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On the Equivalence Between Complementarity Systems, Projected Systems and Unilateral Differential Inclusions

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Thème 4 —Simulation et optimisation
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Abstract: In this note we prove the equivalence, under appropriate conditions, between several dynamical formalisms: projected dynamical systems, two types of unilateral differential inclusions, and a class of complementarity dynamical systems. Each of these dynamical systems can also be considered as a hybrid dynamical system. This work is of interest since it both generalises some previous results and sheds new light on the relationship between known formalisms.

Key-words: Complementarity systems, projected dynamics, unilateral dynamics, hybrid dynamical systems, differential inclusions

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Sur l'équivalence entre les systèmes de complémentarité, les systèmes projetés et les inclusions différentielles

Résumé : Dans cette note, nous prouvons l'équivalence – sous des conditions appropriées – entre plusieurs formalismes dynamiques: systèmes dynamiques projetés, deux types d'inclusions différentielles unilatérales, et une classe de systèmes dynamiques de complémentarité. Chacun de ces systèmes dynamiques peut aussi être vu comme un système dynaique hybride. L'intérêt d'un tel travail est de généraliser des travaux antérieurs, mais aussi de jeter un nouvel éclairage sur les relations entre différents formalismes connus.

Mots-clés : Systèmes de complémentarité, dynamique projetée, dynamique unilatérale, systèmes dynamiques hybrides, inclusion différentielle

1 Introduction, notation

Unilateral dynamical systems have long been studied in the applied mathematics literature [11, 12, 1, 8, 13], because they find important applications in various fields (like mechanics, economics, and electrical circuits as shown recently in [2]). They usually take the form of differential inclusions or variational inequalities. In parallel the theory of complementarity problems has witnessed an impressive development [5], essentially motivated by optimization problems. Recently, complementarity systems, which consist of ordinary differential equations coupled to complementarity conditions, have been the object of deep studies in the control literature [6, 3]. Basic convex analysis tells us that complementarity problems can equivalently be formulated as a special type of generalized equations (i.e. equations of the form $0 \in F(x)$, where $F(x)$ is multivalued). This suggests that there should also exist close links between complementarity systems and unilateral dynamical systems. From a general perspective in the study of hybrid dynamical systems and their control, it seems quite important to clarify the relationships between all these various formalisms. First steps in this direction can be found in [6, 3].

Thus, the object of this note is to study the relationship between four different formalisms: projected dynamical systems [14, §2.2], unilateral differential inclusions [11, 12, 1, 13, 10], and complementarity dynamical systems [6, 3]. We will also discuss existence of solutions to these systems. For other works related with these problems, see [8, 4, 16].

Our material is fairly standard, concerning convex analysis (for example [9], mainly its Chap. III) and differential inclusions ([1]). We work in $\mathbb{R}^n$ considered as a Euclidean space, with scalar product $\langle x, y\rangle$ and associated norm $\|x\|$. Recall that the normal cone to a nonempty closed convex set $K \subset \mathbb{R}^n$ at $x \in K$ is

$$\mathrm{N}_K(x) := \{s \in \mathbb{R}^n : \langle s, y - x\rangle \leqslant 0, \text{ for all } y \in K\},$$

while the tangent cone is the polar of the normal cone, which means

$$\mathrm{T}_K(x) := [\mathrm{N}_K(x)]^\circ := \{d \in \mathbb{R}^n : \langle s, d\rangle \leqslant 0, \text{ for all } s \in \mathrm{N}_K(x)\}$$

(if $x \notin K$, we set $\mathrm{N}_K(x) = \mathrm{T}_K(x) = \emptyset$).

In this paper we are given a nonempty closed convex set $C \subset \mathbb{R}^n$ (the *feasible set*) and two functions $g : \mathbb{R}_+ \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$. Associated with this data, we consider (together with the initial condition $x(0) = x_0 \in C$)

• the Projected Dynamical System

$$\dot{x}(t) \overset{\text{a.e.}}{=} \mathrm{proj}_{\mathrm{T}_C(x(t))}(-f(x(t)) - g(t)) \qquad \text{(PDS)}$$

(a.e. means almost everywhere with respect to t, in the Lebesgue measure),

• the two Unilateral Differential Inclusions

$$-\dot{x}(t) \overset{\text{a.e.}}{\in} f(x(t)) + g(t) + \mathrm{N}_{\mathrm{T}_C(x(t))}(\dot{x}(t)), \qquad \text{(UDI-}T_C\text{)}$$
$$-\dot{x}(t) \overset{\text{a.e.}}{\in} f(x(t)) + g(t) + \mathrm{N}_C(x(t)). \qquad \text{(UDI-}C\text{)}$$

Remark 1 *In [14, 7], (PDS) is presented in different but equivalent ways. Section 2.1 below will clarify the equivalence between the various formulations.* □

Note that any solution to any of the above systems must by necessity lie in C. We will also consider the particular case where C is described explicitly by *constraints*:

$$C = \{x \in \mathbb{R}^n : h(x) \leqslant 0 \in \mathbb{R}^m\}, \qquad (1)$$

where the functions $h_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, m$ are hereby assumed continuously differentiable; we will use $h(\cdot) := (h_1(\cdot), \dots, h_m(\cdot))^\top$. The gradient of h_i at x is $\nabla h_i(x) \in \mathbb{R}^n$, so that

$$h_i(x + d) = h_i(x) + \langle \nabla h_i(x), d\rangle + o(\|d\|) \quad \text{for all } d \in \mathbb{R}^n .$$

The constraint-space $\mathbb{R}^m$ is equipped with the standard dot-product: we write $\lambda^\top h$ for $\sum_{i=1}^m \lambda_i h_i$ and the notation $\nabla h(x)\lambda$ means $\sum_{i=1}^m \lambda_i \nabla h_i(x) \in \mathbb{R}^n$. The nonnegative orthant of $\mathbb{R}^m$ is

$$\mathbb{R}^m_+ := \{\lambda \in \mathbb{R}^m : \lambda_i \geqslant 0, i = 1, \dots, m\},$$

we will also use the notation $\lambda \geqslant 0$; likewise for the nonpositive orthant $\mathbb{R}^m_-$.

With this notation, we consider in addition to (PDS) and (UDI):

• the Complementarity Dynamic System

$$\left\{ \begin{array}{l} -\dot{x}(t) \overset{\text{a.e.}}{=} f(x(t)) + g(t) + \nabla h(x(t))\lambda \\ 0 \geqslant h(x(t)) \perp \lambda \geqslant 0 \end{array} \right. \qquad \text{(CDS)}$$

($\perp$ means orthogonality: $\lambda^\top h(x(t)) = 0$).

We will see that (PDS) and (UDI-T_C) are always equivalent, while (UDI-C) may have more solutions; and (UDI-C) is equivalent to (CDS) if some additional constraint qualification holds. Ultimately, all of these systems are equivalent if (UDI-C) has (no solution at all or) a unique solution which is slow, i.e. $\dot{x}(t)$ is of minimal norm in the set it belongs to[1]:

$$-\dot{x}(t) = f(x(t)) + g(t) + \bar{\nu}\,, \quad \text{with} \quad \bar{\nu} = \operatorname*{argmin}_{\nu \in \mathrm{N}_C(x(t))} \|f(x(t)) + g(t) + \nu\|\,.$$

2 Basic equivalences

We start by establishing some equivalences which do not use the fact that $\dot{x}(t)$ is the (time) derivative of $x(t)$. These two vectors are in fact considered as two independent vectors of $\mathbb{R}^n$.

2.1 Equivalent formulations of (PDS)

In [14, Def. 2.5] and [7, Def. 2.1], (PDS) is defined under different forms. The first one is[2]

$$\dot{x}(t) \overset{\text{a.e.}}{=} \lim_{\delta \downarrow 0} \frac{\mathrm{proj}_C(x(t) + \delta v) - x(t)}{\delta}\,, \tag{2}$$

where $v := -f(x(t)) - g(t)$.

Proposition 1 *The righthand sides of (PDS) and* (2) *are the same.*

Proof. This is Proposition III.5.3.5 of [9]. $\square$

Another form of (PDS) is presented in [14, 7], for which we need special notation: ν^* introduced in the next lemma, and Π_C after it.

Lemma 1 *Given a closed convex cone $N \subset \mathbb{R}^n$ and $v \notin N^\circ$, the optimization problems*

$$\max\{\langle \nu, v\rangle : \nu \in N,\ \|\nu\| \leqslant 1\} \tag{3a}$$

$$\max\{\langle \nu, v\rangle : \nu \in N,\ \|\nu\| = 1\} \tag{3b}$$

have the same value $\beta > 0$ and the same unique solution ν^*.

Besides, $\mathrm{proj}_N(v) \neq 0$ *and* $\nu^* = \dfrac{\mathrm{proj}_N(v)}{\|\mathrm{proj}_N(v)\|}$.

Proof. By definition of N°, there exists at least one ν in (3a) such that $\langle \nu, v\rangle > 0$; hence the optimal value is positive and must be attained at some ν^* of norm 1; strict convexity properties of the unit ball imply that this ν^* must be unique. The first statement is proved.

Now the projection of v onto N is the unique solution of

$$\min_{\nu \in N} \|\nu - v\|^2\,,$$

or equivalently

$$\min_{\nu \in N} \{\|\nu - v\|^2 : \|\nu\| = \|\mathrm{proj}_N(v)\|\}$$

(the extra constraint being redundant!). Up to the constant $\|v\|^2 + \|\mathrm{proj}_N(v)\|^2$, the above optimization problem is equivalent to

$$\min_{\nu \in N} \{-\langle \nu, v\rangle,\ \|\nu\| = \|\mathrm{proj}_N(v)\|\}\,.$$

In view of positive homogeneity, this latter problem has a unique solution collinear to that of (3b), namely $\|\mathrm{proj}_N(v)\|\nu^*$. The proof is complete. $\square$

Noting that $\mathrm{proj}_N(v) = 0$ if $v \in N^\circ$, we see from (3a), (3b) that the operator Π_C used in [14, 7] is equal to

$$\Pi_C(x, v) = v - \langle \nu^*, v\rangle \nu^*\,.$$

[1] Being closed and convex, this set has a unique point of minimum norm.

[2] The writing $\delta \to 0$ of [14, 7] assumes $\delta > 0$, i.e. $\delta \downarrow 0$. In fact, the difference quotient in (2) need not have a limit when δ is allowed to take either sign.

Corollary 1 *Set* $v := -f(x(t)) - g(t)$ *and* $N := \mathrm{N}_C(x(t))$. *Use the notation of Lemma 1, setting* $\nu^* = 0$ *if* $v \in N^\circ = \mathrm{T}_C(x(t))$. *Then the righthand side of (PDS) is*

$$\mathrm{proj}_{T_C(x(t))}(v) = v - \mathrm{proj}_N(v) = v - \langle \nu^*, v\rangle \nu^* \,.$$

Proof. Since $\mathrm{T}_C(x(t)) = [\mathrm{N}_C(x(t))]^\circ$, Moreau's decomposition theorem[3] tells us that the righthand side of (PDS) is $v - \mathrm{proj}_N(v)$. Then use Lemma 1, remembering that $\langle \mathrm{proj}_N(v), v\rangle = \|\mathrm{proj}_N(v)\|^2$. □

This proves the equivalence of (PDS) with the other formulation of [14, 7], namely

$$\dot{x}(t) = \Pi_C(x(t), -f(x(t)) - g(t))\,.$$

2.2 Geometric formulations: Relations between (PDS) and (UDI)

In this subsection, we deal with the case of an abstract set C, not necessarily described by constraints.

Proposition 2 *Let* $K \subset \mathbb{R}^n$ *be a nonempty closed convex cone. For any two vectors* v *and* s *in* $\mathbb{R}^n$, *the following relations are equivalent:*

$$s = \mathrm{proj}_K(v)\,; \tag{4a}$$

$$v - s \in \mathrm{N}_K(s)\,; \tag{4b}$$

$$s \in K,\ v - s \in K^\circ,\ \langle v - s, s\rangle = 0\,; \tag{4c}$$

$$\left.\begin{array}{c} v - s \in K^\circ\,, \\ \forall \nu \in K^\circ,\ \|s\|^2 \leqslant \langle s, v - \nu\rangle\,. \end{array}\right\} \tag{4d}$$

Proof. Use the variational characterization of a projection: (4a) is equivalent to

$$s \in K \quad \text{and} \quad \langle v - s, s' - s\rangle \leqslant 0, \quad \text{for all } s' \in K\,. \tag{5}$$

By definition of a normal cone, this is exactly (4b). Since K is a cone, $\mathrm{N}_K(s) = K^\circ \cap \{s\}^\perp$, so that (4b) is also equivalent to (4c).

Now let v and s satisfy (4c) and take $\nu \in K^\circ$; because $s \in K$, we have $\langle \nu, s\rangle \leqslant 0 = \langle v - s, s\rangle$: (4d) holds. Conversely, let v and s satisfy (4d). In particular, $\langle \nu, s\rangle$ is bounded from above (by $\langle s, v\rangle - \|s\|^2$) when ν describes the cone K°, and therefore cannot be strictly positive[4]; thus, $\langle \nu, s\rangle \leqslant 0$ for all $\nu \in K^\circ$, i.e. $s \in K^{\circ\circ} = K$; combining $v - s \in K^\circ$ with $s \in K$, we have $\langle s, v - s\rangle \leqslant 0$. Besides, take $\nu = 0$ in (4d) to see that $\langle s, s - v\rangle \leqslant 0$. Piecing together, (4c) holds and the proof is complete. □

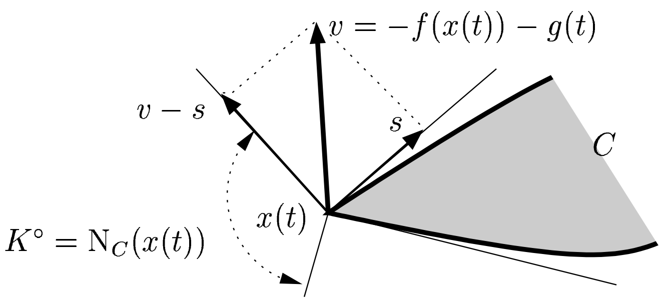

Figure 1: The only possible solution to (PDS) and (UDI-T_C)

The form (4c) clearly reveals that s and $v - s$ make up the Moreau decomposition of $s + v - s = v$; in particular, it follows that $v - s = \mathrm{proj}_{K^\circ}(v)$. Thus, the mere relation (4b) implies that both s and $v - s$ are fairly special points in their respective cones. This is illustrated by Fig. 1, and the link with Problems (PDS) and (UDI) of §1 is clear:

[3] See [9, Thm. III.3.2.5]: two mutually polar cones N and N° decompose the space. More precisely, the projections onto N and N° of a given $v \in \mathbb{R}^n$ are the unique elements in N and N° respectively, mutually orthogonal, and summing up to v:

$$v = \mathrm{proj}_N(v) + \mathrm{proj}_{N^\circ}(v)\,, \quad \langle \mathrm{proj}_N(v), \mathrm{proj}_{N^\circ}(v)\rangle = 0\,.$$

[4] if $\langle \nu_0, s\rangle > 0$ for some $\nu_0 \in K^\circ$, just take $\nu = t\nu_0$ with $t \to +\infty$.

Corollary 2 *Given two vectors $x = x(t) \in C$ and $\dot{x} = \dot{x}(t) \in \mathbb{R}^n$, the following three statements are equivalent:*
(i) *(PDS) holds;*
(ii) *(UDI-T_C) holds;*
(iii) *(UDI-C) holds, together with the following two equivalent properties:*
(iii)$_1$ $-\dot{x}(t) = f(x(t)) + g(t) + \mathrm{proj}_{\mathrm{N}_C(x(t))}(-f(x(t)) - g(t))$.
(iii)$_2$ *the vector $-\dot{x}(t)$ is of minimum norm in $f(x(t)) + g(t) + \mathrm{N}_C(x(t))$,*

Proof. Apply Proposition 2 with $K := \mathrm{T}_C(x(t))$ (which is nonempty), and $v := -f(x(t)) - g(t)$, $s := \dot{x}(t)$; note that $K^\circ = \mathrm{N}_C(x(t))$. Then (4a) = (PDS), (4b) is (UDI-T_C) and the first line of (4d) is (UDI-C).

Now write the second line in (4d) as

$$\langle 0 - (-s), (\nu - v) - (-s) \rangle \leqslant 0 \quad \text{for all } (\nu - v) \in K^\circ - v$$

to see that it means $-s = \mathrm{proj}_{K^\circ - v}(0)$:

$$-s = \operatorname*{argmin}_{\nu \in K^\circ - v} \|\nu\|,$$

which is (iii)$_1$. The form (iii)$_2$ comes with the change of variable $\nu + v = \nu'$:

$$-s = \operatorname*{argmin}_{\nu' \in K^\circ} \|\nu' - v\|. \qquad \square$$

Thus, (UDI-C) is equivalent to the other ones if and only if it has the so-called *slow solution* [4] ($\dot{x}(t)$ is of minimal norm) as only possible solution. Note also that the last form in the statement of Corollary 2 can be written $\dot{x}(t) = v - \mathrm{proj}_{\mathrm{N}_C(x(t))}(v)$ with $v = -f(x(t)) - g(t)$, which is just the formulation of [14, 7] (remember Corollary 1).

2.3 Formulation using constraints explicitly

In this subsection, we turn to (CDS), which requires some more notation and material from convex analysis.

For $x \in C$, we denote by

$$I(x) := \{i \in \{1, \dots, m\} : h_i(x) = 0\} \tag{6}$$

the set of *active constraints* at x and we linearize the constraints, introducing the cones

$$\begin{aligned} T^h(x) &:= \{d \in \mathbb{R}^n : \langle \nabla h_i(x), d \rangle \leqslant 0,\ i \in I(x)\} \\ N^h(x) &:= [T^h(x)]^\circ = \left\{\textstyle\sum_{i \in I(x)} \lambda_i \nabla h_i(x) : \lambda_i \geqslant 0,\ i \in I(x)\right\} \end{aligned} \tag{7}$$

(still with the convention $T^h(x) = N^h(x) = \emptyset$ if $x \notin C$). Beware that they need *not* coincide with the usual tangent and normal cones[5] to C at x. Nevertheless we always have $\mathrm{N}_C(x) \supset N^h(x)$ and $\mathrm{T}_C(x) \subset T^h(x)$.

Actually, a key assumption for what follows is $\mathrm{T}_C = T^h$, or equivalently $\mathrm{N}_C = N^h$. This is guaranteed under any of the following *qualification* conditions (see [9, § VII.2.2] for example):

$$\forall x \in C,\ \exists d \in \mathbb{R}^m \text{ such that } \langle \nabla h_i(x), d \rangle < 0 \text{ for } i \in I(x), \tag{QC.1}$$

which is dually equivalent to the so-called *Mangasarian-Fromowitz* assumption:

$$\left.\begin{array}{l} \sum_{i \in I(x)} \lambda_i \nabla h_i(x) = 0 \\ \text{with } \lambda_i \geqslant 0,\ i \in I(x) \end{array}\right\} \implies \lambda_i = 0,\ i \in I(x). \tag{QC.2}$$

Note that (QC.2) holds in particular if

The gradients of the active constraints at x are linearly independent (QC.3)

(just remove the restriction $\lambda_i \geqslant 0$ in (QC.2)!). When the h_i's are convex, it can be seen that (QC.1) is equivalent to the so-called *Slater* assumption:

$$\exists \bar{x} \in \mathbb{R}^n : h_i(\bar{x}) < 0,\ i = 1, \dots, m. \tag{QC.4}$$

[5] With $m = 1$, the set C of (1) defined by the constraint $h(x) := \|x\|^2$ makes a counterexample: $C = \{0\}$ but $T^h(0) = \mathbb{R}^n$, although $\mathrm{T}_C(0) = \{0\}$.

Proposition 3 *Two vectors $x = x(t)$ and $\dot x = \dot x(t)$ in $\mathbb{R}^n$ satisfy (CDS) if and only if they satisfy*

$$-\dot x(t) \in f(x(t)) + g(t) + N^h(x(t)) \,. \qquad \text{(UDI-}h\text{)}$$

If $\mathrm{T}_C = T^h$, which holds for example under one of the qualification conditions (QC.1-4), this system is in turn equivalent to (UDI-C).

Proof. Since an $x(t) \notin C$ can satisfy neither system, we may assume $x(t) \in C$. The second line in (CDS) means: $\lambda_i \geqslant 0$, $i = 1, \dots, m$ and $\lambda_i = 0$ if $h_i(x(t)) < 0$, i.e. if $i \notin I(x)$. From the definition (7) of N^h, (CDS) is therefore exactly (UDI-h). The result follows. □

Let us summarize this section: the following relations hold – $\mathcal{S}$(P) standing for the solution set of a problem (P):

$$\mathcal{S}(\text{PDS}) = \mathcal{S}(\text{UDI-}T_C) \subset \mathcal{S}(\text{UDI-}C) \supset \mathcal{S}(\text{UDI-}h) = \mathcal{S}(\text{CDS})$$

and the second inclusion becomes an equality if qualification holds. The first inclusion becomes an equality when $\mathcal{S}$(UDI-C) is either empty or reduces to the slow solution.

3 Existence results

The content of the previous results is void if the various systems in §1 have no solution. In fact, our results in §2 show that existence for (PDS) = (UDI-T_C) implies existence for (CDS) = (UDI-C) (under constraint qualification); and uniqueness for (CDS) = (UDI-C) implies for (PDS) = (UDI-T_C)
– either non-existence,
– or uniqueness (with equivalence) in case the solution is slow.

This latter situation occurs for example in the following framework:

Theorem 1 *Assume that $g \in L^1(\mathbb{R}_+)$, and that f is continuous over $\mathbb{R}^n$ and monotone:*

$$\langle f(x) - f(y), x - y\rangle \geqslant 0 \quad \textit{for all } x, y \in \mathbb{R}^n \,.$$

Then, for any initial condition $x(0) \in C$, (UDI-C) has a unique solution over the whole of $\mathbb{R}_+$, which is slow: $\dot x(t)$ is of minimal norm in the set $f(x(t)) + g(t) + \mathrm{N}_C(x(t))$.

Proof. Note that the multivalued mapping $F := f + \mathrm{N}_C$ is maximal monotone ([1, Cor. 2.7])[6] and write (UDI-C) as $\dot x(t) + F(x(t)) \ni -g(t)$. The result is [1, Prop. 3.4]. □

The conclusion is then straightforward:

Theorem 2 *Make the assumptions of Theorem 1. For any $x(0) \in C$, all of the problems stated in §1 have the same unique solution. Furthermore $\dot x(t) \in \mathrm{T}_C(x(t))$.*

Let us conclude with some remarks.

We believe that our results easily extend to a nonconvex but so-called regular set C [15, Def. 6.4]; see also [4, 17] for a study of differential inclusions in this context. In case of explicit constraints, regularity occurs when the differentiable h_i are not necessarily convex – but satisfy the qualification condition.

Note also a consequence of [1, Prop. 3.4]: if g is piecewise continuous, the solution mentioned in Theorem 2 has a right derivative $\dot x_+(t)$ at all t: “almost everywhere” can then be suppressed from the formulations, if $\dot x$ is replaced by $\dot x_+$.

A common approach to derive (UDI-T_C) from (CDS) proceeds as follows: under constraint qualification, (CDS) can be written (see Proposition 3)

$$-\dot x(t) = f(x(t)) + g(t) + \nu \,, \quad \text{with} \quad \nu = \nabla h(x(t))\lambda \in \mathrm{N}_C(x(t)) \,.$$

This is equivalent to (UDI-T_C) = (4c) if and only if $\langle \nu, \dot x(t)\rangle = 0$, i.e.

$$\langle \nabla h(x(t))\lambda, \dot x(t)\rangle = \lambda^\top \dot w(t) = 0 \,,$$

where we have set $w(t) := h(x(t)) \in \mathbb{R}^m$ (writing $\dot w_+(t)$ would be more precise). Our results clearly imply that this extra property holds if and only if the only possible solution of (UDI-C) is slow.

[6]The mapping $x \longmapsto \mathrm{N}_C(x)$ is maximal monotone: it is the subdifferential of the (convex) indicator function of C (0 on C, $+\infty$ outside).

4 Conclusion

The study of relationships between various formalisms of hybrid systems is of interest [3], and this note sheds some new light on the equivalences between complementarity systems, projected dynamical systems, and unilateral differential inclusions. In particular an alternative, simpler proof of a result in [7] is provided in this note. Let us remark in passing that the difference between Moreau's first order sweeping process, i.e. dynamical systems of the form $-\dot{x}(t) \in N_{K(x(t),t)}(x(t))$ [10, 11, 12, 13], and projected dynamical systems, is that the former concerns a "fixed point" that is swept by a moving convex domain, while the latter concerns a "moving point" that is constrained to remain in a fixed convex domain. Merging both concepts is still an open field. The interest of showing such equivalences, apart from a pure mathematical motivation, lies in the fact that techniques developed for one formalism can be used for the others, with a broadened scope of applications.

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