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Thomas Bréhard, Jean-Pierre Le Cadre

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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Thomas Bréhard (IRISA/CNRS), Jean-Pierre Le Cadre (IRISA/CNRS)

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**Abstract:** We here address the classical bearings-only tracking problem (BOT) for a single object, which belongs to the general class of non linear filtering problems. Recently, algorithms based on sequential Monte Carlo methods (particle filtering) have been proposed. However, initializing particle filtering is often the main difficulty, especially if the state is only partially observed (BOT). To remedy for this problem, the problem is immersed in a modified polar coordinate (MP) framework. This approach leads us to consider an original formulation of the BOT problem within the MP system. In particular, it is shown that this problem is relevant to a more general class of problems: non-linear filtering with unknown state covariance. Inside this particular framework, particle filters can be quite convinently initialized by using only observed bearings (optimization problem). The whole algorithm performs quite satisfactorily, avoiding the need of a strong prior about target location and velocity. Simulation results illustrate the benefits of this approach. The Posterior Cramér-Rao Bound (PCRB) provides a lower bound on the mean square error. Original PCRB approximations for the "partial" state target (the observable components) are derived. It is well-known that the "usual" PCRB is (very) over-optimistic. Relaxing the asymptotic unbiasness hypothesis, a new bound is derived, both for partial or complete state vectors, which presents a good agreement with estimated MSE from simulated data.

**Key-words:** bearings-only tracking, initialization, sequential Monte Carlo methods, posterior Cramér-Rao bound, performance analysis.

* {tbrehard}{lecadre}@irisa.fr

Unité de recherche INRIA Rennes
IRISA, Campus universitaire de Beaulieu, 35042 Rennes Cedex (France)
Téléphone : +33 2 99 84 71 00 — Télécopie : +33 2 99 84 71 71

# Initialisation du filtre particulaire et borne de Cramér-Rao à posteriori appliqué au problème de pistage par mesure d'angle en coordonnées polaires modifiées

**Résumé :** Le problème de pistage d'une cible unique par mesure d'angle seule (BOT) est un problème classique appartenant à la classe des problèmes de filtrage non linéaires. Ce type de problème a été résolu récemment à l'aide d'une méthode séquentielle de type Monte Carlo (filtrage particulaire). Toutefois, l'initialisation de cette algorithme reste très délicate notamment lorsque une partie seulement de l'état est observable. L'utilisation des coordonnées polaires modifiées se révéle alors cruciale. En effet, cette approche permet de déduire une nouvelle formulation pour le BOT. En particulier, on montre alors que ce problème appartient à une classe plus générale: les problèmes de filtrage non linéaires avec covariance d'état inconnue. Dans ce cadre, l'algorithme de filtrage particulaire peut-être correctement initialisé en utilisant les mesures d'angles observées (problème d'optimisation). Cette méthode originale est illustrée par des résultats de simulations qui s'avérent très satisfaisants. L'avantage principal de cette dernière est qu'elle ne nécessite pas un a priori fort concernant la localisation et la vitesse de la cible. La borne de Cramér-Rao à posteriori (PCRB) est une borne inférieure pour l'erreur quadratique moyenne. Il est bien connue que la PCRB "usuelle" est très optimiste. Une nouvelle borne pour l'état partiel (les composantes observables) ou l'état complet est obtenue en relâchant l'hypothèse de biais asympotique. Cette nouvelle borne fournit des résultats plus réalistes relativement à l'erreur quadratique calculée à partir de données simulées.

**Mots-clés :** pistage par mesure d'angle seule, initialisation, methodes Monte Carlo sequentielles, borne de Cramér-Rao à posteriori, analyse de performances.

## Notations

MP(C): Modified Polar (Coordinates),

BOT: Bearings Only Tracking,

$n_y$: size of the target state in MP coordinate system,

$n_{y^r}$: size of the partial target state in MP coordinate system called RMP,

$\succcurlyeq$: inequality $A \succcurlyeq B$ means that $A - B$ is a positive semidefinite matrix,

$\otimes$: Kronecker product,

$||X||_Q$: , $||X||_Q = X^T Q^{-1} X$ where $X$ is column vector,

$\Delta$: Laplacian operator,

$\nabla$: gradient operator,

$\xrightarrow{\mathcal{D}}$: convergence in distribution,

$det(X)$: denotes the determinant of matrix $X$,

$X^T$: denotes the transpose of matrix $X$.

# 1 Introduction

The aim of BOT is to determine the trajectory of a target using noisy bearing measurements from a single observer. Let us assume that the target motion may be described by a diffusion model (see [1] for an exhaustive review on dynamic models). The problem is classically composed of two stochastic equations. The first one represents the temporal evolution of the target state (position and velocity) and is called state equation. The second one links the bearing measurement to the state of the target at time t (measurement equation). Non-linearity of the measurement equation is a main difficulty. Particle filtering ([2], [3], [4] and [5]) is now the method of reference.

Particle filtering algorithms are generally composed of three stages at each step of time. First, a particle set representing different possible states of the target is propagated using the state equation. Second, the weights of the particles are updated according Bayes's formula using the measurement equation. The state distribution is a finite weighted sum of Dirac laws centered around the particles. The third stage is a resampling step in order to avoid degeneracy of the particle set. It may be mentioned in passing that many ways have been developed to improve particle filtering algorithms: the use of kernel filter has been studied in [6] as well as the resampling frequency.

However, initializing the particle filter has not been thoroughly investigated yet, reference [7] excepted. In practice it can not be denied that wrong initialization may have disastrous consequences on estimation quality. Intuitively, this problem is the result of range unobservability. However, this fact is not clearly expressed using cartesian coordinate system. Otherwise, other coordinate systems like the MP coordinate system have been proposed by Aidala and Hammel in [8]. Many theoretical considerations testify they are fundamentally relevant in the BOT context, especially the fact that the range is uncoupled with other components of the state vector. In the 3-D context, an hybrid coordinate system has been developed by Grossman in [9].

In section 2, we study the formulation of BOT problem in the MP coordinate system. This section has its own interest because it allows to understand the effect of range for estimating the other components of the state vector. In particular, it appears that range can be assimilated to a state covariance term in the state equation. Moreover, we show that BOT belongs to a class of problems hard to solve namely "non-linear filtering problems with unknown state covariance". From these considerations, an initialization method based on an approximation of the stochastic system is proposed in section 3. Compared with [7], there is no need to have kinematic informations on the target (i.e. maximum and minimum velocity). Simulation results are presented for a typical tracking scenario, involving a manoeuvring observer and a constant velocity target.

Section 4 and 5 are devoted to the performance analysis of MP coordinate system using the recursive PCRB developed by Tichavskỳ et al. [10]. This formulation of the PCRB is now widely used by the tracking community. However, a major and well recognized concern is that this bound is (very) over-optimistic in the BOT context (see [11]). Explanations are provided (asymptotic unbiasness assumption) and a new PCRB is derived. Moreover, as long as the target does not maneuver, only a part of the MP coordinate system is observable. In Section 5, we propose then a general framework for deriving a bound related to the "partial" target state.

## 2 From cartesian to MP system for BOT

Historically, BOT is presented in cartesian system. Let us define

$$X_t = \begin{pmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \\ X_4(t) \end{pmatrix} = \begin{pmatrix} v_x(t) \\ v_y(t) \\ r_x(t) \\ r_y(t) \end{pmatrix}, \tag{1}$$

the state of the target at time $t$ composed of relative velocity and position of the target in the $x-y$ plane. It is assumed that the target follows a nearly constant-velocity model. The discretized state equation[1] is then:

$$X_{t+1} = AX_t + HU_t + W_t \ , \tag{2}$$

[1] For a general review of dynamic models for target tracking see [1].

where:

$$\left|\begin{array}{l} W_t \sim \mathcal{N}(0,Q) \ , \\ A = \begin{pmatrix} 1 & 0 \\ \delta_t & 1 \end{pmatrix} \otimes Id_2 \ , \\ H = \begin{pmatrix} 1 \\ \delta_t \end{pmatrix} \otimes Id_2 \ , \\ Q = \eta^2 \begin{pmatrix} \delta_t & \frac{\delta_t^2}{2} \\ \frac{\delta_t^2}{2} & \frac{\delta_t^3}{3} \end{pmatrix} \otimes Id_2 \ . \end{array}\right. \tag{3}$$

$\delta_t$ is the elementary time period and $U_t$ is the known difference between observer velocity at time $t{+}1$ and $t$. Otherwise, we note $Z_t$ the bearing measurement received at time $t$. The target state is related to this measurement through the following equation:

$$Z_t = \tan^{-1}\left(\frac{r_x(t)}{r_y(t)}\right) + V_t \ , \tag{4}$$

where $V_t \sim \mathcal{N}(0, \sigma_v^2)$. The system (2)-(4) has two components : a linear state equation (2) and a non-linear measurement equation (4). Particle filter techniques (see [3],[4]) are, thus, quite relevant. However, as shown in [12] a problem of observability is hidden in the cartesian formulation. As a matter of fact, no information on range exists, as long as the observer is not maneuvering. So the idea consists in using a coordinate system for which the unobservable component (range) is not coupled with the observable components. This is the motivation of Aidala and Hammel [8] for defining MP system. We denote:

$$Y_t = \begin{pmatrix} Y_1(t) \\ Y_2(t) \\ Y_3(t) \\ Y_4(t) \end{pmatrix} = \begin{pmatrix} \dot{\beta}(t) \\ \frac{\dot{r}(t)}{r(t)} \\ \beta(t) \\ \frac{1}{r(t)} \end{pmatrix} \ , \tag{5}$$

the target state at time t in MP coordinate system where $\beta(t)$ and $r(t)$ are the relative bearing and the target range, respectively. $\dot{\beta}(t)$ and $\dot{r}(t)$ are the time derivative of $\beta(t)$ and $r(t)$. The stochastic system (2)-(4) becomes:

$$\left\{\begin{array}{ll} Y_{t+1} = f_c^{mp}\left[A f_{mp}^c(Y_t) + HU_t + W_t\right] \ , & \text{(6a)} \\ Z_t = H(f_{mp}^c(Y_t)) + V_t \ , & \text{(6b)} \end{array}\right.$$

where $f_c^{mp}$ and $f_{mp}^c$ are cartesian-to-MP and MP-to-cartesian state mapping functions reminded in appendix A. Now we are going to pay more attention to the stochastic system (6) using Aidala and Hammel's formulation of the problem.

Expliciting $f_c^{mp}$ and $f_{mp}^c$, eq.(6) can be rewritten:

$$\left\{ \begin{aligned} &\begin{pmatrix} Y_1(t+1) \\ Y_2(t+1) \\ Y_3(t+1) \end{pmatrix} = \begin{pmatrix} \frac{S_1(t)S_4(t)-S_2(t)S_3(t)}{S_3^2(t)+S_4^2(t)} \\ \frac{S_1(t)S_3(t)+S_2(t)S_4(t)}{S_3^2(t)+S_4^2(t)} \\ Y_3(t)+\tan^{-1}\left(\frac{S_3(t)}{S_4(t)}\right) \end{pmatrix} , && (7a) \\ &Y_4(t+1) = \frac{Y_4(t)}{\sqrt{S_3^2(t)+S_4^2(t)}} , && (7b) \\ &Z_t = Y_3(t) + V_t , && (7c) \end{aligned} \right.$$

where:

$$\begin{pmatrix} S_1(t) \\ S_2(t) \\ S_3(t) \\ S_4(t) \end{pmatrix} = \begin{pmatrix} Y_1(t) \\ Y_2(t) \\ \delta_t Y_1(t) \\ 1+\delta_t Y_2(t) \end{pmatrix} + Y_4(t) \left[ \begin{pmatrix} 1 \\ \delta_t \end{pmatrix} \otimes P_{Y_3(t)} U_t \right] + Y_4(t) W_t , \quad (8)$$

and

$$P_{Y_3(t)} = \begin{pmatrix} \cos(Y_3(t)) & -\sin(Y_3(t)) \\ \sin(Y_3(t)) & \cos(Y_3(t)) \end{pmatrix} . \quad (9)$$

Deliberately, we have split the state vector in observable/unobservable components in (7). We note

$$Y_t^r = \begin{pmatrix} Y_1(t) \\ Y_2(t) \\ Y_3(t) \end{pmatrix} = \begin{pmatrix} \dot{\beta}(t) \\ \frac{\dot{r}(t)}{r(t)} \\ \beta(t) \end{pmatrix} , \quad (10)$$

the observable components of state vector called “Reduced Modified Polar" coordinate system (acronym RMP). Then if the observer does not maneuver ($U_t$ is a zero vector), we can write (7) according to the last notations:

$$\left\{ \begin{aligned} &Y_{t+1}^r = F(Y_t^r, \tilde{W}_t) , && (11a) \\ &Y_4(t+1) = G(Y_t, \tilde{W}_t) , && (11b) \\ &Z_t = Y_3(t) + V_t , && (11c) \end{aligned} \right.$$

where $\tilde{W}_t = Y_4(t) W_t$. Then what conclusions can be drawn from this formulation? We see that if target evolution is deterministic, range disappears from state equation (11a). Moreover, this quantity does not appear in the measurement equation (11c).

Similarly to the deterministic case, range remains unobservable in the stochastic case (non-maneuvering target). Moreover, it is noticeable that in this case range only appears as a covariance term. Then the problem boils down to this one: historically, BOT was presented as a non-linear filtering problem via cartesian coordinate system. However, in this approach the problem of non observability of range is hidden. Within the MP formulation, the problem appears to be more complex, at a first glance. It is a non-linear filtering problem with unknown state covariance ($Y_4(t)$ is the unknown parameter in state covariance) and we will see that despite its *apparent* complexity it is the workhorse for performing a reliable initialization. More precisely, there is no need to have a precise knowledge of the range for estimating the observable components.

# 3 Particle filtering algorithm in MP coordinate system

According to the previous section as long as the observer is not maneuvering only RMP are observable. The fourth component can be fixed without perturbing the estimation of the three first components of the state and must be estimated only when it turns to be observable i.e. when the observer is maneuvering. Consequently, the number of components we have to initialize decreases from 4 to 3; and, more fundamentally, these components are precisely the observable ones. Of course, the particle filtering algorithm can be initialized using a grid approach if "physical" bounds are available for this components. This approach has been proposed by Teuliere in [7]. However, these bounds are not necessary in our initialization context. A first step consists in initializing the observable components. It is presented below.

## 3.1 Initializing observable components

An instrumental remark is that the Maximum Likelihood Estimate (MLE) for the observable components can be infered from an approximation of the stochastic system in the MPC system. More precisely, assuming that the target motion is deterministic, the stochastic system (7) becomes:

$$\left\{\begin{aligned} \begin{pmatrix} Y_1(t+1) \\ Y_2(t+1) \\ Y_3(t+1) \end{pmatrix} &= \begin{pmatrix} \frac{\delta_t Y_1(t)}{(\delta_t Y_1(t))^2+(1+\delta_t Y_2(t))^2} \\ 1-\frac{1+\delta_t Y_2(t)}{(\delta_t Y_1(t))^2+(1+\delta_t Y_2(t))^2} \\ Y_3(t)+\tan^{-1}\left(\frac{\delta_t Y_1(t)}{1+\delta_t Y_2(t)}\right) \end{pmatrix} , && \text{(12a)} \\ Y_4(t+1) &= \frac{Y_4(t)}{\sqrt{(\delta_t Y_1(t))^2+(1+\delta_t Y_2(t))^2}} , && \text{(12b)} \\ Z_t &= Y_3(t)+V_t . && \text{(12c)} \end{aligned}\right.$$

This system modelling results in a non-linear regression problem, i.e. :

$$Z_t = Y_3(k) + \tan^{-1}\left(\frac{(t-k)\delta_t Y_1(k)}{1+(t-k)\delta_t Y_2(k)}\right) + V_t , \ \forall t \geq 0 . \tag{13}$$

Let us denote $\hat{Y}_k^r$, the MLE of the observable components of the state at time $k$ using the $2k+1$ first bearing measurements. It is computed by means of a Gauss-Newton algorithm since eq.(13) is non-linear. This is the approach of Nardone and Graham [13] to solve the BOT problem.

First, let us remark that an advantage of the MPC formalism is that the iterative optimization algorithm (Gauss-Newton) can be easily initialized itself by using the following linearization of eq.(13) [2]:

$$Z_t \approx Y_3(k) + Y_1(k)(t-k)\delta_t - Y_1(k)Y_2(k)(t-k)^2\delta_t^2 + V_t \ , \ \forall t \geq 0 \ . \tag{15}$$

Again, eq.(15) results in a non-linear regression problem. However, it can be (approximately) solved in a linear way by means of a "modified" linear estimate (denoted $\tilde{Y}_k^r$) defined by:

$$\tilde{Y}_k^r = \begin{pmatrix} \hat{\gamma}_2 \\ \frac{\hat{\gamma}_3}{\hat{\gamma}_2} \\ \hat{\gamma}_1 \end{pmatrix} \quad \text{with} \quad \hat{\gamma} = (\mathcal{X}^T\mathcal{X})^{-1}\mathcal{X}^T\mathcal{Z} \ , \tag{16}$$

where:

$$\mathcal{Z} = (Z_0, \ldots, Z_{2k})^T \quad \text{and} \quad \mathcal{X} = \begin{pmatrix} 1 & -k\delta_t & k^2\delta_t^2 \\ \vdots & \vdots & \vdots \\ 1 & k\delta_t & k^2\delta_t^2 \end{pmatrix} \ . \tag{17}$$

This estimator $\tilde{Y}_k^r$ is used for initializing the Gauss-Newton algorithm.

Moreover, using classical convergence results, we can define a confidence area noted $CA(\hat{Y}_k^r)$ for the MLE. Then the observable components of the particles can be initialized by sampling uniformly in $CA(\hat{Y}_k^r)$. Let us now precise the confidence area's formula. If we always assume that the trajectory of the target is not stochastic then the MLE is asymptotically optimal [3] which means that:

$$\sqrt{2k+1}(\hat{Y}_k^r - Y_k^r) \xrightarrow{\mathcal{D}} \mathcal{N}(0, J(Y_k^r)^{-1}) \ , \tag{18}$$

where $J(Y_k^r)$ is the Fisher information matrix defined by:

$$J(Y_k^r) = -E\{\Delta_{Y_k^r}^{Y_k^r} \ln(p_{Y_k^r}(Z_{0:2k}))\} \ . \tag{19}$$

In (19), $p_{Y_k^r}(Z_{0:2k})$ is the density function of $Z_{0:2k}$ conditionally to $Y_k^r$. In this case, the Fisher information matrix $J(Y_k^r)$ can be exactly computed (see Appendix B). From the above convergence result (18), the following result is infered:

$$(2k+1)\left\|\hat{Y}_k^r - Y_k^r\right\|^2_{J(\hat{Y}_k^r)^{-1}} \xrightarrow{\mathcal{D}} \mathcal{X}_3^2 \ , \tag{20}$$

where $\mathcal{X}_3^2$ is a chi-square distribution with 3 degrees of freedom. Actually, this is a non trivial result and we refer to Appendix C for details. Then, we are able to define the confidence area for $Y_k^r$, such that:

$$CA(\hat{Y}_k^r) = \left\{ Y_k^r \ \middle| \ \left\|\hat{Y}_k^r - Y_k^r\right\|^2_{J(\hat{Y}_k^r)^{-1}} \leq \frac{\mathcal{X}_3^2(1-\alpha)}{2k+1} \right\} \ , \tag{21}$$

[2]Notice the analogy between eq.(15) and the following Taylor approximation of $\beta(t)$:

$$\beta_t \approx \beta(k) + \dot{\beta}(k)(t-k)\delta_t + \frac{1}{2}\ddot{\beta}(k)(t-k)^2\delta_t^2 \ , \ \forall t \geq 0 \ . \tag{14}$$

[3]See [14] for more details about the convergence results of such estimator.

where $\mathcal{X}_3^2(\alpha)$ is the $(1-\alpha)$-quantile of the chi-squared distribution with 3 degrees of freedom.
Practically, $\alpha$ is classicaly fixed to 5 percent. It is worth stressing that $CA(\hat{Y}_k^r)$ is an hyperellipsoid. Then the initialization of the observable components of the state of the particles can be done using the algorithm proposed by Dezert and Musso in [15].

It remains finally to fix $Y_4(k)$ the fourth component of each of the particles altough it is unobservable. Let us remark that $Y_4(k)$ is the inverse of the range at time $k$ so an intuitive idea consists in giving to each particle a range value uniformly sampled between a minimum and a maximum relative target range noted $R_{min}$ and $R_{max}$.

## 3.2 Determining the batch initialization time?

It remains now to determine the batch duration (the parameter $k$ with our notations), sufficient for ensuring a good initialization of the particle filtering algorithm. Intuitively, the volume of $CA(\hat{Y}_k^r)$ decreases with the time $k$. If we associate to each of the particles a neighborhood such that the true state of the target is lying in (at least) one of these neighborhoods, then the problem of the choice of $k$ reverts to determining the batch duration which ensures that $N$ particles are sufficient to fill the confidence area.

For a given particle $(i)$, this neighborhood represents the capacity of the particle filter to tend toward the true state. In Cartesian coordinates, the state equation (2) results in:

$$\mathcal{B}(X_k^{(i)}) = \left\{ X_k \;\middle|\; \left\| X_k - AX_{k-1}^{(i)} \right\|_Q^2 \leq \mathcal{X}_4^2(1-\alpha) \right\} \; . \tag{22}$$

Let us denote $\mathcal{V}(\mathcal{B}(X_k^{(i)}))$ the volume of the neighborhood of the particle $(i)$. Obviously we can not compare the set of volumes associated with each of the particle $\{\mathcal{V}(\mathcal{B}(X_k^{(1)}), \dots, \mathcal{V}(\mathcal{B}(X_k^{(N)})\}$ and the volume of the confidence area noted $\mathcal{V}(CA(\hat{Y}_k^r))$. Then the idea consists on the first hand in determining (from $\mathcal{B}(X_k^{(i)})$) the neighborhood in RMPC noted $\mathcal{B}(Y_k^{(i)})$ *for each particle* and on the other hand to define the confidence area for $\hat{Y}_k$ using both $CA(\hat{Y}_k^r)$ and the prior information relative to $Y_4(k)$. Practically, this means that the particle filter can be initialized as soon as the following condition holds:

$$\mathcal{V}(CA(\hat{Y}_k))) \leq \sum_{i=1}^{N} \mathcal{V}(\mathcal{B}(Y_k^{(i)})) \; . \tag{23}$$

To that aim, let us derive, first, a formula for $\mathcal{V}(\mathcal{B}(Y_k^{(i)}))$. A first order approximation of (2), lead us to consider the following approximation for each particle:

$$Y_k^{(i)} \approx Af_{mp}^c(Y_{k-1}^{(i)}) + \nabla_Y f_{mp}^c(Y)\big|_{Y=Y_{k-1}^{(i)}} W_k \; . \tag{24}$$

This allows us to infer the neighborhood associated with a particle $(i)$:

$$\mathcal{B}(Y_k^{(i)}) = \left\{ Y_k \;\middle|\; \left\| Y_k - Af_{mp}^c(Y_{k-1}^{(i)}) \right\|^2_{\nabla_Y f_{mp}^c(Y) Q \nabla_Y^T f_{mp}^c(Y)\big|_{Y=Y_{k-1}^{(i)}}} \leq \mathcal{X}_4^2(1-\alpha) \right\} \; . \tag{25}$$

An elementary calculation of the Jacobian of the MPC to Cartesian coordinates yields:

$$\det\left( \nabla_Y f_{mp}^c(Y) \; Q \; \nabla_Y^t f_{mp}^c(Y)\big|_{Y=Y_{k-1}^{(i)}} \right) = \left( Y_4^{(i)}(k-1) \right)^{10} \quad ,$$

- k=3
- While $\mathcal{V}(CA(\hat{Y}_k)) > \sum_{i=1}^{N} \mathcal{V}(\mathcal{B}(Y_k^{(i)}))$

  1. Estimate $\hat{Y}_k^r$ using a Gauss-Newton iterative algorithm.
  2. Compute $\mathcal{V}(CA(\hat{Y}_k))$ using eq.(28).
  3. Compute $\mathcal{V}(\mathcal{B}(Y_k^{(i)}))$ using eq.(26) for $i = 1, \ldots, N$ .
  4. k=k+1 .

- Initialization of the particles

  1. Initialization of the observable components :

$$\begin{pmatrix} Y_1^{(i)}(k) \\ Y_2^{(i)}(k) \\ Y_3^{(i)}(k) \end{pmatrix} \sim \mathcal{U}(CA(\hat{Y}_k^r)) \text{ for } i = 1, \ldots, N \; . \tag{29}$$

  2. Initialization of the unobservable component :

$$Y_4^{(i)}(k) = \frac{1}{\mathcal{U}([R_{min}, R_{max}])} \text{ for } i = 1, \ldots, N \; . \tag{30}$$

Figure 1: Initialization of particle filtering algorithm in MPC.

then the volume of this hyperellipsoid is given by the following formula:

$$\mathcal{V}(\mathcal{B}(Y_k^{(i)})) = \frac{(\pi \mathcal{X}_4^2(1-\alpha))^2 (Y_4^{(i)}(k-1))^5}{\Gamma(3)} \; , \tag{26}$$

where $\Gamma(.)$ is the classical function. Thus, $\mathcal{V}(\mathcal{B}(Y_{[k/2]}^{(i)}))$ the volume of the neighborhood associated with a particle depends only of the range.

Now, we are paying more attention to $\mathcal{V}(CA(\hat{Y}_k))$. Remark that $\hat{Y}_k^r$ does not depend on the fourth component. Moreover, it has already been assumed that the range is lying between $R_{min}$ and $R_{max}$ so that we can make the following approximation:

$$\mathcal{V}(CA(\hat{Y}_k)) = (\frac{1}{R_{min}} - \frac{1}{R_{max}}) \mathcal{V}(CA(\hat{Y}_k^r)) \; , \tag{27}$$

and the confidence area of $CA(\hat{Y}_k^r)$ (21) is an hyperellipsoid defined by:

$$\mathcal{V}(CA(\hat{Y}_k)) = (\frac{1}{R_{min}} - \frac{1}{R_{max}}) \frac{(\pi \mathcal{X}_4^2(1-\alpha))^{3/2}}{\Gamma(5/2)\sqrt{det(J(\hat{Y}_k^r))}} \; . \tag{28}$$

The algorithm for particle filter algorithm initializion is summarized in figure 1.

### 3.3 Simulation results

Let us now illustrate the performance of initialization method. The following scenario is considered. The initial states of the observer and the target are:

$$X_0^{target} = \begin{pmatrix} 6 \text{ ms}^{-1} \\ 3 \text{ ms}^{-1} \\ 0 \text{ m} \\ 10000 \text{ m} \end{pmatrix}, X_0^{obs} = \begin{pmatrix} -10 \text{ ms}^{-1} \\ 0 \text{ ms}^{-1} \\ 10000 \text{ m} \\ 0 \text{ m} \end{pmatrix}. \tag{31}$$

The relative target state at initial time is then given by $X_0 = X_0^{target} - X_0^{obs}$. The observer follows a leg-by-leg trajectory. Its velocity vector is constant on each leg:

$$2100 \leq t \leq 3900 \begin{pmatrix} v_x^{obs}(t) \\ v_y^{obs}(t) \end{pmatrix} = \begin{pmatrix} 10 \text{ ms}^{-1} \\ 2 \text{ ms}^{-1} \end{pmatrix}, \quad 3900 \leq t \leq \text{end} \begin{pmatrix} v_x^{obs}(t) \\ v_y^{obs}(t) \end{pmatrix} = \begin{pmatrix} -10 \text{ ms}^{-1} \\ 2 \text{ ms}^{-1} \end{pmatrix}. \tag{32}$$

The elementary time period $\delta_t$ is 6 s. The standard deviation of the process noise is fixed to $0.05$ ms$^{-1}$ so that target trajectory strongly departs from a straight line. The standard deviation of the measurement noise is 0.05 rad (about 3 deg.). An example of trajectory is presented in figure 2, while a bearing measurement batch is presented in figure 3. A classical particle filtering algorithm is used with an adaptive resampling. The only difference between concerns the resampling stage. As long as the observer does not perform maneuver, the obervable components of the state of the particle is not correlated with the unobservable component. The resampling threshold is fixed to 0.5 and the number of particles is 10000. As concerns the initialization method, the **only** assumption we make is that target range varies from $R_{min} = 1000$ m to $R_{max} = 40000$ m. In Fig. 4 simulation results are presented. At the beginning of the scenario, the algorithm gives estimes only for the observable components of the state i.e. $\{Y_1(t), Y_2(t), Y_3(t)\}$ as solution of the non-linear regression problem (13). Of course, we do not have an estimate for $Y_4(t)$. At time 420, the "initialization condition" (23) turns true which means that we are able to initialized the particle filtering algorithm at time 210. From this time, the tracking algorithm estimates the full state of the target. The first (three) components of target state are correctly estimated form the beginning thanks to the initialization method. Otherwise, we can see in (d) that the confidence area related to the unobservable component $Y_4(t)$ is very high. At time 2100, the observer is maneuvering so this last component turns to be observable. The effect of this maneuver is a brutal decrease of the confidence area which is also illustrated *at the scenario level* is illustrated by Figure 5. **No prior** about target parameters has been used and the whole algorithm performs quite satisfactorily.

# 4 PCRB for BOT in the MPC setting

We turn now toward performance analysis of observable components of the estimated state vector using the PCRB formulation as given in (see [10]). However, it is recognized that in the BOT context this bound is over-optimistic.
We show that this result is due to the asymptotic unbiasness assumption which is not a sensible assumption in our

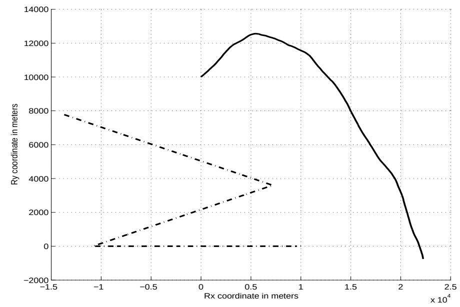



Figure 2: *trajectories of the observer (dashed line) and the target (solid line).*

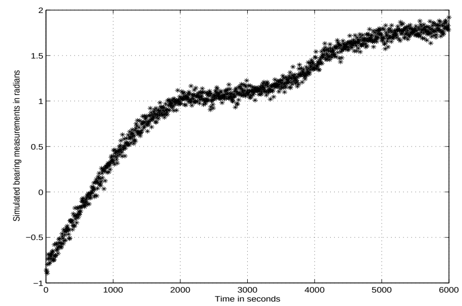



Figure 3: *Simulated bearing measurements.*

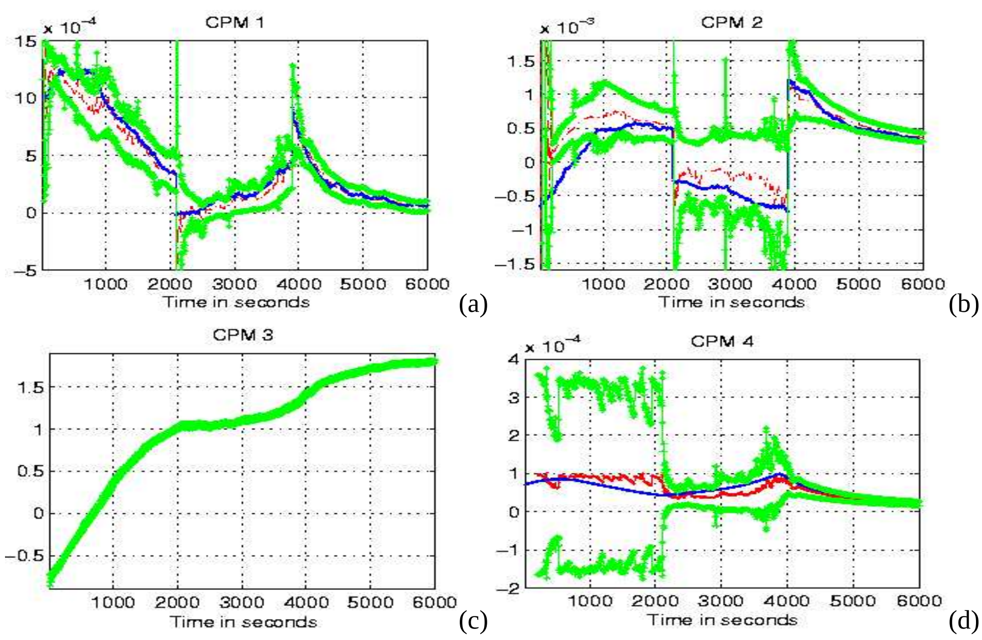



Figure 4: *Estimates for one particular run (dashed lines), $2\sigma$ confidence bounds area in grey. The solid line stands for the true values. (a): $Y_1(t)$, (b): $Y_2(t)$, (c): $Y_3(t)$, (d): $Y_4(t)$.*

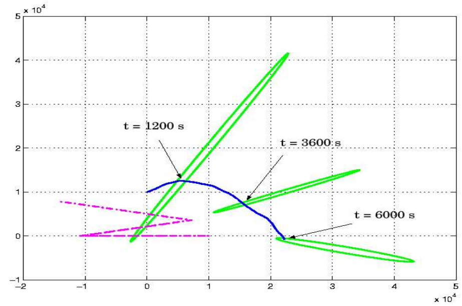



Figure 5: *$2\sigma$ confidence ellipse obstained for one particular run at time* $1200$, $3600$ *and* $6000$*s. The dashed line and the solid line stand for the observer and observer trajectories.*

case. Actually, an "asymptotic unbiasness assumption" has been implicitly made; which is generally not valid in our case. The remedy will be to incorporate the "bias" terms in a new version of the PCRB.

## 4.1 PCRB for state estimation

Let $Y_{0:t}$ and $Z_{0:t}$ be the trajectory and the set bearing measurements up to time t which are random vectors of size $n_y(t+1)$ and $(t+1)$, respectively. Let $g(Z_{0:t})$ be an estimator of $Y_{0:t}$. We focus here on the Error Covariance Matrix, denoted:

$$\text{ECM} = \mathbb{E}\left\{(g(Z_{0:t}) - Y_{0:t})(g(Z_{0:t}) - Y_{0:t})^T\right\} \ , \tag{33}$$

and the diagonal terms of the ECM matrix:

$$\begin{aligned} \text{ECM}_{i,j} &= \mathbb{E}\left\{(g_{i,j}(Z_{0:t}) - Y_i(j))^2\right\} \ , \\ &\quad \forall\, j \in \{0, \dots, t\}, \forall\, i \in \{1, \dots, n_y\} \ , \end{aligned} \tag{34}$$

where $g_{i,j}(Z_{0:t})$ is the estimator of $Y_i(j)$ (the coordinate $i$ of the target state at time $j$). $\text{ECM}_{i,j}$ is the mean square error related to the estimation of $Y_i(j)$. First, let us recall the Fisher Information Matrix (FIM) and bias definitions.

***Definition 1 (FIM)*** *Let $J(t)$ be the Fisher Information Matrix at time t:*

$$J(t) = \mathbb{E}\left\{-\Delta_{Y_{0:t}}^{Y_{0:t}} \ln p(Z_{0:t}, Y_{0:t})\right\} \ , \tag{35}$$

*where $p(Z_t, Y_t)$ is the probability distribution of $(Z_t, Y_t)$.*

***Definition 2 (bias)*** *Bias is defined as:*

$$B(Y_{0:t}) = \mathbb{E}\left\{\,(g(Z_{0:t}) - Y_{0:t})\,|Y_{0:t}\right\} \ . \tag{36}$$

*$B_{i,j}(Y_{0:t})$ is the bias related to the estimation of $Y_i(j)$ such that:*

$$B_{i,j}(Y_{0:t}) = \mathbb{E}\left\{g_{i,j}(Z_{0:t}) - Y_i(j)|Y_{0:t}\right\} \ . \tag{37}$$

*The estimator of the trajectory $g(Z_{0:t})$ is unbiased if $B(Y_{0:t})$ is almost surely equal to zero.*

This choice of the bias definition is justified in the Appendix C. The following theorem ensures that the FIM gives a lower bound for the mean square error under a specific assumption called “asymptotic unbiasness assumption”.

***Assumption 1 (Asymptotic unbiasness)*** *The asymptotic unbiasness assumption is defined as:*

$$\lim_{Y_i(j)\to\mathcal{Y}_i^+} B(Y_{0:t})p(Y_{0:t}) = \lim_{Y_i(j)\to\mathcal{Y}_i^-} B(Y_{0:t})p(Y_{0:t}) \ , \quad \forall j \in \{0,\dots,t\},\ \forall i \in \{1,\dots,n_y\} \ , \tag{38}$$

*where $\mathcal{Y}_i$ is the (connected) domain of $Y_i(j)$ for all $j$ in $\{0,\dots,t\}$, while $\left\{\mathcal{Y}_i^- \ , \ \mathcal{Y}_i^+\right\}$ are its bounds.*

***Proposition 1 (PCRB)*** *Under assumption 1,*

$$ECM \succcurlyeq J(t)^{-1} \ . \tag{39}$$

Now, let us give a proof of proposition 1. Two elementary Lemmas are needed and recalled below for the sake of completeness.

***Lemma 1***

$$J(t) = \mathbb{E}\left\{\nabla_{Y_{0:t}} \ln p(Z_{0:t}, Y_{0:t}) \nabla^T_{Y_{0:t}} \ln p(Z_{0:t}, Y_{0:t})\right\} \ . \tag{40}$$

***Lemma 2*** *Let $S$ be a symmetric matrix defined such that:*

$$S = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix} \ , \tag{41}$$

*with*

- *A non negative real symmetric matrix,*
- *B positive real symmetric matrix,*
- *C real matrix.*

*Then*

$$S \succcurlyeq 0 \textit{ implies } A - CB^{-1}C^T \succcurlyeq 0 \ .$$

**Proof of proposition 1** We build the $S$ matrix such that:

$$S = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix} \ , \tag{42}$$

where:

$$\begin{cases} A = \text{ECM} \ , \\ B = J(t) \ , \\ C = \mathbb{E}\left\{(g(Z_{0:t}) - Y_{0:t}) \nabla^T_{Y_{0:t}} \ln p(Z_{0:t}, Y_{0:t})\right\} \ . \end{cases} \tag{43}$$

From this definition, $C$ is non negative definite. Using lemmas 1 and 2, one remarks that we just have to prove that $C$ is equal to identity.

$$\begin{aligned} C &= \mathbb{E}\left\{(g(Z_{0:t}) - Y_{0:t})\nabla^T_{Y_{0:t}} \ln p(Z_{0:t}, Y_{0:t})\right\} \\ &= \int (g(Z_{0:t}) - Y_{0:t})\nabla^T_{Y_{0:t}} p(Z_{0:t}, Y_{0:t}) d(Y_{0:t}, Z_{0:t}) \ . \end{aligned} \tag{44}$$

C is a $n_y(t+1) \times n_y(t+1)$ matrix. Integrating by parts, an element of C can be rewritten:

$$\begin{aligned} &C(\{i,j\},\{i',j'\}) = \mathbb{1}_{\{i,j\}=\{i',j'\}} + \\ &\int \left[(g_{i,j}(Z_{0:t}) - Y_i(j))p(Z_{0:t}, Y_{0:t})\right]^{\mathcal{Y}^+_{i'}}_{\mathcal{Y}^-_{i'}} d(Y_{0:t}^{-\{i',j'\}}, Z_{0:t}) \ , \end{aligned} \tag{45}$$

where $Y_{0:t}^{-\{i,j\}}$ is a whole target trajectory, the term $Y_i(j)$ excepted. Now, if limit and integral operators can be interverted, we have:

$$\begin{aligned} &C(\{i,j\},\{i',j'\}) = \mathbb{1}_{\{i,j\}=\{i',j'\}} + \\ &\int \left[\int (g_{i,j}(Z_{0:t}) - Y_i(j))p(Z_{0:t}, Y_{0:t}) dZ_{0:t}\right]^{\mathcal{Y}^+_{i'}}_{\mathcal{Y}^-_{i'}} dY_{0:t}^{-\{i',j'\}} \ . \end{aligned} \tag{46}$$

Using bias notation previously introduced, we obtain finally:

$$\begin{aligned} C(\{i,j\},\{i',j'\}) &= \mathbb{1}_{\{i,j\}=\{i',j'\}} + \\ &\int \left[B_{i,j}(Y_{0:t})p(Y_{0:t})\right]^{\mathcal{Y}^+_{i'}}_{\mathcal{Y}^-_{i'}} dY_{0:t}^{-\{i',j'\}} \ . \end{aligned} \tag{47}$$

Thus, under assumption 1, C is the identity matrix.

□

However, objections can be expressed regarding theorem 1. While recognizing that the asymptotic bias assumption is very convenient in that PCRB is simple to compute and independent of the choice of the estimator $g(Z_{0:t})$, one must however admit that this assumption is not always sensible. Even if unbiasness is a desirable property, such assumption may strongly departs from reality.

Otherwise, we may assume that $p(Y_{0:t})$ converges to zero at the endpoints of $Y_i(j)$ for all $i$ in $\{1, \ldots, n_y\}$ and $j$ in $\{0, \ldots, t\}$. This assumption may be hardly fulfilled. In our specific context, let us remark that $\mathcal{Y}^+_3 = \frac{\pi}{2}$ and $\mathcal{Y}^-_3 = -\frac{\pi}{2}$, so that using (90) in Appendix D, we can see that:

$$\lim_{Y_3(t)\to\mathcal{Y}^+_3} p(Y_{t+1}|Y_t) \neq \lim_{Y_3(t)\to\mathcal{Y}^-_3} p(Y_{t+1}|Y_t) \ , \tag{48}$$

$$\lim_{Y_3(t+1)\to\mathcal{Y}^+_3} p(Y_{t+1}|Y_t) \neq \lim_{Y_3(t+1)\to .\mathcal{Y}^-_3} p(Y_{t+1}|Y_t) \ . \tag{49}$$

Then, the $C(t)$ matrix cannot be considered equal to the identity matrix. Thus, to remedy drawbacks, we propose a more general theorem:

***Proposition 2 (PCRB)***

$$\mathbb{E}\left\{(g(Z_{0:t})-Y_{0:t})(g(Z_{0:t})-Y_{0:t})^T\right\} \succcurlyeq C(t)J(t)^{-1}C(t)^T \ , \tag{50}$$

*where:*

$$C(t) \quad = \quad \mathbb{E}\left\{(g(Z_{0:t})-Y_{0:t})\nabla^T_{Y_{0:t}} \ln p(Z_{0:t},Y_{0:t})\right\} \ . \tag{51}$$

Now remark that it is not necessary to compute $J^{-1}(t)$. We only have to estimate the two matrix $C(t)$ and $J(t)$ and to solve a linear system to approximate the PCRB. To estime $J(t)$, the key idea is to use the following relation:

$$\ln p(Z_{0:t},Y_{0:t}) = \ln p(Z_t|Y_t) + \ln p(Y_t|Y_{t-1}) + \ln p(Z_{0:t-1},Y_{0:t-1}) \ , \tag{52}$$

which is true under two assumptions. First, the measurement at time $t$ depends only from the target state at time $t$. Second, $\{Y_t\}_{t\in\mathbb{N}+}$ is a Markovian process. These two assumptions are easily deduced from the formulation of the BOT problem (11). We deduce from (52) that $J(t)$ and $J(t+1)$ can be rewritten:

$$J(t) = \begin{pmatrix} R_t & S_t \\ S_t^T & T_t \end{pmatrix} \ , \tag{53}$$

where:

$$\begin{cases} R_t &= \mathbb{E}\left\{\nabla_{Y_{0:t-1}} \ln p(Z_{0:t},Y_{0:t})\nabla^T_{Y_{0:t-1}} \ln p(Z_{0:t},Y_{0:t})\right\} \ , \\ S_t &= \mathbb{E}\left\{\nabla_{Y_t} \ln p(Z_{0:t},Y_{0:t})\nabla^T_{Y_{0:t-1}} \ln p(Z_{0:t},Y_{0:t})\right\} \ , \\ T_t &= \mathbb{E}\left\{\nabla_{Y_t} \ln p(Z_{0:t},Y_{0:t})\nabla^T_{Y_t} \ln p(Z_{0:t},Y_{0:t})\right\} \ , \end{cases} \tag{54}$$

and

$$J(t+1) = \begin{pmatrix} R_t & S_t & 0 \\ S_t^T & T_t + D_t^{11} & D_t^{12} \\ 0 & D_t^{21} & D_t^{22} \end{pmatrix} \ , \tag{55}$$

where:

$$\begin{cases} D_t^{11} &= E\{\nabla_{Y_t} \ln p(Y_{t+1}|Y_t)\nabla^T_{Y_t} \ln p(Y_{t+1}|Y_t)\}\} \ , \\ D_t^{21} &= E\{\nabla_{Y_{t+1}} \ln p(Y_{t+1}|Y_t)\nabla^T_{Y_t} \ln p(Y_{t+1}|Y_t)\} \ , \\ D_t^{12} &= E\{\nabla_{Y_t} \ln p(Y_{t+1}|Y_t)\nabla^T_{Y_{t+1}} \ln p(Y_{t+1}|Y_t)\} \ , \\ D_t^{22} &= E\{\nabla_{Y_{t+1}} \ln p(Y_{t+1}|Y_t)\nabla^T_{Y_{t+1}} \ln p(Y_{t+1}|Y_t)\} \\ &+ \ E\{\nabla_{Y_{t+1}} \ln p(Z_{t+1}|Y_{t+1})\nabla^T_{Y_{t+1}} \ln p(Z_{t+1}|Y_{t+1})\} \ . \end{cases} \tag{56}$$

Then $J(t+1)$ can be estimated using $J(t)$ and approximations of $D_t^{11}$, $D_t^{21}$, $D_t^{12}$ and $D_t^{22}$. Let us remark that we are able to approximate $D_t^{11}$, $D_t^{21}$, $D_t^{12}$ and $D_t^{22}$ using Monte Carlo methods because we have analytic formulas for $p(Z_t|Y_t)$ and $p(Y_{t+1}|Y_t)$ which are given in Appendix C. Otherwise, $C(t)$ can be rewritten using (52) again, i.e. :

$$C(t) = \mathbb{E}\left\{(g(Z_{0:t})-Y_{0:t})\nabla^T_{Y_{0:t}}\left\{\sum_{k=0}^{t} \ln p(Z_k|Y_k) + \sum_{k=1}^{t} \ln p(Y_k|Y_{k-1})\right\}\right\} \ . \tag{57}$$

In the same way, $C(t)$ can be approximated using Monte Carlo methods.

## 4.2 Simulation Results

Based on the previous section, we illustrate the accuracy of the two bounds through the scenario previously introduced in section 3. The estimated mean square error (computed with 10 target trajectories, while for each trajectory 10 set of measurements are simulated) is compared with the lower bounds (the "classical PCRB" and the "modified PCRB" with C estimated) on figure 6. As expected, for all the components of the target state the mean square error is upper than the two bounds.

We can see that the two lower bounds have a different evolution particularly when the observer is maneuvering. This confirms that the asymptotic unbiasness assumption is dubious, *in this context*. Clearly, the classical PCRB is over optimistic, while the modified PCRB appears to be tight and reliable. However, we can express a last objection about this two lower bounds. Notice that the component $Y_4(t)$ is unobservable as long as the observer does not maneuver, thus it is likely that $J(t)$ be an ill-conditioned matrix. Then the problem is to derive a PCRB, for the **observable-only** components of the target state. This will be the aim of the next section.

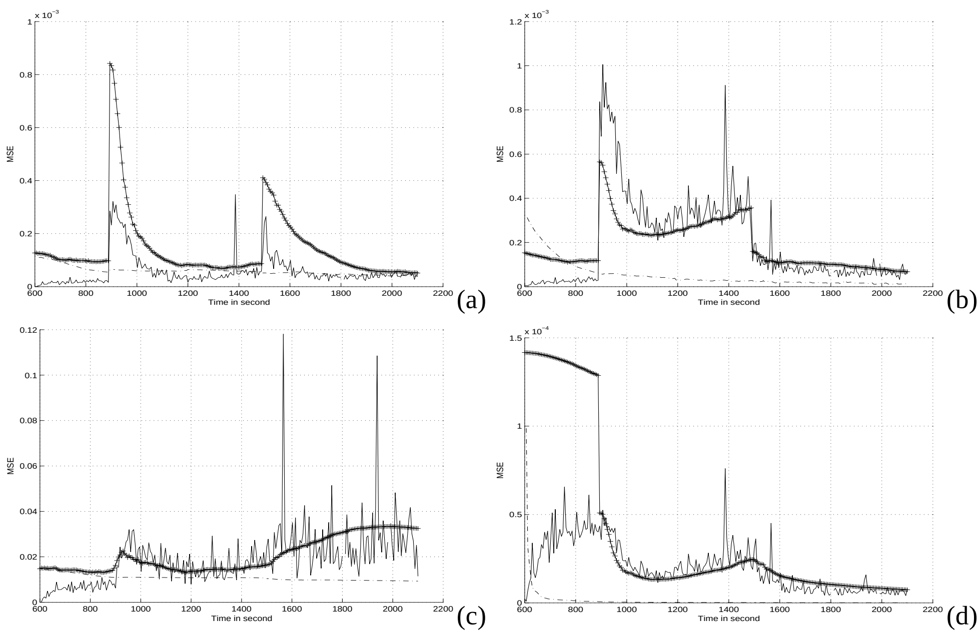



Figure 6: *Comparison between classical PCRB (dashed line), PCRB with C estimated (solid line) and mean square error (cross line) for (a) $Y_1(t)$, (b) $Y_2(t)$, (c) $Y_3(t)$, (d) $Y_4(t)$.*

# 5 PCRB for RMP coordinates system

We focus here on the computation of the PCRB for the partial target state, i.e. the observable components of the target state in MP coordinates system named RMP.

### 5.1 PCRB and Conditional FIM

Let $g^r(Z_{0:t})$ be an estimator of $Y^r_{0:t}$. Let us define the Error Covariance Matrix for RMP:

$$\mathrm{ECM}^r = \mathbb{E}\left\{(g^r(Z_{0:t}) - Y^r_{0:t})(g^r(Z_{0:t}) - Y^r_{0:t})^T\right\} \ , \tag{58}$$

and the diagonal terms of ECM:

$$\begin{aligned}\mathrm{ECM}^r_{i,j} &= \mathbb{E}\left\{(g^r_{i,j}(Z_{0:t}) - Y_i(j))^2\right\}, \\ &\forall i \in \{1,\dots,n_{y^r}\}, \forall\, j \in \{0,\dots,t\} \ .\end{aligned} \tag{59}$$

The term $\mathrm{ECM}^r_{i,j}$ is the mean square error related to the estimation of $Y_i(j)$. We have shown in section 2 that BOT problem belongs to the class of non linear filtering problem with unknown state covariance. Considering system (11), we remark that $\{Y^r_t\}_{t\in\mathbb{N}^+}$ is not a Markovian process. So, derivation of the PCRB theory have to be thoroughly changed. Let us first define the Conditional Error Covariance Matrix for the RMP:

$$\mathrm{CECM}^r = \mathbb{E}_{Y_4(0)}\left\{(g^r(Z_{0:t}) - Y^r_{0:t})(g^r(Z_{0:t}) - Y^r_{0:t})^T\right\} \ , \tag{60}$$

where $Y_4(0)$ is the initial unobservable component of the target state. In the same way, we can define the Fisher Information Matrix given $Y_4(0)$:

***Definition 3 (Conditional FIM)*** *Let $J_{Y_4(0)}(t)$ be the FIM given $Y_4(0)$ at time t:*

$$J_{Y_4(0)}(t) = \mathbb{E}_{Y_4(0)}\left\{-\Delta^{Y^r_{0:t}}_{Y^r_{0:t}} \ln p_{Y_4(0)}(Z_{0:t}, Y^r_{0:t})\right\} \ , \tag{61}$$

*where $p_{Y_4(0)}(Z_t, Y^r_t)$ is the probability distribution of $(Z_t, Y^r_t)$ given $Y_4(0)$.*

***Definition 4 (Conditional bias)*** *Bias given $Y_4(0)$ is defined as:*

$$B_{Y_4(0)}(Y^r_{0:t}) = \mathbb{E}_{Y_4(0)}\left\{g^r(Z_{0:t}) - Y^r_{0:t} | Y^r_{0:t}\right\} \ . \tag{62}$$

***Assumption 2 (Conditional Asymptotic unbiasness)*** *The conditional asymptotic unbiasness assumption is defined as:*

$$\lim_{Y_i(j)\to\mathcal{Y}_i^+} B_{Y_4(0)}(Y^r_{0:t})p(Y^r_{0:t}) = \lim_{Y_i(j)\to\mathcal{Y}_i^-} B_{Y_4(0)}(Y_{0:t})p(Y^r_{0:t}) \ , \tag{63}$$

$$\forall j \in \{0,\dots,t\}, \forall i \in \{1,\dots,n_y\} \ , \tag{64}$$

*where $\mathcal{Y}_i$ is the domain of $Y_i(j)$ for all $j$ in $\{0,\dots,t\}$. Moreover, $\mathcal{Y}_i^-$ and $\mathcal{Y}_i^+$ are the endpoints of $\mathcal{Y}_i$.*

***Proposition 3 (PCRB)*** *Under assumption 2,*

$$CECM^r \succcurlyeq J^{-1}_{Y_4(0)}(t) \ . \tag{65}$$

The proof of proposition 3 is identical to the proof of proposition 1. We deduce from this theorem that:

$$ECM^r \succcurlyeq \int J_{Y_4(0)}^{-1}(t) p(Y_4(0)) dY_4(0) \ . \tag{66}$$

Then, if we are able to compute $J_{Y_4(0)}(t)$ at each step of time, a lower bound is given by:

$$ECM^r \succcurlyeq \frac{1}{n_{Y_4(0)}} \sum_{n=1}^{n_{Y_4(0)}} J_{Y_4^n(0)}(t)^{-1} \ , \tag{67}$$

where $Y_4^n(0)$ is a sample from $p(Y_4(0))$ for all $n$ in $\{1, \dots, n_{Y_4(0)}\}$. Again, the following result holds :

**Proposition 4 (PCRB)**

$$CECM^r \succcurlyeq C_{Y_4(0)}(t) J_{Y_4(0)}^{-1}(t) C_{Y_4(0)}(t)^T \ , \tag{68}$$

*where:*

$$C_{Y_4(0)}(t) \quad = \quad \mathbb{E}_{Y_4(0)} \Big\{ (g^r(Z_{0:t}) - Y_{0:t}^r) \nabla_{Y_{0:t}^r}^T \ln p_{Y_4(0)}(Z_{0:t}, Y_{0:t}^r) \Big\} \ . \tag{69}$$

In this case, we have:

$$ECM^r \succcurlyeq \int C_{Y_4(0)}(t) J_{Y_4(0)}^{-1}(t) C_{Y_4(0)}^T(t) p(Y_4(0)) dY_4(0) \ . \tag{70}$$

If we are able to compute $J_{Y_4(0)}(t)$ and $C_{Y_4(0)}(t)$ at each step of time, then a lower bound is given by:

$$ECM^r \succcurlyeq \frac{1}{n_{Y_4(0)}} \sum_{n=1}^{n_{Y_4(0)}} C_{Y_4(0)}(t) J_{Y_4^n(0)}(t)^{-1} C_{Y_4(0)}^T(t) \ , \tag{71}$$

where $Y_4^n(0)$ is sampled from $p(Y_4(0))$ for each value of $n$ within $\{1, \dots, n_{Y_4(0)}\}$. Consequently, we can construct the PCRB with or without the asymptotic unbiasness assumption if we can estimate the conditional FIM and $C_{Y_4(0)}(t)$. Without any assumption we have:

$$J_{Y_4(0)}(t+1) = \begin{pmatrix} J_{Y_4(0)}(t) + D_t^{11} & D_t^{12} \\ D_t^{21} & D_t^{22} \end{pmatrix} \ , \tag{72}$$

where:

$$\left\{ \begin{array}{lcl} D_t^{11} & = & \mathbb{E}_{Y_4(0)} \left\{ \nabla_{Y_{0:t}^r} \ln p_{Y_4(0)}(Y_{t+1}^r | Y_{0:t}^r) \ \nabla_{Y_{0:t}^r}^T \ln p_{Y_4(0)}(Y_{t+1}^r | Y_{0:t}^r) \} \right\}, \\ D_t^{21} & = & \mathbb{E}_{Y_4(0)} \left\{ \nabla_{Y_{t+1}^r} \ln p_{Y_4(0)}(Y_{t+1}^r | Y_{0:t}^r) \ \nabla_{Y_{0:t}^r}^T \ln p_{Y_4(0)}(Y_{t+1}^r | Y_{0:t}^r) \right\}, \\ D_t^{12} & = & \mathbb{E}_{Y_4(0)} \left\{ \nabla_{Y_{t+1}^r} \ln p_{Y_4(0)}(Y_{t+1}^r | Y_{0:t}^r) \ \nabla_{Y_t^r}^T \ln p_{Y_4(0)}(Y_{t+1}^r | Y_{0:t}^r) \right\}, \\ D_t^{22} & = & \mathbb{E}_{Y_4(0)} \left\{ \nabla_{Y_{t+1}^r} \ln p_{Y_4(0)}(Y_{t+1}^r | Y_{0:t}^r) \ \nabla_{Y_{t+1}^r}^T \ln p_{Y_4(0)}(Y_{t+1}^r | Y_{0:t}^r) \right\} \ , \\ & & + \mathbb{E}_{Y_4(0)} \left\{ \nabla_{Y_{t+1}^r} \ln p(Z_{t+1} | Y_{t+1}^r) \nabla_{Y_{t+1}^r}^T \ln p(Z_{t+1} | Y_{t+1}^r) \right\} \end{array} \right. \tag{73}$$

and

$$\begin{aligned} C_{Y_4(0)}(t) \quad &= \quad \mathbb{E}_{Y_4(0)} \left[ (g^r(Z_{0:t}) - Y_{0:t}^r) \ \nabla_{Y_{0:t}^r}^T \left\{ \sum_{k=0}^{t} \ln p(Z_k | Y_k^r) \right\} \right] + \\ &\quad + \quad \mathbb{E} \left[ (g^r(Z_{0:t}) - Y_{0:t}^r) \nabla_{Y_{0:t}^r}^T \left\{ \sum_{k=1}^{t} \ln p_{Y_4(0)}(Y_k^r | Y_{0:k-1}^r) \right\} \right] \ . \end{aligned} \tag{74}$$

At this point, an analytic formulation for $\ln p_{Y_4(0)}(Y^r_{t+1}|Y^r_{0:t})$ is required for performing Monte Carlo evaluation of the above quantities. To that aim, an approximated formulation of the BOT problem (11) is intrumental. We propose the two following approximations:

**Approximation 1** *$Y_4(t)$ given target state at time $t-1$ has a deterministic evolution i.e. :*

$$Y_4(t) \approx \tilde{G}(Y_{t-1}) \;, \tag{75}$$

*where:*

$$\tilde{G}(Y_{t-1}) = \frac{Y_4(t-1)}{(\delta_t Y_1(t-1))^2 + (1+\delta_t Y_2(t-1))^2} \;. \tag{76}$$

*More specifically, this is the approximation of (11b) by (12b).*

**Approximation 2** *$\{Y_4(t)\}_{t\in\mathbb{N}^+}$ is constant i.e.:*

$$Y_4(t) \approx Y_4(0) \;. \tag{77}$$

We have seen in section 2 that $Y_4(t)$ appears as a covariance term in the system (11), so that we can assume that these approximations make sense.

The crucial point is to be able to reconstruct $\{Y_4(k)\}_{0\le k\le t}$ in a deterministic way given $Y_4(0)$ and $Y^r_{0:t}$. With the first approximation, we obtain:

$$p_{Y_4(0)}(Y^r_{t+1}|Y^r_{0:t}) \approx \; p(Y^r_{t+1}|Y_t)\,\mathbb{1}_{\{Y_4(k)=\tilde{G}(Y(k-1)),\forall k\in\{1,\dots,t\}\}} \;, \tag{78}$$

and with the second one:

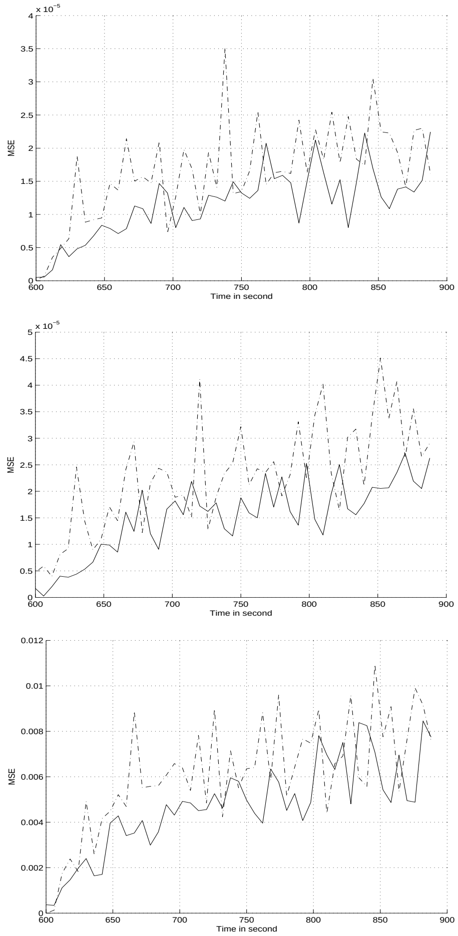
$$p_{Y_4(0)}(Y^r_{t+1}|Y^r_{0:t}) \approx \; p(Y^r_{t+1}|Y_t)\,\mathbb{1}_{\{Y_4(k)=Y_4(0),\forall k\in\{1,\dots,t\}\}} \;. \tag{79}$$

In the second case, we can notice that $\{Y^r(t)\}_{t\in\mathbb{N}^+}$ becomes a Markovian process. Furthermore, an analytic formula for $p(Y^r_{t+1}|Y_t)$ is given in appendix D such that we have now an analytic formulation for $p_{Y_4(0)}(Y^r_{t+1}|Y^r_{0:t})$. Consequently, we can estimate the conditional FIM and $C_{Y_4(0)}(t)$.

## 5.2 Simulation results

We illustrate the PCRB for the partial target state using the scenario previously introduced before the observer has done any maneuver. $C_{Y_4(0)}(t)$ and $J_{Y_4(0)}(t)$ are computed via Monte Carlo methods and sampling at each step of time of 10 target trajectories, while for each trajectory 10 set of measurements are simulated. $n_{Y_4(0)}$ is taken equal to 5 and the initial probability distribution for the fourth initial component is $Y_4(0) \sim \frac{1}{\mathcal{U}([R_{min},R_{max}])}$.

We compare the PCRB with an estimated C matrix (see section 4) with the PCRB for RMP with approximations 1 and 2 (see section 5) in figures 7 and 8 respectively. In the two cases, we can see that the lower bound are close. This is a bit surprising and means that we can directly use the PCRB computed for the full target state.




(a)

(b)

(c)

Figure 7: *Comparison between PCRB with C estimated (dashed line) and PCRB with C estimated and assumption 1 (solid line) for (a)* $Y_1(t)$, *(b)* $Y_2(t)$, *(c)* $Y_3(t)$.

# 6 Conclusion

We have proposed an original initialization method for the particle filtering algorithm in the context of the BOT problem. No prior about target parameters is used in this method and the whole algorithm performs quite satisfactory. Moreover, we have shown that the asymptotic unbiasness assumption does not make sense in the BOT context and renders the "classical" PCRB over optimistic. Relaxing this assumption, a “realistic” lower bound for the estimation of target state in MP coordinates system has been derived and subsequently extended to the estimation of the observable part of the state vector. Future developments include the study of the speed of convergence toward the bound.

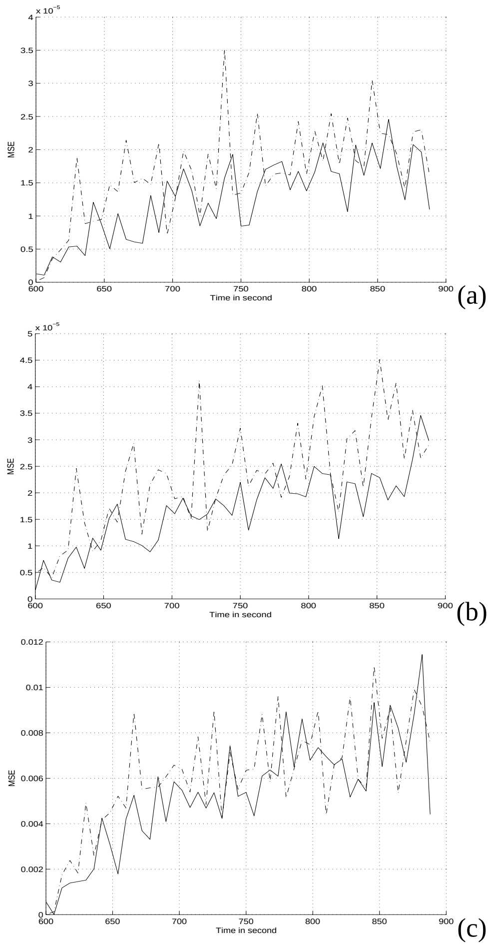



Figure 8: *Comparison between PCRB with C estimated (dashed line) and PCRB with C estimated and assumption 2 (solid line) for (a) $Y_1(t)$, (b) $Y_2(t)$, (c) $Y_3(t)$.*

# APPENDIX

## Appendix A : MP coordinate system.

This brief review of MP coordinate system is based on [8]. MP to cartesian state mapping function is given by

$$\begin{aligned} X(t) &= f^c_{mp}(Y(t)) \\ &= \frac{1}{Y_4(t)} \begin{pmatrix} Y_2(t)\sin(Y_3(t)) + Y_1(t)\cos(Y_3(t)) \\ Y_2(t)\cos(Y_3(t)) - Y_1(t)\sin(Y_3(t)) \\ \sin(Y_3(t)) \\ \cos(Y_3(t)) \end{pmatrix} . \end{aligned} \tag{80}$$

Cartesian to MP state mapping function is given by

$$\begin{aligned} Y(t) &= f_c^{mp}(X(t)) \\ &= \begin{pmatrix} \frac{X_1(t)X_4(t)-X_2(t)X_3(t)}{X_3^2(t)+X_4^2(t)} \\ \frac{X_1(t)X_3(t)+X_2(t)X_4(t)}{X_3^2(t)+X_4^2(t)} \\ \tan^{-1}\left(\frac{X_3(t)}{X_4(t)}\right) \\ \frac{1}{\sqrt{X_3^2(t)+X_4^2(t)}} \end{pmatrix} . \end{aligned} \tag{81}$$

## Appendix B: Fisher information matrix $J(Y_k^r)$ in the deterministic case.

The object of this section is to give a more precise statement of the Fischer Information Matrix for the observable component of the state at time $k$, when the target follows a **deterministic** trajectory using the $2k$ first bearing measurements. Let us remind the definition (19) of the Fisher information matrix in this case:

$$J(Y_k^r) = -E\{\Delta_{Y_k^r}^{Y_k^r} \ln(p_{Y_k^r}(Z_{0:2k}))\} . \tag{82}$$

Assuming independent observation, we have $p_{Y_k^r}(Z_{0:2k}) = \prod_{t=0}^{2k} p_{Y_k^r}(Z_t)$ and using (13), we obtain:

$$p_{Y_k^r}(Z_{0:2k}) = \prod_{t=0}^{2k} \frac{1}{\sqrt{2\pi}\sigma_v} \, \exp\left[-\frac{1}{2\sigma_v^2}\left(Z_t - Y_3(k) - \tan^{-1}\left(\frac{(t-k)Y_1(k)}{1+(t-k)Y_2(k)}\right)\right)^2\right] . \tag{83}$$

Substituting (83) in (19) yields:

$$\begin{aligned} &J(Y_k^r) = \tfrac{1}{\sigma_v^2} \textstyle\sum_{t=0}^{t=2k} \; G(k)\, G(k)^t \;, \\ &G(k)^t = \begin{pmatrix} \frac{(t-k)(1+(t-k)Y_2(k))}{(1+(t-k)Y_2(k))^2+((t-k)Y_1(k))^2} & -\frac{(t-k)^2 Y_1(k))}{(1+(t-k)Y_2(k))^2+((t-k)Y_1(k))^2} & 1 \end{pmatrix} . \end{aligned} \tag{84}$$

Consequently, the Fisher information matrix in this case can be computed exactly.

## Appendix B: Convergence of $(2k+1)\|\hat{Y}_k^r - Y_k^r\|^2_{J(\hat{Y}_k^r)^{-1}}$.

Since $\hat{Y}_k^r$ is the MLE, it converges in probability[4] to $Y^r_{[K/2]}$. Then by the Delta Method, $J(\hat{Y}_k^r)$ converges in probability to $J(Y_k^r)$. Moreover, let us remind that $(2k+1)\|\hat{Y}_k^r - Y_k^r\|^2_{J(Y_k^r)^{-1}}$ converges in law (see result (20)). Consequently using Slutsky's lemma, we have the following result:

$$(2k+1)\|\hat{Y}_k^r - Y_k^r\|^2_{J(\hat{Y}_k^r)^{-1}} \xrightarrow{\mathcal{D}} \mathcal{X}_3^2 . \tag{85}$$

## Appendix C : About the bias

Our bias definition may appear surprising at first. A more natural definition could be $\mathbb{E}\{g(Z_{0:t}) - Y_{0:t}\}$. This is this point that we are now going to enlighten through a decomposition of the mean square error related to the estimation of

[4] See [14] for more details about the convergence results of such estimator.

$Y_i(j)$. When estimating a deterministic parameter, the mean square error can be classically decomposed in estimation variance and bias. However, in stochastic case, we only have the following relation:

$$\begin{aligned} ECM_{i,j} &= \mathbb{E}\left\{(g_{i,j}(Z_{0:t}) - \mathbb{E}\{g_{i,j}(Z_{0:t})|Y_{0:t}\})^2\right\} + \\ & \mathbb{E}\left\{(\mathbb{E}\{g_{i,j}(Z_{0:t})|Y_{0:t}\} - Y_i(j))^2\right\} \ . \end{aligned} \tag{86}$$

The mean square error is then equal to the covariance estimation error if and only if

$$\mathbb{E}\left\{(\mathbb{E}\{g_{i,j}(Z_{0:t})|Y_{0:t}\} - Y_i(j))^2\right\} = 0 \ . \tag{87}$$

Assumption (87) is equivalent to:

$$\mathbb{E}\left\{g_{i,j}(Z_{0:t}) - Y_i(j)|Y_{0:t}\right\} = 0, \text{ for almost } Y_{0:t} \ . \tag{88}$$

which is the retained definition of an unbiased estimator.

## Appendix D: Analytic formulas for $p(Y_{t+1}|Y_t)$ and $p(Z_t|Y_t)$

We express here $p(Y_{t+1}|Y_t)$. Remind that (2) implies that:

$$p(X_{t+1}|X_t) = \frac{1}{(2\pi)^2\sqrt{det(Q)}} \exp\left(-\frac{1}{2}\|X_{t+1} - AX_t - HU_t\|_Q^2\right) \ , \tag{89}$$

using the change of variable theorem, we obtain:

$$\begin{aligned} p(Y_{t+1}|Y_t) = & \frac{1}{(2\pi)^2\sqrt{det(Q)}y_4^5(t+1)} \times \\ & \exp\left(-\frac{1}{2}\|f^c_{mp}(Y_{t+1}) - Af^c_{mp}(Y_t) - HU_t\|_Q^2\right) \mathbb{1}_{\mathcal{Y}\times\mathcal{Y}}(Y_{t+1}, Y_t) \ , \end{aligned} \tag{90}$$

where $\mathcal{Y} = \prod_{i=1}^{n_y} \mathcal{Y}_i$. In our case, the size of the target state $n_y$ is 4 and

$$\left| \begin{array}{lcl} \mathcal{Y}_1 &=& \mathbb{R} \ , \\ \mathcal{Y}_2 &=& \mathbb{R} \ , \\ \mathcal{Y}_3 &=& ]-\frac{\pi}{2}; \frac{\pi}{2}[ \ , \\ \mathcal{Y}_4 &=& ]0, +\infty[ \end{array} \right. \ . \tag{91}$$

Otherwise, it is trivial that:

$$p(Z_t|Y_t) = \frac{1}{\sqrt{2\pi}\sigma_v} e^{-\frac{(Z_t - Y_3(t))^2}{2\sigma_v^2}} \ . \tag{92}$$

## Appendix E: Analytic formula for $p(Y^r_{t+1}|Y_t)$

First, keep in mind the classical result:

$$p(Y^r_{t+1}|Y_t) = \int_{]0,+\infty[} p(Y_{t+1}|Y_t) dY_4(t+1) \ , \tag{93}$$

using (90), we then obtain:

$$p(Y^r_{t+1}|Y_t) = \frac{e^{-\frac{c}{2}}}{(2\pi)^2\sqrt{det(Q)}} \int_{]0,+\infty[} \frac{1}{Y_4^5(t+1)} e^{-\frac{a}{2Y_4^2(t+1)}+\frac{b}{Y_4(t+1)}} dY_4(t+1) \ , \tag{94}$$

where:

$$\left\{ \begin{array}{lcl} a & = & \|f(Y^r_{t+1})\|_Q \ , \\ b & = & f^T(Y^r_{t+1})Q^{-1}(Af^c_{mp}(Y^r_t) - HU_t) \ , \\ c & = & \|Af^c_{mp}(Y^r_t - HU_t\|_Q \ , \end{array} \right. \tag{95}$$

and

$$f(Y^r(t)) = \left( \begin{array}{c} Y_2(t)\sin(Y_3(t)) + Y_1(t)\cos(Y_3(t)) \\ Y_2(t)\cos(Y_3(t)) - Y_1(t)\sin(Y_3(t)) \\ \sin(Y_3(t)) \\ \cos(Y_3(t)) \end{array} \right) . \tag{96}$$

We deduce by changing the variable

$$p(Y^r_{t+1}|Y_t) = \frac{e^{-\frac{1}{2}(c-\frac{b^2}{a})}}{(2a\pi)^2\sqrt{det(Q)}} \int_{]0,+\infty[} x^3 e^{-\frac{1}{2}(x-\frac{b}{\sqrt{a}})^2} dx \ . \tag{97}$$

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Unité de recherche INRIA Rennes
IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex (France)

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