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Near-Optimal Parameterization of the Intersection of Quadrics: III. Parameterizing Singular Intersections

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Abstract: In Part II [3] of this paper, we have shown, using a classification of pencils of quadrics over the reals, how to determine quickly and efficiently the real type of the intersection of two given quadrics.

For each real type of intersection, we design, in this third part, an algorithm for computing a near-optimal parameterization. We also give here examples covering all the possible situations, in terms of both the real type of intersection and the number and depth of square roots appearing in the coefficients.

Key-words: Intersection of surfaces, quadrics, pencils of quadrics, curve parameterization, singular intersections.

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Paramétrisation quasi-optimale de l'intersection de quadriques : III. Paramétrisation des courbes d'intersection singulières

Résumé : Dans la partie II [3] de cet article, nous avons montré, en se basant sur une classification des faisceaux de quadriques de $\mathbb{P}^3(\mathbb{R})$ sur les réels, comment déterminer rapidement et efficacement le type réel de l'intersection de deux quadriques données.

Dans cette troisième partie, nous utilisons l'information collectée pendant la phase de détection du type pour diriger le paramétrage de l'intersection. Dans chaque cas possible, nous donnons la trame d'un algorithme quasi-optimal pour paramétrer la partie réelle de l'intersection et donnons des exemples couvrant toutes les situations possibles en terme de nombre et de profondeur de radicaux impliqués.

Mots-clés : Intersection de surfaces, quadriques, faisceaux de quadriques, paramétrisation, intersections singulières.

1 Introduction

Building on the classification of pencils of quadrics of $\mathbb{P}^3(\mathbb{R})$ over the reals achieved in Part II [3] and the type-detection algorithm that we deduced from this classification, we now are ready to present near-optimal parameterization algorithms for all the possible types of real intersection.

Since the smooth quartic case has already been thoroughly studied in Part I [2], we focus here on the singular cases. For each case, we prove the near-optimality of the parameterization and, when there is possibly an extra square root, we describe the test needed to assert the full optimality which always boils down to finding a rational point on a (possibly non-rational) conic.

In what follows, Q_S, Q_T refer to the initial quadrics and Q_R (assumed to be distinct from Q_S) to the intermediate quadric used to parameterize the intersection C of Q_S and Q_T. As in Section I.7.1, denote by Ω the equation in the parameters:

$$\Omega : \mathbf{X}^T S \mathbf{X} = 0,$$

where $\mathbf{X}$ is the parameterization of Q_R. Denote also by C_Ω the curve zero-set of Ω. Recall that the parameterization of Q_R defines an isomorphism between C and the plane curve C_Ω. When C is singular, its genus is 0 so it can be parameterized by rational functions (i.e. $\sqrt{\Delta}$ can be avoided).

Our general philosophy is to use for Q_R the rational quadric of the pencil of smallest rank. This will lead us to use repeatedly the results of Section I.6 on the optimality of parameterizations of projective quadrics and to parameterize cones without a rational point, cones with a rational point, pairs of planes, etc. As will be seen, this philosophy has the double advantage of (i) avoiding $\sqrt{\Delta}$ in all singular cases, and (ii) minimizing the number of radicals. As an additional benefit, it helps keep the size of the numbers involved in intermediate computations and in the final parameterizations to a minimum (see Part IV [4]).

For every type of real intersection, we give a set of worst-case examples where the maximum number of square roots is reached, both in the optimal and near-optimal situations (the best-case examples are those given by the canonical forms of Section II.3). Examples covering all possible situations are gathered in Appendix C.

A summary of the results of this part is given in Table 1.

The rest of this third part is as follows. Section 2 gives near-optimal parameterization algorithms for all types of real intersection when the pencil is regular. Section 3 does the same for singular pencils (i.e., when the determinantal equation vanishes identically). Several examples are detailed in Section 4 and it is shown how our implementation fares on these examples. Finally, we conclude in Section 5 and give a few perspectives.

2 Parameterizing degenerate intersections: regular pencils

In this section, we outline parameterization algorithms for all cases of regular pencils, i.e. when the determinantal equation does not identically vanish. Information gathered in the type-detection phase (Part II) is used as input; see in particular Table II.4 but also the details of the classification of pencils over the reals. In each case, we study optimality issues and give worst-case examples.

Segre string	real type of intersection	worst case format of parameterization	worst-case optimality of parameterization
$[1111]$	nonsingular quartic (see part I)	$\mathbb{Q}(\sqrt{\delta})[\xi,\sqrt{\Delta}]$, $\Delta\in\mathbb{Q}(\sqrt{\delta})[\xi]$	rational point on degree-8 surface
$[112]$	point	$\mathbb{Q}$	optimal
	nodal quartic	$\mathbb{Q}(\sqrt{\delta})[\xi]$	rational point on conic
$[13]$	cuspidal quartic	$\mathbb{Q}[\xi]$	optimal
$[22]$	cubic and non-tangent line	$\mathbb{Q}[\xi]$	optimal
$[4]$	cubic and tangent line	$\mathbb{Q}[\xi]$	optimal
$[11(11)]$	two points	$\mathbb{Q}(\sqrt{\delta})$	optimal
	conic	$\mathbb{Q}(\sqrt{\delta},\sqrt{\mu})[\xi]$, $\mu\in\mathbb{Q}(\sqrt{\delta})$	optimal if $\sqrt{\delta}\notin\mathbb{Q}$; rational point on conic if $\sqrt{\delta}\in\mathbb{Q}$
	two non-tangent conics	$\mathbb{Q}(\sqrt{\delta},\sqrt{\delta'})[\xi]$	$\mathbb{Q}(\sqrt{\delta'})$-rational point on $\mathbb{Q}(\sqrt{\delta'})$-conic
$[1(21)]$	point	$\mathbb{Q}$	optimal
	two tangent conics	$\mathbb{Q}(\sqrt{\delta})[\xi]$	optimal
$[1(111)]$	double conic	$\mathbb{Q}(\sqrt{\delta})[\xi]$	rational point on conic
$[2(11)]$	point	$\mathbb{Q}$	optimal
	conic and point	$\mathbb{Q}(\sqrt{\delta})[\xi]$	rational point on conic
	conic and two lines not crossing on the conic	$\mathbb{Q}(\sqrt{\delta})[\xi]$	rational point on conic
$[(31)]$	conic	$\mathbb{Q}[\xi]$	optimal
	conic and two lines crossing on the conic	$\mathbb{Q}(\sqrt{\delta})[\xi]$	optimal
$[(11)(11)]$	two points	$\mathbb{K}[\xi]$, degree $(\mathbb{K})=4$	optimal
	two skew lines	$\mathbb{K}[\xi]$, degree $(\mathbb{K})=4$	optimal
	four skew lines	$\mathbb{K}[\xi]$, degree $(\mathbb{K})=4$	optimal
$[(22)]$	double line	$\mathbb{Q}[\xi]$	optimal
	two simple skew lines cutting a double line	$\mathbb{Q}(\sqrt{\delta})[\xi]$	optimal
$[(211)]$	point	$\mathbb{Q}$	optimal
	two double concurrent lines	$\mathbb{Q}(\sqrt{\delta})[\xi]$	optimal
$[1\{3\}]$	conic and double line	$\mathbb{Q}[\xi]$	optimal
$[111]$	point	$\mathbb{Q}$	optimal
	two concurrent lines	$\mathbb{K}[\xi]$, degree $(\mathbb{K})=4$	optimal
	four concurrent lines	$\mathbb{K}[\xi]$, degree $(\mathbb{K})=4$	optimal
$[12]$	double line	$\mathbb{Q}[\xi]$	optimal
	two simple and a double concurrent lines	$\mathbb{Q}(\sqrt{\delta})[\xi]$	optimal
$[3]$	concurrent simple and triple lines	$\mathbb{Q}[\xi]$	optimal
$[1(11)]$	point	$\mathbb{Q}$	optimal
	two concurrent double lines	$\mathbb{Q}(\sqrt{\delta})[\xi]$	optimal
$[(21)]$	quadruple line	$\mathbb{Q}[\xi]$	optimal
$[11]$	quadruple line	$\mathbb{Q}[\xi]$	optimal

Table 1: Ring of definition of the projective coordinates of the parameterization of each component of the intersection and optimality, in all cases where the real part of the intersection is 0- or 1-dimensional. $\delta,\delta'\in\mathbb{Q}$.

In the following, we often need to compute the parameterization of the intermediate quadric Q_R and this is achieved using the normal form of Q_R. Recall that a rational congruence sending a quadric with rational coefficients into normal form can be computed using Gauss reduction of quadratic forms into sums of squares (see Part I).

Recall also that the discriminant of a quadric is the determinant of the associated matrix. In the following, we also call *discriminant of a pair of planes* Q_R the product ab where $ax^2 - by^2 = 0$ is the canonical equation of a pair of planes obtained from Q_R by a real rational congruence transformation; the discriminant is defined up to a rational square factor.

2.1 Nodal quartic in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [112]$

If we parameterize C using the generic algorithm (see Part I), we will not be able to avoid the appearance of $\sqrt{\Delta}$ because C_Ω (as C) is irreducible. However, since the intersection curve is singular, we we know that $\sqrt{\Delta}$ is avoidable by Proposition I.7.1. We thus proceed differently.

2.1.1 Algorithms

Let λ_1 be the real and rational double root of the determinantal equation. Let Q_R be the rational cone associated with λ_1. As we have found in Section II.3, there are essentially two cases depending on the real type of the intersection.

Point. Q_R is an imaginary cone. The intersection is reduced to a point, which is the apex of Q_R. Since λ_1 is rational, this apex is rational (otherwise its algebraic conjugate would also be a singular point of the cone). Thus the intersection in this case is defined in $\mathbb{Q}$.

Real nodal quartic (with or without isolated singularity). Let P be a real rational congruence transformation sending the apex of Q_R to $(0,0,0,1)$. The parameterization $\mathbf{X}(u,v,s) = P\,(x_1(u,v), x_2(u,v), x_3(u,v), s)^T, (u,v,s) \in \mathbb{P}^{\star 2}$ of the cone (see Table I.3) introduces a square root $\sqrt{\delta}$. Equation Ω in the parameters is

$$as^2 + b(u,v)s + c(u,v) = 0,$$

with a and the coefficients of b, c defined in $\mathbb{Q}(\sqrt{\delta})$. The nodal quartic passes through the vertex of Q_R and this point corresponds to the value $u = v = 0$ of the parameters. At this point $s \neq 0$ because $(u,v,s) \in \mathbb{P}^{\star 2}$. Thus $a = 0$, Ω is linear in s and it can be solved rationally for s. This leads to the parameterization of the quartic

$$\mathbf{X}(u,v) = P\,(b(u,v)x_1(u,v), b(u,v)x_2(u,v), b(u,v)x_3(u,v), -c(u,v))^T.$$

The coefficients of $\mathbf{X}(u,v)$ clearly live in $\mathbb{Q}(\sqrt{\delta})[\xi]$, where $\xi = (u,v)$.

When the node of the quartic is not isolated, the singularity is now reached by two different values of $(u,v) \in \mathbb{P}^1(\mathbb{R})$ which are precisely those values such that $b(u,v) = 0$. When the node of the quartic is isolated, the singularity is not reached by $(u,v) \in \mathbb{P}^1(\mathbb{R})$, i.e. $b(u,v) = 0$ has no real solution. In that situation, the node has to be added to the output. Since this point is the vertex of the cone Q_R, it is rational.

2.1.2 Optimality

By Proposition I.6.3, if the cone Q_R contains a rational point other than its vertex, it can be parameterized with rational coefficients and thus the parameterization of the nodal quartic is defined over $\mathbb{Q}[\xi]$. Otherwise, if Q_R contains no rational point other than its vertex, then the nodal quartic also contains no rational point other than its singular point. Hence the nodal quartic admits no parameterization over $\mathbb{Q}[\xi]$. Therefore, testing whether $\sqrt{\delta}$ can be avoided in the parameterization of real nodal quartics is akin to deciding whether Q_R has a rational point outside its singular locus; furthermore, finding a parameterization in $\mathbb{Q}[\xi]$ amounts to finding a rational point on Q_R outside its singular locus.

There are cases where $\sqrt{\delta}$ cannot be avoided. Example of these are

$$\begin{cases} x^2+y^2-3z^2=0, \\ xw+z^2=0 \end{cases}$$

when the singularity is not isolated and

$$\begin{cases} x^2+y^2-3z^2=0, \\ zw+x^2=0 \end{cases}$$

when the singularity is isolated. In both cases, the projective cone corresponding to the double root of the determinantal equation is the first equation. By Proposition I.6.3, this cone has no rational point except its singular point and $\sqrt{\delta}$ cannot be avoided in the parameterization of the intersection.

2.2 Cuspidal quartic in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4=[13]$

The intersection in this case is always a real cuspidal quartic. As above, using the generic algorithm is not good idea: it would introduce an unnecessary and unwanted $\sqrt{\Delta}$.

We consider instead the cone Q_R associated with the real and rational triple root of the determinantal equation. The singular point of the quartic is the vertex $\mathbf{p}$ of Q_R. The intersection of Q_R with the tangent plane of Q_S at $\mathbf{p}$ consists of the double line tangent to C at the cusp. Since it is double, this line is necessarily rational. So we have a rational cone containing a rational line. By Theorem I.6.1, this cone admits a rational parameterization.

So we are left with an equation $\Omega: as^2+b(u,v)s+c(u,v)=0$ whose coefficients are defined on $\mathbb{Q}$. As above, the singularity is reached at $(u,v)=(0,0)$ and at this point $s\neq 0$, so $a=0$. Thus Ω can be solved rationally for s and the intersection is in $\mathbb{Q}[\xi], \xi=(u,v)$. This is optimal.

2.3 Cubic and secant line in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4=[22]$

The real intersection consists of a cubic and a line. The cubic and the line are either secant or skew. Note that the line of the intersection is necessarily rational, otherwise its algebraic conjugate would also belong to the intersection.

When the double roots of the determinantal equation are real and rational, the pencil contains two rational cones Q_{R_1} and Q_{R_2}. The line of C is the rational line joining the vertices of Q_{R_1} and Q_{R_2}.

Also, the vertex of Q_{R_2} is a rational point on Q_{R_1}, and vice versa, so the two cones can be rationally parameterized (see Theorem I.6.1). Setting up Ω, we have again that it is linear in s, because the line and the cubic intersect at the vertex of the cone, corresponding to $(u,v) = (0,0)$. But here the content of Ω in s is linear in (u,v) and it corresponds to the line of C. The cubic is found after dividing by this content and rationally solving for s. The parameterization of the cubic is defined in $\mathbb{Q}[\xi]$.

When the double roots of the determinantal equation are either complex conjugate (the cubic and the line are not secant) or real algebraic conjugate (the cubic and the line are secant), there exists quadrics of inertia $(2,2)$ in the pencil (by Theorems I.4.1 and I.4.3). We use the generic algorithm of Part I: first find a quadric Q_R of inertia $(2,2)$ of the pencil through a rational point. Since C contains a rational line, the discriminant of this quadric is a square by Lemma I.7.4 and Q_R can be rationally parameterized by Theorem I.6.1. Now compute the bidegree $(2,2)$ equation Ω. The line of C corresponds to a fixed value of one of the parameters and the contents provide factors of bidegree $(1,0)$ and $(1,2)$ (or $(0,1)$ and $(2,1)$), which are linear in one of the parameters and thus easy to solve rationally for getting a parameterization of the intersection. The parameterization of C is defined in $\mathbb{Q}[\xi]$.

2.4 Cubic and tangent line in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [4]$

The real intersection consists of a cubic and a tangent line. The line is necessarily rational, by the same argument as above. The determinantal equation has a real and rational quadruple root. To it corresponds a real rational projective cone. Since this cone contains a rational line, it can be rationally parameterized (by Theorem I.6.1). The rest is as in the cubic and secant line case when the two roots are rational. The parameterization of the cubic is defined in $\mathbb{Q}[\xi]$.

2.5 Two secant conics in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [11(11)]$

In this case, the determinantal equation has a double root corresponding to a rational pair of planes Q_R. There are several cases depending on the real type of the intersection.

2.5.1 Two points

The pair of planes Q_R is imaginary. Its rational singular line intersects any other quadric of the pencil in two points. So parameterize the line and intersect it with any quadric of the pencil having rational coefficients. A square root is needed to parameterize the two points if and only if the equation in the parameters of the line has irrational roots.

This situation can happen as the following example shows:

$$\begin{cases} z^2 + w^2 = 0, \\ x^2 - 2y^2 + w^2 = 0. \end{cases}$$

Clearly, the two points are defined by $z = w = 0$ and $x^2 - 2y^2 = 0$ so they live in $\mathbb{Q}[\sqrt{2}]$.

2.5.2 One conic

In this case, the pair of planes is real, the pencil has no quadric of inertia $(2,2)$ and only one of the planes of Q_R intersects the other quadrics of the pencil.

The algorithm is as follows. First parameterize the pair of planes and separate the two individual planes. Plugging the parameterization of each plane into the equation of Q_S gives two equations of conics in parameter space, with coefficients in $\mathbb{Q}(\sqrt{\delta})$ where δ is the discriminant of the pair of planes. The conics in parameter space correspond to the components of the intersection, thus one of these conics is real and the other is imaginary. Determine the real conic, that is the one with inertia $(2,1)$, and parameterize it. Substituting this parameterization into the parameterization of the corresponding plane gives a parameterization of the conic of intersection. The parameterization is in $\mathbb{Q}(\sqrt{\delta},\sqrt{\mu})$, where δ is the discriminant of the pair of planes Q_R and $\sqrt{\mu}$ is the square root needed to parameterize the conic in parameter space, $\mu \in \mathbb{Q}(\sqrt{\delta})$.

If δ is not a square, the parameterization is optimal. Indeed, if the intersection had a real $\mathbb{Q}(\sqrt{\delta})$-rational point, the conjugate of that point would be on the conjugate conic which is not real. So such a point does not exist and the parameterization is optimal. If δ is a square, the parameterization is defined in $\mathbb{Q}(\sqrt{\mu})[\xi]$ with $\mu \in \mathbb{Q}$. By Proposition I.6.3, the parameterization is optimal if and only if the (rational) conic contains no rational point; moreover, testing if the parameterization is non-optimal and, if so, finding an optimal parameterization is equivalent to finding a rational point on this rational conic.

The situation where δ is a square but the conic has no rational point (the field of the coefficients is of degree two) can be attained for instance with the following pair of quadrics:

$$\left\{\begin{array}{l}(x-w)(x-3w)=0,\\ x^2+y^2+z^2-4w^2=0.\end{array}\right.$$

The two planes of the first quadric are rational. The plane $x-w=0$ cuts the second quadric in the conic $x-w=y^2+z^2-3w^2=0$. By Proposition I.6.3, this conic has no rational point, so $\sqrt{\delta}$ cannot be avoided and the parameterization of the conic is in $\mathbb{Q}(\sqrt{3})$.

A field extension of degree 4 is obtained with the following quadrics:

$$\left\{\begin{array}{l}x^2-4xw-3w^2=0,\\ x^2+y^2+z^2-w^2=0.\end{array}\right.$$

The pair of planes is defined on $\mathbb{Q}(\sqrt{7})$, so, by the above argument, a field extension of degree 4 is unavoidable.

2.5.3 Two (secant or non-secant) conics

By contrast to the one conic case, the pencil now contains quadrics of inertia $(2,2)$. But going through the generic algorithm and factoring C_Ω directly in two curves of bidegree $(1,1)$ can induce nested radicals. So we proceed as follows. First, find a rational quadric Q_R of inertia $(2,2)$ through a rational point. This introduces one square root, say $\sqrt{\delta}$. Independently, factor the pair of planes, which introduces another square root $\sqrt{\delta'}$. Now plug the parameterization of Q_R in each of the planes. This gives linear equations in the parameters of Q_R which can be solved without introducing nested radicals. The two conics have a parameterization defined in $\mathbb{Q}(\sqrt{\delta}, \sqrt{\delta'})$.

Note that when the two simple roots of the determinantal equation are rational, an alternate approach is to parameterize one of the two rational cones of the pencil instead of a quadric of inertia $(2,2)$, and then proceed as above.

In terms of optimality, $\sqrt{\delta'}$ cannot be avoided if the planes are irrational. As for the other square root, it can be avoided if and only if the conics contain a point that is rational in $\mathbb{Q}(\sqrt{\delta'})$ (by Proposition I.6.3 in which the field $\mathbb{Q}$ can be replaced by $\mathbb{Q}(\sqrt{\delta'})$); moreover, testing if this square root can be avoided and, if so, finding a parameterization avoiding it is equivalent to finding a $\mathbb{Q}(\sqrt{\delta'})$-rational point on this conic whose coefficients are in $\mathbb{Q}(\sqrt{\delta'})$.

All cases can happen. We illustrate this in the non-secant case. An extension of $\mathbb{Q}$ of degree 4 is needed to parameterize the intersection of the following pair:

$$\begin{cases} x^2 - 33w^2 = 0, \\ y^2 + z^2 - 3w^2 = 0. \end{cases}$$

Indeed, $\sqrt{\delta'} = \sqrt{33}$ cannot be avoided. In addition, by Proposition I.6.6, $y^2 + z^2 - 3w^2 - 11x^2 = 0$ has no rational point on $\mathbb{Q}(\sqrt{33})$, thus its intersection with the plane $x = 0$, the conic $y^2 + z^2 - 3w^2 = 0$, also has no rational point on $\mathbb{Q}(\sqrt{33})$; hence the cone $y^2 + z^2 - 3w^2 = 0$ has no rational point on $\mathbb{Q}(\sqrt{33})$ except for its singular locus.

An extension field of degree 2 can be obtained by having conics without rational point, but living in rational planes, as in this example:

$$\begin{cases} x^2 - w^2 = 0, \\ y^2 + z^2 - 3w^2 = 0. \end{cases}$$

It can also be attained by having conics living in non-rational planes but having rational points in the extension of $\mathbb{Q}$ defined by the planes:

$$\begin{cases} x^2 - 3w^2 = 0, \\ y^2 + z^2 - 3w^2 = 0. \end{cases}$$

As can be seen, the points of coordinates $(\sqrt{3}, 0, \pm\sqrt{3}, 1)$ belong to the intersection. So the conic has a parameterization in $\mathbb{Q}(\sqrt{3})[\xi]$.

2.6 Two tangent conics in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [1(21)]$

Here, the determinantal equation has a real and rational triple root, corresponding to a pair of planes Q_R. The other (real and rational) root corresponds to a real projective cone. There are two types of intersection over the reals.

Point. The pair of planes is imaginary and its rational singular line intersects the cone in a double point, which is the only component of the intersection. This point is necessarily rational, otherwise its conjugate would also be in the intersection. One way to compute it is to parameterize the singular line, plug the parameterization in the rational equation of the cone and solve the resulting equation in the parameters.

Two real tangent conics. The pair of planes is real and each of the planes intersects the cone. The singular line of Q_R is tangent to the cone. As above, the point of tangency of the two conics is rational. So, by Proposition I.6.3, the conics have a rational parameterization in the extension of $\mathbb{Q}$ defined by the planes. In other words, the conics have a parameterization defined in $\mathbb{Q}(\sqrt{\delta})[\xi]$, where δ is the discriminant of the pair of planes Q_R, if and only if δ is not a square.

One situation where $\sqrt{\delta}$ cannot be avoided is the following:

$$\begin{cases} x^2 - 2w^2 = 0, \\ xy + z^2 = 0. \end{cases}$$

2.7 Double conic in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [1(111)]$

The determinantal equation has a real rational triple root, corresponding to a double plane. The other root gives a rational cone. Assume this cone is real (otherwise the intersection is empty).

To obtain the parameterization of the double conic, first parameterize the double plane. Then plug this parameterization in the equation of the cone. This gives the rational equation of the conic (in the parameters of the plane). If the conic has a rational point, then it can be rationally parameterized. Otherwise, one square root is needed.

One worst-case situation where a square root is always needed is the following:

$$\begin{cases} x^2 = 0, \\ y^2 + z^2 - 3w^2 = 0. \end{cases}$$

By Proposition I.6.3, the second quadric (a cone) has no rational point outside its vertex. Thus the conic cannot be parameterized rationally.

2.8 Conic and two lines not crossing on the conic in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [2(11)]$

The determinantal equation has two double roots, corresponding to a cone and a pair of planes which is always real. The two roots are necessarily real and rational, otherwise the quadrics associated with them in the pencil would have the same rank. So both the cone and the pair of planes are rational. Also, the vertex of the cone falls on the pair of planes outside its singular line. Thus, by Proposition I.6.2, the discriminant of the pair of planes is a square and each individual plane has a rational parameterization.

Over the reals, there are three cases.

Point. The projective cone is imaginary. The intersection is limited to its real vertex. Since the cone is rational, its vertex is rational.

Point and conic. The cone is now real. One of the planes cuts the cone in a conic living in a rational plane, the other plane cuts the cone in its vertex. The point of the intersection is this vertex and it is rational. To parameterize the conic of the intersection, plug the parameterization of the plane that does not go through the vertex of the cone. This gives a rational conic in the parameters of the plane. One square root is possibly needed to parameterize this conic. It can be avoided if and only if the conic has a rational point.

One example where the square root cannot be avoided is the following:

$$\begin{cases} xw = 0, \\ y^2 + z^2 - 3w^2 = 0. \end{cases}$$

By Proposition I.6.3, the projective cone has no rational point other than its vertex $(1,0,0,0)$. So the conic $x = y^2 + z^2 - 3w^2 = 0$ has no rational point.

Two lines and conic. Again, the cone is real and one plane cuts it in a rational nonsingular conic. But now the second plane, going through the vertex of the cone, further cuts the cone in two lines. The parameterization of the conic goes as above. To represent the lines, we plug the second plane in the equation of the cone and parameterize.

Note that if the lines are rational, then the cone contains a rational line and can be rationally parameterized. Since the conic is the intersection of this cone with a rational plane, it has a rational parameterization. So in that case all three components have parameterizations in $\mathbb{Q}[\xi]$. If the lines are irrational, it can still happen that the conic has a rational point and thus a rational parameterization.

We give examples for the three situations we just outlined. First, the pair

$$\begin{cases} xy = 0, \\ y^2 + z^2 - w^2 = 0 \end{cases}$$

gives birth to the rational lines $y = z \pm w = 0$ and the rational conic $x = y^2 + z^2 - w^2 = 0$ which contains the rational point $(0,0,1,1)$ and can be rationally parameterized. Second, the pair of quadrics

$$\begin{cases} xy = 0, \\ 2y^2 + z^2 - 3w^2 = 0 \end{cases}$$

has as intersection the two irrational lines $y = z \pm \sqrt{3}w = 0$ and the conic $x = 2y^2 + z^2 - 3w^2 = 0$ which contains the rational point $(0,1,1,1)$ so can be rationally parameterized. Finally, the lines and the conic making the intersection of the quadrics

$$\begin{cases} xy = 0, \\ y^2 + z^2 - 3w^2 = 0 \end{cases}$$

cannot be rationally parameterized. Indeed, by Proposition I.6.3, the cone has no rational point outside the vertex $(1,0,0,0)$, so the conic $x = y^2 + z^2 - 3w^2 = 0$ has no rational point.

2.9 Conic and two lines crossing on the conic in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [(31)]$

The determinantal equation has a real pair of planes Q_R corresponding to a real and rational quadruple root. The asymmetry in the sizes of the Jordan blocks associated with this root (the two blocks have size 1 and 3) implies that the individual planes of this pair are rational. The conic of the intersection is always real and the two lines (real or imaginary) cross on the conic.

There are two types of intersection over the reals.

Conic. The point at which the two lines cross is the double point that is the intersection of the singular line of Q_R with any other quadric of the pencil. This point is necessarily rational. So the conic can be rationally parameterized by Proposition I.6.3.

Conic and two lines. To parameterize the intersection, first compute the parameterization of the two planes of Q_R. Plugging these parameterizations in the equation of any other quadric of the pencil yields a conic on one side and a pair of lines on the other side. As above, the conic can be rationally parameterized. As for the two lines, they have a rational parameterization if and only if the discriminant of the pair of lines is a square.

One situation where this discriminant is not a square is as follows:

$$\begin{cases} yz = 0, \\ y^2 + xz - 2w^2 = 0. \end{cases}$$

The conic is given by $y = xz - 2w^2 = 0$ which contains the rational point $(1,0,0,0)$ and can be rationally parameterized. The lines are defined by $z = y^2 - 2w^2 = 0$. But the pair of planes $y^2 - 2w^2 = 0$ has no rational point outside its singular locus so the lines are defined in $\mathbb{Q}(\sqrt{2})$.

2.10 Two skew lines and a double line in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [(22)]$

The determinantal equation has a real and rational quadruple root, which corresponds to a pair of planes. The singular line of the pair of planes is contained in all the quadrics of the pencil. There are two cases.

Double line. The pair of planes is imaginary. The intersection is reduced to the rational singular line of the pair of planes. So C is defined in $\mathbb{Q}[\xi]$.

Two simple lines and a double line. The pair of planes is real. We can factor it into simple planes, parameterize these planes and plug them in any other quadric of the pencil. The two resulting equations in the parameters of the planes are pairs of lines, each pair containing the double line of the intersection and one of the simple lines. The simple lines are rational if and only if the discriminant of the pair of planes is a square.

A situation where the two simple lines are irrational is the following:

$$\begin{cases} y^2 - 2w^2 = 0, \\ xy - zw = 0. \end{cases}$$

2.11 Two double lines in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [(211)]$

The determinantal equation has a real rational quadruple root, which corresponds to a double plane. The double plane cuts any other quadric of the pencil in two double lines in $\mathbb{P}^3(\mathbb{C})$. There are two cases.

Point. Except for the double plane, the pencil consists of quadrics of inertia $(3,1)$. The two lines are imaginary. The intersection is reduced to their rational intersection point, i.e. the point at which the double plane is tangent to the other quadrics of the pencil.

Two real double lines. Except for the double plane, the pencil consists of quadrics of inertia $(2,2)$. The two lines are real. To parameterize them, first compute a parameterization of the double plane and then plug it in any quadric of inertia $(2,2)$ of the pencil. The resulting pair of lines can easily be parameterized. The intersection is thus parameterized with one square root if and only if the lines are irrational.

One case where the square root cannot be avoided is as follows:

$$\begin{cases} w^2 = 0, \\ x^2 - 2y^2 + zw = 0. \end{cases}$$

The lines $w = x^2 - 2y^2 = 0$ have no rational point except for their singular point $(0,0,1,0)$ so their parameterization is in $\mathbb{Q}(\sqrt{2})[\xi]$.

2.12 Four skew lines in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [(11)(11)]$

We start by describing the algorithms we use in this case. We then prove the optimality of the parameterizations and conclude the section by giving examples of pairs of rational quadrics for all possible types of real intersections and extension fields.

2.12.1 Algorithms

In this case the determinantal equation has two double roots that correspond to (possibly imaginary) pairs of planes. It can be written in the form

$$\mathcal{D}(\lambda,\mu) = \gamma\left(a\lambda^2 + b\lambda\mu + c\mu^2\right)^2 = 0, \tag{1}$$

with $\gamma, a, b,$ and c in $\mathbb{Q}$.

In order to minimize the number and depth of square roots in the coefficients of the parameterization of the intersection, we proceed differently depending on the type of the real intersection and the values of γ and $\delta = b^2 - 4ac$.

Note that the roots of the determinantal equation are defined in $\mathbb{Q}(\sqrt{\delta})$ and thus the coefficients of the pairs of planes in the pencil also live in $\mathbb{Q}(\sqrt{\delta})$. Let $d^+, d^- \in \mathbb{Q}(\sqrt{\delta})$ be the discriminants of the two pairs of planes, with $d^+ > d^-$. When $d^+ > 0$ (resp. $d^- > 0$), the corresponding pair of planes is real and can be factored into two planes that are defined over $\mathbb{Q}(\sqrt{d^+})$ (resp. $\mathbb{Q}(\sqrt{d^-})$). The algorithms in the different cases are as follows.

Two points. In this case one pair of planes of the pencil is real (the one with discriminant d^+) and the other is imaginary. We factor the two real planes and substitute in each a parameterization of the (real) singular line of the imaginary pair of planes. The singular line is defined in $\mathbb{Q}(\sqrt{\delta})$

and each of the real planes are defined in $\mathbb{Q}(\sqrt{d^+})$. We thus obtain the two points of intersection with coordinates in $\mathbb{Q}(\sqrt{\delta},\sqrt{d^+})$. The two points are thus defined over $\mathbb{Q}(\sqrt{d^+})$, $d^+ \in \mathbb{Q}(\sqrt{\delta})$, an extension field of degree 4 (in the worst case) with one nested square root.

Two or four lines. Since the intersection is contained in every quadric of the pencil, there are no quadric of inertia $(3,1)$ in the pencil in this case (such quadrics contain no line) and thus $\gamma > 0$. Furthermore all the non-singular quadrics of the pencil have inertia $(2,2)$ (by Theorem I.4.3) and their discriminant is equal to γ, up to a square factor (by Eq. (1)). Hence we can parameterize a quadric Q_R of inertia $(2,2)$ in the pencil using the parameterization of Table I.3 with coefficients in $\mathbb{Q}(\sqrt{\gamma})$ (see Part I).

There are three subcases.

$\sqrt{\delta} \in \mathbb{Q}$**.** The roots of the determinantal equation are real (since $\delta > 0$), thus the intersection consists of four real lines and the two pairs of planes of the pencil are real (see Table II.4). We factor the two pairs of planes into four planes with coefficients in $\mathbb{Q}(\sqrt{d^\pm})$ and intersect them pairwise. We thus obtain a parameterization of the four lines over $\mathbb{Q}(\sqrt{d^+},\sqrt{d^-})$ with $d^\pm \in \mathbb{Q}$ (since δ is a square), an extension field of degree 4 (in the worst case) with no nested square root.

$\sqrt{\delta} \notin \mathbb{Q}$ **and** $\sqrt{\gamma\delta} \in \mathbb{Q}$**.** Here again $\delta > 0$ thus the intersection consists of four real lines and the two pairs of planes of the pencil are real. We factor one of these pairs of planes (say the one with discriminant d^+) in two planes with coefficients in $\mathbb{Q}(\sqrt{d^+})$; if the discriminant of one of the pair of planes is a square, we choose this pair of planes for the factorization. We then substitute the parameterization of the quadric Q_R into each plane. This leads to an equation of bidegree $(1,1)$ in the parameters with coefficients in $\mathbb{Q}(\sqrt{d^+},\sqrt{\gamma})$. This field is equal to $\mathbb{Q}(\sqrt{d^+})$ because $d^+ \in \mathbb{Q}(\sqrt{\delta})$ and $\gamma\delta$ is a square. We finally obtain each line by factoring the equation in the parameters into to terms of bidegree $(1,0)$ and $(0,1)$ and by substituting the solutions of these factors into the parameterization of Q_R. We thus obtain a parameterization of the four lines defined over $\mathbb{Q}(\sqrt{d^+})$, $d^+ \in \mathbb{Q}(\sqrt{\delta})$, an extension field of degree 4 (in the worst case) with one nested square root.

$\sqrt{\delta} \notin \mathbb{Q}$ **and** $\sqrt{\gamma\delta} \notin \mathbb{Q}$**.** In this case we apply the generic algorithm of Part I: we substitute the parameterization of Q_R into the equation of another quadric of the pencil (with rational coefficients). The resulting equation in the parameters of bidegree $(2,2)$ has coefficients in $\mathbb{Q}(\sqrt{\gamma})$. We factor it into two terms of bidegree $(2,0)$ and $(0,2)$, whose coefficients also live in $\mathbb{Q}(\sqrt{\gamma})$. We solve each term separately and each real solution leads to a real line. At least one of the two factors has two real solutions, which are defined in an extension field of the form $\mathbb{Q}(\sqrt{\alpha_1+\alpha_2\sqrt{\gamma}})$, $\alpha_i \in \mathbb{Q}$. If the other factor has real solutions, they are defined in $\mathbb{Q}(\sqrt{\alpha_1-\alpha_2\sqrt{\gamma}})$. Thus in the case where the intersection consists of two real lines, we obtain parameterization defined over an extension field $\mathbb{Q}(\sqrt{\alpha_1+\alpha_2\sqrt{\gamma}})$ of degree 4 (in the worst case), with one nested square root. In the case where the intersection consists of four real lines, the parameterization of the four lines altogether is defined over an extension field of degree 8 (in the worst case) but each of the lines is parameterized over an extension $\mathbb{Q}(\sqrt{\alpha_1+\alpha_2\sqrt{\gamma}})$ or $\mathbb{Q}(\sqrt{\alpha_1-\alpha_2\sqrt{\gamma}})$ of degree 4 (in the worst case), with one nested square root.

2.12.2 Optimality

We prove that the algorithms described above output parameterizations that are always optimal in the number and depth of square roots appearing in their coefficients. This proof needs some considerations of Galois theory that can be found in Appendix A.

The two input quadrics intersect here in four lines in $\mathbb{P}^3(\mathbb{C})$. The pencil contains two (possibly complex) pair of planes and these lines are the intersections between two planes taken in two different pairs of planes. Let $\mathbf{p}_1, \ldots, \mathbf{p}_4$ be their pairwise intersection points of the four lines. These points are the singular points of the intersection. These points are also the intersections of the singular line of a pair of planes with the other pair of planes, and vice versa. Let the points be numbered such that $\mathbf{p}_1$ and $\mathbf{p}_3$ are on the singular line of one pair of planes of the pencil; $\mathbf{p}_2$ and $\mathbf{p}_4$ are then on the singular line of the other pair of planes of the pencil. The four lines of intersection are thus $\mathbf{p}_1\mathbf{p}_2$, $\mathbf{p}_2\mathbf{p}_3$, $\mathbf{p}_3\mathbf{p}_4$, and $\mathbf{p}_4\mathbf{p}_1$.

Let $\mathbb{K}$ be the field of smallest degree on which the four points $\mathbf{p}_i$ are rational. The above algorithms show that $\mathbb{K}$ has degree 1, 2, 4 or 8 (since two rational lines in $\mathbb{K}$ intersect in a rational point in $\mathbb{K}$). Let G be its Galois group, which acts by permutations on the points $\mathbf{p}_i$. It follows that G is a subgroup of the dihedral group D_4 of order 8 of the symmetries of the square. This group D_4 acts on the four points $\mathbf{p}_i$ and on the lines joining them the way the 8 isometries of a square act on its vertices and edges. We show that the optimal number of square roots needed for parameterizing the four lines and the way this optimal number is reached only depend on G and on its action on the $\mathbf{p}_i$.

The eight elements of D_4 are the identity, the transpositions τ_{13} and τ_{24} which exchange $\mathbf{p}_1$ and $\mathbf{p}_3$ or $\mathbf{p}_2$ and $\mathbf{p}_4$ (symmetries with respect to the diagonal), the permutation $\tau_{12,34}$ (resp. $\tau_{14,23}$) of order 2 which exchange $\mathbf{p}_1$ with $\mathbf{p}_2$ and $\mathbf{p}_3$ with $\mathbf{p}_4$ (resp. $\mathbf{p}_1$ with $\mathbf{p}_4$ and $\mathbf{p}_2$ with $\mathbf{p}_3$), the circular permutations ρ and ρ^{-1} of order 4, and the permutation $\rho^2 = \tau_{13}\tau_{24} = \tau_{12,34}\tau_{14,23}$ of order 2.

If G is included in the group $G_{\mathcal{L}}$ of order 4 generated by τ_{13} and τ_{24} (symmetries of the lozenge), its action leaves fixed the pairs $\{\mathbf{p}_1, \mathbf{p}_3\}$ and $\{\mathbf{p}_2, \mathbf{p}_4\}$ and thus also the lines $\mathbf{p}_1\mathbf{p}_3$ and $\mathbf{p}_2\mathbf{p}_4$ and the two singular quadrics of the pencil (the two pairs of planes). It follows that the roots of the determinantal equation $\mathcal{D}$ are rational. Conversely, if these roots are rational, the singular quadrics and their singular lines are invariant under the action of G, as well as the pairs $\{\mathbf{p}_1, \mathbf{p}_3\}$ and $\{\mathbf{p}_2, \mathbf{p}_4\}$, which implies that G is included in $G_{\mathcal{L}}$. A similar argument shows that G is the identity (resp. is generated by τ_{13} (or τ_{24}), or contains $\tau_{13}\tau_{24}$), if and only if 0 (resp. 1 or 2) of the singular quadrics consist of irrational planes. Moreover, in the case where G contains $\tau_{13}\tau_{24}$, the group is different from $G_{\mathcal{L}}$ if and only if any element which exchanges $\mathbf{p}_1$ and $\mathbf{p}_3$ also exchanges $\mathbf{p}_2$ and $\mathbf{p}_4$, i.e. if and only if the conjugations exchanging the planes in each singular quadric is the same (implying that the square roots needed for factoring them are one and the same). As the degree of $\mathbb{K}$ is the order of G, this shows that the number of square roots needed in our algorithm is always optimal if the roots of $\mathcal{D}$ are rational (i.e. δ is a square).

When the roots of $\mathcal{D}$ are not rational, we consider, in the algorithm, a rational quadric Q_R passing through a rational point $\mathbf{p}$. Let D be the line of Q_R passing through $\mathbf{p}$ and intersecting the lines $\mathbf{p}_1\mathbf{p}_2$ and $\mathbf{p}_3\mathbf{p}_4$ in two points $\mathbf{q}_1$ and $\mathbf{q}_2$. If the discriminant of Q_R is a square (and its parameterization is rational), then D is rational and is fixed by any Galois automorphism. It follows that the lines $\mathbf{p}_1\mathbf{p}_2$ and $\mathbf{p}_3\mathbf{p}_4$ are either fixed or exchanged, which implies that G is included in the group $G_{\mathcal{R}}$ of order 4 generated by $\tau_{12,34}$ and $\tau_{14,23}$ (symmetries of the rectangle). Conversely, if $G \subset G_{\mathcal{R}}$, the lines $\mathbf{p}_1\mathbf{p}_2$ and $\mathbf{p}_3\mathbf{p}_4$ are fixed or exchanged by any Galois automorphism; the image of D by such an automorphism is D itself or the other line of Q_R passing through $\mathbf{p}$; as this image contains the images of $\mathbf{q}_1$ and $\mathbf{q}_2$ which are on $\mathbf{p}_1\mathbf{p}_2$ or $\mathbf{p}_3\mathbf{p}_4$, we may conclude that D is fixed by any Galois automorphism, and is rational; this shows that the discriminant of Q_R is a square by Lemma I.7.4. Pushing these arguments a little more, it is easy to see that G is generated by $\tau_{12,34}$ or $\tau_{14,23}$ (or is the identity) if and only if the roots of either or both of the factors of bidegree $(2,0)$ and $(0,2)$ of the equation Ω in the parameters are rational.

By similar arguments of invariance, we may also conclude that the group G is generated by the circular permutation ρ if and only if any Galois automorphism which exchanges the lines $\mathbf{p}_1\mathbf{p}_3$ and $\mathbf{p}_2\mathbf{p}_4$ exchanges also the lines of Q_R passing through $\mathbf{p}$ (and if there is such an automorphism). It follows that this case occurs when the square root of the discriminant of Q_R and the roots of $\mathcal{D}$ generate the same field.

Finally, G is of order 8 if none of the preceding cases occur.

Optimality in all cases is proved by checking that, for each possible group, the algorithm involves exactly 0, 1 or 2 square roots for parameterizing the lines if the size of the orbit of a line is 1, 2 or 4 respectively.

2.12.3 Examples

We now give examples in all the possible cases outlined in Section 2.12.1. These examples are obtained using the following proposition which gives a rational canonical form for pencils having two double roots corresponding to quadrics of rank 2. Its proof is postponed to Appendix B and needs again elements of Galois theory that can be found in Appendix A.

Proposition 2.1. *Let $R(\lambda,\mu)$ be a rational pencil of quadrics whose determinantal equation has two double roots corresponding to two quadrics of rank 2. Then there is a rational change of frame such that the pencil is generated by quadrics (Q_S, Q_T) of equation*

$$\begin{cases} x^2-\gamma y^2-2wz=0, \\ \alpha x^2+2\gamma xy+\alpha\gamma y^2-z^2-(\alpha^2-\gamma)w^2=0, \end{cases}$$

where $\alpha,\gamma\in\mathbb{Q}, \delta=\alpha^2-\gamma\neq 0$. Moreover, a field $\mathbb{K}$ of smallest degree on which the four lines of the intersection are rationally parameterized is generated by the roots of $t^4-2\alpha t+\gamma=0$.

The different cases are as follows:

- $\delta>0,\gamma>0$, *and* $\alpha<0$: *the real intersection is empty.*
- $\delta>0,\gamma>0$, *and* $\alpha>0$: *the real intersection consists of four lines.*
- $\delta>0$ *and* $\gamma<0$: *the real intersection consists of two points.*
- $\delta<0$: *the real intersection consists of two lines.*

Note that the determinantal equation for the reduced pair of quadrics Q_S and Q_T is

$$\mathcal{D}(\lambda,\mu)=\gamma\left(\lambda^2-\delta\mu^2\right)^2=0.$$

Its roots are $(\lambda_0,\mu_0) = (\pm\sqrt{\delta},1)$ and, when $\delta > 0$, the associated quadrics of the pencil are the pairs of planes of equations

$$\lambda_0 Q_S + \mu_0 Q_T = (\alpha \pm \sqrt{\delta})\left(x + (\alpha \mp \sqrt{\delta})y\right)^2 - (z \pm \sqrt{\delta}w)^2 = 0$$

and of discriminants $d^{\pm} = \alpha \pm \sqrt{\delta}$. Note also that, when $\gamma > 0$, the quadric Q_S has inertia $(2,2)$ and can be parameterized, using the parameterization of Table I.3, by:

$$\mathbf{X} = \left(ut + vs, \frac{ut - vs}{\sqrt{\gamma}}, vt, 2us\right), \quad (u,v),(s,t) \in \mathbb{P}^1(\mathbb{R}).$$

Plugging $\mathbf{X}$ into the equation of T gives the following biquadratic equation in the parameters:

$$\Omega : \left(2\,(\alpha + \sqrt{\gamma})\,u^2 - v^2\right)\left(2\,(\alpha - \sqrt{\gamma})\,s^2 - t^2\right) = 0.$$

We can now give examples in all the possible cases outlined cases outlined in Section 2.12.1. We start with the four real lines case:

- δ is a square:
  - If $(\alpha,\gamma) = (5,9)$, then $\sqrt{\delta} = 4$, the discriminants of the pairs of planes are $d^{\pm} = 5 \pm 4$, so $\sqrt{d^{\pm}} \in \mathbb{Q}$, and the four lines are defined in $\mathbb{Q}$.
  - If $(\alpha,\gamma) = (3,5)$, then $\sqrt{\delta} = 2$, the discriminants of the pairs of planes are $d^{\pm} = 3 \pm 2$, so $\sqrt{d^+} \notin \mathbb{Q}$ and $\sqrt{d^-} \in \mathbb{Q}$, and the four lines are defined in $\mathbb{Q}(\sqrt{5})$.
  - If $(\alpha,\gamma) = (5,16)$, then $\sqrt{\delta} = 3$, the discriminants of the pair of planes are $d^{\pm} = 5 \pm 3$, so $\sqrt{d^{\pm}} \notin \mathbb{Q}$ but $\sqrt{d^-}$ and $\sqrt{d^+}$ generate the same field $\mathbb{Q}(\sqrt{2})$, and the four lines are defined in $\mathbb{Q}(\sqrt{2})$.
  - If $(\alpha,\gamma) = (6,20)$, then $\sqrt{\delta} = 4$, the discriminants of the pairs of planes are $d^{\pm} = 6 \pm 4$, so $\sqrt{d^{\pm}} \notin \mathbb{Q}$, $\sqrt{d^+}$ and $\sqrt{d^-}$ do not generate the same field and the four lines are defined in $\mathbb{Q}(\sqrt{2},\sqrt{10})$.
- δ is not a square but $\gamma\delta$ is a square: let $(\alpha,\gamma) = (2,2)$, then $\delta = 2$, $\sqrt{\delta} \notin \mathbb{Q}$ but $\sqrt{\gamma\delta} = 2 \in \mathbb{Q}$. The discriminant of the pairs of planes are $d^{\pm} = 2 \pm \sqrt{2}$, so the four lines are defined in $\mathbb{Q}(\sqrt{2+\sqrt{2}})$.
- Neither δ nor $\gamma\delta$ are squares:
  - If $(\alpha,\gamma) = (3,1)$, then $\delta = \gamma\delta = 8$ is not a square, $\sqrt{\gamma} \in \mathbb{Q}$ so Q_R can be rationally parameterized and the factors of bidegree $(2,0)$ and $(0,2)$ of Ω have rational coefficients. Since $2(\alpha - \sqrt{\gamma}) = 4$ is a square, one of those factors splits in two rational linear factors. Thus two lines have a rational parameterization. Since $2(\alpha + \sqrt{\gamma}) = 8$, the other two lines are defined in $\mathbb{Q}(\sqrt{8}) = \mathbb{Q}(\sqrt{2})$.

- If $(\alpha, \gamma) = (2, 1)$, then $\delta = \gamma\delta = 3$ is not a square, $\sqrt{\gamma} \in \mathbb{Q}$ so Q_R can be rationally parameterized. Since $2(\alpha + \sqrt{\gamma}) = 6$ and $2(\alpha - \sqrt{\gamma}) = 2$, two lines have a rational parameterization in $\mathbb{Q}(\sqrt{2})$ and the other two lines have a rational parameterization in $\mathbb{Q}(\sqrt{6})$.
- If $(\alpha, \gamma) = (3, 3)$, then $\delta = 6$ and $\gamma\delta = 18$ are not squares, $\sqrt{\gamma} \notin \mathbb{Q}$ so Q_R cannot be rationally parameterized. Two lines are defined in $\mathbb{Q}(\sqrt{6 + 2\sqrt{3}})$ and the other two lines are rational in $\mathbb{Q}(\sqrt{6 - 2\sqrt{3}})$.

Now we give examples for the two real lines case:

- If $(\alpha, \gamma) = (3, 25)$, then $\delta = -16 < 0$, $\sqrt{\gamma} = 5$ and $\sqrt{2(\alpha + \sqrt{\gamma})} = 4$ are rational, so the two lines are rational.
- If $(\alpha, \gamma) = (1, 4)$, then $\delta = -3 < 0$, $\sqrt{\gamma} \in \mathbb{Q}$ and $\sqrt{2(\alpha + \sqrt{\gamma})} = \sqrt{6} \notin \mathbb{Q}$, so the two lines are defined in $\mathbb{Q}(\sqrt{6})$.
- If $(\alpha, \gamma) = (1, 3)$, then $\delta = -2 < 0$, $\sqrt{\gamma}$ and $\sqrt{2(\alpha + \sqrt{\gamma})}$ are not rational, so the two lines are defined in $\mathbb{Q}(\sqrt{\sqrt{3} - 1})$.

Finally, here are examples for the two points case:

- If $(\alpha, \gamma) = (0, -1)$, then $\sqrt{\delta} = 1$ is rational and the discriminant $\alpha + \sqrt{\delta} = 1$ of the real pair of planes is a square, so the two points are in $\mathbb{Q}$.
- If $(\alpha, \gamma) = (1, -3)$, then $\sqrt{\delta} = 2$ is rational but the discriminant $\alpha + \sqrt{\delta} = 3$ of the real pair is not a square, so the two points are in $\mathbb{Q}(\sqrt{3})$.
- If $(\alpha, \gamma) = (1, -2)$, then $\sqrt{\delta}$ is not rational and the two points are in $\mathbb{Q}(\sqrt{1 + \sqrt{3}})$.

3 Parameterizing degenerate intersections: singular pencils

We now turn to singular pencils. Except when the intersection consists of four concurrent lines, the parameterization algorithms are straightforward and therefore only briefly sketched.

3.1 Conic and double line in $\mathbb{P}^3(\mathbb{C})$, $\sigma_4 = [1\{3\}]$

As we have seen in Section II.4.1, the pencil contains in this case one pair of planes. Furthermore each of the planes is rational by Proposition I.6.2 because the pair of planes contains a rational point outside its singular locus (by Proposition II.4.1). One plane is tangent to all the cones of the pencil, giving a rational double line. The other plane intersects all the cones transversally, giving a conic. The conic contains a rational point (its intersection with the singular line of the planes), so it can be rationally parameterized.

To actually parameterize the line and the conic, we proceed as follows. If Q_S is a pair of planes, replace Q_S by $Q_S + Q_T$. Now, Q_S is a real projective cone whose vertex is on Q_T. Use this rational vertex to obtain a rational parameterization of Q_T. Plug this parameterization into the equation of Q_S. This equation in the parameters factors in a squared linear factor (corresponding to the double line) and a bilinear factor, corresponding to the conic. It can rationally be solved. The parameterization of C is defined in $\mathbb{Q}[\xi]$.

3.2 Four concurrent lines in $\mathbb{P}^3(\mathbb{C})$, $\sigma_3 = [111]$

In this case, and in the three following cases, the two quadrics Q_S and Q_T have a singular point in common. So first compute this singular point **p**, which is rational, and compute the rational transformation sending this point to $(0,0,0,1)$. In this new frame, Q_S and Q_T are functions of x,y,z only and we can look at the restricted determinantal equation of the 3×3 upper left matrices to determine the real type of the intersection.

When the restricted determinantal equation has three simple roots (in $\mathbb{C}$), there are three types of intersection over the reals: a point, two concurrent lines, or four concurrent lines (see Table I.5). In the first case, the four lines are imaginary and the real part of the intersection consists of their common rational point, i.e. the point **p**.

We now look at the two other cases.

3.2.1 Algorithm and optimality

The algorithm for computing the lines is as follows. Determine a plane $x = 0, y = 0, z = 0$, or $w = 0$, that does not contain the singular point **p**. Substitute the equation of that plane (say $x = 0$) into the equations of $Q_S(x,y,z,w)$ and $Q_T(x,y,z,w)$. This gives a system of two non-homogeneous degree-two equations in three variables having four distinct complex projective solutions $\mathbf{q}_i$. The real lines of C are then the two or four lines going through **p** and one of the $\mathbf{q}_i$ with real coordinates, $i = 1,\dots,4$.

This algorithm outputs an optimal parameterization of C. Indeed, since the common singular point **p** of Q_S and Q_T is rational and the plane ($x = 0$) used to cut Q_S and Q_T is rational, the lines are rational if and only if their intersection with the planes (the points $\mathbf{q}_i$) are rational.

3.2.2 Degree of the extension

The following result shows that the roots of any polynomial of degree 4 without multiple root may be needed to parameterize four real concurrent lines. It uses notions of Galois theory that can be found in Appendix A.

Proposition 3.1. *For any rational univariate polynomial of degree 4 without multiple root, there are rational pencils of quadrics whose intersection is four (real or imaginary) concurrent lines, such that each of them is rational on the field generated by one of the roots of the polynomial and is not rational on any smaller field (for the inclusion and the degree).*

Proof. Let us consider a polynomial of degree 4 with rational coefficients and without multiple factors. Let us consider its four real or complex roots $t_1,\dots,t_4$ and the four points $\mathbf{q}_i$ of coordinates $(1,t_i,t_i^2,0)$. Let us consider also two rational points $\mathbf{r}_j = (a_j,b_j,c_j,0)$, $j = 1,2$. Exactly one conic exists in the plane $w = 0$, which passes through the four points $\mathbf{q}_i$ and one of the $\mathbf{r}_j$. Each of these conics has necessarily a rational equation, because, if it were not, the conjugate conics (under the action of the Galois group of the field containing the coefficients) would pass through the same five points. In other words, the equation of the conic is invariant under the Galois group and is thus rational. Now the rational cones containing these conics and having the point $(0,0,0,1)$ as vertex intersect in four (real or imaginary) lines passing through this vertex and the points $\mathbf{q}_i$.

The equations of these conics are easy to compute explicitly. Consider a conic with generic coefficients. Expressing that it passes through 5 points induces five linear equations in the coefficients of the equation of the conic. Solving this linear system expresses these coefficients as symmetric functions of the t_i, and thus as rational functions of the coefficients of the polynomial of degree 4. □

3.2.3 Examples

The proof of Proposition 3.1 gives a way of constructing examples of pencils of quadrics whose intersection is four concurrent lines for any quartic f without multiple root. Table 2 shows an exhaustive list of examples covering the possible degrees of field extension on which the lines of intersection are defined. We here focus on the cases where f has two or four real roots, corresponding, respectively, to the two and four concurrent lines cases.

When f has four real roots, the degree of the extension of $\mathbb{Q}$ needed to parameterize the four lines together is the order of the Galois group of f, in view of Proposition 3.1 and Appendix A. In other words, this degree is either 1, 2, 3, 4, 6, 8, 12 or 24. However, each line is defined individually on an extension of degree at most 4. For instance, when the Galois group is the dihedral group D_4 of order 8, the four lines are collectively defined in an extension of degree 8 but each line is defined in an extension of degree 4.

When f has two real and two complex roots, the degree of the extension on which the four lines are defined is again the order of the Galois group of f, but the degree of the extension on which the two real lines are collectively defined is only half the order of the Galois group. This degree is 1, 2, 3, 4 or 12.

Every extension degree can be attained by picking the right polynomial f. To build examples in all cases, it is sufficient to find the equations of two distinct rational cones containing the four points $\mathbf{q}_i = (1, t_i, t_i^2, 0)$ and having the same vertex. Assume f is given by:

$$f = t^4 + \alpha t^3 + \beta t^2 + \gamma t + \delta.$$

Then the following pair (Q_S, Q_T) satisfies the constraints:

$$\begin{cases} xz - y^2 = 0, \\ \delta x^2 + \gamma xy + \beta y^2 + \alpha yz + z^2 = 0. \end{cases}$$

Any two distinct quadrics of the pencil generated by Q_S and Q_T intersect in four (real or imaginary) concurrent lines defined collectively on an extension of $\mathbb{Q}$ of degree equal to the order of the Galois group of f.

By picking the right polynomial, we can generate pairs of quadrics intersecting in four concurrent lines for all types of Galois groups of a quartic. For instance, taking $f = t^4 + t + 1$, we build the two quadrics

$$\begin{cases} xz - y^2 = 0, \\ x^2 + xy + z^2 = 0. \end{cases}$$

The four real concurrent lines of the intersection are defined on an extension of $\mathbb{Q}$ of degree 24, since the Galois group of f is the group S_4 of permutations of four elements (of order 24). Each line is defined in an extension of degree 4.

3.3 Two concurrent lines and a double line in $\mathbb{P}^3(\mathbb{C})$, $\sigma_3 = [12]$

In this case, the restricted pencil has a real and rational double root corresponding to a pair of planes Q_R and a rational simple root corresponding to another pair of planes. The second pair is always real. There are two cases.

Double line. The pair of planes Q_R is imaginary. The intersection is reduced to the singular line of this pair, which is clearly rational.

Double line and two simple lines. The double line is rational (otherwise its conjugate would also be in the intersection). The two simple lines are contained in Q_R and go through the common singular point $\mathbf{p}$ of Q_S and Q_T. They are rational if and only if Q_R has a rational point outside its singular line, i.e. if the discriminant of Q_R is a square.

A simple example where this is not the case is as follows:

$$\begin{cases} xy = 0, \\ y^2 - 2z^2 = 0. \end{cases}$$

3.4 Two double lines in $\mathbb{P}^3(\mathbb{C})$, $\sigma_3 = [1(11)]$

The determinantal equation has a real and rational double root corresponding to a double plane Q_R. The other root corresponds to a (real or imaginary) pair of planes. There are two cases.

Point. The pair of planes is imaginary. The intersection is reduced to the intersection of its singular line with Q_R, i.e. the rational point $\mathbf{p}$.

Two real double lines. The two double lines are conjugate. They contain the rational point $\mathbf{p}$ and are rational if they go through another rational point. This happens when the discriminant of the pair of planes is a square.

This situation does not necessarily happen, as the following example shows:

$$\begin{cases} z^2 = 0, \\ x^2 - 2y^2 = 0. \end{cases}$$

3.5 Line and triple line in $\mathbb{P}^3(\mathbb{C})$, $\sigma_3 = [3]$

The determinantal equation has a real and rational triple root corresponding to a real pair of planes. The intersection consists of the (triple) singular line of this pair of planes, which is clearly rational, and of a single line which is also rational, otherwise its conjugate would also be in the intersection. The simple line is found by intersecting one of the planes with any other cone of the pencil.

3.6 Quadruple line in $\mathbb{P}^3(\mathbb{C})$, $\sigma_3 = [(21)]$

Here, the determinantal equation has a real and rational triple root, corresponding to a double plane. The intersection consists of the (quadruple) line of tangency of this double plane with a cone of the pencil. This line is clearly rational.

3.7 Remaining cases

In the remaining cases, that is when the restricted determinantal equation identically vanishes, the description of the possibles cases given in Section II.4.2.6 directly yields algorithms for computing parameterizations of the intersection over $\mathbb{Q}[\xi]$.

4 Examples

The algorithm described in this paper for computing a near-optimal parameterization of the intersection of two arbitrary quadrics with integer coefficients has been fully implemented in C++. The implementation details as well as an analysis of the complexity (i.e., the height) of the integer coefficients appearing in the parameterizations are presented in Part IV [4].

In this section, we illustrate our algorithm with three examples covering different situations. The output given is the actual output of our implementation, with debug information turned on so as to follow what the algorithm is doing.

4.1 Example 1

Our first example is as shown in Output 1. As explained in Section II.5, we first determine the type of the intersection by looking at the multiple roots of the determinantal equation and the ranks of the associated quadrics. Here, we find two real double roots corresponding to quadrics of rank 3: Algorithm II.6 tells us that the real type of the intersection is "cubic and secant line". We have rational roots, so we can parameterize the intersection using cones. One of the two cones of the pencil is

$$Q_R = -2Q_S + Q_T = -25wx - 30wy - 5wz - 20xy - 30w^2 - 5x^2 - 20y^2.$$

This cone has the point $(-2,1,4,0)$ as vertex and contains the vertex of the second cone, i.e. $(-4,1,4,1)$. The line of the intersection is the line joining these two points. Here, we have applied a simple reparameterization to the line by picking two other representative points with smaller "height" than the original two. On the reparameterized line, a very simple point is $(-2,0,0,1)$ which

Output 1 Execution trace for Example 1.

```
>> quadric 1: 3*x^2 + 5*x*y + 2*x*z + 13*x*w + 14*y^2 - 4*y*z + 4*y*w + z^2 + 6*z*w + 14*w^2
>> quadric 2: x^2 - 10*x*y + 4*x*z + x*w + 8*y^2 - 8*y*z - 22*y*w + 2*z^2 + 7*z*w - 2*w^2

>> launching intersection
>> determinantal equation: 4*l^4 + 12*l^3*m + l^2*m^2 - 12*l*m^3 + 4*m^4
>> gcd of derivatives of determinantal equation: 2*l^2 + 3*l*m - 2*m^2
>> ranks of singular quadrics: 3 and 3
>> two real rational double roots: [ 1 2 ] and [ -2 1 ]
>> complex intersection: cubic and secant line
>> real intersection: cubic and secant line
>> reparameterization of line
>> singular point of cone: [ -2 1 4 0 ]
>> rational point on cone: [ -2 0 0 1 ]
>> parameterization of cone with rational point
>> cubic and line intersect at [ -4 1 4 1 ] and [ -2 1 4 0 ]
>> status of intersection param: optimal
>> end of intersection

>> parameterization of cubic:
[4*u^3 + u^2*v + 2*v^3, - u^3 - v^3, - 4*u^3 - u^2*v + u*v^2 - 4*v^3, - u^3]
>> parameterization of line:
[- 2*u, v, 4*v, u - v]

>> time spent: 10 ms
```

we use as rational point for parameterizing Q_R. Plugging the parameterization of Q_R in the second cone and leaving aside the linear factor corresponding to the line gives the cubic.

4.2 Example 2

Our second example is displayed in Output 2. Here, the determinantal equation has a double real root at $(\lambda, \mu) = (0, 1)$, the two other simple roots are real (solution of $\mathcal{E}(\lambda, \mu) = 4\mu^2 - \lambda^2 = 0$), the singular quadric $R = R(0, 1)$ is a real pair of planes and $\mathcal{E}(0, 1) > 0$, so Algorithm II.3 tells us that the intersection is two secant conics, the singularities of the intersection being convex. Here, the two planes of R are rational:

$$Q_R = (w - y + 2z)(w - 4x + 3y - 2z).$$

We can parameterize each of these planes and plug their parameterization in turn in any other quadric of the pencil. This gives the implicit equations of the two conics which we can parameterize. As explained in Section 2.5, we are here in one of the few situations where we cannot guarantee that what we output is optimal: the square root in the parameterization of the conics might well be unnecessary. (It turns out that in this particular example it is necessary.) This explains why the implementation reports that the parameterizations of the two conics are only near-optimal.

4.3 Example 3

Our last example is presented in Output 3. Here, the determinantal equation vanishes identically and all the quadrics of the pencil share a common singular point, with coordinates $(1, 3, -1, -2)$. We apply to all the quadrics of the pencil a projective transformation sending this point to the point

Output 2 Execution trace for Example 2.

```
>> quadric 1: 16*x^2 - 12*x*y + 8*x*z + 4*x*w - y^2 - 20*y*z + 2*y*w - 2*z^2 + 3*w^2
>> quadric 2: 4*x*y - 8*x*z - 4*x*w - 3*y^2 + 8*y*z + 2*y*w - 4*z^2 + w^2

>> launching intersection
>> determinantal equation: - l^4 + 4*l^2*m^2
>> gcd of derivatives of determinantal equation: l
>> double real root: [ 0 1 ]
>> inertia: [ 1 1 ]
>> parameterization of pair of planes
>> complex intersection: two secant conics
>> real intersection: two secant conics, convex singularities
>> parameterization of rational conic
>> status of conic 1 param: near-optimal
>> parameterization of rational conic
>> status of conic 2 param: near-optimal
>> end of intersection

>> parameterization of conic 1:
[- 3*u^2 - 9*v^2 - 14*u*v*sqrt(2), - 12*u^2 - 36*v^2 + (- 7*u^2 - 14*u*v + 21*v^2)*sqrt(2),
- 2*u^2 - 6*v^2, - 8*u^2 - 24*v^2 + (- 7*u^2 - 14*u*v + 21*v^2)*sqrt(2)]
>> parameterization of conic 2:
[(u^2 - 28*u*v - 42*v^2)*sqrt(2), - 2*u^2 - 84*v^2 - 28*u*v*sqrt(2), - 8*u^2 - 336*v^2
+ (- 6*u^2 + 252*v^2)*sqrt(2), - 10*u^2 - 420*v^2 + (- 8*u^2 - 28*u*v + 336*v^2)*sqrt(2)]

>> time spent: 10 ms
```

Output 3 Execution trace for Example 3.

```
>> quadric 1: 17*x^2 - 12*x*y + 14*x*z - 8*x*w - 4*y^2 - 18*y*w + 5*z^2 + 2*z*w - 16*w^2
>> quadric 2: - 3*x^2 + 28*x*y + 30*x*z + 24*x*w - 4*y^2 + 8*y*z - 2*y*w + 9*z^2 + 18*z*w

>> launching intersection
>> vanishing 4 x 4 determinantal equation
>> dimension of common singular locus: 0
>> common singular point of quadrics: [ 1 3 -1 -2 ]
>> computing matrix sending singular point to [ 0 0 0 1 ]
>> 3 x 3 determinantal equation: - l^3 + l^2*m + 5*l*m^2 + 3*m^3
>> gcd of derivatives of 3 x 3 determinantal equation: l + m
>> double real root: [ -1 1 ]
>> second root: [ 3 1 ]
>> complex intersection: two concurrent lines and double line
>> real intersection: two concurrent lines and double line
>> reparameterization of line
>> parameterization of pair of lines
>> reparameterization of lines
>> the three lines meet at [ 1 3 -1 -2 ]
>> status of intersection param: optimal
>> end of intersection

>> parameterization of double line:
[v, 3*v, - 2*u - v, u - 2*v]
>> parameterization of line 1:
[- 3*u + v, 3*v, u - v, - 2*u - 2*v]
>> parameterization of line 2:
[u - v, 3*u + 6*v, - u + 5*v, - 2*u]

>> time spent: 10 ms
```

$(0,0,0,1)$. The determinantal equation of the pencil restricted to the upper left 3×3 part has a double real root at $(-1,1)$. Corresponding to this double root is a real pair of planes Q_R and Algorithm II.2 tells us that the real type of the intersection is “two concurrent lines and a double line”.

As we have seen in Section 3.3, the double line of the intersection is the singular line of Q_R. To parameterize the other two lines, we first parameterize Q_R and plug the result in the equation of the other pair of planes of the pencil, corresponding to the second root $(3,1)$ of the restricted determinantal equation. The result follows.

5 Conclusion

We have presented in Parts I, II, and III of this paper a new algorithm for computing an exact parametric representation of the intersection of two quadrics in three-dimensional real space given by implicit equations with rational coefficients. We have shown that our algorithm computes projective parameterizations that are optimal in terms of the functions used in the sense that they are polynomials whenever it is possible and contain the square root of some polynomial otherwise. The parameterizations are also near-optimal in the sense that the number of square roots appearing in the coefficients of these functions is minimal except in a small number of cases (characterized by the real type of the intersection) where there may be an extra square root (see Table 1 for a summary). Furthermore, we have shown that in the latter cases, testing whether the extra square root is unnecessary and, if so, finding an optimal parameterization are equivalent to finding a rational point on a curve or a surface. Hence, leaving for a moment that well-known problem aside, our algorithm closes the problem of finding parameterizations of the intersection that are optimal in the senses discussed above. It should be emphasized that our algorithm is not only theoretically powerful but is also practical: a complete, robust and efficient C++ implementation is described in Part IV [4].

For most applications, the near-optimal parameterizations of intersections of quadrics computed by our algorithm are good enough since they are at most one square root away from being optimal. However, there may be situations where one is interested in fully asserting the optimality of a parameterization and, if a given parameterization is not optimal, in obtaining one. As we have seen, this is akin to deciding whether a given curve or surface has a rational point and to computing such a point. The problem of finding integer (or rational) points on an algebraic variety is known to be hard in general, and many instances of the problem are undecidable [6]. When the intersection is a smooth quartic, deciding whether the extra square root can be avoided amounts to finding a rational point on a surface of degree 8 (see Section I.7) and very little is known about this problem, to the best of our knowledge. The situation is, however, better for the other near-optimal cases, which boil down to finding a rational point on a (possibly non-rational) conic. Indeed, when the conic is rational, Cremona and Rusin [1] recently gave an efficient algorithm for solving this problem, which has been implemented in recent releases of the Magma computational algebra system [5]. As an example, this implementation solves the problem for an equation of the form $ax^2+by^2=cz^2$, where a,b and c are 200-digit primes, in less than 2 seconds on a mainstream PC. In the future, we plan to use this algorithm in our intersection software.

Finally, it should be stressed that the classification, presented in Part II, of pairs of quadrics depending on the type of their real intersection is of independent interest. For instance, it could be used in a collision detection context to predict when collisions between two moving quadrics will occur.

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A A primer on Galois theory

Galois theory was introduced in the 19th century for deciding when a polynomial is solvable by radicals. We give here a brief introduction to this theory, which is especially geared towards geometric objects.

Let K be a finite field extension of the field $\mathbb{Q}$ of the rational numbers. Its dimension as $\mathbb{Q}$-vector space is called the *degree* of K. In our context, K is usually the smallest field containing the coefficients of the equations or of the parameterization of a geometric object such as a point, a line, a curve or a plane. This field K may always be defined as $\mathbb{Q}(\alpha)$, where α is a root of some polynomial f of degree $n = \mathrm{degree}(K)$.

The *splitting field* K' of K is defined as the smallest field containing all the roots of f. It may be proved that it is independent from the choice of f and α. The *Galois group* G of K and K' is the group of the field automorphisms of K'. It is immediate that the elements of G permute the roots of f, and this allows to identify G to a subgroup of the group S_n of all the permutations of the n roots of f.

The important fact for geometric considerations is that any element g of G acts on any object defined from the elements of K by the four field operations $(+,-,*,/)$ simply by replacing any element of K appearing in the definition of the object by its image by the automorphism g, exactly as the complex conjugation acts on any object defined with complex numbers. The different images of an object under the action of the elements of G are called the *conjugates* of this object.

If H is a subgroup of the Galois group G, one may define K'^H, the field of the elements of K' such that $g(x) = x$ for any $g \in H$. The main result of Galois theory is that the map $H \mapsto K'^H$ is a bijection between the subgroups of G and the subfields of K'^H. Moreover, we have degree $\left(K'^H\right) = \mathrm{order}(G)/\mathrm{order}(H)$. It follows that an element of K which has k conjugates lies in a subfield of K of degree k.

This may be extended to the

> **Galois principle**: *Two conjugate objects are isomorphic and any object which has k conjugates (including itself) may be defined on a field of degree k and may not be defined on a smaller field.*

This principle is behind all our proofs of optimality. We describe briefly some other consequences in the context of intersection of quadrics.

One of its first consequences is that any object which has no conjugate except itself may be rationally defined. For example this is the case for the singular point of a singular quartic appearing as the intersection of two rational quadrics. If the intersection of two quadrics is decomposed in a cubic and a line, both may be rationally defined, because their conjugates have to be components of the intersection and the conjugate of a line is a line. Similarly, if the determinantal equation of a pencil of quadrics has two double roots and if the corresponding singular quadrics are a cone and a pair of planes, both are rational because they are not isomorphic.

A more involved application of the Galois principle occurs when the intersection of two quadrics consists in four non concurrent lines. As a conjugate of a line in the intersection may only be another line of the intersection, each line may be defined on a field of degree 4. Moreover, if a point lies on a line, any conjugate of the point lies on the corresponding conjugate of the line. Thus the arrangement of the four lines and their four intersection points is preserved by the action of the Galois group. It follows that the Galois group is included in a group of order 8 which acts on the lines and their intersections as the 8 isometries of a square act on its edges and its vertices. By looking at the subgroups of this group, the Galois principle says that the field K of definition of any of the lines has a degree which divides 4, and that if its degree is 4 it has a subfield of degree 2. Therefore each line may be parameterized with at most 2 square roots. As the splitting field has degree at most 8, it is possibly generated by another square root. This shows, without any explicit computation, that at most 3 square roots are needed to define all the four intersection points and the four lines.

B Rational canonical form for the four skew lines case

We here give a proof of Proposition 2.1.

Proof. Let $\mathbb{K}$ be a $\mathbb{Q}$-extension field of smallest degree on which the four lines of the intersection are rationally parameterized, and let $\mathbb{L}$ be the field generated by the roots of the determinantal equation. To decompose the singular quadrics of the pencil in two planes, we have to extract the square roots of two elements d_1 and d_2 of $\mathbb{L}$, which are algebraically conjugate on $\mathbb{Q}$ if $\mathbb{L}$ is different from $\mathbb{Q}$. Let $t^2 - 2\alpha t + \gamma$ be the (rational) polynomial having d_1 and d_2 as roots. It is easy to verify that $\mathbb{K}$ is generated by the roots of the biquadratic polynomial $t^4 - 2\alpha t^2 + \gamma$.

Let the points $\mathbf{p}_i$, $i = 1, \ldots, 4$, be the singular points of the intersection in $\mathbb{P}^3(\mathbb{C})$ of any two distinct quadrics of the pencil. Now, let us choose four points $\mathbf{q}_i$ on which the Galois group G of this polynomial acts in the same way as on the $\mathbf{p}_i$. For instance, take for $\mathbf{q}_i$, $i = 1, \ldots, 4$, the points of coordinates $(1, t_i, t_i^2, t_i^3)$, where the t_i are the roots of the biquadratic polynomial, numbered such that $t_1 = -t_3$. It is now easy to compute the equations H_j of the planes containing all the points $\mathbf{q}_i$ except $\mathbf{q}_j$. The quadrics of equations

$$\left\{ \begin{array}{l} H_1H_3 + H_2H_4 = 0, \\ \sqrt{\alpha^2 - \gamma}(H_1H_3 - H_2H_4) = 0 \end{array} \right.$$

are rational and their intersection consists in the four (not necessarily real) lines $\mathbf{q}_1\mathbf{q}_2$, $\mathbf{q}_2\mathbf{q}_3$, $\mathbf{q}_3\mathbf{q}_4$, and $\mathbf{q}_4\mathbf{q}_1$. Now, the change of frame sending the $\mathbf{q}_i$ to the points of coordinates $(t_i, -1/t_i, t_i^2 - \alpha, 1)$ is rational and leads to the equations

$$\left\{ \begin{array}{l} x^2 - \gamma y^2 - 2wz = 0, \\ \alpha x^2 + 2\gamma xy + \alpha\gamma y^2 - z^2 - (\alpha^2 - \gamma)w^2 = 0. \end{array} \right.$$

There is a unique projective transformation sending the points $\mathbf{p}_i$ on the $\mathbf{q}_i$ and leaving fixed some rational point which is not on any of the planes defined by the $\mathbf{p}_i$ or by the $\mathbf{q}_i$. This transformation is invariant under the action of the Galois group G. Thus it is rational, showing the existence of the rational change of frame which is sought.

We now show how the different types of real intersection follow from the signs of $\delta = \alpha^2 - \gamma, \alpha$, and γ. First note that the determinantal equation for the rational canonical pencil above is

$$\mathcal{D}(\lambda, \mu) = \gamma\left(\lambda^2 - \delta\mu^2\right)^2.$$

Furthermore, the double roots of $\mathcal{D}$ are $(\lambda_0, \mu_0) = (\pm\sqrt{\delta}, 1)$ and, when $\delta > 0$, the associated pairs of planes have equations

$$\lambda_0 Q_S + \mu_0 Q_T = (\alpha \pm \sqrt{\delta})\left(x + (\alpha \mp \sqrt{\delta})y\right)^2 - (z \pm \sqrt{\delta}w)^2.$$

The discriminants of the two pairs of planes of the pencil are thus $d^{\pm} = \alpha \pm \sqrt{\delta}$.

We know from Table I.4 that, when the real intersection consists of two lines, the double roots are complex, hence $\delta < 0$. In the other cases, the double roots are real and distinct thus $\delta > 0$. Note that d^+d^- is then equal to c. When the real intersection is empty, the inertia of the two pairs of planes is $(2,0)$, hence $d^{\pm} < 0$; thus $d^+ + d^- = 2\alpha < 0$ and $d^+d^- = \gamma > 0$. When the real intersection consists of two points, the discriminants of the two pairs of planes have opposite signs, thus $d^+d^- = \gamma < 0$. Finally, when the intersection consists of four skew lines, the discriminants of the two pairs of planes are both positive, thus $d^+ + d^- = 2\alpha > 0$ and $d^+d^- = \gamma > 0$. This completes the proof since these cases for the signs of δ, α, and γ are disjoint. □

C Examples in all cases

Table 2 gives an exhaustive list of examples covering all possible degrees of extension fields on which the components of the intersection are defined, for all real types of intersection. The next-to-last column gives the optimal ring of definition on which a parameterization of the given example is known to exist (ξ is the parameter of the parameterization). When the parameterization output by our algorithm is optimal, the last column gives the degree of the $\mathbb{Q}$-extension field on which the coefficients of the parameterization of each real component of the intersection is defined. When our algorithm is only near-optimal, the last column gives both the optimal degree and the near-optimal one.

Table 2: Exhaustive list of examples when the intersection is 0- or 1-dimensional over $\mathbb{C}$.

complex type	real type	example	fi eld of defi nition	degree
smooth quartic, $\sigma_4 = [1111]$	$\emptyset$	$\begin{cases} 6xy+5y^2+2z^2+6zw-w^2=0 \\ 3x^2+y^2-z^2+11w^2=0 \end{cases}$		

Table 2: (continued)

complex type	real type	example	field of definition	degree
	smooth quartic, two finite components	$\begin{cases} x^2+y^2-z^2-w^2=0 \\ xy-2zw=0 \end{cases}$	quartic in $\mathbb{Q}[\xi,\sqrt{\Delta}]$, $\Delta\in\mathbb{Q}[\xi]$	**1 / 2**
		$\begin{cases} 6xy+5y^2+2z^2+6zw-w^2=0 \\ 3x^2+y^2-z^2-w^2=0 \end{cases}$	quartic in $\mathbb{Q}(\sqrt{\delta})[\xi,\sqrt{\Delta}]$, $\Delta\in\mathbb{Q}(\sqrt{\delta})[\xi]$	2
	smooth quartic, one finite component	$\begin{cases} x^2+y^2+2zw=0 \\ x^2+z^2+zw-w^2=0 \end{cases}$	quartic in $\mathbb{Q}[\xi,\sqrt{\Delta}]$	**1 / 2**
		$\begin{cases} 2x^2-2xy+2xz-2xw+y^2 \\ \quad +4yz-4yw+2z^2-4zw=0 \\ x^2-2xy+4xz+4xw-y^2 \\ \quad +2yz+4yw+4zw-2w^2=0 \end{cases}$	quartic in $\mathbb{Q}(\sqrt{\delta})[\xi,\sqrt{\Delta}]$, $\Delta\in\mathbb{Q}(\sqrt{\delta})[\xi]$	2
	smooth quartic, two infinite components	$\begin{cases} xy+zw=0 \\ x^2-y^2+z^2+2zw-w^2=0 \end{cases}$	quartic in $\mathbb{Q}[\xi,\sqrt{\Delta}]$, $\Delta\in\mathbb{Q}[\xi]$	**1 / 2**
		$\begin{cases} x^2-2y^2+4zw=0 \\ xy+z^2+2zw-w^2=0 \end{cases}$	quartic in $\mathbb{Q}(\sqrt{\delta})[\xi,\sqrt{\Delta}]$, $\Delta\in\mathbb{Q}(\sqrt{\delta})[\xi]$	2
nodal quartic, $\sigma_4=[112]$	point	$\begin{cases} y^2+z^2+w^2=0 \\ xy+w^2=0 \end{cases}$	point in $\mathbb{Q}$	1
	nodal quartic	$\begin{cases} y^2+z^2-w^2=0 \\ xy+w^2=0 \end{cases}$	quartic in $\mathbb{Q}[\xi]$	**1 / 2**
		$\begin{cases} y^2+z^2-3w^2=0 \\ xy+w^2=0 \end{cases}$	quartic in $\mathbb{Q}(\sqrt{3})[\xi]$	2
	nodal quartic with isolated singularity	$\begin{cases} y^2-z^2-w^2=0 \\ xy+w^2=0 \end{cases}$	quartic in $\mathbb{Q}[\xi]$, point in $\mathbb{Q}$	**1 / 2** 1
		$\begin{cases} y^2+z^2-3w^2=0 \\ xw+y^2=0 \end{cases}$	quartic in $\mathbb{Q}(\sqrt{3})[\xi]$, point in $\mathbb{Q}$	2 1
cuspidal quartic, $\sigma_4=[13]$	cuspidal quartic	$\begin{cases} yz+w^2=0 \\ xz+y^2=0 \end{cases}$	quartic in $\mathbb{Q}[\xi]$	1
cubic and secant line, $\sigma_4=[22]$	cubic and secant line	$\begin{cases} y^2+zw=0 \\ xy+w^2=0 \end{cases}$	cubic in $\mathbb{Q}[\xi]$, line in $\mathbb{Q}[\xi]$	1 1
	cubic and non-secant line	$\begin{cases} xw+yz=0 \\ xz-yw+zw=0 \end{cases}$	cubic in $\mathbb{Q}[\xi]$, line in $\mathbb{Q}[\xi]$	1 1
cubic and tangent line, $\sigma_4=[4]$	cubic and tangent line	$\begin{cases} yw+z^2=0 \\ xw+yz=0 \end{cases}$	cubic in $\mathbb{Q}[\xi]$, line in $\mathbb{Q}[\xi]$	1 1
two secant conics, $\sigma_4=[11(11)]$	$\emptyset$	$\begin{cases} z^2-w^2=0 \\ x^2+y^2+w^2=0 \end{cases}$		

Table 2: (continued)

complex type	real type	example	field of definition	degree
	two points	$\begin{cases} z^2+w^2=0 \\ x^2-y^2+w^2=0 \end{cases}$	two points in $\mathbb{Q}$	1
		$\begin{cases} z^2+w^2=0 \\ x^2-2y^2+w^2=0 \end{cases}$	two points in $\mathbb{Q}(\sqrt{2})$	2
	one conic	$\begin{cases} zw=0 \\ x^2+y^2+z^2-w^2=0 \end{cases}$	one conic in $\mathbb{Q}[\xi]$	**1** / **2**
		$\begin{cases} x^2-4xw+3w^2=0 \\ x^2+y^2+z^2-4w^2=0 \end{cases}$	one conic in $\mathbb{Q}(\sqrt{3})[\xi]$	2
		$\begin{cases} x^2-4xw-3w^2=0 \\ x^2+y^2+z^2-w^2=0 \end{cases}$	one conic in $\mathbb{Q}(\sqrt{4\sqrt{7}-10})$	4
	two non-secant conics	$\begin{cases} x^2-w^2=0 \\ y^2+z^2-w^2=0 \end{cases}$	two conics in $\mathbb{Q}[\xi]$	**1** / **2**
		$\begin{cases} x^2-w^2=0 \\ y^2+z^2-3w^2=0 \end{cases}$	two conics in $\mathbb{Q}(\sqrt{3})[\xi]$	2
		$\begin{cases} x^2-3w^2=0 \\ y^2+z^2-3w^2=0 \end{cases}$	two conics in $\mathbb{Q}(\sqrt{3})[\xi]$	**2** / **4**
		$\begin{cases} x^2-33w^2=0 \\ y^2+z^2-3w^2=0 \end{cases}$	two conics in $\mathbb{Q}(\sqrt{3},\sqrt{11})[\xi]$	4
	two secant conics	$\begin{cases} x^2-y^2=0 \\ y^2+z^2-w^2=0 \end{cases}$	two conics in $\mathbb{Q}[\xi]$	**1** / **2**
		$\begin{cases} x^2-y^2=0 \\ y^2+z^2-3w^2=0 \end{cases}$	two conics in $\mathbb{Q}(\sqrt{3})[\xi]$	2
		$\begin{cases} x^2-3y^2=0 \\ y^2+z^2-3w^2=0 \end{cases}$	two conics in $\mathbb{Q}(\sqrt{3})[\xi]$	**2** / **4**
		$\begin{cases} x^2-33y^2=0 \\ y^2+z^2-3w^2=0 \end{cases}$	two conics in $\mathbb{Q}(\sqrt{3},\sqrt{11})[\xi]$	4
two tangent conics, $\sigma_4=[1(21)]$	point	$\begin{cases} x^2+w^2=0 \\ xy+z^2=0 \end{cases}$	point in $\mathbb{Q}$	1
	two conics	$\begin{cases} x^2-w^2=0 \\ xy+z^2=0 \end{cases}$	two conics in $\mathbb{Q}[\xi]$	1
		$\begin{cases} x^2-2w^2=0 \\ xy+z^2=0 \end{cases}$	two conics in $\mathbb{Q}(\sqrt{2})[\xi]$	2
double conic, $\sigma_4=[1(111)]$	$\emptyset$	$\begin{cases} w^2=0 \\ x^2+y^2+z^2=0 \end{cases}$		
	double conic	$\begin{cases} x^2=0 \\ y^2+z^2-w^2=0 \end{cases}$	conic in $\mathbb{Q}[\xi]$	**1** / **2**
		$\begin{cases} x^2=0 \\ y^2+z^2-3w^2=0 \end{cases}$	conic in $\mathbb{Q}(\sqrt{3})[\xi]$	2

Table 2: (continued)

complex type	real type	example	field of definition	degree
conic and two lines not crossing, $\sigma_4 = [2(11)]$	point	$\begin{cases} xy=0 \\ y^2+z^2+w^2=0 \end{cases}$	point in $\mathbb{Q}$	1
	conic and point	$\begin{cases} xw=0 \\ y^2+z^2-w^2=0 \end{cases}$	point in $\mathbb{Q}$, conic in $\mathbb{Q}[\xi]$	1 **1 / 2**
		$\begin{cases} xw=0 \\ y^2+z^2-3w^2=0 \end{cases}$	point in $\mathbb{Q}$, conic in $\mathbb{Q}(\sqrt{3})[\xi]$	1 2
	conic and two lines	$\begin{cases} xy=0 \\ y^2+z^2-w^2=0 \end{cases}$	two lines in $\mathbb{Q}[\xi]$, conic in $\mathbb{Q}[\xi]$	1 1
		$\begin{cases} xy=0 \\ 2y^2+z^2-3w^2=0 \end{cases}$	two lines in $\mathbb{Q}(\sqrt{3})[\xi]$, conic in $\mathbb{Q}[\xi]$	2 **1 / 2**
		$\begin{cases} xy=0 \\ y^2+z^2-3w^2=0 \end{cases}$	two lines in $\mathbb{Q}(\sqrt{3})[\xi]$, conic in $\mathbb{Q}(\sqrt{3})[\xi]$	2 2
conic and two lines crossing, $\sigma_4 = [(31)]$	conic	$\begin{cases} yz=0 \\ xz+y^2+w^2=0 \end{cases}$	conic in $\mathbb{Q}[\xi]$	1
	conic and two lines	$\begin{cases} yz=0 \\ xz+y^2-w^2=0 \end{cases}$	conic in $\mathbb{Q}[\xi]$, two lines in $\mathbb{Q}[\xi]$	1 1
		$\begin{cases} yz=0 \\ xz+y^2-2w^2=0 \end{cases}$	conic in $\mathbb{Q}[\xi]$, two lines in $\mathbb{Q}(\sqrt{2})[\xi]$	1 2
four skew lines, $\sigma_4 = [(11)(11)]$	$\emptyset$	$\begin{cases} x^2+y^2=0 \\ z^2+w^2=0 \end{cases}$		
	two points	$\begin{cases} x^2+y^2-2zw=0 \\ 2xy+z^2+w^2=0 \end{cases}$	two points in $\mathbb{Q}$	1
		$\begin{cases} x^2+3y^2-2zw=0 \\ x^2-6xy-3y^2-z^2-4w^2=0 \end{cases}$	two points in $\mathbb{Q}(\sqrt{3})$	2
		$\begin{cases} x^2+2y^2-2zw=0 \\ x^2-4xy-2y^2-z^2-3w^2=0 \end{cases}$	two points in $\mathbb{Q}(\sqrt{1+\sqrt{3}})$	4
	two skew lines	$\begin{cases} x^2-25y^2-2zw=0 \\ 3x^2+50xy+75y^2-z^2+16w^2=0 \end{cases}$	two lines in $\mathbb{Q}[\xi]$	1
		$\begin{cases} x^2-4y^2-2zw=0 \\ x^2+8xy+4y^2-z^2+3w^2=0 \end{cases}$	two lines in $\mathbb{Q}(\sqrt{6})[\xi]$	2
		$\begin{cases} x^2-3y^2-2zw=0 \\ x^2+6xy+3y^2-z^2+2w^2=0 \end{cases}$	two lines in $\mathbb{Q}(\sqrt{\sqrt{3}-1})[\xi]$	4
	four skew lines	$\begin{cases} x^2-9y^2-2zw=0 \\ 5x^2+18xy+45y^2-z^2-16w^2=0 \end{cases}$	four lines in $\mathbb{Q}[\xi]$	1
		$\begin{cases} x^2-y^2-2zw=0 \\ 3x^2+2xy+3y^2-z^2-8w^2=0 \end{cases}$	two lines in $\mathbb{Q}[\xi]$, two lines in $\mathbb{Q}(\sqrt{2})[\xi]$	1 2

Table 2: (continued)

complex type	real type	example	field of definition	degree
		$\begin{cases} x^2-16y^2-2zw=0 \\ 5x^2+32xy+80y^2-z^2-9w^2=0 \end{cases}$	four lines in $\mathbb{Q}(\sqrt{2})[\xi]$	2
		$\begin{cases} x^2-y^2-2zw=0 \\ 2x^2+2xy+2y^2-z^2-3w^2=0 \end{cases}$	two lines in $\mathbb{Q}(\sqrt{2})[\xi]$, two lines in $\mathbb{Q}(\sqrt{6})[\xi]$	2 2
		$\begin{cases} x^2-20y^2-2zw=0 \\ 6x^2+40xy+120y^2-z^2-16w^2=0 \end{cases}$	four lines in $\mathbb{Q}(\sqrt{2},\sqrt{10})[\xi]$	4
		$\begin{cases} x^2-2y^2-2zw=0 \\ 2x^2+4xy+4y^2-z^2-2w^2=0 \end{cases}$	four lines in $\mathbb{Q}(\sqrt{2+\sqrt{2}})[\xi]$	4
		$\begin{cases} x^2-3y^2-2zw=0 \\ 3x^2+6xy+9y^2-z^2-6w^2=0 \end{cases}$	two lines in $\mathbb{Q}(\sqrt{3-\sqrt{3}})[\xi]$, two lines in $\mathbb{Q}(\sqrt{3+\sqrt{3}})[\xi]$	4 4
two skew lines and a double line, $\sigma_4=[(22)]$	double line	$\begin{cases} y^2+w^2=0 \\ xy+zw=0 \end{cases}$	double line in $\mathbb{Q}[\xi]$	1
	two skew lines and a double line	$\begin{cases} y^2-w^2=0 \\ xy-zw=0 \end{cases}$	double line in $\mathbb{Q}[\xi]$ two simple lines in $\mathbb{Q}[\xi]$	1 1
		$\begin{cases} y^2-2w^2=0 \\ xy-zw=0 \end{cases}$	double line in $\mathbb{Q}[\xi]$ two simple lines in $\mathbb{Q}(\sqrt{2})[\xi]$	1 2
two concurrent double lines, $\sigma_4=[(211)]$	point	$\begin{cases} w^2=0 \\ x^2+y^2+zw=0 \end{cases}$	point in $\mathbb{Q}$	1
	two double lines	$\begin{cases} w^2=0 \\ x^2-y^2+zw=0 \end{cases}$	two lines in $\mathbb{Q}[\xi]$	1
		$\begin{cases} w^2=0 \\ x^2-2y^2+zw=0 \end{cases}$	two lines in $\mathbb{Q}(\sqrt{2})[\xi]$	2
conic and double line, $\sigma_4=[1\{3\}]$	conic and double line	$\begin{cases} xw=0 \\ xz+y^2=0 \end{cases}$	conic in $\mathbb{Q}[\xi]$, line in $\mathbb{Q}[\xi]$	1 1
four concurrent lines, $\sigma_3=[111]$	point	$\begin{cases} x^2+z^2=0 \\ y^2+z^2=0 \end{cases}$	point in $\mathbb{Q}$	1
	two concurrent lines	$\begin{cases} xz-y^2=0 \\ -x^2+z^2=0 \end{cases}$	two lines in $\mathbb{Q}[\xi]$	1
		$\begin{cases} xz-y^2=0 \\ -2x^2-y^2+z^2=0 \end{cases}$	two lines in $\mathbb{Q}(\sqrt{2})[\xi]$	2
		$\begin{cases} xz-y^2=0 \\ 2xy+z^2=0 \end{cases}$	one line in $\mathbb{Q}[\xi]$, one line in $\mathbb{K}[\xi]$, degree$(\mathbb{K})=3$	1 3
		$\begin{cases} xz-y^2=0 \\ -3x^2+z^2=0 \end{cases}$	two lines in $\mathbb{Q}(\sqrt{3\sqrt{3}})[\xi]$	4
		$\begin{cases} xz-y^2=0 \\ -3x^2-3xy+z^2=0 \end{cases}$	one line in $\mathbb{K}[\xi]$, degree$(\mathbb{K})=4$ one line in $\mathbb{K}'[\xi]$, degree$(\mathbb{K}')=4$	4 4

Table 2: (continued)

complex type	real type	example	field of definition	degree
	four concurrent lines	$\begin{cases} xz-y^2=0 \\ 4x^2-5y^2+z^2=0 \end{cases}$	four lines in $\mathbb{Q}[\xi]$	1
		$\begin{cases} xz-y^2=0 \\ 2x^2-3y^2+z^2=0 \end{cases}$	two lines in $\mathbb{Q}[\xi]$, two lines in $\mathbb{Q}(\sqrt{2})[\xi]$	1 2
		$\begin{cases} xz-y^2=0 \\ -4x^2+8xy-4yz+z^2=0 \end{cases}$	four lines in $\mathbb{Q}(\sqrt{2})[\xi]$	2
		$\begin{cases} xz-y^2=0 \\ xy-3y^2+z^2=0 \end{cases}$	one line in $\mathbb{Q}[\xi]$, three lines l_i in $\mathbb{K}_i[\xi], \mathrm{degree}(\mathbb{K}_i)=3$	1 3
		$\begin{cases} xz-y^2=0 \\ 2x^2-4y^2+z^2=0 \end{cases}$	four lines in $\mathbb{Q}(\sqrt{2+\sqrt{2}})[\xi]$	4
		$\begin{cases} xz-y^2=0 \\ 4x^2-10y^2+z^2=0 \end{cases}$	four lines in $\mathbb{Q}(\sqrt{6},\sqrt{14})[\xi]$	4
		$\begin{cases} xz-y^2=0 \\ 2x^2+10xy+15y^2+7yz+z^2=0 \end{cases}$	two lines in $\mathbb{Q}(\sqrt{2})[\xi]$, two lines in $\mathbb{Q}(\sqrt{5})[\xi]$	2 2
		$\begin{cases} xz-y^2=0 \\ xy-4y^2+z^2=0 \end{cases}$	one line in $\mathbb{Q}[\xi]$, three lines l_i in $\mathbb{K}_i[\xi], \mathrm{degree}(\mathbb{K})=3$	1 3
		$\begin{cases} xz-y^2=0 \\ 2x^2-5y^2+z^2=0 \end{cases}$	two lines in $\mathbb{Q}(\sqrt{5+\sqrt{17}})[\xi]$, two lines in $\mathbb{Q}(\sqrt{5-\sqrt{17}})[\xi]$	4 4
		$\begin{cases} xz-y^2=0 \\ x^2-3xy-7y^2+z^2=0 \end{cases}$	four lines l_i in $\mathbb{K}_i[\xi], \mathrm{degree}(\mathbb{K}_i)=4$	4
		$\begin{cases} xz-y^2=0 \\ x^2+xy+z^2=0 \end{cases}$	four lines l_i in $\mathbb{K}_i[\xi], \mathrm{degree}(\mathbb{K}_i)=4$	4
two concurrent lines and a double line, $\sigma_3=[12]$	double line	$\begin{cases} xy=0 \\ y^2+z^2=0 \end{cases}$	double line in $\mathbb{Q}[\xi]$	1
	two concurrent lines and a double line	$\begin{cases} xy=0 \\ y^2-z^2=0 \end{cases}$	double line in $\mathbb{Q}[\xi]$, two simple lines in $\mathbb{Q}[\xi]$	1 1
		$\begin{cases} xy=0 \\ y^2-2z^2=0 \end{cases}$	double line in $\mathbb{Q}[\xi]$, two simple lines in $\mathbb{Q}(\sqrt{2})[\xi]$	1 2
line and triple line, $\sigma_3=[3]$	line and triple line	$\begin{cases} xz+y^2=0 \\ yz=0 \end{cases}$	simple line in $\mathbb{Q}[\xi]$, triple line in $\mathbb{Q}[\xi]$	1 1
two concurrent double lines, $\sigma_3=[1(11)]$	point	$\begin{cases} z^2=0 \\ x^2+y^2=0 \end{cases}$	point in $\mathbb{Q}$	1
	two double lines	$\begin{cases} z^2=0 \\ x^2-y^2=0 \end{cases}$	two lines in $\mathbb{Q}[\xi]$	1
		$\begin{cases} z^2=0 \\ x^2-2y^2=0 \end{cases}$	two lines in $\mathbb{Q}(\sqrt{2})[\xi]$	2

Table 2: (continued)

complex type	real type	example	field of definition	degree
quadruple line, $\sigma_3 = [(21)]$	quadruple line	$\begin{cases} y^2=0 \\ xy+z^2=0 \end{cases}$	line in $\mathbb{Q}[\xi]$	1
quadruple line, $\sigma_2 = [11]$	quadruple line	$\begin{cases} x^2=0 \\ y^2=0 \end{cases}$	line in $\mathbb{Q}[\xi]$	1

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