Strong Bi-homogeneous Bézout's Theorem and degree bounds for algebraic optimization

Mohab Safey El Din, Philippe Trebuchet

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Strong Bi-homogeneous Bézout's Theorem and degree bounds for algebraic optimization

Mohab Safey El Din — Philippe Trébuchet

N° 5071

Janvier 2004

THÈME 2

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--5071--FR+ENG

Strong Bi-homogeneous Bézout's Theorem and degree bounds for algebraic optimization

Mohab Safey El Din , Philippe Trébuchet

Thème 2 — Génie logiciel
et calcul symbolique

Projet SPACES

Rapport de recherche n° 5071 — Janvier 2004 — 21 pages

Abstract: Let $(f_1, \ldots, f_s)$ be a polynomial family in $\mathbb{Q}[X_1, \ldots, X_n]$ (with $s \leq n-1$) of degree bounded by D, generating a radical ideal, and defining a smooth algebraic variety $\mathcal{V} \subset \mathbb{C}^n$. Consider a *generic* projection $\pi : \mathbb{C}^n \to \mathbb{C}$, its restriction to $\mathcal{V}$ and its critical locus which is supposed to be zero-dimensional. We state that the number of critical points of π restricted to $\mathcal{V}$ is bounded by $D^s(D-1)^{n-s}\binom{n}{n-s}$. This result is obtained in two steps. First the critical points of π restricted to $\mathcal{V}$ are characterized as projections of the solutions of the Lagrange system for which a bi-homogeneous structure is exhibited. Secondly we apply a bi-homogeneous Bézout Theorem, for which we give a proof and which bounds the sum of the degrees of the isolated primary components of an ideal generated by a bi-homogeneous family for which we give a proof. This result is improved in the case where $(f_1, \ldots, f_s)$ is a regular sequence. Moreover, we use Lagrange's system to generalize the algorithm due to Safey El Din and Schost for computing at least one point in each connected component of a smooth real algebraic set to the non equidimensional case. Then, evaluating the size of the output of this algorithm gives new upper bounds on the first Betti number of a smooth real algebraic set.

Key-words: Polynomial systems, Real solutions

Unité de recherche INRIA Rocquencourt
Domaine de Voluceau, Rocquencourt, BP 105, 78153 Le Chesnay Cedex (France)
Téléphone : +33 1 39 63 55 11 — Télécopie : +33 1 39 63 53 30

Théorème de Bézout bi-homogène fort et nouvelles bornes pour l'optimisation algébrique

Résumé : Soit $(f_1, \ldots, f_s)$ une famille de polynômes dans $\mathbb{Q}[X_1, \ldots, X_n]$ (with $s \leq n-1$) de degré borné par D, engendrant un idéal radical et définissant une variété algébrique lisse $\mathcal{V} \subset \mathbb{C}^n$. Considérons une projection *générique* $\pi : \mathbb{C}^n \to \mathbb{C}$, sa restriction à $\mathcal{V}$ et son lieu critique supposé être de dimension nulle. Nous établissons que le nombre de points critiques de π restreinte à $\mathcal{V}$ est borné par $D^s(D-1)^{n-s}\binom{n}{n-s}$. Ce résultat est obtenu en deux temps. Tout d'abord les points critiques de π restreinte à $\mathcal{V}$ sont caractérisés comme projections de l'ensemble des solutions du système de Lagrange pour lequel nous exhibons une structure «presque bi-homogène». Puis, nous appliquons un théorème de Bézout bi-homogène, que nous démontrons au préalable et qui borne la somme des degrés des composantes primaires isolées d'un idéal engendré par un système de polynômes bi-homogènes. Ce résultat est amélioré dans le cas où $(f_1, \ldots, f_s)$ forme une suite régulière. De plus, grâce au système de Lagrange nous généralisons l'algorithme de Safey El Din et Schost calculant au moins un point par composante connexe sur une variété algébrique rélle au cas non équi-dimensionnelle. L'évaluation de la taille de la sortie de cet algorithme au moyen des bornes démontrées précédemment permet de déomntrer de nouvelles bornes sur le premier nombre de Betti d'une variété algébrique réelle lisse.

Mots-clés : Systèmes polynomiaux, Solutions réelles

1 Introduction

Consider polynomials $(f_1, \ldots, f_s)$ in $\mathbb{Q}[X_1, \ldots, X_n]$ (with $s \leq n-1$) of degree bounded by D generating a radical ideal and defining a smooth algebraic variety $\mathcal{V} \subset \mathbb{C}^n$. Given a projection $\pi : \mathbb{C}^n \to \mathbb{C}$, we study in this paper the degree of the critical locus of π restricted to $\mathcal{V}$ and give optimal bounds on this quantity.

Motivation and description of the problem. Since computing critical points solves the problem of algebraic optimization, it has many applications in chemistry, electronics, financial mathematics (see [9] for a non-exhaustive list of problems and applications). More traditionally, computations of critical points are used in effective real algebraic geometry to compute at least one point in each connected components of a real algebraic set. Indeed, every polynomial mapping restricted to a compact real algebraic set reaches its extrema. Moreover, the critical locus of a generic projection restricted to a smooth algebraic set is zero-dimensional or empty. Thus, computing the critical locus of a generic projection allows to intersect each connected component of a compact real algebraic set which is the real counterpart of a smooth algebraic variety.

In [11, 13, 14, 6, 7], the authors consider the hypersurface defined by $f_1^2 + \cdots + f_s^2 = 0$ to study the real algebraic set $\mathcal{V} \cap \mathbb{R}^n$ and, via several infinitesimal deformations, reduce the study to a smooth and compact situation. Such techniques yield algorithms returning zero-dimensional algebraic sets encoded by rational parametrizations of degree $\mathcal{O}(D)^n$ (the better bound is obtained in [6, 7] and is $(4D)^n$). Similar techniques based on the use of a distance function to a generic point and a single infinitesimal deformation (see [16]) yield the bound $(2D)^n$ on the output.

More recently, other algorithms, avoiding the sum of squares and the associated growth of degree, have been proposed (see [1, 17, 5, 2, 3, 20, 19]). They are based on the computation of critical loci of polynomial mappings restricted to *equidimensional* algebraic varieties of dimension d, defined by polynomial systems generating radical ideals. Indeed, under these assumptions, critical points can be algebraically defined as points where a jacobian matrix has rank $n-d$, and then vanishing some minors of the considered jacobian matrix. On the one hand, some of these algorithms allow to obtain efficient implementations (see [18]) while the algorithms mentionned in the above paragraph do not allow to obtain usable implementations. On the other hand, applying the classical Bézout bound to the polynomial systems containing minors of a jacobian matrix which define the critical locus of a projection gives in the case of a regular sequence $D^{n-d}\left((n-d)(D-1)\right)^d$ (see [19, 5] where such a bound is explicitely mentionned). This bound is worse than the aforementionned bounds, but it has never been reached in the experiments we performed with our implementations.

Moreover, remark that these polynomial systems are not generic: they are overdetermined, and the extracted minors from the jacobian matrix depend on $f_1, \ldots, f_s$, so that one can hope that the classical Bézout bound is pessimistic. Understanding the structure and the geometry of polynomial systems defining critical points by evaluating with accuracy the number of computed points in the worst case can lead to further practical improvements.

The goal of this paper is to give optimal bounds on the number of critical points of a projection function restricted to a smooth algebraic variety, in the sense that these bounds are reached in the worst case.

Main contributions. To reach this goal, we substitute the classical algebraic characterization of critical points described above by *Lagrange's characterization* which consists in writing that, at a critical point, there exists a linear combination between the vectors $(\mathbf{grad}(f_1), \ldots, \mathbf{grad}(f_s), \mathbf{e})$, where $\mathbf{e} \in \mathbb{C}^n$ is the vector supporting the line which is the image of π. This polynomial system is called in the sequel *Lagrange's system* and the additional variables are classically called *Lagrange multipliers*. Equipped with such a characterization, critical points can be geometrically interpreted as *projections of the complex solution set of Lagrange's system*, and we prove that *such a characterization remains valid in the non equidimensional situations*, which is not the case of the algebraic characterization of critical points used in [1, 17, 5, 2, 3, 20, 19]. Additionally, we prove that if $f_1, \ldots, f_s$ is a regular sequence and if the critical locus of π restricted to $\mathcal{V}$ is zero-dimensional, then the complex solution set of Lagrange's system is also zero-dimensional (which is not the case in the non equidimensional case).

Since Lagrange multipliers appear with degree 1 in Lagrange's system, evaluating and bounding the degree of the critical locus (which is supposed to be zero-dimensional) of π restricted to $\mathcal{V}$ is equivalent to bounding the sum of the degrees of the isolated primary components of the ideal generated by Lagrange's system. Lagrange's system can be easily transformed by a bi-homogeneisation process which distinguish the variables $X_1, \ldots, X_n$ and the Lagrange multipliers into a bi-homogeneous polynomial system. Thus, the problem is reduced to prove a strong bi-homogeneous Bézout Theorem, i.e. a bound on the sum of the *degrees* of the isolated primary components of an ideal generated by a bi-homogeneous system. We prove such a result, by defining a convenient notion of *bi-degree* of such ideals.

This allows to prove that the critical locus of a *generic* projection $\pi : \mathbb{C}^n \rightarrow \mathbb{C}$ restricted to $\mathcal{V}$ has degree $D^s(D-1)^{n-s}\binom{n}{n-s}$. This bound becomes $D^s(D-1)^{n-s}\binom{n-1}{n-s}$ in the case where $(f_1, \ldots, f_s)$ is a regular sequence, and some computer simulations show it is optimal, since it is reached.

At last, we use the aforementionned properties of Lagrange's systems to generalize the algorithm due to Safey and Schost for computing at least one point in each connected component of a smooth and equidimensional real algebraic set (see [19]) to non equidimensional situations. Then, by evaluating the size of the output of this generalised algorithm, we obtain some improved upper bounds on the first Betti number of a smooth real algebraic set.

Organization of the paper. The paper is organized as follows. In Section 2, we prove the strong Bézout bi-homogeneous Theorem. Then, in Section 3, we focus on the properties of Lagrange's system and use our Bézout Theorem to prove some bounds on the degree of critical loci of generic projections on a line. At last, in Section 4, we generalize the algorithm provided in [19] to the non equidimensional case. Moreover, using the results of the preceeding sections, we prove some improved upper bounds on the first Betti number of a smooth real algebraic set.

Acknowledgements. We thank D. Lazard for its helpful remarks and encouragements, and J. Heintz who helped us by sending us the manuscript [12], without which it would have been difficult to obtain these results.

2 Bi-homogeneous Bézout bound

We denote by R the polynomial ring $\mathbb{Q}[X_1, \ldots, X_n, \ell_1, \ldots, \ell_k]$.

Definition 1 *A polynomial f in R is said to be bi-homogeneous if and only if for all $(u, v) \in \mathbb{Q} \times \mathbb{Q}$:*

$$f(uX_1, \ldots, uX_n, v\ell_1, \ldots, v\ell_k) = u^\alpha v^\beta f(X_1, \ldots, X_n, \ell_1, \ldots, \ell_n).$$

Then, the couple (α, β) is called the bi-degree *of f.*

An ideal $I \subset R$ is said to be bi-homogeneous *if there exists a bi-homogeneous polynomial family generating it. A primary component of a bi-homogeneous ideal is said to be* admissible *if and only if it does contain neither a power of $\langle X_1, \ldots, X_n \rangle$ nor a power of $\langle \ell_1, \ldots, \ell_k \rangle$.*

Remark that the admissible primary components of bi-homogeneous ideals define a bi-projective variety $\mathcal{V}$ in $\mathbb{P}^{n-1}(\mathbb{C}) \times \mathbb{P}^{k-1}(\mathbb{C})$.

Definition 2 *Let $I \subset R$ be a bi-homogeneous polynomial, and (d, e) be a couple in $\mathbb{N} \times \mathbb{N}$. The couple (d, e) is an* admissible bi-dimension *of I if and only if $d+e+2$ equals the maximum of the dimensions of the admissible primary components of I.*

Lemma 1 *Let $I \subset R$ be a bi-homogeneous ideal and $(d, e) \in \mathbb{N} \times \mathbb{N}$ an admissible bi-dimension of I. Then, for a generic choice of $d+1$ homogeneous linear forms $u_1, \ldots, u_d, X$ in $\mathbb{Q}[X_1, \ldots, X_n]$ and a generic choice of $e+1$ homogeneous linear forms $v_1, \ldots, v_e, L$ in $\mathbb{Q}[\ell_1, \ldots, \ell_k]$, the ideal*

$$I + \langle u_1 \rangle + \ldots + \langle u_d \rangle + \langle X - 1 \rangle + \langle v_1 \rangle + \ldots + \langle v_e \rangle + \langle L - 1 \rangle \subset R$$

either equals R or is zero-dimensional.

Proof. The proof is done by induction on the dimension p of $I \subset R$. Suppose I has dimension 2 and let C be a primary component of I having dimension 2. Suppose C contains a power of $\langle X_1, \ldots, X_n \rangle$ (resp. $\langle \ell_1, \ldots, \ell_k \rangle$). Consequently, for any linear homogeneous forms X and L lying respectively in $\mathbb{Q}[X_1, \ldots, X_n]$ and $\mathbb{Q}[\ell_1, \ldots, \ell_k]$, $C + \langle X - 1 \rangle + \langle L - 1 \rangle = R$. Suppose now C does contain neither a power of $\langle X_1, \ldots, X_n \rangle$ nor a power of $\langle \ell_1, \ldots, \ell_k \rangle$. Then, for a generic choice of two linear homogeneous forms X and L lying respectively in $\mathbb{Q}[X_1, \ldots, X_n]$ and $\mathbb{Q}[\ell_1, \ldots, \ell_k]$, $C + \langle X - 1 \rangle + \langle L - 1 \rangle \subset R$ is zero-dimensional.

Now, suppose the assertion to be true for $p - 1$ and let $I \subset R$ be an ideal of dimension p. Then, for each primary component C of I having maximal dimension, given a generic linear homogeneous form u lying in $\mathbb{Q}[X_1, \dots, X_n]$ or $\mathbb{Q}[\ell_1, \dots, \ell_k]$, $C + \langle u \rangle$ (or $C + \langle v \rangle$) either contains a power of $\langle X_1, \dots, X_n \rangle$ or a power of $\langle \ell_1, \dots, \ell_k \rangle$, and then $C + \langle u \rangle + \langle X - 1 \rangle + \langle L - 1 \rangle$ (resp. $C + \langle v \rangle + \langle X - 1 \rangle + \langle L - 1 \rangle$) equals $\mathbb{Q}[X_1, \dots, X_n]$ (resp. $\mathbb{Q}[\ell_1, \dots, \ell_k]$), or has dimension equal to $p - 1$.

□

In the next section, we define Hilbert bi-series of bi-homogeneous ideals. This tool is used to relate the number of roots of a bi-homogeneous ideal of bi-dimension $(0, 0)$ to the degree of a zero-dimensional ideal in R. This will lead to define *algebraic bi-degrees* of bi-homogeneous ideals, by applying iteratively the above lemma.

2.1 Preliminaries and definitions

Notations Let R be a polynomial ring. Given a couple $(i, j) \in \mathbb{N} \times \mathbb{N}$, we denote by $R_{i,j}$ the subset of polynomials in R of degrees i in the set of variables $X_1, \dots, X_n$ and j in the set of variables $\ell_1, \dots, \ell_k$. Given an ideal I in R and a couple $(i, j) \in \mathbb{N} \times \mathbb{N}$, we denote by $I_{i,j}$ the intersection of I with $R_{i,j}$.

Given a couple $(i, j) \in \mathbb{N} \times \mathbb{N}$, we denote by $R_{\leq i, \leq j}$ the subset of polynomials in R of degrees less than or equal to i in the set of variables $X_1, \dots, X_n$ and less than or equal to j in the set of variables $\ell_1, \dots, \ell_k$. Given an ideal I in R and a couple $(i, j) \in \mathbb{N} \times \mathbb{N}$, we denote by $I_{\leq i, \leq j}$ the intersection of I with $R_{\leq i, \leq j}$.

Definition 3 *The Hilbert bi-series of an ideal $I \subset R$ generated by a* bi-homogeneous *polynomial family is the series $\sum_{i,j} \dim (R_{i,j}/I_{i,j})\, t_1^i t_2^j$.*

*The affine Hilbert bi-series of an ideal $I \subset R$ generated by a (*non-homogeneous*) polynomial family is the series $\sum_{i,j} \dim (R_{\leq i, \leq j}/I_{\leq i, \leq j})\, t_1^i t_2^j$.*

Consider a bi-homogeneous polynomial system in R defining a non-empty bi-projective variety in $\mathbb{P}^{n-1}(\mathbb{C}) \times \mathbb{P}^{k-1}(\mathbb{C})$ of bi-dimension $(0, 0)$, and I the ideal it generates. This section is devoted to prove that, for i *and* j *large enough*, the coefficient appearing in the Hilbert bi-series of I is constant. define a notion of

The following result relates the cardinality of a bi-homogeneous ideal of bi-dimension $(0, 0)$ with the degree (in the classical meaning) of a zero-dimensional ideal in R.

Lemma 2 *Let $I \subset R$ be an ideal generated by a family of bi-homogeneous polynomial, defining a zero-dimensional bi-projective variety $\mathcal{V} \subset \mathbb{P}^{n-1}(\mathbb{C}) \times \mathbb{P}^{k-1}(\mathbb{C})$.*

For a couple of homogeneous linear forms $X \in \mathbb{Q}[X_1, \dots, X_n]$ and $L \in \mathbb{Q}[\ell_1, \dots, \ell_k]$ chosen outside a Zariski closed subset, the ideal $I + \langle X - 1 \rangle + \langle L - 1 \rangle \subset R$ is zero-dimensional and its degree does not depend on the choice of X and L.

Proof. Consider the admissible primary components of the primary decomposition of the ideal I in $\mathbb{C}[X_1, \dots, X_n, \ell_1, \dots, \ell_k]$.

Since each point in $\mathbb{P}^{n-1}(\mathbb{C})$ (resp. $\mathbb{P}^{k-1}(\mathbb{C})$) is in one-to-one correspondence with a line in $\mathbb{C}^n$ (resp. $\mathbb{C}^k$) containing the origin, each point in $\mathcal{V}$, of multiplicity the one of the corresponding primary component in the above primary decomposition, is in one-to-one correspondence with a couple of lines in $\mathbb{C}^n \times \mathbb{C}^k \simeq \mathbb{C}^{n+k}$ (with the same multiplicity). In the sequel, a homogeneous linear form $X \in \mathbb{Q}[X_1, \ldots, X_n]$ (resp. $L \in \mathbb{Q}[\ell_1, \ldots, \ell_k]$) is said to be *generic* if the hyperplane defined by $X = 0$ (resp. $L = 0$) does not contain any of these lines in $\mathbb{C}^n$ (resp. $\mathbb{C}^k$).

Thus, for a *generic* choice of a homogeneous linear form (resp. L) in $\mathbb{Q}[X_1, \ldots, X_n]$ (resp. $\mathbb{Q}[\ell_1, \ldots, \ell_k]$), the ideal $I + \langle X - 1\rangle + \langle L - 1\rangle$ is zero-dimensional, the components of its primary decomposition in $\mathbb{C}[X_1, \ldots, X_n, \ell_1, \ldots, \ell_k]$ are contained neither in $\langle X_1, \ldots, X_n\rangle$ nor in $\langle \ell_1, \ldots, \ell_k\rangle$. Moreover, the ideals $(I + \langle X - 1\rangle) \cap \mathbb{Q}[X_1, \ldots, X_n]$ and $(I + \langle L - 1\rangle) \cap \mathbb{Q}[\ell_1, \ldots, \ell_k])$ are zero-dimensional and have the same degree, which does not depend on the choice of X and L.

$\square$

Remark 1 *Notice that if $X \in \mathbb{Q}[X_1, \ldots, X_n]$ and $L \in \mathbb{Q}[\ell_1, \ldots, \ell_k]$ are not chosen* generically *(in the meaning of the above proof), the ideal $I + \langle X - 1\rangle + \langle L - 1\rangle$ has a degree less than the one obtained when X and L are generic.*

Denote by R' the polynomial ring $\mathbb{Q}[X_1, \ldots, X_{n-1}, \ell_1, \ldots, \ell_{k-1}]$. Given a polynomial f in $R'_{\leq i, \leq j}$ of total degree $\alpha \leq i$ (resp. $\beta \leq j$) when considered as a polynomial with coefficients in $\mathbb{Q}[\ell_1, \ldots, \ell_{k-1}]$ (resp. $\mathbb{Q}[X_1, \ldots, X_{n-1}]$), consider the application $\phi_{i,j} : R'_{\leq i, \leq j} \to R_{i,j}$ sending $f \in R'_{\leq i, \leq j}$ to the polynomial $\phi_{i,j}(f) = X_n^{i-\alpha} \ell_k^{j-\beta} f(\frac{X_1}{X_n}, \ldots, \frac{X_{n-1}}{X_n}, \frac{\ell_1}{\ell_k}, \ldots, \frac{\ell_{k-1}}{\ell_k})$. Given $I \subset R'$, $\phi_{i,j}(I)$ denotes $\{\phi_{i,j}(f) \mid f \in I_{\leq i, \leq j}\}$. Additionally, consider the mapping $\psi_{i,j} : R_{i,j} \to R'_{\leq i, \leq j}$ (which takes place of a dehomogeneization process in a homogeneous context) by sending a polynomial $f \in R_{i,j}$ to

$$\psi_{i,j}(f) = f(X_1, \ldots, X_{n-1}, 1, \ell_1, \ldots, \ell_{k-1}, 1) \in R'.$$

Lemma 3 *Consider two integers i and j, an ideal I of R' and $\phi_{i,j}$ the above application from R' to R. Then*

$$\dim(R'_{\leq i, \leq j}/I_{\leq i, \leq j}) = \dim(R_{i,j}/\phi_{i,j}(I_{\leq i, \leq j})).$$

Proof. Checking that ϕ and ψ are linear mappings which are invert of each other is immediate. Thus, $R_{i,j}$ and $R'_{\leq i, \leq j}$ on the one hand, and $I_{\leq i, \leq j}$ and $\phi(I)_{i,j}$ on the other hand, are finite-dimensional isomorphic vector spaces. Since for vector spaces E, F with $F \subset E$, $\dim(E) = \dim(F) + \dim(E/F)$, we are done. $\square$

Given a polynomial $f \in \mathbb{Q}[X_1, \ldots, X_n]$, and $\mathbf{A} \in GL_{n+k}(\mathbb{Q})$, we denote by $f^{\mathbf{A}}$ the polynomial obtained by performing the change of variables induced by $\mathbf{A}$ on f. We denote by $I^{\mathbf{A}}$ the ideal generated by $f_1^{\mathbf{A}}, \ldots, f_s^{\mathbf{A}}$ and by $\mathcal{V}^{\mathbf{A}}$ the algebraic variety associated to $I^{\mathbf{A}}$. In the sequel, we consider exclusively matrices $\mathbf{A}$ such that the action of $\mathbf{A}$ on R is a bi-graded isomorphism on R with respect to the variables $X_1, \ldots, X_n$ and $\ell_1, \ldots, \ell_k$, i.e. for all homogeneous linear forms $X \in \mathbb{Q}[X_1, \ldots, X_n]$ (resp. $L \in \mathbb{Q}[\ell_1, \ldots, \ell_k]$), $\mathbf{A}(X)$ (resp. $\mathbf{A}(L)$) is linear form in $\mathbb{Q}[X_1, \ldots, X_n]$ (resp. $\mathbb{Q}[\ell_1, \ldots, \ell_k]$).

Lemma 4 *Let $I \subset R$ be an ideal, then I and $I^{\mathbf{A}}$ have the same Hilbert bi-series.*

Proof. The action of $\mathbf{A}$ on R is an isomorphism of bi-graded ring of degree $(0,0)$ for which the inverse is the action of $\mathbf{A}^{-1}$. Thus, $R_{i,j}$ equals $R_{i,j}^{\mathbf{A}}$ and, if $E \subset R$ is a $\mathbb{Q}$-vector space, $\dim(V) = \dim(E^{\mathbf{A}})$. Since for vector spaces E, F with $F \subset E$, $\dim(E) = \dim(F) + \dim(E/F)$, we have $\dim(R_{i,j}/I_{i,j}) = \dim(R_{i,j}/I_{i,j}^{\mathbf{A}})$ which implies the equality of the Hilbert bi-series of I and $I^{\mathbf{A}}$. □

Proposition 1 *Let $I \subset R$ be an ideal generated by bi-homogeneous polynomials, defining a zero-dimensional bi-projective variety $\mathcal{V}$ in $\mathbb{P}^{n-1}(\mathbb{C}) \times \mathbb{P}^{k-1}(\mathbb{C})$. For i and j large enough, the coefficient of index i,j in the Hilbert bi-series of I: $dim(R_{i,j}/I_{i,j})$ is constant.*

Its value is the degree of the ideal $I + \langle X-1\rangle + \langle L-1\rangle \subset R$, where X and L are generic homogeneous linear forms chosen respectively in $\mathbb{Q}[X_1, \ldots, X_n]$ and $\mathbb{Q}[\ell_1, \ldots, \ell_k]$.

Proof. Given generic homogeneous linear forms $X \in \mathbb{Q}[X_1, \ldots, X_n]$ and $L \in \mathbb{Q}[\ell_1, \ldots, \ell_k]$ satisfying the assumptions of Lemma 2, the ideal $I+\langle X-1\rangle+\langle L-1\rangle \subset R$ is zero-dimensional and its degree does not depend on the choice of X and L. Now, choosing $\mathbf{A} \in GL_{n+k}(\mathbb{Q})$ such that $\mathbf{A}(X) = X_n$ and $\mathbf{A}(L) = \ell_k$, and such that the canonical action of $\mathbf{A}$ on R is a bi-graded isomorphism on R of degree $(0,0)$ with respect to the variables $X_1, \ldots, X_n$ and $\ell_1, \ldots, \ell_k$, the conclusion follows immediately from Lemmata 4 and 3.

□

2.2 Definition and properties of an algebraic bi-degree

From the preceeding paragraph, the *algebraic bi-degree* of a bi-homogeneous ideal defining a bi-projective variety in $\mathbb{P}^{n-1}(\mathbb{C}) \times \mathbb{P}^{k-1}(\mathbb{C})$ can be defined as the classical degree of the same ideal augmented by two affine linear forms, the first one lying in $\mathbb{Q}[X_1, \ldots, X_n]$ on the one hand, and the other one lying in $\mathbb{Q}[\ell_1, \ldots, \ell_k]$ on the other hand. Remark that this algebraic bi-degree is also the sum of the algebraic bi-degrees of the admissible primary components of the considered ideal. To define a notion of algebraic bi-degree available in the positive dimensional case, we use the preceeding by augmenting the ideal under consideration with homogeneous linear forms in $\mathbb{Q}[X_1, \ldots, X_n]$ and $\mathbb{Q}[\ell_1, \ldots, \ell_k]$ reducing our study to the case of bi-homogeneous ideals of bi-dimension $(0,0)$. As mentionned above, this can be done in several ways. The result below is crucial to understand the canonicity of the further definitions of algebraic bi-degree given in this paragraph.

Lemma 5 *Let $I \subset R$ be a bi-homogeneous ideal and suppose the maximal dimension of the admissible primary components of I to be 3. Consider $f \in R$ of bi-degree (α, β) which is generic for I (i.e. f does not belong to any prime associated to a primary component of I). Then, for generic linear homogeneous forms u, X in $\mathbb{Q}[X_1, \ldots, X_n]$ and v, L in $\mathbb{Q}[\ell_1, \ldots, \ell_k]$:*

$$\deg(I+\langle f\rangle+\langle X-1\rangle+\langle L-1\rangle) = \alpha \deg(I+u+\langle X-1\rangle+\langle L-1\rangle)+\beta \deg(I+v+\langle X-1\rangle+\langle L-1\rangle).$$

Proof. Let $\mathcal{H}(I) = \sum_{i,j} a_{i,j} t_1^i t_2^j$ denote the Hilbert bi-series of I, $\mathrm{ann}_{R/I}(f)$ the annihilator of f with respect to I. Remark that since f is generic for I, $\mathrm{ann}_{R/I}(f)$ is 0 and that since

$\dim(I + \langle f\rangle + \langle X-1\rangle + \langle L-1\rangle) = 0$. Since the sequence below

$$0 \to \mathrm{ann}_{R/I}(f) \longrightarrow R/I \xrightarrow{f} R/I \longrightarrow R/(I+\langle f\rangle) \to R$$

is exact, the following holds for any generic element $f \in R$ of bi-degree (α, β):

$$(1 - t_1^\alpha t_2^\beta)\mathcal{H}(I) = \mathcal{H}(I + \langle f\rangle)$$

The same way, remark also that for a generic linear form u (resp. v) in $\mathbb{Q}[X_1,\ldots,X_n]$ (resp. $\mathbb{Q}[\ell_1,\ldots,\ell_k]$) the bi-dimension of $I+\langle u\rangle$ (resp. $I+\langle v\rangle$) is $(0,0)$ and then the Hilbert bi-series $\mathcal{H}(I+\langle u\rangle)$ and $\mathcal{H}(I+\langle v\rangle)$ have a constant term when both $i, j \to \infty$, say S_1 and S_2.

Thus, the coefficient of index (i,j) in $\mathcal{H}(I+\langle f\rangle)$ is constant and equals: $a_{i,j} - a_{i-\alpha,j-\beta}$, which can be rewritten as:

$$\sum_{p=0}^{\alpha-1}(a_{i-p,j} - a_{i-p-1,j}) + \sum_{p=0}^{\beta-1}(a_{i-\alpha,j-p} - a_{i-\alpha,j-p-1})$$

Remark now, that the first term of the above sum equals $\alpha.S_1$ while the second equals $\beta.S_2$. □

Remark that if $(I + \langle X-1\rangle + \langle L-1\rangle) \cap \mathbb{Q}[X_1,\ldots,X_n]$ (resp. $(I + \langle X-1\rangle + \langle L-1\rangle) \cap \mathbb{Q}[\ell_1,\ldots,\ell_k]$) is zero-dimensional in $\mathbb{Q}[X_1,\ldots,X_n]$ (resp. $\mathbb{Q}[\ell_1,\ldots,\ell_k]$), $\deg(I + \langle u\rangle + \langle X-1\rangle + \langle L-1\rangle) = 0$ (resp. $\deg(I+\langle v\rangle + \langle X-1\rangle + \langle L-1\rangle) = 0$).

Definition 4 *Let $I \subset R$ be a bi-homogeneous ideal, $\mathcal{D} \subset \mathbb{N}\times\mathbb{N}$ the subset of couples (d,e) such that $d \le n-1$, $e \le k-1$ and $d+e+2$ equals the maximal dimension of the admissible primary components of I. The algebraic bi-degree of I, denoted by* bideg(I) *is the sum*

$$\sum_{(d,e)\in\mathcal{D}} \deg(I + \langle u_1\rangle + \cdots + \langle u_d\rangle + \langle v_1\rangle + \cdots + \langle v_e\rangle + \langle X-1\rangle + \langle L-1\rangle)$$

where $u_1,\ldots,u_d,X$ (resp. $v_1,\ldots,v_e,L$) are generic linear homogenenous forms in the polynomial ring $\mathbb{Q}[X_1,\ldots,X_n]$ (resp. $\mathbb{Q}[\ell_1,\ldots,\ell_k]$).

The strong algebraic bi-degree *of I is the sum of the algebraic bi-degrees of the admissible isolated primary components of I.*

Notations : In the sequel, given a bi-homogeneous ideal $I \subset R$ and an admissible bi-dimension $(d,e) \in \mathbb{N}\times\mathbb{N}$ of I, we denote by $C_{d,e}(I)$ the degree of:

$$I + \langle u_1\rangle + \cdots + \langle u_d\rangle + \langle v_1\rangle + \cdots + \langle v_e\rangle + \langle X-1\rangle + \langle L-1\rangle$$

where $u_1,\ldots,u_d,X$ (resp. $v_1,\ldots,v_e,L$) are generic linear homogenenous forms in the polynomial ring $\mathbb{Q}[X_1,\ldots,X_n]$ (resp. $\mathbb{Q}[\ell_1,\ldots,\ell_k]$). By convention, if $d > n-1$ or $e > k-1$, $C_{d,e}(I)$ is null.

In [21], the author uses Intersection Theory to prove that in the case of a bi-projective variety $\mathcal{V} \subset \mathbb{P}^{n-1}(\mathbb{C}) \times \mathbb{P}^{k-1}(\mathbb{C})$ of bi-dimension $(0,0)$, the cardinality of $\mathcal{V}$ (which coincides in this case with the bi-degree of the ideal defining $\mathcal{V}$) is bounded by the classical bi-homogeneous Bézout bound.

In [12], the authors give a similar statement using deformation techniques. We extend this statement to the strong bi-degree of a bi-projective variety $\mathcal{V} \subset \mathbb{P}^{n-1}(\mathbb{C}) \times \mathbb{P}^{k-1}(\mathbb{C})$ defined by a bi-homogeneous polynomial system $S \subset R$. To this end, we follow [15] which gives a similar statement in the homogeneous case.

2.3 Proof of a bi-homogeneous Bézout-Theorem

Proposition 2 *Let $I \subset R$ be a bi-homogeneous ideal, D the maximal dimension of its admissible primary components, and $f \in R$ be a* generic *bi-homogeneous polynomial of bi-degree (α, β) (i.e. f is not a zero-divisor in R/C for any admissible primary component C of I having maximal dimension).*

Consider $\mathcal{D} \subset \mathbb{N} \times \mathbb{N}$ the subset such that $(d,e) \in \mathcal{D}$ if and only if $d+e+2$ equals $D-1$ and $0 \leq d \leq n-1$ and $0 \leq e \leq k-1$. Then, the bi-degree of $I + \langle f \rangle$ equals:

$$\alpha \left(\sum_{(d,e)\in\mathcal{D}} C_{d+1,e}(I)\right) + \beta \left(\sum_{(d,e)\in\mathcal{D}} C_{d,e+1}(I)\right)$$

Proof. Consider a bi-homogeneous ideal I whose admissible primary components have maximal dimension D, a generic element f for I, and generic linear homogeneous forms u, X (resp. v, L) in $\mathbb{Q}[X_1, \ldots, X_n]$ (resp. $\mathbb{Q}[\ell_1, \ldots, \ell_k]$). From Definition 4, the bi-degree of $I + \langle f \rangle$ equals:

$$\sum_{\substack{(d,e) \in \mathbb{N}^2 \\ d+e+2=D-1}} C_{d,e}(I + \langle f \rangle)$$

Given generic linear homogeneous forms $u_1, \ldots, u_d, X$ (resp. $v_1, \ldots, v_e, L$) in $\mathbb{Q}[X_1, \ldots, X_n]$ (resp. $\mathbb{Q}[\ell_1, \ldots, \ell_k]$), from Lemma 5 applied to $I + \langle u_1 \rangle + \cdots + \langle u_d \rangle + \langle v_1 \rangle + \cdots + \langle v_e \rangle + \langle X-1 \rangle + \langle L-1 \rangle$, which has dimension 3, the following holds for any $(d,e) \in \mathbb{N}^2$ such that $d+e+2 = D-1$:

$$C_{d,e}(I + \langle f \rangle) = \alpha . C_{d+1,e}(I) + \beta . C_{d,e+1}(I)$$

Adding these terms ends the proof.

□

Applying recursively this result, one obtains immediately the following corollary.

Corollary 1 *Let $f_1, \ldots, f_s$ be a regular sequence of R of respective bi-degrees $(\alpha_1, \beta_1), \ldots, (\alpha_s, \beta_s)$. Then, the bi-degree of the ideal $\langle f_1, \ldots, f_s \rangle$ is bounded by:*

$$\mathcal{B}(f_1, \ldots, f_s) = \sum_{\mathcal{I},\mathcal{J}} (\Pi_{i\in\mathcal{I}}\alpha_i) . (\Pi_{j\in\mathcal{J}}\beta_j)$$

where $\mathcal{I}$ and $\mathcal{J}$ are disjoint sets for which the union is $\{1, \ldots, \min(s, n+k-2)\}$ such that the cardinality of $\mathcal{I}$ (resp. $\mathcal{J}$) is bounded by $n-1$ (resp. $k-1$).

Lemma 6 *Let $I \subset R$ be a primary ideal of dimension D defining a non-empty bi-projective variety in $\mathbb{P}^{n-1}(\mathbb{C}) \times \mathbb{P}^{k-1}(\mathbb{C})$, $f \in R$ of bi-degree (α, β) such that I contains a power of f, and $\widetilde{f}$ a generic element for I. Then, if both α and β are not null or if $\alpha = 0$ and $D > k+1$ or if $\beta = 0$ and $D > n+1$, the bi-degree of $I + \langle f \rangle$ is less or equal to the bi-degree of $I + \widetilde{f}$.*

Proof. Since f is not a generic element for I, $\dim(I + \langle f \rangle) = \dim(I)$, and then the bi-degree of $I + \langle f \rangle$ equals the sum of $C_{d,e}(I + \langle f \rangle)$ where (d, e) lies in the set of admissible bi-dimensions of I. Remark that $C_{d,e}(I + \langle f \rangle)$ is the degree of I augmented by generic linear forms and f. Since the forms are generic, I augmented by these forms is primary also, and contains a power of f. Thus, we are lead to consider to evaluate the degree of a zero-dimensional primary ideal augmented by an element f of R such that a power of f is contained in the considered primary ideal. Thus, $C_{d,e}(I + \langle f \rangle) \leq C_{d,e}(I)$.

On the other hand, consider a generic element $\widetilde{f} \in R$ for I. Then, if both α and β are not null, or if $\beta = 0$ and $D > k + 1$, or if $\alpha = 0$ and $D > n - 1$, Proposition 2 implies that $\text{bideg}(I + \langle \widetilde{f} \rangle) \geq \text{bideg}(I) \geq \text{bideg}(I + \langle f \rangle)$.

□

Theorem 1 *Let $f_1, \ldots, f_s$ be bi-homogenous polynomials of R of respective bi-degree (α_1, β_1), $\ldots, (\alpha_s, \beta_s)$, with $s \leq n + k - 2$. Suppose that for all $i = 2, \ldots, s$:*

- *either both α_i and β_i are not null,*
- *or if $\beta_i = 0$ the maximal dimension of the admissible primary components of the ideal $\langle f_1, \ldots, f_{i-1} \rangle$ is greater than $k + 1$,*
- *or if $\alpha_i = 0$ the maximal dimension of the admissible primary components of the ideal $\langle f_1, \ldots, f_{i-1} \rangle$ is greater than $n + 1$.*

Let $C_1, \ldots, C_p$ be the isolated primary components of I. Then, the following inequality holds:

$$\sum_{j=1}^{p} \text{bideg}(C_j) \leq \mathcal{B}(f_1, \ldots, f_s)$$

where $\mathcal{B}(f_1, \ldots, f_s) = \sum_{\mathcal{I},\mathcal{J}} (\Pi_{i \in \mathcal{I}} \alpha_i) . (\Pi_{j \in \mathcal{J}} \beta_j)$ where $\mathcal{I}$ and $\mathcal{J}$ are disjoint subsets for which the union is $\{1, \ldots, s\}$ such that the cardinality of $\mathcal{I}$ (resp. $\mathcal{J}$) is bounded by $n - 1$ (resp. $k - 1$).

Proof. Consider the ideal $I = \langle f_1, \ldots, f_{s-1} \rangle \subset R$ and $f_s \in R$. The strong bi-degree of $I + \langle f_s \rangle$ is the sum of the bi-degrees of the isolated primary components of $I + \langle f_s \rangle$ which can be obtained as the isolated primary components of $C + \langle f_s \rangle$, where C goes through the set of isolated primary components of I.

Consider now an isolated primary component C of I. If it does not contain a power of f_s, then $C + \langle f_s \rangle$ is equidimensional and then the strong bi-degree of $C + \langle f_s \rangle$ equals its bi-degree. Suppose now C contains a power of f_s. Then, f_s belongs to the prime ideal $\mathcal{P}$ associated to C and since $\mathcal{P} + \langle f_s \rangle = \mathcal{P}$, the strong bi-degree of $C + \langle f_s \rangle$ is $\text{bideg}(C + \langle f_s \rangle)$, since it is also equidimensional. Thus, we are interested in bounding $\sum_{i=1}^{p} \text{bideg}(C_i + \langle f_s \rangle)$ where p denotes the number of isolated primary components of I.

We distinguish the primary components containing a power of f_s from the others. Suppose C is an isolated primary component of I which does contain a power of f_s. Moreover, since $s \leq n + k - 2$, there exists $\widetilde{f} \in R$ of bi-degree (α_s, β_s), which is not a zero-divisor in R/I. Then, from Lemma 6, $\text{bideg}(C + \langle f_s \rangle) \leq \text{bideg}(C + \langle \widetilde{f} \rangle)$.

Thus, the strong bi-degree of $I + \langle f_s \rangle$ is bounded by the strong bi-degree of $I + \langle \widetilde{f} \rangle$ where $\widetilde{f}$ is a non-zero-divisor in R/I of bi-degree (α_s, β_s). Applying inductively this argument, the strong bi-degree of $\langle f_1, \ldots, f_s \rangle$ is bounded by the strong bi-degree of a regular sequence $\langle \widetilde{f_1}, \ldots, \widetilde{f_s} \rangle$ of respective bi-degrees $(\alpha_1, \beta_1), \ldots, (\alpha_s, \beta_s)$, and then by $\mathcal{B}(f_1, \ldots, f_s)$, following Corollary 1.

□

3 Upper bound on the critical locus of a generic projection

Definition 5 *Consider an algebraic variety $\mathcal{V} \subset \mathbb{C}^n$, and denote by $I(\mathcal{V})$ the ideal associated to $\mathcal{V}$.*

- *If f is a polynomial in $\mathbb{Q}[X_1, \ldots, X_n]$, the* linear part *of f at a point $p = (p_1, \ldots, p_n) \in \mathbb{C}^n$, denoted by $d_p(f)$, is defined to be: $d_p(f) = \frac{\partial f}{\partial X_1}(X_1 - p_1) + \ldots + \frac{\partial f}{\partial X_n}(X_n - p_n)$.*
- *The tangent space of $\mathcal{V}$ at p, denoted by $T_p(\mathcal{V})$, is the variety: $T_p(\mathcal{V}) = V(d_p(f) : f \in I(\mathcal{V}))$.*
- *For $p \in \mathcal{V}$, the dimension of $\mathcal{V}$ at p, denoted by $\dim_p(\mathcal{V})$ is the maximum dimension of an irreducible component of $\mathcal{V}$ containing p.*
- *A point $y \in \mathcal{V}$ is said to be* smooth *(or* nonsingular*) if $\dim(T_p(\mathcal{V})) = \dim_p(\mathcal{V})$.*
- *An algebraic variety $\mathcal{V} \subset \mathbb{C}^n$ is smooth if and only if all points $p \in \mathcal{V}$ are smooth points.*

Lemma 7 *Let $\mathcal{V} \subset \mathbb{C}^n$ be a smooth algebraic variety defined by s polynomials $f_1, \ldots, f_s$ lying in $\mathbb{Q}[X_1, \ldots, X_n]$ generating a* radical *ideal, π the canonical projection:*

$$\begin{array}{cccc} \pi : & \mathbb{C}^n & \longrightarrow & \mathbb{C} \\ & (x_1, \ldots, x_n) & \mapsto & x_1 \end{array}$$

Given $p \in \mathcal{V}$, the point p is a critical point of π restricted to $\mathcal{V}$ if and only if there exists a point $(\lambda_1, \ldots, \lambda_s)$ in $\mathbb{C}^s$ such that $(\lambda_1, \ldots, \lambda_s, p) \in \mathbb{C}^s \times \mathbb{C}^n$ is a solution of the polynomial *system in* $\mathbb{Q}[\ell_1, \ldots, \ell_s, X_1, \ldots, X_n]$:

$$\left\{ \begin{array}{l} f_1 = \ldots = f_s = 0 \\ \ell_1 \frac{\partial f_1}{\partial X_1} + \ldots + \ell_s \frac{\partial f_s}{\partial X_1} = 1 \\ \ell_1 \frac{\partial f_1}{\partial X_2} + \ldots + \ell_s \frac{\partial f_s}{\partial X_2} = 0 \\ \vdots \\ \ell_1 \frac{\partial f_1}{\partial X_n} + \ldots + \ell_s \frac{\partial f_s}{\partial X_n} = 0 \end{array} \right.$$

where $\ell_1, \ldots, \ell_s$*, are new variables.*

Proof. By definition, a point $p \in \mathcal{V}$ is a critical point of π restricted to $\mathcal{V}$ if and only if the differential of π in p, denoted by $d_p(\pi)$ projecting the vectors of $T_p(\mathcal{V})$ on their first coordinate, is not surjective. This is equivalent to say that the vectors whom coordinates are null except the first are normal to $T_p(\mathcal{V})$. On the other hand, from the second item of Definition 5 and since $\langle f_1, \ldots, f_s \rangle$ is a radical ideal, the vector space generated by $\mathrm{Span}(\mathbf{grad}_p(f_1), \ldots, \mathbf{grad}_p(f_s))$ is supplementar with $T_p(\mathcal{V})$.

Thus, p is a critical point of π restricted to $\mathcal{V}$ if and only if the vectors whom coordinates are null except the first one belongs to $\mathrm{Span}(\mathbf{grad}_p(f_1), \ldots, \mathbf{grad}_p(f_s))$.

This means exactly there exist complex numbers $\lambda_1, \ldots, \lambda_s$ such that:

$$\begin{array}{l} f_1(p) = \ldots = f_s(p) = 0 \\ \lambda_1 \frac{\partial f_1}{\partial X_1} + \ldots + \lambda_s \frac{\partial f_s}{\partial X_1} = 1 \\ \lambda_1 \frac{\partial f_1}{\partial X_2} + \ldots + \lambda_s \frac{\partial f_s}{\partial X_2} = 0 \\ \vdots \\ \lambda_1 \frac{\partial f_1}{\partial X_n} + \ldots + \lambda_s \frac{\partial f_s}{\partial X_n} = 0 \end{array}$$

which ends the proof.

□

Remark 2 *This algebraic characterization is well known as* Lagrange's characterization *(or* Lagrange's system*).*

Notice also that the above Lemma defines critical points of a projection function restricted to an algebraic variety as roots of an elimination ideal. Remark that the considered algebraic variety is not supposed to be equidimensional contrarily to the algebraic characterization of critical points which is used in [1, 17, 19, 5, 2, 3].

This fact is a key point to generalize Safey-Schost's algorithm [19] computing at least one point in each connected component of a real algebraic variety to the non equidimensional case.

From now on, we consider a smooth algebraic variety $\mathcal{V} \subset \mathbb{C}^n$ defined by s polynomials $f_1, \ldots, f_s$ lying in $\mathbb{Q}[X_1, \ldots, X_n]$ generating a *radical* ideal. In the sequel, we use linear change of variables. Given a polynomial $f \in \mathbb{Q}[X_1, \ldots, X_n]$, and $\mathbf{A} \in GL_n(\mathbb{Q})$, we denote by

$f^{\mathbf{A}}$ the polynomial obtained by performing the change of variables induced by $\mathbf{A}$ on f. We denote by $I^{\mathbf{A}}$ the ideal generated by $f_1^{\mathbf{A}}, \ldots, f_s^{\mathbf{A}}$ and by $\mathcal{V}^{\mathbf{A}}$ the algebraic variety associated to $I^{\mathbf{A}}$.

Lemma 8 *Let $\mathcal{V} \subset \mathbb{C}^n$ be a smooth algebraic variety defined by s polynomials $f_1, \ldots, f_s$ lying in $\mathbb{Q}[X_1, \ldots, X_n]$ generating a* radical *ideal. Given $\mathbf{A} \in GL_n(\mathbb{Q})$, consider $I^{\mathbf{A}} \subset \mathbb{Q}[\ell_1, \ldots, \ell_s, X_1, \ldots, X_n]$ the ideal generated by the polynomial system:*

$$\begin{cases} f_1^{\mathbf{A}} = \ldots = f_s^{\mathbf{A}} = 0 \\ \ell_1 \frac{\partial f_1^{\mathbf{A}}}{\partial X_1} + \ldots + \ell_s \frac{\partial f_s^{\mathbf{A}}}{\partial X_1} = 1 \\ \ell_1 \frac{\partial f_1^{\mathbf{A}}}{\partial X_2} + \ldots + \ell_s \frac{\partial f_s^{\mathbf{A}}}{\partial X_2} = 0 \\ \vdots \\ \ell_1 \frac{\partial f_1^{\mathbf{A}}}{\partial X_n} + \ldots + \ell_s \frac{\partial f_s^{\mathbf{A}}}{\partial X_n} = 0 \end{cases}$$

There exists a proper Zariski-closed subset $\mathcal{H} \subset GL_n(\mathbb{C})$ such that if $\mathbf{A} \notin \mathcal{H}$, the elimination ideal $I^{\mathbf{A}} \cap \mathbb{Q}[X_1, \ldots, X_n]$ is zero-dimensional.

Moreover, if $f_1, \ldots, f_s$ is a regular sequence in $\mathbb{Q}[X_1, \ldots, X_n]$ generating a radical ideal, there exists a proper Zariski-closed subset $\mathcal{H}' \subset GL_n(\mathbb{C})$ such that if $\mathbf{A} \notin \mathcal{H}'$, then $I^{\mathbf{A}}$ is zero-dimensional.

Proof. Consider a family of polynomials $(g_1, \ldots, g_k)$ in $\mathbb{Q}[X_1, \ldots, X_n]$ generating a radical equidimensional ideal whose associated algebraic variety is an equidimensional component C_d of $\mathcal{V} \subset \mathbb{C}^n$. Let $\mathcal{P}^{\mathbf{A}} \subset \mathbb{C}^n$ be the algebraic variety associated to $I^{\mathbf{A}} \cap \mathbb{Q}[X_1, \ldots, X_n]$ and $\mathcal{P}_d^{\mathbf{A}}$ a subset of $\mathcal{P}^{\mathbf{A}}$ such that $\mathcal{P}_d^{\mathbf{A}}$ is the intersection of $\mathcal{P}^{\mathbf{A}}$ and $C_d^{\mathbf{A}}$, the complex solution set of:

$$g_1^{\mathbf{A}} = \ldots = g_k^{\mathbf{A}}.$$

From [19, Theorem 2], there exists a Zariski proper closed subset of $GL_n(\mathbb{C})$ such that if $\mathbf{A} \notin \mathcal{H}_d$, the critical locus of the canonical projection π:

$$\begin{array}{llll} \pi : & \mathbb{C}^n & \longrightarrow & \mathbb{C} \\ & (x_1, \ldots, x_n) & \mapsto & x_1 \end{array}$$

restricted to $C_d^{\mathbf{A}}$ is zero-dimensional. Now, consider the polynomial system in $\mathbb{Q}[m_1, \ldots, m_k, X_1, \ldots, X_n]$:

$$\begin{cases} g_1^{\mathbf{A}} = \ldots = g_k^{\mathbf{A}} = 0 \\ m_1 \frac{\partial g_1^{\mathbf{A}}}{\partial X_1} + \ldots + m_k \frac{\partial g_k^{\mathbf{A}}}{\partial X_1} = 1 \\ m_1 \frac{\partial g_1^{\mathbf{A}}}{\partial X_2} + \ldots + m_k \frac{\partial g_k^{\mathbf{A}}}{\partial X_2} = 0 \\ \vdots \\ m_1 \frac{\partial g_1^{\mathbf{A}}}{\partial X_n} + \ldots + m_k \frac{\partial g_k^{\mathbf{A}}}{\partial X_n} = 0 \end{cases}$$

and $J^{\mathbf{A}}$ the ideal it generates. From Lemma 7, the critical locus of π restricted to $C_d^{\mathbf{A}}$ is the algebraic variety associated to $J^{\mathbf{A}} \cap \mathbb{Q}[X_1, \ldots, X_n]$, which is consequently zero-dimensional if $\mathbf{A} \notin \mathcal{H}_d$.

Since the tangent space to $\mathcal{V}$ at any point p in $C_d \subset \mathcal{V}$ is equalled to the tangent space to C_d at p, and since $\langle g_1, \ldots, g_k \rangle$ is a radical ideal, the following holds:

$$\mathrm{Span}(\mathbf{grad}_p(g_1), \ldots, \mathbf{grad}_p(g_k)) = \mathrm{Span}(\mathbf{grad}_p(f_1), \ldots, \mathbf{grad}_p(f_s))$$

Moreover, for $\mathbf{A}.p \in C_d^{\mathbf{A}} \subset \mathcal{V}^{\mathbf{A}}$:

$$\mathbf{A}.\left(\mathrm{Span}(\mathbf{grad}_p(g_1), \ldots, \mathbf{grad}_p(g_k))\right) = \mathrm{Span}(\mathbf{grad}_{\mathbf{A}.p}(g_1^{\mathbf{A}}), \ldots, \mathbf{grad}_{\mathbf{A}.p}(g_k^{\mathbf{A}}))$$

and

$$\mathbf{A}.\left(\mathrm{Span}(\mathbf{grad}_p(f_1), \ldots, \mathbf{grad}_p(f_s))\right) = \mathrm{Span}(\mathbf{grad}_{\mathbf{A}.p}(f_1^{\mathbf{A}}), \ldots, \mathbf{grad}_{\mathbf{A}.p}(f_s^{\mathbf{A}}))$$

Thus, at each point $p_{\mathbf{A}} \in \mathcal{P}_d^{\mathbf{A}}$,

$$\mathrm{Span}(\mathbf{grad}_{p_{\mathbf{A}}}(g_1^{\mathbf{A}}), \ldots, \mathbf{grad}_{p_{\mathbf{A}}}(g_k^{\mathbf{A}})) = \mathrm{Span}(\mathbf{grad}_{p_{\mathbf{A}}}(f_1^{\mathbf{A}}), \ldots, \mathbf{grad}_{p_{\mathbf{A}}}(f_s^{\mathbf{A}}))$$

Thus, $\mathcal{P}_d^{\mathbf{A}}$ is exactly the algebraic variety associated to $J^{\mathbf{A}} \cap \mathbb{Q}[X_1, \ldots, X_n]$, which is zero-dimensional if $\mathbf{A} \notin \mathcal{H}_d$. Iterating the above on each equidimensional component of $\mathcal{V}$, we are done.

Now, suppose $f_1, \ldots, f_s$ to be a regular sequence in $\mathbb{Q}[X_1, \ldots, X_n]$ generating a radical ideal. Then the ideal $\langle f_1, \ldots, f_s \rangle$ is equidimensional of dimension $n - s$. Thus, at any point of $\mathcal{V}$ the jacobian matrix $\mathrm{Jac}(f_1, \ldots, f_s)$ associated to $f_1, \ldots, f_s$ has rank s. Consequently, at any point p in the algebraic variety associated to $I^{\mathbf{A}} \cap \mathbb{Q}[X_1, \ldots, X_n]$, the jacobian matrix $\mathrm{Jac}(f_1^{\mathbf{A}}, \ldots, f_s^{\mathbf{A}})$ has rank s since it equals $\mathbf{A}^{-1}.\mathrm{Jac}(f_1^{\mathbf{A}}, \ldots, f_s^{\mathbf{A}})$. Moreover, if $\mathbf{A}$ does not belong to a proper Zariski closed subset of $GL_n(\mathbb{C})$, the algebraic variety $\mathcal{P}^{\mathbf{A}}$ associated to $I^{\mathbf{A}} \cap \mathbb{Q}[X_1, \ldots, X_n]$ is zero-dimensional. Thus, for any point $p \in \mathcal{P}^{\mathbf{A}}$ there exists at most a finite set of points $(\lambda_1, \ldots, \lambda_s)$ in $\mathbb{P}(\mathbb{C})^s$ which is a solution of the linear system:

$$\lambda_1.\mathbf{grad}_p(f_1^{\mathbf{A}}) + \ldots + \lambda_s.\mathbf{grad}_p(f_s^{\mathbf{A}}) = \mathbf{u}$$

(where $\mathbf{u}$ has null coordinates except the first one), which ends the proof.

$\square$

We are now ready to state the main result of this section.

Theorem 2 *Let $f_1, \ldots, f_s$ in $\mathbb{Q}[X_1, \ldots, X_n]$ be s polynomials (with $s \leq n - 1$) generating a radical ideal, and $\mathcal{V} \subset \mathbb{C}^n$ the algebraic variety it defines and $\pi : \mathbb{C}^n \to \mathbb{C}$ the canonical projection on the first coordinate. Denote by $D \in \mathbb{N}$ an integer bounding the degree of $f_1, \ldots, f_s$, and d the dimension of $\langle f_1, \ldots, f_s \rangle$. There exists a hypersurface $\mathcal{H} \subset GL_n(\mathbb{Q})$ such that if $\mathbf{A} \notin \mathcal{H}$, the critical locus of π restricted to $\mathcal{V}^{\mathbf{A}}$ is zero-dimensional. Moreover :*

- *the number of critical points of π restricted to $\mathcal{V}^{\mathbf{A}}$ is bounded by:*

$$D^s(D-1)^{n-s}\binom{n}{n-s}$$

- *if $f_1, \ldots, f_s$ is a regular sequence in R, the number of critical points of π restricted to $\mathcal{V}^{\mathbf{A}}$ is bounded by:*

$$D^s(D-1)^{n-s}\binom{n-1}{n-s}$$

Proof. The fact that, up to a generic choice of $\mathbf{A} \in GL_n(\mathbb{Q})$, the critical locus of π restricted to $\mathcal{V}^{\mathbf{A}}$ is zero-dimensional is an immediate consequence of Lemma 8. Then, replacing each X_i by X_i/X_0 for $i = 1, \ldots, n$, each ℓ_j by ℓ_j/ℓ_0 for $j = 1, \ldots, s$ in the polynomials of the system given in Lemma 8, and taking the numerators, one obtains a bi-homogeneized system in $\mathbb{Q}[X_0, \ldots, X_n, \ell_0, \ldots, \ell_s]$. Since the Lagrange multipliers appear with degree 1, bounding the cardinality of the critical locus of π is equivalent to bound the strong bi-degree of the ideal generated by this polynomial system. Then, a simple application of Theorem 1 proves the first item of the Theorem.

Now, suppose that $f_1, \ldots, f_s$ is a regular sequence. Since $\mathcal{V}$ is smooth the jacobian matrix of $f_1, \ldots, f_s$ has rank $n-s$ at any point of $\mathcal{V}$. Then, the solutions $(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_s)$ of the polynomial system:

$$\left\{\begin{array}{l} f_1^{\mathbf{A}} = \ldots = f_s^{\mathbf{A}} = 0 \\ \ell_1 \frac{\partial f_1^{\mathbf{A}}}{\partial X_2} + \ldots + \ell_s \frac{\partial f_s^{\mathbf{A}}}{\partial X_2} = 0 \\ \\ \vdots \\ \ell_1 \frac{\partial f_1^{\mathbf{A}}}{\partial X_n} + \ldots + \ell_s \frac{\partial f_s^{\mathbf{A}}}{\partial X_n} = 0 \end{array}\right.$$

for which there exists $i \in \{1, \ldots, s\}$ such that $\lambda_i \neq 0$ are the critical points of π restricted to $\mathcal{V}$. One can then apply the bi-homogeneization process exhibited above and apply Theorem 1 on this system defining a bi-projective variety in $\mathbb{P}^n(\mathbb{C}) \times \mathbb{P}^{s-1}(\mathbb{C})$.

□

Remark 3 *Several corollaries can be obtained from the above result:*

- *In the case where $D = 2$, the bound we obtain is single exponential in the number of polynomials s and not in the number of variables. It explains the complexity of the algorithm provided in [10], which is also single exponential in s and computes the extrema of a quadratic mapping restreicted to an algebraic variety defined by quadratic equations.*

- *According to the proof of Lemma 7, in the case where $s = n-d$, and where $(f_1, \ldots, f_{n-d})$ defines a regular sequence in $\mathbb{Q}[X_1, \ldots, X_n]$ and generates a radical ideal, the critical locus of the projection π on the first coordinate can be defined as the complex solution set of the polynomial system:*

$$f_1 = \ldots = f_{n-d} = 0$$

and the vanishing of all the $(n-d, n-d)$ minors of the jacobian matrix associated to the above polynomial family with respect to the variables $X_2, \ldots, X_n$. Since these determinants have degree $(n-d)(D-1)$, applying the classical Bezout bound to such a system allows to bound the number of critical points of π by $D^{n-d}\left((n-d)(D-1)\right)^d$.

This quantity, which is greater than the bound given in Theorem 2, is used in [3, 2] to bound, in the worst case, the number of critical points computed by the algorithms proposed in these papers. A similar approach is used in [17, 5, 4] to give similar bounds on the number of critical points of a distance function in the worst case.

Our bound, given in Theorem 2, shows that the previous ones were not accurate enough, in particular in the case where $d = n/2$.

We show in the following section how our bound can be used to improve the already known bounds on the first Betti number of a smooth real algebraic set defined by a polynomial system generating a radical ideal.

4 Generalisation of Safey/Schost's Algorithm

In this section, we first generalize Safey-Schost's algorithm provided in [19] to the non equidimensional case. Then, we estimate the number of computed points by the algorithm we propose using Theorem 2 to bound the first Betti number of a smooth real algebraic set defined by a polynomial system generating a radical ideal.

Given a smooth algebraic variety $\mathcal{V} \subset \mathbb{C}^n$ of dimension d, we denote by Π_i (for i in $\{1, \ldots, d\}$) the canonical projection:

$$\begin{array}{cccc} \Pi_i : & \mathbb{C}^n & \longrightarrow & \mathbb{C}^i \\ & (x_1, \ldots, x_n) & \mapsto & (x_1, \ldots, x_i) \end{array}$$

and by $W_{n-(i-1)}(\mathcal{V})$ the critical locus of Π_i restricted to $\mathcal{V}$, i.e. the union of the critical points of Π_i restricted to each equidimensional component of $\mathcal{V}$. Following [19], we set $W_{n-d}(\mathcal{V}) = \mathcal{V}$ and we have:

$$W_n(\mathcal{V}) \subset W_{n-1}(\mathcal{V}) \subset \ldots \subset W_{n-d+1}(\mathcal{V}) \subset W_{n-d}(\mathcal{V})$$

In the equidimensional case, Safey Schost's algorithm is based on the following geometric result:

Theorem 3 *[19] Let $\mathcal{V} \subset \mathbb{C}^n$ be a smooth equidimensional algebraic variety of dimension d. Up to a generic linear change of variables, given an arbitrary point $p = (p_1, \ldots, p_d) \in \mathbb{R}^d$, $W_{n-(i-1)}(\mathcal{V}) \cap \Pi_{i-1}^{-1}(p_1, \ldots, p_{i-1})$ is zero-dimensional for i in $\{1, \ldots, d+1\}$, and the finite algebraic set:*

$$W_{n-d}(\mathcal{V}) \cap \Pi_d^{-1}(p_1, \ldots, p_d), \ldots, W_{n-(i-1)}(\mathcal{V}) \cap \Pi_{i-1}^{-1}(p_1, \ldots, p_{i-1}), \ldots, W_n(\mathcal{V})$$

intersects each connected component of the real counterpart of $\mathcal{V}$.

A naive way of using this result to compute at least one point in each connected component of a real algebraic set defined by a polynomial system

$$f_1 = \ldots = f_s = 0$$

generating a non equidimensional ideal is to compute an equidimensional decomposition of the ideal $\langle f_1, \ldots, f_s \rangle$ and apply Theorem 3 to each computed equidimensional component. This technique is underlying in many recent algorithms computing at least one point in each connected component of a real algebraic set (see [1, 20, 17]) and does not allow to prove satisfactory complexity results neither on the output of the algorithms nor on the arithmetic complexity, since the degree of the polynomials defining each equidimensional component is not well controlled.

Notations We denote by π_i the canonical projection:

$$\begin{array}{rccc} \pi_i : & \mathbb{C}^n & \longrightarrow & \mathbb{C} \\ & (x_1, \ldots, x_n) & \mapsto & x_i \end{array}$$

Lemma 9 *Let $(f_1, \ldots, f_s)$ be a polynomial family in $\mathbb{Q}[X_1, \ldots, X_n]$ generating a radical ideal of dimension d. Given a generic point $(p_1, \ldots, p_d)$ in $\mathbb{Q}^d$, consider the polynomial system in $\mathbb{Q}[\ell_1, \ldots, \ell_s, X_1, \ldots, X_n]$:*

$$\begin{array}{l} f_1 = \ldots = f_s = 0,\ X_1 - p_1 = 0, \ldots X_i - p_i = 0 \\ \ell_1 \frac{\partial f_1}{\partial X_1} + \ldots + \ell_s \frac{\partial f_s}{\partial X_1} = 1 \\ \ell_1 \frac{\partial f_1}{\partial X_2} + \ldots + \ell_s \frac{\partial f_s}{\partial X_2} = 0 \\ \vdots \\ \ell_1 \frac{\partial f_1}{\partial X_n} + \ldots + \ell_s \frac{\partial f_s}{\partial X_n} = 0 \end{array}$$

The projection of its complex solution set on $X_1, \ldots, X_n$ is $\Pi_i^{-1}(p_1, \ldots, p_i) \cap W_{n-i}(\mathcal{V})$.

Proof. Consider $(g_1, \ldots, g_k) \in \mathbb{Q}[X_1, \ldots, X_n]$ a polynomial family generating a radical ideal whose associated algebraic variety C_d is an equidimensional component of $\mathcal{V} \subset \mathbb{C}^n$. Let y be a point in $\Pi_i^{-1}(p_1, \ldots, p_i) \cap W_{n-i}(C_d)$ and $\mathbf{e}_1, \ldots, \mathbf{e}_{i+1}$ the gradient vectors of $X_1, \ldots, X_i$. Since $y \in W_{n-i}(C_d)$:

$$\dim(\mathrm{Span}(\mathbf{e}_1, \ldots, \mathbf{e}_{i+1}) + \mathrm{Span}(\mathbf{grad}_y(g_1), \ldots, \mathbf{grad}_y(g_k))) \leq n - d + i.$$

Since y is a regular point of C_d, $\dim(\mathrm{Span}(\mathbf{grad}_y(g_1), \ldots, \mathbf{grad}_y(g_k))) = n - d$.

Moreover, y satisfies the equations $X_i = p_i, \ldots, X_1 = p_1$, and since $(p_1, \ldots, p_i)$ is generic, $\Pi_i^{-1}(p_1, \ldots, p_i) \cap C_d$ has dimension $d - i$. This implies :

$$\dim(\mathrm{Span}(\mathbf{grad}_y(g_1), \ldots, \mathbf{grad}_y(g_k), \mathbf{e}_1, \ldots, \mathbf{e}_i)) = n - d + i$$

and then $\mathbf{e}_{i+1} \in \mathrm{Span}(\mathbf{grad}_y(g_1), \ldots, \mathbf{grad}_y(g_k), \mathbf{e}_1, \ldots, \mathbf{e}_i)$.

Since $(f_1, \ldots, f_s)$ generates a radical ideal and since C_d is an equidimensional component of $\mathcal{V}$,

$$\mathrm{Span}(\mathbf{grad}_y(g_1), \ldots, \mathbf{grad}_y(g_k)) = \mathrm{Span}(\mathbf{grad}_y(f_1), \ldots, \mathbf{grad}_y(f_s)).$$

which ends the proof.

□

Given a polynomial family $(f_1, \ldots, f_s) \subset \mathbb{Q}[X_1, \ldots, X_n]$ and d the dimension of the ideal they generate, and $\mathbf{A} \in GL_n(\mathbb{Q})$, for $i \in \{1, \ldots, d\}$ we denote by $I_i^{\mathbf{A}} \subset \mathbb{Q}[X_1, \ldots, X_n, \ell_1, \ldots, \ell_k]$ the ideal generated by:

$$\left\{\begin{array}{l} f_1^{\mathbf{A}} = \ldots = f_s^{\mathbf{A}} = 0, \\ X_1 - p_1 = 0, \ldots, X_i - p_i = 0 \\ \ell_1 \frac{\partial f_1^{\mathbf{A}}}{\partial X_1} + \ldots + \ell_1 \frac{\partial f_s^{\mathbf{A}}}{\partial X_1} = 1 \\ \ell_1 \frac{\partial f_1^{\mathbf{A}}}{\partial X_2} + \ldots + \ell_1 \frac{\partial f_s^{\mathbf{A}}}{\partial X_2} = 0 \\ \vdots \\ \ell_1 \frac{\partial f_1^{\mathbf{A}}}{\partial X_n} + \ldots + \ell_1 \frac{\partial f_s^{\mathbf{A}}}{\partial X_n} = 0 \end{array}\right.$$

Theorem 4 *Let $(f_1, \ldots, f_s) \subset \mathbb{Q}[X_1, \ldots, X_n]$ be a polynomial family generating a radical ideal and defining a smooth algebraic variety $\mathcal{V} \subset \mathbb{C}^n$ of dimension d, and $(p_1, \ldots, p_d) \in \mathbb{Q}^d$ a generic point. Then, there exists a hypersurface $\mathcal{H} \subset GL_n(\mathbb{Q})$ such that if $\mathbf{A} \notin \mathcal{H}$, for all $i \in \{1, \ldots, d\}$ the ideals $I_i^{\mathbf{A}} \cap \mathbb{Q}[X_1, \ldots, X_n]$ are either zero-dimensional or equal to $\langle 1 \rangle$ and the set of their real roots intersect each connected component of $\mathcal{V} \cap \mathbb{R}^n$.*

Proof. Let C_d be an equidimensional component of $\mathcal{V} \subset \mathbb{C}^n$ having dimension d. From Theorem 3, given an arbitrary point $(p_1, \ldots, p_d) \in \mathbb{Q}^d$, there exists a proper Zariski closed subset $\mathcal{H}_d \subset GL_n(\mathbb{Q})$ such that if $\mathbf{A} \notin \mathcal{H}_d$, then the union $\Pi_i^{-1}(p_1, \ldots, p_i) \cap W_{n-i}(C_d^{\mathbf{A}})$ for $i = 1, \ldots, d$ and $W_n^{\mathbf{A}}$ is zero-dimensional and its real locus intersect each connected component of the real counterpart of $\mathcal{V}$. The conclusion follows by applying Lemma 9.

□

Following the above result, after a generic choice of $\mathbf{A} \in GL_n(\mathbb{Q})$, the elimination ideals $I_i^{\mathbf{A}} \cap \mathbb{Q}[X_1, \ldots, X_n]$ are zero-dimensional or $\langle 1 \rangle$ and encode at least one point in each connected component of $\mathcal{V} \cap \mathbb{R}^n$. To obtain new bounds on the first Betti number of a smooth real algebraic variety, it suffices to sum the bound on the number of the critical points which are computed by applying Theorem 2 to each polynomial system defining the ideals I_i. This proves the following result.

Theorem 5 *Let $(f_1, \ldots, f_s) \subset \mathbb{Q}[X_1, \ldots, X_n]$ (with $s \leq n-1$) generating a radical ideal and defining a smooth algebraic variety $\mathcal{V} \subset \mathbb{C}^n$ of dimension d. Then, the number of connected components of $\mathcal{V} \cap \mathbb{R}^n$ is bounded by:*

$$D^s \sum_{i=0}^{d} (D-1)^{n-s-i} \binom{n-i}{n-i-s}$$

Moreover, if $(f_1, \ldots, f_s)$ is a regular sequence, the number of connected components of $\mathcal{V} \cap \mathbb{R}^n$ is bounded by:

$$D^s \sum_{i=0}^{n-s} (D-1)^{n-s-i} \binom{n-1-i}{n-i-s}$$

Is is easy to prove that $D^s \sum_{i=0}^{n-s} (D-1)^{n-s-i} \binom{n-1-i}{n-i-s}$ is less or equal to $D.(2D-1)^{n-1}$ which is the best known bound on the first Betti number (see [8]). Computer simulations show that $D^s \sum_{i=0}^{d} (D-1)^{n-s-i} \binom{n-i}{n-i-s}$ is less or equal to $D.(2D-1)^{n-1}$ for values D, n and s between 2 and 200.

References

[1] P. Aubry, F. Rouillier, and M. Safey El Din. Real solving for positive dimensional systems. *Journal of Symbolic Computation*, 34(6):543–560, 2002.

[2] B. Bank, M. Giusti, J. Heintz, and G.-M. Mbakop. Polar varieties and efficient real equation solving: the hypersurface case. *Journal of Complexity*, 13(1):5–27, 1997.

[3] B. Bank, M. Giusti, J. Heintz, and G.-M. Mbakop. Polar varieties and efficient real elimination. *Mathematische Zeitschrift*, 238(1):115–144, 2001.

[4] B. Bank, M. Giusti, J. Heintz, and L.-M. Pardo. Generalized polar varieties: geometry and algorithms. Technical report, Humboldt Universität, 2003.

[5] B. Bank, M. Giusti, J. Heintz, and L.-M. Pardo. The light is polar. *Unpublished manuscript*, 2003.

[6] S. Basu, R. Pollack, and M.-F. Roy. On the combinatorial and algebraic complexity of quantifier elimination. *Journal of ACM*, 43(6):1002–1045, 1996.

[7] S. Basu, R. Pollack, and M.-F. Roy. A new algorithm to find a point in every cell defined by a family of polynomials. In *Quantifier elimination and cylindrical algebraic decomposition*. Springer-Verlag, 1998.

[8] M. Coste. Introduction à la géométrie semi-algébrique. Polycopié, Institut de Recherche Mathématique de Rennes.

[9] C. A. Floudas, P. M. Pardalos, C. S. Adjiman, W. R. Esposito, Z. H. Gumus, S. T. Harding, J. L. Klepeis, C. A. Meyer, and C. A. Schweiger. *Handbook of test problems in local and global optimization*. Kluwer Academic Publishers, 1999.

[10] D. Grigoriev and D. De Klerk, E. Pasechnik. Finding optimum subject to few quadratic constraints in polynomial time. In *Proceedings of MEGA'2003*, 2003.

[11] D. Grigoriev and N. Vorobjov. Solving systems of polynomials inequalities in subexponential time. *Journal of Symbolic Computation*, 5:37–64, 1988.

[12] J. Heintz, G. Jerónimo, J. Sabia, J. San Martin, and P. Solerno. Intersection theory and deformation algorithm. the multi-homogeneous case. Manuscript, 2002.

[13] J. Heintz, M.-F. Roy, and P. Solernò. On the complexity of semi-algebraic sets. In *Proceedings IFIP'89 San Francisco, North-Holland*, 1989.

[14] J. Heintz, M.-F. Roy, and P. Solernò. On the theoretical and practical complexity of the existential theory of the reals. *The Computer Journal*, 36(5):427–431, 1993.

[15] D. Lazard. On the Bézout theorem. Unpublished manuscript.

[16] F. Rouillier, M.-F. Roy, and M. Safey El Din. Finding at least one point in each connected component of a real algebraic set defined by a single equation. *Journal of Complexity*, 16:716–750, 2000.

[17] M. Safey El Din. *Résolution réelle des systèmes polynomiaux de dimension positive.* PhD thesis, Université Paris 6, January 2001.

[18] M. Safey El Din. RAGLib. available at `http://www-calfor.lip6.fr/~safey/RAGLib`, 2003.

[19] M. Safey El Din and É. Schost. Polar varieties and computation of one point in each connected component of a smooth real algebraic set. ISSAC'03 Proceedings, 2003.

[20] M. Safey El Din and É. Schost. Properness defects of projections and computation of one point in each connected component of a real algebraic set. *Journal of Discrete and Computational Geometry*, 2004.

[21] I. Shafarevich. *Basic Algebraic Geometry 1.* Springer Verlag, 1977.

Unité de recherche INRIA Rocquencourt
Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)

Unité de recherche INRIA Lorraine : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex (France)
Unité de recherche INRIA Rennes : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex (France)
Unité de recherche INRIA Rhône-Alpes : 655, avenue de l'Europe - 38330 Montbonnot-St-Martin (France)
Unité de recherche INRIA Sophia Antipolis : 2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399