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# Uniqueness for a Scalar Conservation Law with Discontinuous Flux via Adapted Entropies

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**Abstract:** We prove uniqueness of solutions to scalar conservation laws with space discontinuous fluxes. To do so, we introduce a partial adaptation of Kruzkov's entropies which naturally takes into account the space dependency of the flux. The advantage of this approach is that the proof turns out to be a simple variant of Kruzkov's original method. Especially, we do not need traces, interface condition, Bounded Variation assumptions (neither on the solution nor on the flux), or convex fluxes. However we use a special 'local uniform invertibility' structure of the flux which applies to cases where different interface condiftions are known to yield different solutions.

**Key-words:** Scalar conservation law, uniqueness, entropies, discontinuous flux

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# Unicité pour les lois de conservation scalaires à coefficients discontinus via des entropies adaptées

**Résumé :** Nous établissons dans ce rapport un théorème d'unicité des solutions des lois de conservations scalaires avec flux discontinu en espace. L'idée est d'introduire des entropies de Kruzkov adaptées qui prennent en compte la dépendance spatiale du flux. L'intérêt de cette approche est que la preuve deveint une simple variante de la méthode de Kruzkov. En particulier, la preuve ne requiert ni notion de trace, ni condition d'interface, et les hypothèses de convexité du flux et de variation bornée du flux et des conditions initiales sont levées. Cependant il est nécessaire d'introduire une hypopthèse sur la structure 'uniformément localement inversible' du flux. A titre d'exemple, nous appliquons la méthode à un cas où différentes conditions d'interface sont connues pour sélectionner différentes solutions.

**Mots-clés :** Loi de conservation scalaire, unicité, entropies, flux discontinu

# 1 Introduction

We consider the Cauchy problem associated with a scalar conservation law where the flux depends discontinuously on space

$$\begin{cases} \partial_t u + \partial_x [A(x,u)] = 0 \qquad x \in \mathbb{R},\ t \in \mathbb{R}_+, \\ u(0,x) = u_0(x) \in L^\infty(\mathbb{R}). \end{cases} \tag{1.1}$$

We propose a new method to prove a $L^1$ contraction principle for a class of solutions to this equation when the space dependence of the flux is discontinuous. As in most of the recent papers that deal with this subject [14, 1] our proof is based on Kruzkov's framework. But our idea is to adapt the definition of Kruzkov entropies to the discontinuous case and thus to avoid a special treatment of the interface.

When the space dependence of the flux is sufficiently smooth, this scalar equation is quite well known. In particular, Kruzkov's theory applies and provides existence and uniqueness of a weak solution to (1.1) that satisfies Kruzkov entropy inequalities - see [5, 18, 21].
We consider in this article the case where the flux is a discontinuous function of $x$, not necessarily of bounded variation. First existence results for such a problem were obtained through an analogy with $2 \times 2$ hyperbolic systems and the study of the related Riemann problem. Indeed if we assume that the flux is on the form $A(x,u) = f(\gamma(x), u)$, the equation (1.1) can be written as a system in $(u, \gamma)$ by adding the trivial equation $\partial_t \gamma = 0$. In the 80's and considering particular forms of the flux, [22, 8, 9] established the global existence of a solution for the corresponding Cauchy problems by proving the convergence of different numerical methods. Later on existence results were extended to more general fluxes by using convergence of numerical schemes [16, 10, 23, 24, 17, 12, 11, 14, 4, 1] or regularization of the coefficients [15, 19, 20, 13].
Here we are interested in the problem of the uniqueness of solutions. First results about this topic were obtained in the middle of the 90's and different methods have been investigated. In [6], Diehl considered a flux on the form $A(x,u) = H(x)f(u) + (1 - H(x))g(u)$ where $H(x)$ is the Heavyside function and proved existence and uniqueness locally in time by introducing a coupling condition $\Gamma$ at the interface. In the same year, in [16], Klingenberg and Risebro considered a multiplicative flux $A(x,u) = k(x)f(u)$ such that $f$ is a convex function that satisfies $f(0) = f(1) = 0$ and $k(x) \geq k > 0$ is a BV piecewise smooth function with a finite number of discontinuity points ; they proved uniqueness for a solution that satisfies a wave entropy condition - see also [15] where the authors proved continuous dependence on the coefficient $k$ and on the initial data for the same problem. In [10], Greenberg et al. considered a convex additive flux $A(x,u) = f(u) + a(x)$ where $f$ is even and convex and $a$ is piecewise constant and proved a contraction principle for the solution that they constructed by solving Riemann problems and studying interactions of waves in a right way. In [19] Ostrov proposed an other approach : he proved uniqueness of a solution of the Hamilton Jacobi equation obtained as the limit of viscosity solutions for regularized coefficients cases. Then he concluded for (1.1) by using the equivalency between

Hamilton Jacobi equations and scalar conservation laws. He extended uniqueness result to fluxes on the form $A(t, x, u) = f(k(t, x), u)$ where $f$ is convex in $u$ and satisfies a superlinear growth condition and $k$ is bounded and discontinuous along a finite number of curves and is Lipschitz continuous away from these curves. Towers [23] came back to the multiplicative case and established a $L^1$ contraction principle for a class of solutions that satisfies Kruzkov type entropy inequalities - it means that the solution satisfies classical Kruzkov entropy inequalities away from the discontinuities of the flux and satisfies a geometric condition at the discontinuity points that can be interpreted as an interface entropy condition. Note that to give meaning to this new entropy condition he needed to assume some additional regularity conditions on the solution, namely that $u$ is piecewise $C^1$ and possesses traces on the discontinuities of $k$. In this earlier work the flux was assumed to be convex in $u$, but this approach was further investigated by Karlsen, Risebro and Towers in [14] and the uniqueness result has been extended to non convex fluxes on the form $A(x, u) = f(k(x), u)$ where $k$ is a piecewise $C^1$ BV function with a finite number of discontinuities and $f$ is Lipschitz continuous in $u$ and $k$ and satisfies a given crossing condition. In that paper and in [20] the existence of traces for $u$ is proved for particular additive / multiplicative fluxes but should be assumed in the general case. Very recently, Adimurthi, Jaffre and Veerappa Gowda [1] introduced an other interface entropy condition - still coupled with classical Kruzkov entropy inequalities away from the discontinuity - and proved also a $L^1$ contraction principle for this new class of solutions. They considered a Heavyside flux type where $f$ and $g$ have only one global minimum and no local minimum and assumed the existence of traces on the discontinuity. For particular fluxes and initial data it can be proved that the interface conditions in [14] and [1] do not select the same solution. We prove in Section 5 that our *interface condition free* method selects the solution derived from the interface condition of [1].

In [23, 20, 14, 1] the uniqueness proof is based on the use of classical Kruzkov entropies which leads to the following entropies inequalities

$$\partial_t |u - k| + \partial_x [(A(x, u) - A(x, k)) \text{ sgn}(u - k)] + \text{sgn}(u - k)\partial_x A(x, k) \leq 0. \tag{1.2}$$

Thus an interface entropy condition has to be introduced by the authors to deal with the discontinuities of the flux and to give meaning to the last term of the left hand side. Here we propose to adapt the definition of Kruzkov entropies to the discontinuous case by introducing partially adapted Kruzkov entropies

$$E_\alpha(x, u) = |u - k_\alpha(x)|,$$

where $k_\alpha(x)$ satisfies

$$A(x, k_\alpha(x)) = \alpha.$$

This new definition allows us to remove the problematic term in the entropy inequalities (1.2) since we obtain

$$\partial_t |u - k_\alpha(x)| + \partial_x [(A(x, u) - A(x, k_\alpha(x))) \text{ sgn}(u - k_\alpha(x))] \leq 0. \tag{1.3}$$

Thus the interface does not need a special treatment and no interface entropy condition is needed. Uniqueness then follows from arguments very close to Kruzkov's original proof and the main difficulty is now to deal with the family(ies) of functions $k_\alpha(x)$.
This new method allows us to remove the hypothesis about the traces of the solution on the discontinuities of the flux and the BV bounds on the space dependence of the flux and on the initial data. Also we can deal with an infinite number of discontinuity points and we do not need convexity assumptions or crossing conditions. However we need some other hypothesis on the flux - and more particurlarly on the $u$ dependence of the flux - to be able to define our partially adapted Kruzkov entropies.

The outline of the paper is the following. In Section 2 we list the hypothesis on the flux and we comment them with some examples. In Section 3 we define the partially adapted Kruzkov entropies and in Section 4 we prove the $L^1$ contraction principle. Finally in Section 5, and for a particular flux, we study the selected solution and we compare it with the existing results [1, 14].

## 2 Hypothesis on the flux

In this work we assume the following hypothesis on the flux $A$

• (H1) $A(x,u)$ is continuous at all points of $\mathbb{R}\backslash\mathcal{N} \times \mathbb{R}$ where $\mathcal{N}$ is a closed zero measure set,

• (H2) $\exists (f,g) \in \left(C^0(\mathbb{R})\right)^2$ such that $\forall x \in \mathbb{R} \quad f(u) \leq |A(x,u)| \leq g(u)$. We assume that $f$ is a non-negative (non strictly) decreasing then increasing function and $|f(\pm\infty)| = +\infty$.

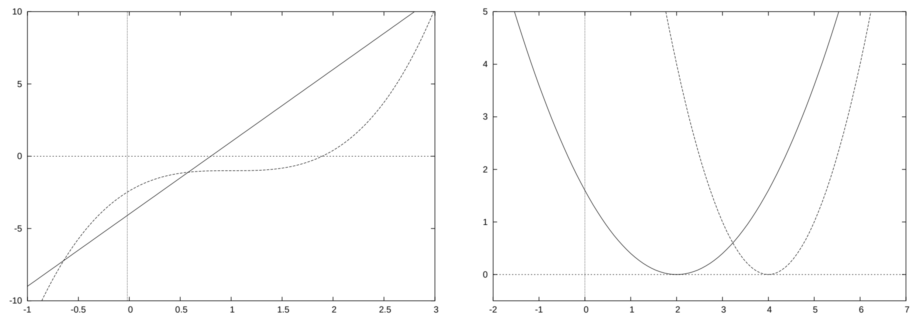
• (H3) For $x \in \mathbb{R}\backslash\mathcal{N}$, $A(x,.)$ is a locally Lipschitz one to one function from $\mathbb{R}$ to $\mathbb{R}$.

All along the article we also consider an alternative case by replacing the hypothesis (H3) by

• (H3') There is a function $u_M(x)$ from $\mathbb{R}$ to $\mathbb{R}$ such that for $x \in \mathbb{R}\backslash\mathcal{N}$, $A(x,.)$ is a locally Lipschitz one to one function from $[-\infty, u_M(x)]$ and $[u_M(x), +\infty]$ to $[0,+\infty]$ that satisfies $A(x,u_M(x)) = 0$.

Two examples of Heavyside type fluxes $A(x,u) = H(x)f(u) + (1 - H(x))g(u)$ that satisfy hypothesis (H3) and (H3') are presented in Figure 2.1-(a) and Figure 2.1-(b) respectively. For the case of (H3), it is enough that $f$ anf $g$ are increasing one-to-one functions.

An example of application - with hypothesis (H3) - is the classical transport equation but




(a) Hypothesis (H3) (b) Hypothesis (H3')

Figure 2.1: Admissible fluxes (Heavyside type)

with a (positive) discontinuous coefficient $S(x)$ in the flux

$$\partial_t u + \partial_x \left[S(x)u\right] = 0. \tag{2.1}$$

However notice that in this case our result is less general than those in [3].

The alternative case involving hypothesis (H3') is obviously related to discontinuous Burger-Hopf's equation - here also $S(x)$ is supposed to be positive

$$\partial_t u + \partial_x \left[S(x)u^2\right] = 0. \tag{2.2}$$

In both examples (2.1) and (2.2), hypothesis (H3) (resp. (H3')) is satisfied, hypothesis (H1) gives the admissible discontinuous form of $S$ and hypothesis (H2) is equivalent to

$$\exists (m_S, M_S) \quad \text{for a.e. } x \in \mathbb{R} \quad 0 < m_S \leq S(x) \leq M_S < +\infty.$$

But hypothesis (H3') is also able to cover more general crossing convex fluxes on the form $A(x,u) = k_\pm (u - \beta_\pm)^2$ - see Figure 2.1-b ans Section 5.

The discontinuous flux case is concerned with numerous applications : sedimentation process, two phase flow in porous media, road traffic... Let us also mention that the discontinuous flux case has natural links with Saint-Venant models : the modelization of blood flow with the Saint-Venant system exhibits the Young modulus of arteries - that can be discontinuous after chirurgical acts - as a coefficient in the pressure flux - see [7] ; for a stationary flow, the coupled transport equation is of the form $A(x,u) = a(x)u$ where $a(x)$ is the velocity of the flow, that can be discontinuous - see [2].

# 3 Partially Adapted Kruzkov Entropies

Given $\alpha \in \mathbb{R}$, an immediate consequence of hypothesis (H1) and (H3) is the existence and the uniqueness of a function $k_\alpha$ from $\mathbb{R}$ to $\mathbb{R}$ such that

$$A\left(x, k_\alpha(x)\right) = \alpha \qquad \text{for a.e. } x \in \mathbb{R}. \tag{3.1}$$

The alternative case (H3') leads to similar conclusions : given $\alpha \in [0, +\infty]$ and $x \in \mathbb{R}\backslash\mathcal{N}$, there are two unique real numbers $k_\alpha^+(x) \in [u_M(x), +\infty]$ and $k_\alpha^-(x) \in [-\infty, u_M(x)]$ such that

$$A\left(x, k_\alpha^\pm(x)\right) = \alpha. \tag{3.2}$$

In the following, when relations are valid under hypothesis (H3) or (H3'), and to avoid unuseful repetitions, $k_\alpha(x)$ will denote either $k_\alpha(x)$ or $k_\alpha^\pm(x)$.

Let us notice that the hypothesis (H2) implies that, for each $\alpha$, $k_\alpha \in L^\infty(\mathbb{R})$.

These definitions allow us to introduce partially adapted Kruzkov entropies, which are a natural way to extend classical Kruzkov entropies to the discontinuous flux case.

**Definition 3.1** *Let $u$ (resp. $v$) $\in L^\infty([0,T] \times \mathbb{R}) \cap C^0([0,T], L^1_{loc}(\mathbb{R}))$. We say it is an entropy subsolution -resp. supersolution- of (1.1) if and only if for all $\alpha \in \mathbb{R}$ (or $\mathbb{R}^+$ under hypothesis (H3'))*

$$\partial_t(u - k_\alpha(x))_+ + \partial_x\left[(A(x,u) - A(x,k_\alpha(x)))\ \mathrm{sgn}_+(u - k_\alpha(x))\right] \le 0, \tag{3.3}$$

*-resp.*

$$\partial_t(v - k_\alpha(x))_- + \partial_x\left[(A(x,v) - A(x,k_\alpha(x)))\ \mathrm{sgn}_-(v - k_\alpha(x))\right] \ge 0. \tag{3.4}$$

Our motivation to introduce these adapted entropies comes from the contraction property which still holds true under the form (1.3). It is natural to state Kruzkov entropy with the steady state solution

$$\frac{\partial}{\partial_t}k_\alpha(x) + \frac{\partial}{\partial_x}A\left(x, k_\alpha(x)\right) = 0, \tag{3.5}$$

but not with constants.

In a future work, we will prove that this condition can be derived from the vanishing viscosity method and thus is a natural entropy condition.

# 4 Uniqueness Theorem

**Theorem 4.1** *Let $u$ and $v \in L^\infty([0,T], \mathbb{R}) \cap C^0([0,T], L^1_{loc}(\mathbb{R}))$ be respectively an entropy sub- and supersolution to the initial value problem (1.1) with initial data $u_0, v_0 \in L^\infty(\mathbb{R})$.*

*Assume hypothesis (H1)-(H2)-(H3) or (H1)-(H2)-(H3') on the flux are true. Then for a.e.* $t \in [0,T]$

$$\int_a^b (u(x,t) - v(x,t))_+ \, dx \leq \int_{a-Mt}^{b+Mt} (u_0(x) - v_0(x))_+ \, dx. \tag{4.1}$$

**Proof of Theorem 4.1.** We denote by $Q = \mathbb{R} \times [0,T]$. Proving the theorem is equivalent to establishing the following inequality for all $\phi \in C_0^\infty(Q)$ - see [5, 21]

$$\begin{aligned} &\int_Q (u(t,x) - v(t,x))_+ \partial_t \phi dx dt \\ &\quad + \int_Q \big(A(x,u(t,x)) - A(x,v(t,x))\big) \mathrm{sgn}_+(u(t,x) - v(t,x)) \partial_x \phi dx dt \\ &\qquad + \int_{\mathbb{R}} (u_0(x) - v_0(x))_+ \phi(x,0) dx \geq 0. \end{aligned} \tag{4.2}$$

Since $A(.,u)$ is continuous for $x \in \mathbb{R} \backslash \mathcal{N}$, we can define for a.e. $(x,y,s,t) \in Q^2$ two functions $\tilde{u}(t,x,y)$ and $\tilde{v}(s,y,x)$ from $[0,T] \times \mathbb{R}^2$ to $\mathbb{R}$ such that

$$\begin{aligned} A(y, \tilde{u}(t,x,y)) &= A(x, u(t,x)), \\ A(x, \tilde{v}(s,y,x)) &= A(y, v(s,y)). \end{aligned} \tag{4.3}$$

According to the notations of Section 3, and under hypothesis (H3), it is equivalent to

$$\begin{aligned} \tilde{u}(t,x,y) &= k_{A(x,u(t,x))}(y), \\ \tilde{v}(s,y,x) &= k_{A(y,v(s,y))}(x). \end{aligned} \tag{4.4}$$

In the case (H3'), we impose also that

$$\begin{aligned} \mathrm{sgn}(\tilde{u}(t,x,y) - u_M(y)) &= \mathrm{sgn}(u(t,x) - u_M(x)), \\ \mathrm{sgn}(\tilde{v}(s,y,x) - u_M(x)) &= \mathrm{sgn}(v(s,y) - u_M(y)). \end{aligned}$$

We denote the new sign function by $\widetilde{\mathrm{sgn}}(x,u) = \mathrm{sgn}(u - u_M(x))$. According to the previous notations it means that

$$\begin{aligned} \tilde{u}(t,x,y) &= k^+_{A(x,u(t,x))}(y)\, \widetilde{\mathrm{sgn}}_+(x,u(t,x)) \;+\; k^-_{A(x,u(t,x))}(y)\, \widetilde{\mathrm{sgn}}_-(x,u(t,x)), \\ \tilde{v}(s,y,x) &= k^+_{A(y,v(s,y))}(x)\, \widetilde{\mathrm{sgn}}_+(y,v(s,y)) \;+\; k^-_{A(y,v(s,y))}(x)\, \widetilde{\mathrm{sgn}}_-(y,v(s,y)). \end{aligned} \tag{4.5}$$

Now we write the entropy condition (3.3) for $u(t,x)$ with $\alpha = A(y, v(s,y))$

$$\begin{aligned} &\partial_t (u(t,x) - k_{A(y,v(s,y))}(x))_+ \\ &\quad + \partial_x \left[ \big(A(x,u(t,x)) - A(x, k_{A(y,v(s,y))}(x))\big)\ \mathrm{sgn}_+\big(u(t,x) - k_{A(y,v(s,y))}(x)\big) \right] \leq 0. \end{aligned}$$

It leads to

$$\begin{aligned}&\partial_t \left(u(t,x) - \tilde{v}(s,y,x)\right)_+ \\ &\qquad + \partial_x \left[ \left(A(x,u(t,x)) - A(y,v(s,y))\right) \operatorname{sgn}_+\left(u(t,x) - \tilde{v}(s,y,x)\right)\right] \leq 0.\end{aligned} \tag{4.6}$$

We obtain a similar inequality when we write partially adapted Kruzkov entropy relation (3.4) for $v(s,y)$ with $\alpha = A(x,u(t,x))$

$$\begin{aligned}&\partial_s \left(v(s,y) - \tilde{u}(t,x,y)\right)_- \\ &\qquad + \partial_y \left[ \left(A(y,v(s,y)) - A(x,u(t,x))\right) \operatorname{sgn}_-\left(v(s,y) - \tilde{u}(t,x,y)\right)\right] \leq 0.\end{aligned} \tag{4.7}$$

Now for $\epsilon > 0, \eta > 0$, we introduce two positive functions $\rho, \xi \in C_0^\infty(\mathbb{R})$ such that

$$\int_{\mathbb{R}} \rho(z)dz = \int_{\mathbb{R}} \xi(z)dz = 1, \tag{4.8}$$

and, for $\eta, \epsilon > 0$, we define two families of functions $\rho_\epsilon, \xi_\eta \in C_0^\infty(\mathbb{R})$ such that

$$\xi_\eta(z) = \frac{1}{\eta}\xi(\frac{z}{\eta}), \qquad \rho_\epsilon(z) = \frac{1}{\epsilon}\rho(\frac{z}{\epsilon}),$$

which provide two approximations of the Dirac mass $\delta_0$. Moreover we impose that the support of $\rho$ is included in $]-2,-1[$. Then we add (4.7) to (4.6) and we integrate in $y, s, x, t$ against a function $\Phi_{\eta\epsilon}(x,t,y,s) \in C_0^\infty(Q^2)$ with $\Phi_{\eta\epsilon}(x,t,y,s) = \phi(x,t)\rho_\epsilon(t-s)\xi_\eta(x-y)$. Finally we obtain

$$\begin{aligned}
&\text{(I)} && \int_{Q^2} (u(t,x) - \tilde{v}(s,y,x))_+ \partial_t \phi(x,t)\rho_\epsilon(t-s)\xi_\eta(x-y)dydsdxdt \\
&\text{(II)} && -\int_{Q^2} ((u(t,x) - \tilde{v}(s,y,x))_+ - (v(s,y) - \tilde{u}(t,x,y))_-) \\
& && \qquad\qquad \phi(x,t)\rho_\epsilon'(t-s)\xi_\eta(x-y)dydsdxdt \\
&\text{(III)} && +\int_{Q^2} (A(x,u(t,x)) - A(x,\tilde{v}(s,y,x)))\partial_x\phi(x,t)\rho_\epsilon(t-s)\xi_\eta(x-y) \\
& && \qquad\qquad (\operatorname{sgn}_+(u(t,x) - \tilde{v}(s,y,x))dydsdxdt \\
&\text{(IV)} && -\int_{Q^2} (A(x,u(t,x)) - A(y,v(s,y)))\phi(x,t)\rho_\epsilon(t-s)\xi_\eta'(x-y) \\
& && \qquad (\operatorname{sgn}_+(u(t,x) - \tilde{v}(s,y,x)) + \operatorname{sgn}_-(v(s,y) - \tilde{u}(t,x,y)))dydsdxdt \\
&\text{(V)} && +\int_{Q\times\mathbb{R}} (u_0(x) - \tilde{v}(s,y,x))_+\phi(x,0)\rho_\epsilon(-s)\xi_\eta(x-y)dydsdx \\
&\text{(VI)} && +\int_{Q\times\mathbb{R}} (v_0(y) - \tilde{u}(t,x,y))_-\phi(x,t)\rho_\epsilon(t)\xi_\eta(x-y)dydxdt \geq 0.
\end{aligned} \tag{4.9}$$

The main difference with classical Kruzkov's proof - see [18, 5, 21] - is the Terms (II) and (IV). Notice that the derivatives that appear in these terms are derivatives of functions that tend to Dirac masses. We will prove that Term (IV) is equal to zero for all $(\eta, \epsilon)$. For Term (II), the main idea of the proof is to first consider the limit when $\eta$ tends to zero - with a fixed $\epsilon$ - and to show that this limit is equal to zero for all $\epsilon$.

Let us first establish that $\tilde{v}$ and $\tilde{u}$ belong to $L^\infty([0,T] \times \mathbb{R}^2)$. We give the proof for $\tilde{v}$. By hypothesis $v \in L^\infty([0,T] \times \mathbb{R})$. It follows from (H2) that for a.e. $s, y$

$$|A(y, v(s,y))| \leq \max_{-\|v\|_{L^\infty} \leq \sigma \leq \|v\|_{L^\infty}} g(\sigma) = M.$$

Since $A(x, \tilde{v}(s,y,x)) = A(y, v(s,y))$ and using (H2) again we obtain

$$f(\tilde{v}(s,y,x)) \leq M.$$

Finally, since $f$ is a decreasing then increasing function such that $|f(\pm\infty)| = +\infty$, we conclude that $\tilde{v} \in L^\infty\left([0,T] \times \mathbb{R}^2\right)$.

**Term (IV).** We now treat the part involving the sign functions. We prove that

$$\operatorname{sgn}(u(t,x) - \tilde{v}(s,y,x)) = \operatorname{sgn}(\tilde{u}(t,x,y) - v(s,y)) \quad \text{for a.e. } t,x,s,y \in Q^2. \tag{4.10}$$

By definition of $\tilde{u}$ and $\tilde{v}$ in (4.3), we have for a.e. $t,x,s,y \in Q^2$

$$A(x, u(t,x)) - A(x, \tilde{v}(s,y,x)) = A(y, \tilde{u}(t,x,y)) - A(y, v(s,y)). \tag{4.11}$$

Under the hypothesis (H3), $A(x,.)$ is monotonous and therefore (4.11) implies the result (4.10). Under hypothesis (H3'), it follows from (4.5) that

$$\widetilde{\operatorname{sgn}}(x,u) - \widetilde{\operatorname{sgn}}(y,\tilde{u}) = \widetilde{\operatorname{sgn}}(y,v) - \widetilde{\operatorname{sgn}}(x,\tilde{v}) = 0. \tag{4.12}$$

The case $\widetilde{\operatorname{sgn}}(x,u) = \widetilde{\operatorname{sgn}}(y,v)$ reduces to hypothesis (H3) since $A(x,.)$ is monotonous on each semi-space $[-\infty, u_M(x)]$ and $[u_M(x), +\infty]$. If $\widetilde{\operatorname{sgn}}(x,u) \neq \widetilde{\operatorname{sgn}}(y,v)$, the result (4.10) is an immediate consequence of (4.12).
Then from (4.10) we deduce that for a.e. $t,x,s,y \in Q^2$

$$(A(x,u) - A(y,v))\phi(x,t)\rho_\epsilon(t-s)\xi_\eta^{'}(x-y)(\operatorname{sgn}_+(u-\tilde{v}) + \operatorname{sgn}_-(v-\tilde{u})) = 0.$$

Since $u, v, \tilde{u}, \tilde{v} \in L^\infty$ and, for $\eta, \epsilon > 0$, $\phi, \xi_\eta, \rho_\epsilon \in C_0^\infty$, we can apply Lebesgue's theorem and conclude that, for every $\eta, \epsilon > 0$, Term (IV) is equal to zero.

**Term(II).** We first observe that

$$\begin{aligned} &|(u(t,x) - \tilde{v}(s,y,x))_+ - (v(s,y) - \tilde{u}(t,x,y))_-| \\ &\qquad \leq |u(t,x) - \tilde{u}(t,x,y)| + |v(s,y) - \tilde{v}(s,y,x)|, \end{aligned}$$

and then it is sufficient to prove that

$$\int_{\mathbb{R}} |v(s,y) - \tilde{v}(s,y,x)| \xi_\eta(x-y) dx \xrightarrow[\eta \to 0]{} 0 \quad \text{for a.e. } s, y \in Q, \tag{4.13}$$

to establish that, for every $\epsilon > 0$, the limit in $\eta$ of Term (II) is equal to zero. Indeed, once we have (4.13), and since, for $\epsilon > 0$, all functions are bounded, we can apply dominated convergence to conclude that the integral in $s, y, t, x$ tends to zero. The results for $|u(t,x) - \tilde{u}(t,x,y)|$ is obviously similar. Thus the absolute value of Term (II) is bounded by an expression that vanishes with $\eta$.
In order to prove (4.13) we now establish that

$$\tilde{v}(s,y,x) \xrightarrow[x \to y]{} \tilde{v}(s,y,y) = v(s,y) \quad \text{for a.e. } s, y \in Q. \tag{4.14}$$

Here we use the assumtion (H3) or (H3'), i.e. that $A$ is continuous outside a negligeable set. Then for $y \in \mathbb{R} \backslash \mathcal{N}$

$$A(x, \tilde{v}(s,y,y)) \xrightarrow[x \to y]{} A(y, \tilde{v}(s,y,y)).$$

On an other hand, we have by hypothesis

$$A(y, \tilde{v}(s,y,y)) = A(y, v(s,y)) = A(x, \tilde{v}(s,y,x)) \quad \text{for a.e. } s, y, x \in Q \times \mathbb{R}.$$

Thus

$$A(x, \tilde{v}(s,y,x)) - A(x, \tilde{v}(s,y,y)) \xrightarrow[x \to y]{} 0,$$

and (4.14) is a consequence of the fact that $A(x,.)$ is a one to one function.
Now we claim that the integral in (4.13) can be written

$$\int_{\mathbb{R}} |v(s,y) - \tilde{v}(s,y,y+\eta z)| \xi(z) dz,$$

and then, since all functions are bounded and since the support of $\xi$ is bounded also, we can use the result (4.14) and dominated convergence to conclude (4.13).

Terms (I) and (III) are more classical. The only key point is that we must deal with "tilda functions", but we will use the result (4.13) to recover classical Kruzkov's proof.

**Term (I).** We first observe that

$$\begin{aligned} \Big| \int_{Q^2} & (u(t,x) - \tilde{v}(s,y,x))_+ \partial_t \phi(x,t) \rho_\epsilon(t-s) \xi_\eta(x-y) dy ds dx dt \\ & - \int_{Q^2} (u(t,x) - v(s,y))_+ \partial_t \phi(x,t) \rho_\epsilon(t-s) \xi_\eta(x-y) dy ds dx dt \Big| \\ & \qquad \le \int_{Q^2} |\tilde{v}(s,y,x) - v(s,y)| \, \partial_t \phi(x,t) \rho_\epsilon(t-s) \xi_\eta(x-y) dy ds dx dt, \end{aligned}$$

and we use the previous computation - see (4.13) - to claim that the limit in $\eta, \epsilon$ of term (I) is the same as the limit of

$$\int_{Q^2} (u(t,x) - v(s,y))_+ \, \partial_t \phi(x,t) \rho_\epsilon(t-s) \xi_\eta(x-y) dy ds dx dt.$$

Now we claim it is enough to prove that

$$\int_{Q^2} |v(t,x) - v(s,y)| \partial_t \phi(x,t) \rho_\epsilon(t-s) \xi_\eta(x-y) dy ds dx dt \xrightarrow[\eta\to 0,\ \epsilon \to 0]{} 0, \tag{4.15}$$

to conclude that the limit of Term (I), when $\eta$ and $\epsilon$ tend to zero, is

$$\int_{Q} (u(t,x) - v(t,x))_+ \, \partial_t \phi(x,t) dt dx.$$

The proof of (4.15) is also a crucial step of the uniqueness proof when the flux does not depend on the space variable and we refer to [5, 21] for the details.

**Term (III).** Finally we consider the term that contains the fluxes. We define

$$G(x,u,w) = (A(x,u) - A(x,w)) \mathrm{sgn}(u-w).$$

Hypothesis (H2) implies that $G$ is a locally Lipschitz function of the third variable. Since $\tilde{v} \in L^\infty([0,T] \times \mathbb{R} \times \mathbb{R})$, it follows that

$$\begin{aligned} |G(x,u(t,x),\tilde{v}(s,y,x)) - G(x,u(t,x),\tilde{v}(s,y,y))| &\le C|\tilde{v}(s,y,x) - \tilde{v}(s,y,y)| \\ &= C|\tilde{v}(s,y,x) - v(s,y)|, \end{aligned}$$

and it follows from (4.13) that the limit of Term (III) is the same as the limit of

$$\int_{Q^2} (A(x,u(t,x)) - A(x,v(s,y))) \ \mathrm{sgn}(u(t,x) - v(s,y)) \\ \partial_x \phi(x,t) \rho_\epsilon(t-s) \xi_\eta(x-y) dy ds dx dt.$$

We use a second time the Lipschitz property on $G$. Now $v \in L^\infty([0,T] \times \mathbb{R})$ and

$$|G(x,u(t,x),v(s,y)) - G(x,u(t,x),v(t,x))| \le C|v(s,y) - v(t,x)|.$$

Thus it is sufficient to prove that

$$\int_{Q^2} |v(t,x) - v(s,y)| \partial_x \phi(x,t) \rho_\epsilon(t-s) \xi_\eta(x-y) dy ds dx dt \xrightarrow[\eta\to 0,\ \epsilon \to 0]{} 0, \tag{4.16}$$

to conclude that Term (III) tends to

$$\int_{Q} \big(A(x,u(t,x)) - A(x,v(t,x))\big) \mathrm{sgn}_+(u(t,x) - v(t,x)) \partial_x \phi dx dt.$$

The integral in (4.16) appears in classical Kruzkov's proof. It is very similar to the one in (4.15) and same arguments lead to the result (4.16).

The computation of Terms (V) and (VI) are classical. Thanks to the hypothesis on the support of $\rho$, Term (VI) is equal to zero. For Term (V), we claim that the result (4.13) allows us to consider only

$$\int_{Q\times\mathbb{R}} (u_0(x) - v(s,y))_+ \phi(x,0)\rho_\epsilon(-s)\xi_\eta(x-y)dydsdx \to \int_{\mathbb{R}} (u_0(x) - v_0(x))_+ \phi(x,0)dx,$$

and then the end of the proof is standard - see [5, 21]. □

# 5 Application : Discontinuous convex flux

We noticed at the end of the introduction that different interface conditions can select different unique solutions. Since our method does not require an interface condition it can be used to discriminate the existing interface conditions, at least for the cases where our theory can be applied.

Here we propose to study a particular Heavyside type flux $A(x,u) = H(x)f_+(u) + (1 - H(x))f_-(u)$ where $f_\pm(u) = k_\pm(u-\beta_\pm)^2$ for which the interface conditions in [14, 1] do not select the same solution. This convex flux satisfies hypothesis (H3') and thus we can exhibit the solution that is selected by our method.

We study the Riemann problem associated with the very simple discontinuous convex flux of Heavyside type

$$A(x,u) = H(x)\frac{u^2}{2} + (1 - H(x))\frac{(u-1)^2}{2}, \tag{5.1}$$

and with the constant initial data

$$u_0(x) = \frac{1}{2}. \tag{5.2}$$

It is obvious that

$$u(t,x) = \frac{1}{2}, \tag{5.3}$$

is a weak solution of the Riemann problem (5.1)-(5.2). But more generally, for $u_i \in [0, \frac{1}{2}]$ the function defined by

$$u(t,x) = \begin{cases} \frac{1}{2} & x \le -\frac{t}{2} \\ 1 + \frac{x}{t} & -\frac{t}{2} < x \le -u_i t \\ 1 - u_i & -u_i t < x \le 0 \\ u_i & 0 < x \le u_i t \\ \frac{x}{t} & u_i t < x \le \frac{t}{2} \\ \frac{1}{2} & \frac{t}{2} < x \end{cases}, \tag{5.4}$$

is also a weak solution of the Riemann problem (5.1)-(5.2).
Now let us apply our entropy theory to this case. It follows from the definition (3.2) that

$$k_\alpha^\pm(x) = \pm\sqrt{2\alpha} + 1 - H(x). \tag{5.5}$$

Thus the entropy inequality (3.3) becomes

$$\begin{aligned} &\partial_t |u - (\pm\sqrt{2\alpha} + 1 - H(x))| \\ &+\partial_x \left[ (H(x)\frac{u^2}{2} + (1 - H(x))\frac{(u-1)^2}{2} - \alpha) \ \mathrm{sgn}(u - (\pm\sqrt{2\alpha} + 1 - H(x))) \right] \le 0. \end{aligned} \tag{5.6}$$

Let us choose a solution on the form (5.4) with $u_i = 0$

$$u(t,x) = \begin{cases} \frac{1}{2} & x \le -\frac{t}{2} \\ 1 + \frac{x}{t} & -\frac{t}{2} < x \le 0 \\ \frac{x}{t} & 0 < x \le \frac{t}{2} \\ \frac{1}{2} & \frac{t}{2} < x \end{cases}. \tag{5.7}$$

For $(x,t)$ such that $2x \le -t$ or $2x > t$ the entropy inequality (5.6) is obviously satisfied. Now for $(x,t)$ such that $2x \in ]-t, t]$, the solution can be denoted by

$$u(t,x) = \frac{x}{t} + 1 - H(x).$$

The entropy inequality (5.6) becomes

$$\partial_t \left| \frac{x}{t} \pm \sqrt{2\alpha} \right| + \frac{1}{2}\partial_x \left[ \left( \left(\frac{x}{t}\right)^2 - 2\alpha \right) \mathrm{sgn}(\frac{x}{t} \pm \sqrt{2\alpha}) \right] \leq 0,$$

and is also obviously satisfied. Thus the solution (5.7) is the entropy solution of the Riemann problem (5.1)-(5.2).

Notice that for this particular crossing convex flux, the interface condition in [14] selects the constant solution (5.3) whereas the interface condition in [1] selects the solution (5.7).

# Contents

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