u)z)\, e^{-z^2/2} + \gamma_1(u)v_1(u;z) + \gamma_2(u)v_2(u;z), \quad (34)$$

where the functions v_1 and v_2 are defined by Equation (28) and the constants $\gamma_1(u)$ and $\gamma_2(u)$ depend upon u.

By differentiating once Equation (34) with respect to z and using the fact that the Wronskian of the functions $v_1(u;z)$ and $v_2(u;z)$ is $\exp[(\nu(u)+1)\pi i/2]$, we can easily express

$\gamma_1(u)$ and $\gamma_2(u)$ by means of $K_u(z)$, $a(u)$, and $b(u)$. This shows that $\gamma_1(u)$ and $\gamma_2(u)$ are analytic in the open unit disk deprived of the points 0 and u_1. From the asymptotic properties satisfied by the functions v_1 and v_2, we know that $\gamma_1(u) = 0$ and $\gamma_2(u) = 0$ for u such that $\nu(u) > 1$. It follows that $\gamma_1(u) \equiv \gamma_2(u) \equiv 0$ for $|u| < 1$.

By using the fact that $g_u^{(1)}(x)$ has to be analytic in variable u is the unit disk, we necessarily have

$$c_1 = -\frac{\rho u_1(1-\rho)}{\sqrt{\pi}(1-\rho u_1)}$$

and then,

$$b(u) = \frac{(1-\rho)(1-u)}{\sqrt{\pi}(u-\tilde{u}_1)(1-\rho u_1)(1-\rho u)}.$$

Moreover, since $g_1^{(1)}(x) \equiv 0$, we have

$$c_0 = -\frac{1}{\sigma}\sqrt{\frac{\alpha}{\pi}}\rho m$$

and then,

$$a(u) = \frac{1}{\sigma}\sqrt{\frac{\alpha}{\pi}}\frac{\rho(u-1)}{(1-\rho u)^2}m.$$

By using the expressions of $a(u)$ and $b(u)$, the result follows. □

5.2 Higher order terms

We assume that $g_u^{(n)}(x)$ can be expressed as

$$g_u^{(n)}(x) = \sum_{k=0}^{n} c_{n,k}(u)\varphi_k(x), \tag{35}$$

where the function φ_n is defined by Equation (19) and the coefficients $c_{n,k}$ are analytic functions in variable u. From previous sections, this representation is valid for $n = 0, 1$. If it is valid for $n-1$, then the function $g_u^{(n)}(x)$, $n \geq 1$, satisfies the equation

$$\frac{\sigma^2}{2}\frac{\partial^2 g_u^{(n)}}{\partial x^2} + \alpha(x-m)\frac{\partial g_u^{(n)}}{\partial x} + \alpha(\nu(u)+1)g_u^{(n)}(x) = \mu\left(\frac{1}{u}-1\right)\left(g_0^{(n)}(x) - x(g_0^{(n-1)}(x) - g_u^{(n-1)}(x))\right). \tag{36}$$

First note that by using the recurrence relation

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0$$

satisfied by Hermite polynomials, it is easily checked that

$$x(g_u^{(n-1)}(x) - g_0^{(n-1)}(x)) = \sum_{k=0}^{n} d_{n,k}(u)\varphi_k(x),$$

where

$$d_{n,n}(u) = \frac{\sigma}{2\sqrt{\alpha}}(c_{n-1,n-1}(u) - c_{n-1,n-1}(0)),$$

and for $0 \leq k \leq n-1$,

$$d_{n,k}(u) = \frac{\sigma}{2\sqrt{\alpha}}(c_{n-1,k-1}(u) - c_{n-1,k-1}(0)) + m(c_{n-1,k}(u) - c_{n-1,k}(0)) + \frac{(k+1)\sigma}{\sqrt{\alpha}}(c_{n-1,k+1}(u) - c_{n-1,k+1}(0)).$$

By using the above notation, we have the following result.

Proposition 7. *The coefficients $c_{n,k}$ appearing in the representation* (35) *of $g_u^{(n)}(x)$ are recursively defined as follows: we have*

$$c_{0,0}(u) = \frac{1}{\sigma}\sqrt{\frac{\alpha}{\pi}}\frac{1-\rho}{1-\rho u},$$

and for $n \geq 1$,

$$\begin{aligned} c_{n,0}(u) &= \frac{d_{n,0}(u) - d_{n,0}(1)}{1-\rho u}, \\ c_{n,k}(u) &= \frac{\mu}{\alpha}\left(\frac{1}{u} - 1\right)\frac{d_{n,k}(u) - d_{n,k}(u_k)}{\nu(u) - k} \quad 1 \leq k \leq n, \end{aligned}$$

where for $k \geq 1$, u_k and $\tilde{u}_k$ are the two real solutions to the quadratic equation $\nu(u) = k$, i.e.

$$\rho u^2 - \left(1 + \rho + \frac{k\alpha}{\mu}\right)u + 1 = 0$$

with $0 < u_k < 1 < \tilde{u}_k$.

Proof. As in the previous section, we first search for a solution to the equation

$$\frac{\sigma^2}{2}\frac{\partial^2 \xi_u^{(n)}}{\partial x^2} + \alpha(x-m)\frac{\partial \xi_u^{(n)}}{\partial x} + \alpha(\nu(u)+1)\xi_u^{(n)}(x) = \mu\left(\frac{1}{u} - 1\right)x(g_u^{(n-1)}(x) - g_0^{(n-1)}(x)).$$

Assuming that the function $\xi_u^{(n)}(x)$ is of the form

$$\xi_u^{(n)}(x) = \sum_{k=0}^{n} \delta_{n,k}(u)\varphi_n(x),$$

and we have, by using the fact that the functions $\varphi_n(x)$ are eigenfunctions of the operator A associated with the eigenvalues $-\alpha n$ and that these functions are linearly independent, for $k = 0, \dots, n$,

$$\delta_{n,k} = \frac{\mu}{\alpha}\left(\frac{1}{u} - 1\right)\frac{d_{n,k}(u)}{\nu(u) - k}.$$

It is easily checked that $\xi_0^{(n)}(x) \equiv 0$. We can then decompose $g_u^{(n)}(x)$ as

$$g_u^{(n)}(x) = \psi_u^{(n)}(x) + \xi_u^{(n)}(x),$$

where the function $\psi_u^{(n)}(x)$ is solution to the equation

$$\frac{\sigma^2}{2}\frac{\partial^2 \psi_u^{(n)}}{\partial x^2} + \alpha(x - m)\frac{\partial \psi_u^{(n)}}{\partial x} + \alpha(\nu(u) + 1)\psi_u^{(n)}(x) = \mu\left(\frac{1}{u} - 1\right)\psi_0(x).$$

By using the same arguments as in the proof of Lemma 5, we can easily show that $\psi_0(x)$ has the form

$$\psi_0(x) = \sum_{k=0}^{n} c_k \varphi_n(x),$$

where the coefficients $c_k \in \mathbb{C}$ for $k = 0, ..., n$. It follows that the function $g_u^{(n)}(x)$ is solution to the ordinary differential equation

$$\frac{\sigma^2}{2}\frac{\partial^2 g_u^{(n)}}{\partial x^2} + \alpha(x - m)\frac{\partial g_u^{(n)}}{\partial x} + \alpha(\nu(u) + 1)g_u^{(n)}(x) = \mu\left(\frac{1}{u} - 1\right)\sum_{k=0}^{n}(c_k + d_{n,k}(u))\varphi_n(x).$$

By using the same arguments as in the proof of Proposition 6, we come up with the conclusion that $g_u^{(n)}(x)$ is of the form (35) with the coefficients $c_{n,k}(u)$ given by

$$c_{n,k}(u) = \frac{\mu}{\alpha}\left(\frac{1}{u} - 1\right)\frac{c_k + d_{n,k}(u)}{\nu(u) - k}.$$

Since the function $g_u^{(n)}(x)$ has to be analytic in the open unit disk, we have for $k \geq 1$

$$c_k = -d_{n,k}(u_k)$$

In addition, since $g_1^{(n)}(x) \equiv 0$, we have $c_0 = -d_{n,k}(1)$. □

5.3 Radius of convergence

In this section, we examine under which conditions the expansion (20) defines an element of $\mathcal{H}$. In a first step, note that as a consequence of Proposition 7, the function $g_u^{(n)}(x)$ can be written as

$$g_u^{(n)}(x) = x\Theta\left(g_u^{(n-1)}(x)\right) = \Theta\left(xg_u^{(n-1)}(x)\right)$$

where the operator Θ is defined in $\mathcal{H}$ as follows: for $f \in \mathcal{H}$, which gives rise to the function

$$f_u(x) = \sum_{n=0}^{\infty} c_n(u)\varphi_n(x),$$

the element $h = \Theta f$ is defined by the function

$$h_u(x) = \sum_{n=0}^{\infty} \mu\left(\frac{1}{u} - 1\right)\frac{c_n(u) - c_n(u_n)}{\nu(u) - n}\varphi_n(x).$$

It is easily checked that for $n \geq 0$, $0 < u_n < 1 < 1/\sqrt{\rho} < \tilde{u}_n$. Moreover, the function $c_n(u)$ appearing in the expression of f_u is analytic in the disk $D_\rho = \{z : |z| < 1/\sqrt{\rho}\}$ and continuous in the closed disk $\overline{D}_\rho = \{z : |z| \leq 1/\sqrt{\rho}\}$ for $n \geq 0$. Similarly, for all $n \geq 0$, the function

$$u \to \frac{\mu}{\alpha}\left(\frac{1}{u} - 1\right)\frac{c_n(u) - c_n(u_n)}{\nu(u) - n}\varphi_n(x)$$

is analytic in D_ρ and continuous in $\overline{D}_\rho$. With the above notation, we can state the main result of this section.

Proposition 8. *The operator Θ is bounded and if $\varepsilon < 1/(m\|\Theta\|)$, where $\|\Theta\|$ denotes the norm of Θ, then the sequence defined by Equation* (20) *is in $\mathcal{H}$.*

Proof. Let $f \in \mathcal{H}$ be defined by the function

$$f_u(x) = \sum_{n=0}^{\infty} c_n(u)\varphi_n(x).$$

For $(c_n) \in L^2_\rho(\mathbb{N})$ associated with the generating function

$$c(u) = \sum_{n=0}^{\infty} c_n u^n \text{ then } \|c\|^2_\rho = \frac{1}{2\pi}\int_0^{2\pi}\left|c\left(\frac{1}{\sqrt{\rho}}e^{i\theta}\right)\right|^2 d\theta,$$

and define the sequence $(\tilde{c}_n)$ associated with the generating function

$$\tilde{c}(u) = \frac{\mu}{\alpha}\left(\frac{1}{u} - 1\right)\frac{c(u) - c(u_n)}{\nu(u) - n}.$$

Assume first that $n \geq 1$, then

$$\tilde{c}(u) = \frac{1}{\rho}(1-u)\frac{1}{u-\tilde{u}_n}\frac{c(u)-c(u_n)}{u-u_n}.$$

and then

$$\|\tilde{c}\|_\rho^2 \leq \frac{1}{\rho^2}\left(1+\frac{1}{\sqrt{\rho}}\right)^2 \frac{1}{(\tilde{u}_n - 1/\sqrt{\rho})^2}\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{c(e^{i\theta}/\sqrt{\rho})-c(u_n)}{e^{i\theta}/\sqrt{\rho}-u_n}\right|^2 d\theta.$$

Simple manipulations show that

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{c(e^{i\theta}/\sqrt{\rho})-c(u_n)}{e^{i\theta}/\sqrt{\rho}-u_n}\right|^2 d\theta \leq \|c\|_\rho^2 \frac{1}{(1/\sqrt{\rho}-u_n)^2}\left(1+\sqrt{\frac{1}{1-\rho u_n^2}}\right)^2.$$

It follows that $\|\tilde{c}\|_\rho \leq \kappa_n \|c\|_\rho$, where

$$\begin{aligned}\kappa_n &= \frac{1}{\rho}\left(1+\frac{1}{\sqrt{\rho}}\right)\frac{1}{(\tilde{u}_n - 1/\sqrt{\rho})(1/\sqrt{\rho}-u_n)}\left(1+\sqrt{\frac{1}{1-\rho u_n^2}}\right)\\ &= \frac{1+\sqrt{\rho}}{(1-\sqrt{\rho})^2+\frac{n\alpha}{\mu}}\left(1+\sqrt{\frac{1}{1-\rho u_n^2}}\right).\end{aligned}$$

It is easily checked that the sequence (κ_n) for $n \geq 1$ is decreasing.

When $n = 0$, we define

$$\tilde{c}(u) = \frac{\mu}{\alpha}\left(\frac{1}{u}-1\right)\frac{c(u)-c(1)}{\nu(u)} = \frac{c(u)-c(1)}{1-\rho u}.$$

It is then easily checked that $\|\tilde{c}\|_\rho \leq \kappa_0 \|c\|_\rho$, where

$$\kappa_0 = \frac{1}{1-\sqrt{\rho}}\left(1+\sqrt{\frac{1}{1-\rho}}\right).$$

Define $\kappa = \max\{\kappa_0, \kappa_1\}$. The above computations show that for all $f \in L^2_\rho(\mathbb{N})$, $\|\Theta f\| \leq \kappa \|f\|$. It follows that the operator Θ is bounded; its norm is denoted by $\|\Theta\| \stackrel{\text{def.}}{=} \inf\{k > 0 : \forall f \in \mathcal{H},\ \|\Theta f\| \leq k\|f\|\}$. The above computations shows that

$$\|\Theta\| \leq \frac{1+\sqrt{\rho}}{(1-\sqrt{\rho})^2}\left(1+\sqrt{\frac{1}{1-\rho}}\right).$$

We immediately deduce that the sequence $c^{(n)} = (c^{(n)}_{k,\ell})$ associated with the function $g^{(n)}_u(x)$, in the sense that

$$g^{(n)}_u(x) = \sum_{k=0}^{n}\sum_{\ell=0}^{\infty} c_{k,\ell} u^\ell \varphi_k(x),$$

is such that

$$\|c^{(n)}\| \leq \|\Theta\|^n \|c^{(0)*n}\|$$

where the sequence $c^{(0)*n}$ is associated with the function

$$\frac{1-\rho}{1-\rho u} x^n p(x),$$

where the function $p(x)$ is defined by Equation (2).

Straightforward computations show that

$$\|c^{(0)*n}\|^2 = \left(\frac{\sigma}{2\sqrt{\alpha}}\right)^{2n} H_{2n}\left(\frac{\sqrt{\alpha}m}{\sigma}\right),$$

where $H_n(x)$ is the nth Hermite polynomial. Using the asymptotic estimate (30), we have

$$\|c^{(0)*n}\| \sim m^n$$

when $n \to \infty$. It follows that $\|c^{(n)}\| \leq a_n$ with $a_n \sim (\|\Theta\| m)^n$ as n tends to infinity. It follows that the sequence defined by the expansion (20) is convergent in $\mathcal{H}$ if $\varepsilon\|\Theta\|m < 1$. □

By using all the above results, we are now ready to prove the validity of the reduced service rate approximation given by Theorem 1.

Proof of Theorem 1. From the above result, we deduce that, under the assumption $\varepsilon < 1/(m\|\Theta\|)$, the first order expansion of the generating function of the stationary distribution of $(L(t))$

$$\mathbb{E}\left(u^L\right) = \frac{1-\rho}{1-\rho u} - \frac{\rho(1-u)}{(1-\rho u)^2} m\varepsilon + o(\varepsilon).$$

holds. Theorem 1 is proved. □

6 Concluding remarks

The perturbation results presented in this paper have been obtained for a particular form of the perturbation function $\phi(x)$. Of course, the same approach could be extended to more complicated perturbation functions of the form $\phi(x) = 1-\varepsilon p(x)$ for some function $p(x)$. The key point consists of determining how the operator corresponding to the multiplication by $p(x)$ acts on the basic functions φ_n. For computing explicit expressions, however, the main difficulty is in solving the differential equations satisfied by the coefficients of the expansion. When $p(x)$ is a polynomial, a particular solution to the equations similar to Equations (22) and (36) is obtained in the form of a polynomial times the function $\exp(-\alpha(x-m)^2/\sigma^2)$ and in that case, explicit computations can be carried out.

The perturbation function $\phi(x) = 1-\varepsilon x$ correspond to the case when unresponsive flows have a peak bit rate ε much smaller than the transmission capacity of the link. The results of this paper show that the reduced service rate approximation yields accurate results for the performance of responsive flows.

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