Differential Algebra for Derivations with Nontrivial Commutation Rules

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Abstract: The classical assumption of differential algebra, differential elimination theory and formal integrability theory is that the derivations do commute. That is the standard case arising from systems of partial differential equations written in terms of the derivations w.r.t. the independant variables. We inspect here the case where the derivations satisfy nontrivial commutation rules. That situation arises for instance when we consider a system of equations on the differential invariants of a Lie group action. We develop the algebraic foundations for such a situation. They lead to algorithms for completion to formal integrability and differential elimination.

Key-words: Differential Algebra - Differential Elimination - Differential Invariants

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Algèbre Différentielle avec des Dérivations Satisfaisant des Règles de Commutation non Trivial

Résumé : L'algèbre différentielle, les théories classiques de d'élimination différentielle et de l'intégrabilité formelle reposent sur l'hypothèse que les dérivations utilisées commutent. Cela reflète la situation des systèmes différntiels décrits à l'aide des dérivations par rapport aux variables indépendantes. Nous étudions ici la théorie dans le cas où les dérivations satisfont des règles de commuation non triviales. Cette situation apparait par exemple lorsqu'on considère un système d'équations portant sur les invariants différentiels de l'action d'un groupe de Lie. Nous développons les fondements algébriques pour une telle situation. Ceux-ci permettent de developper des algorithmes de complétion pour l'intégrabilité formelle et d'élimination.

Mots-clés : Algèbre Différentielle - Elimination Différentielle - Invariants Différentiels.

1 Introduction

We establish the bases of a differential algebra theory where the derivations do not commute but satisfy some non trivial relationships.

Classically [20, 12], to treat algebraic differential systems with independent variables $(t_1, \dots, t_m)$ and dependent variables $\mathcal{Y} = \{y_1, \dots, y_n\}$ we introduce the ring of differential polynomials $\mathcal{F}[\,y_\alpha \,|\, y \in \mathcal{Y}, \alpha \in \mathbb{N}^m\,]$ where $\mathcal{F}$ is a field of rational or meromorphic functions in $(t_1, \dots, t_m)$. $\mathcal{F}$ is naturally endowed with the commuting derivations $\delta_1 = \frac{\partial}{\partial t_1}, \dots, \delta_m = \frac{\partial}{\partial t_m}$. These derivations are extended to $\mathcal{F}[\,y_\alpha \,|\, y \in \mathcal{Y}, \alpha \in \mathbb{N}^m\,]$ by the formula $\delta_k(y_\alpha) = y_{\alpha+\epsilon_k}$ where ϵ_k is the m-uplet having 1 as k^{th} component and 0 otherwise. We call $\mathcal{Y}$ the set of differential indeterminates while $\{\, y_\alpha \,|\, y \in \mathcal{Y}, \alpha \in \mathbb{N}^m \,\}$ is the set of their derivatives.

In this paper we treat differential system in some differential indeterminates $\mathcal{Y} = \{y_1, \dots, y_n\}$ with derivations $\{\delta_1, \dots, \delta_m\}$ that do not commute but rather satisfy commutation rules of the type

$$\delta_i \delta_j = \delta_i \delta_j + \sum_{l=1}^{m} c_{ijl}\, \delta_l$$

where the coefficients c_{ijl} are polynomials in the derivatives of $\mathcal{Y}$.

Actually many a differential problem are better expressed with non commuting derivations. Our motivation for this generalization of differential algebra takes its root in a project initiated by E.L.Mansfield. One has to reckon that differential systems that are *too* symmetric lead to untractable computations for differential elimination software, as for instance [17, 22, 2]. The introduction of a ranking in the underlying algorithms indeed breaks the symmetry instead of using it to reduce the problem. Mansfield's original idea was to factor systems invariant under the action of a Lie group by their symmetry before their treatment. The moving frame construction proposed by M. Fels and P.J. Olver [6] provides the ingredients of a reduction by the symmetry. A first reduction was proposed in [18]. The present paper offers the differential algebra foundations for a different reduction. We shall give a quick example of this reduction as a motivating example. The complete description of the reduction used is presented and compared to the reduction of [18] in [10].

Differential elimination software were extended to deal with non commuting derivations [14, 15, 16] that satisfy commutation rules where the coefficients belong to a function field of the independent variables $(t_1, \dots, t_m)$. That choice alleviate the computation load as is demonstrated in [14] where classification problems are solved that way. A model theoretic treatment for such a situation is given in[23]. The algebraic version of Froebenius theorem [13] shows that this situation is isomorphic to the commutative case. The theoretical grounds we shall give in this paper have in fact a more general setting.

In Section 2 we outline on an example how the differential algebras we want to study arise. In Section 3 we define the ring of differential polynomials when the derivations do not commute nor satisfy commutation rules. In Section 4 we study the quotient of that formal ring with the relationships induced by the commutation rules that we wish to satisfy. We establish sufficient conditions for that quotient to be isomorphic to $\mathcal{F}[\, y_\alpha \,|\, y \in \mathcal{Y}, \alpha \in \mathbb{N}^m]$, as is the classical case, but with a different action of the derivations. The chapter concludes on an explicit definition of the derivations that implement the commutation rules. In Section 5 we outline the constructions leading to a characteristic decomposition algorithm. Indeed, once a couple of fundamental properties are exhibited there is a close parallel between the constructions in classical differential algebra and the extension of it presented in this paper. Recent developments were given in great details in the tutorial [9] and we shall avoid repetition.

Acknowledgement: E. Mansfield is the one who dragged me into this project and it has been such an exciting experience. I wish to express here my deep gratitude to her. She and I. Kogan spent precious time explaining me the moving frame constructions. It's been fun and I want to thank them both for that. Discussion with I. Kogan lead to the reduction for which I am developping this theory and the material in this paper took a definite turn while I visited M. Singer at MSRI. Discussions with him were absolutely decisive in finding the right way to look at the problem. I am greatful for that opportunity.

2 A motivating example

G. Metivier's request is to determine if the following system constrains s by two independent differential equations.

$$S \left\{ \begin{array}{rcl} s\,(\phi_{xx}+\phi_{yy}) + s_x\,\phi_x + s_y\,\phi_y + \phi & = & 0 \\ s\,(\psi_{xx}+\psi_{yy}) + s_x\,\psi_x + s_y\,\psi_y + \psi & = & 0 \\ \psi_x\,\phi_x + \psi_y\,\phi_y & = & 0. \end{array} \right.$$

To look at this system algebraically, we place ourselves in a *differential polynomial ring* with *differential indeterminates* $\{s, \phi, \psi\}$ and *derivations* $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ [20, 12]. The coefficient field $\mathcal{K}$ can be taken as anything between $\mathbb{Q}$ and $\mathbb{C}(x, y)$. To answer the question one could compute a *characteristic decomposition* of the radical differential ideal generated by the underlying differential polynomials w.r.t. a *ranking* that *eliminates* ψ and ϕ [9].

We can see that the system is rather symmetric and we want to exploit that fact to obtain the answer. One can check that the system is indeed left invariant by the following 7-dimensional Lie group action of the zeroth order jet space. This symmetry was in fact computed with the help of the MAPLE *Desolv* package [5]. A group element g is determined by a 7-tuple $(t, \rho, a, b, \mu, \nu) \in \mathcal{K}^7$. Its pull back action on the coordinate functions $((x, y), (s, \phi, \psi))$ of

$J^0(\mathcal{K}^2, \mathcal{K}^4)$ is given by the following expressions:

$$\begin{aligned} g^*x &= \frac{\left(1-t^2\right)}{\rho\,(1+t^2)}\,x - \frac{2\,t}{\rho\,(1+t^2)}\,y + \frac{a}{\rho} \\ g^*y &= \frac{2\,t}{\rho\,(1+t^2)}\,x + \frac{\left(1-t^2\right)}{\rho\,(1+t^2)}\,y + \frac{b}{\rho} \end{aligned}$$

$$g^*s = \frac{s}{\rho^2} \qquad g^*\phi = \frac{\phi}{\mu} \qquad g^*\psi = \frac{\psi}{\nu}$$

The system can thus be rewritten in terms of a set of *fundamental differential invariants* and the related two *invariant derivations* constructed by the moving frame method of [6].

A fundamental set of differential invariants $\{s_1, s_2, s_3, \psi_1, \psi_2, \phi_1, \phi_2\}$ was computed with the *Vessiot* package [1] and the *Groebner* library of MAPLE. The detailed computations are presented in [10].

$$s_1^2 := \frac{s_x^2 + s_y^2}{4\,s},$$

$$\begin{aligned} s_2 &:= \frac{s_{xy}(s_y^2 - s_x^2) + s_x s_y (s_{xx} - s_{yy})}{8\,s\,s_1^3}, & s_3 &:= \frac{s_x^2 s_{yy} + s_y^2 s_{xx} - 2\,s_x s_y s_{xy}}{8\,s\,s_1^3}, \\ \psi_1 &:= \frac{s_y\psi_x - s_x\psi_y}{2\,s_1\,\psi}, & \psi_2 &:= \frac{s_x\psi_x + s_y\psi_y}{2\,s_1\,\psi}, \\ \phi_1 &:= \frac{s_y\phi_x - s_x\phi_y}{2\,s_1\,\phi}, & \phi_2 &:= \frac{s_x\phi_x + s_y\phi_y}{2\,s_1\,\phi}. \end{aligned}$$

We can write down the two invariant derivations in terms of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$:

$$\begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \frac{\pm\sqrt{s\,(s_y^2 + s_x^2)}}{s_x^2 - s_y^2} \begin{pmatrix} -s_y & s_x \\ s_x & -s_y \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$$

What is needed though is the commutator of these derivations. We use the IVB program by I. Kogan [11] to compute it and we programmed an extension to rewrite the system in terms of the derivatives of the fundamental differential invariants. We obtain:

$$\delta_1\delta_2 - \delta_2\delta_1 = (s_3 - s_1)\delta_1 + s_2\delta_2$$

and

$$\mathcal{S}\left\{\begin{array}{c} \delta_1(\phi_1) + \delta_2(\phi_2) + \phi_1^2 + \phi_2^2 - \phi_1 s_2 + (s_1 + s_3)\phi_2 + 1 = 0, \\ \delta_1(\psi_1) + \delta_2(\psi_2) + \psi_1^2 + \psi_2^2 - \psi_1 s_2 + (s_1 + s_3)\psi_2 + 1 = 0, \\ \phi_1\,\psi_1 + \phi_2\,\psi_2 = 0. \end{array}\right.$$

Now, if generating, the set of differential invariants given above is not differentially independent. The construction of [6] allows to compute a generating sets of differential relationships among them.

$$\mathcal{Z}\left\{\begin{array}{rcl} \delta_1(s_1) & = & s_1 s_2 \\ \delta_1(s_2)-\delta_2(s_3) & = & s_3^2+s_2^2-s_1\,s_3 \\ \delta_1(\phi_2)-\delta_2(\phi_1) & = & \phi_1\,(s_3-s_1)+\phi_2\,s_2, \\ \delta_1(\psi_2)-\delta_2(\psi_1) & = & \psi_1\,(s_3-s_1)+\psi_2\,s_2. \end{array}\right.$$

What is suggested by this example is to consider the problem in the differential algebra where the set of differential indeterminates is $\mathcal{Y}=\{s_1,s_2,s_3,\phi_1,\phi_2,\psi_1,\psi_2\}$ and the set of derivations is $\Delta=\{\delta_1,\delta_2\}$. We consider the system $\mathcal{S}\cup\mathcal{Z}$. We face two obvious difficulties:

- the derivations do not commute, contrary to the basic assumption made in the classical differential algebra setting [20, 12, 9].
- the coefficients in the commutators are differential polynomials, contrary to the assumption of [14, 23].

Let us note now two *lucky* properties[1] that are important to the particular problem presented above. First, s_1,s_2,s_3 depend only on the original function s and its derivatives. Thus a differential relationship in s_1,s_2,s_3 will mean a differential relationship on the original differential indeterminate s. Second, the coefficients of the commutator of δ_1 and δ_2 depends only on (s_1,s_2,s_3). So, if we use a block elimination ranking $s_1,s_2,s_3\ll\phi_1,\phi_2,\psi_1,\psi_2$ we will be able to find additional relationships in (s_1,s_2,s_3) if there are any.

In this paper we present the algebraic foundations for the treatment of the problem in terms of those new differential indeterminates and the non commuting new derivations.

3 Differential polynomial rings with non commuting derivations

In this section we introduce a formal differential polynomial ring for strictly non commuting derivations. In classical differential polynomial rings the indeterminates are indexed by *terms*. Here the indeterminates here are indexed by *words*. We shall be concerned, in next section, with the quotient of this differential polynomial ring by the relationships induced by the commutation rules on the derivations.

3.1 Words and terms

Let $m\in\mathbb{N}\setminus\{0\}$ and $\mathbb{N}_m=\{1,\ldots,m\}$. We consider $\mathcal{W}_m$ the semi-group of *words* formed on $\mathbb{N}_m$: an element $I\in\mathcal{W}_m$ can be represented by an empty tuple () or a tuple $(i_1,i_2,\ldots,i_p)$ for some $p\in\mathbb{N}\setminus\{0\}$ and $i_k\in\mathbb{N}_m$. The length of a word $I=(i_1,\ldots,i_p)$ is $|I|=p$. For

[1] We in fact made the appropriate choices in the moving frame construction so that those arise

$I = (i_1, \ldots, i_p)$ and $J = (j_1, \ldots, j_q)$ two words of $\mathcal{W}_m$ we denote $I.J = (i_1, \ldots, i_p, j_1, \ldots, j_q)$ the *concatenation* of I and J. By extension, if $i_0 \in \mathbb{N}_m$ and $I = (i_1, i_2, \ldots, i_p) \in \mathcal{W}_m$ we write $i_0.I$ for the element $(i_0, i_1, \ldots, i_p)$ of $\mathcal{W}_m$. The word $()$ is the neutral element of the concatenation and therefore $(\mathcal{W}_m, .)$ is a monoid.

We have a natural semi-group morphism λ from $\mathcal{W}_m$ to the commutative semi-group $\mathbb{N}^m$ that associates to the word $I = (i_1, \ldots, i_p)$ the m-tuple $\alpha = (\alpha_1, \ldots, \alpha_m)$ where α_k is the cardinal of the set $\{j \in \mathbb{N}_m \,|\, i_j = k\}$. $\mathbb{N}^m$ is isomorphic to the semi-group of terms $\mathcal{T}_m(\xi) = \{\xi_1^{\alpha_1} \ldots \xi_m^{\alpha_m} \mid (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m\}$ in a set of m indeterminates $\xi = \{\xi_1, \ldots, \xi_m\}$. To avoid confusion with the different types of tuples, we call an element of $\mathbb{N}^m$ a *term* and we shall note $\mathcal{T}_m = \mathbb{N}^m$ for a unified notation system. As a general rule, words will be denoted by capital letter while terms will be denoted by Greek lower case letter.

A word $(i_1, \ldots, i_p)$ is *monotone* if $i_1 \leq i_2 \leq \ldots \leq i_p$. We note $\mathcal{M}_m$ the set of monotone words. The restriction of λ to $\mathcal{M}_m$ is one-to-one. For a word $I \in \mathcal{W}_m$ we shall note $\hat{I}$ the monotone word that has the same image as I by λ.

Example 3.1 If $m = 4$ and $I = (1,4,2,1,4,1) \in \mathcal{W}_m$ then $\lambda(I) = (3,1,0,2) \in \mathcal{T}_m$ and $\hat{I} = (1,1,1,2,4,4) \in \mathcal{M}_m \subset \mathcal{W}_m$.

We recall the definition of the *lexicographical order* on $\mathcal{W}_m$. Let $I = (i_1, \ldots, i_p)$ and $J = (j_1, \ldots, j_q)$ be elements of $\mathcal{W}_m$.

$$I \prec_{\text{lex}} J \text{ iff } \left\{ \begin{array}{l} \exists K \in \mathcal{W}_m,\ J = I.K \text{ and } |K| > 0 \\ \text{or} \\ \exists K, I_1, J_1 \in \mathcal{W}_m,\ i, j \in \mathbb{N}_m \text{ s.t. } I = K.i.I_1 \text{ and } J = K.j.J_1 \text{ and } i < j \end{array} \right.$$

A monotone word is lower, with respect to the lexicographical order, to any word obtained from it by permutation of the components.

3.2 Indexed indeterminates and ranking

To a finite set $\mathcal{Y}$ of indeterminates we associate two infinite sets of indexed indeterminates, on the one hand the ones indexed by terms $\mathcal{T}_m(\mathcal{Y}) = \{y_\alpha \mid y \in \mathcal{Y},\ \alpha \in \mathcal{T}_m\}$ called *derivatives* and on the other hand the ones indexed by words $\mathcal{W}_m(\mathcal{Y}) = \{y_I \mid y \in \mathcal{Y},\ I \in \mathcal{W}_m\}$ called *formal derivatives*. Let $\mathcal{K}$ be a field (of characteristic zero). We shall consider the polynomial rings $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]$ and $\mathcal{K}[\mathcal{T}_m(\mathcal{Y})]$.

We shall often abbreviate elements of $\mathcal{W}_m(\mathcal{Y})$. For instance $y_{()}$ becomes y and $y_{(i,j,k)}$ becomes $y_{i.j.k}$.

The semi-group morphism λ is extended to a $\mathcal{K}$-algebra morphism

$$\begin{array}{rccc} \lambda : & \mathcal{K}[\mathcal{W}_m(\mathcal{Y})] & \rightarrow & \mathcal{K}[\mathcal{T}_m(\mathcal{Y})] \\ & y_I & \mapsto & y_{\lambda(I)}. \end{array}$$

For clarity we shall also use the subset $\mathcal{M}_m(\mathcal{Y}) = \{y_I \mid y \in \mathcal{Y},\ I \in \mathcal{M}_m\} \subset \mathcal{W}_m(\mathcal{Y})$ of indeterminates indexed by monotone words that we may call *monotone derivatives.* The associated polynomial ring $\mathcal{K}[\mathcal{M}_m(\mathcal{Y})]$ is isomorphic to $\mathcal{K}[\mathcal{T}_m(\mathcal{Y})]$ through λ.

A *ranking*[2] on $\mathcal{T}_m(\mathcal{Y})$ is a total order $\prec$ on $\mathcal{T}_m(\mathcal{Y})$ such that

- $y_\alpha \prec y_{\alpha+\beta},\ \forall \alpha, \beta \in \mathcal{T}_m,\ \forall y \in \mathcal{Y}.$
- $y_\alpha \prec z_\beta \ \Rightarrow\ y_{\alpha+\gamma} \prec z_{\beta+\gamma},\ \forall \alpha, \beta, \gamma \in \mathcal{T}_m,\ \forall y, z \in \mathcal{Y}$

A ranking on $\mathcal{T}_m(\mathcal{Y})$ is extended to $\mathcal{W}_m(\mathcal{Y})$ by lexicographical order, that is:

$$y_I \prec z_J \ \Leftrightarrow \ \left\{ \begin{array}{l} y_{\lambda(I)} \prec_{\mathcal{T}_m} z_{\lambda(J)} \\ \text{or} \\ y_{\lambda(I)} = z_{\lambda(J)} \text{ and } I \prec_{\text{lex}} J \end{array} \right.$$

We shall speak directly of a *ranking on* $\mathcal{W}_m(\mathcal{Y})$. This assumes that if $y_{\hat{I}} \prec z_{\hat{J}}$ then $y_I \prec z_J$.

Proposition 3.2 *A ranking on $\mathcal{W}_m(\mathcal{Y})$ is a well order i.e. any strictly decreasing sequence of elements of $\mathcal{W}_m(\mathcal{Y})$ is finite.*

proof: A ranking on $\mathcal{T}_m(\mathcal{Y})$ refines the product order on $\mathbb{N}^m$. So by Dickson's lemma it is a well order. The preimage of an element of $\mathcal{T}_m(\mathcal{Y})$ by λ has a finite cardinal. So the result is true also of a ranking on $\mathcal{W}_m(\mathcal{Y})$ □

Given a ranking on $\mathcal{W}_m(\mathcal{Y})$ we can define for any element $p \in \mathcal{K}[\mathcal{W}_m(\mathcal{Y})] \setminus \mathcal{K}$, as is usual in differential algebra and in the theory of triangular sets [8, 9]:

- $\mathsf{lead}(p)$: the *leader* of p that is the highest ranking derivative in p
- $\mathsf{rank}(p)$: the *rank* of p that is the leader raised at the highest power appearing in p.
- $\mathsf{init}(p)$: the *initial* of p that is the coefficient of $\mathsf{rank}(p)$ in p considered as a polynomial in $\mathsf{lead}(p)$.
- $\mathsf{sep}(p)$: the *separant* of p that is the formal derivative of p w.r.t. $\mathsf{lead}(p)$
- $\mathsf{tail}(p)$: that is $p - \mathsf{init}(p)\,\mathsf{rank}(p)$.

A ranking on $\mathcal{W}_m(\mathcal{Y})$ induces a partial order on $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]$:

$$p \prec q \text{ if } \left\{ \begin{array}{l} p \in \mathcal{K} \text{ and } q \notin \mathcal{K} \\ \mathsf{lead}(p) \prec \mathsf{lead}(q) \\ \mathsf{lead}(p) = \mathsf{lead}(q) \text{ and } \deg(p, \mathsf{lead}(p)) < \deg(p, \mathsf{lead}(q)) \end{array} \right.$$

[2] in the usual language of Gröbner bases theory it is an admissible term ordering on the free module $\mathcal{K}[\xi]^{|\mathcal{Y}|}$.

DEFINITION 3.3 *A ranking is* orderly[3] *if whenever* $|I| < |J|$, *for* $I, J \in \mathcal{W}_m$, *then* $y_I < z_J$ *for any* $y, z \in \mathcal{Y}$.

A ranking is semi-orderly *if whenever* $|I| < |J|$ *then* $y_I \prec y_J$, *for all* $y \in \mathcal{Y}$.

3.3 Derivations

A derivation δ on $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]$ is a map from $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]$ to $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]$ that is $\mathcal{K}$-linear and s.t. $\delta(a\,b) = a\,\delta(b) + \delta(a)\,b$. On $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]$ we define m derivations $\delta_1, \ldots, \delta_m$ by

$$\delta_i|_{\mathcal{K}} = 0 \quad \text{and} \quad \delta_i(y_I) = y_{i.I}, \quad \forall i \in \mathbb{N}_m, \forall y_I \in \mathcal{W}_m(\mathcal{Y})$$

If $\prec$ is a ranking on $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]$ then we retrieve the classical conditions of compatibility with derivation:

- $y_I \prec \delta_i(y_I)$ for all $i \in \mathbb{N}_m$ and $y_I \in \mathcal{W}_m(\mathcal{Y})$
- $y_I \prec z_J \Rightarrow \delta_i(y_I) \prec \delta_i(z_J)$ for all $i \in \mathbb{N}_m$ and $y_I, z_J \in \mathcal{W}_m(\mathcal{Y})$,

For $I = (i_1, \ldots, i_p) \in \mathcal{W}_m$ we shall denote $\delta_{i_1} \circ \ldots \circ \delta_{i_p}$ by δ^I.

4 Differential polynomial rings with non trivial commutation rules

Our goal here is to define a differential polynomial ring in the differential indeterminates $\mathcal{Y} = \{y_1, \ldots, y_n\}$ where the derivations satisfy non trivial commutation rules. We gave an example of such a structure in Section 2.

That differential polynomial ring is obtained by considering the quotient of the formal differential polynomial ring $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]$ by the ideal generated by the set Ω of all relationships induced by the commutation rules on the derivations. We exhibit the conditions under which that quotient is an integral ring isomorphic to $\mathcal{K}[\mathcal{T}_m(\mathcal{Y})]$. The theorem is reminiscent of the Poincaré-Birkhoff-de Witt theorem for enveloping algebras. In the same line, we could not apply the results of [19] but it was a source of understanding and inspiration.

The proof is as follow. We first select a subset Γ of Ω. We show that, provided there exists a ranking that is compatible with the commutation rules, the quotient of the formal differential polynomial ring by (Γ) is isomorphic to $\mathcal{K}[\mathcal{T}_m(\mathcal{Y})]$. Also Γ defines a constructive normalization. We then prove that $(\Omega) = (\Gamma)$ under some natural conditions on the commutation rules.

That entitles us to define the sought differential structure on $\mathcal{K}[\mathcal{T}_m(\mathcal{Y})]$. In particular we shall give the action of the derivations explicitly.

[3] the definition of [12] is here adapted to indeterminates indexed by words.

4.1 Commutation rules and relationships

Let $\mathcal{M}$ be the free $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]$-module $\oplus_{i=1}^m \mathcal{K}[\mathcal{W}_m(\mathcal{Y})]\,\bar{\delta}_i$. To a family $\mathcal{C} = \{c_{ijk} \mid i,j,k \in \mathbb{N}_m\}$ of elements of $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]$ we associate the $\mathcal{K}$-bilinear map $[\cdot,\cdot] : \mathcal{M} \to \mathcal{M}$ with the following rules for $i,j \in \mathbb{N}_m$ and $p \in \mathcal{K}[\mathcal{W}_m(\mathcal{Y})]$:

$$\begin{array}{rcl} [\bar{\delta}_i, \bar{\delta}_j] & = & \sum_{k=1}^m c_{ijk}\,\bar{\delta}_k \\ [p\,\bar{\delta}_i, \bar{\delta}_j] & = & p\,[\bar{\delta}_i, \bar{\delta}_j] - \delta_j(p)\bar{\delta}_i \\ [\bar{\delta}_i, p\bar{\delta}_j] & = & p\,[\bar{\delta}_i, \bar{\delta}_j] + \delta_i(\alpha)\bar{\delta}_j \end{array}$$

We can consider the associative $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]$-algebra $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]\langle\bar{\Delta}\rangle$, where $\bar{\Delta} = \{\bar{\delta}_1, \ldots, \bar{\delta}_2\}$, in which the inner product (composition) $\circ$ satisfies

$$\bar{\delta}_i \circ (p \cdot \theta) = \delta_i(p) \cdot \theta + p \cdot \bar{\delta}_i \circ \theta, \quad \forall p \in \mathcal{K}[\mathcal{W}_m(\mathcal{Y})], \forall \theta \in \mathcal{K}[\mathcal{W}_m(\mathcal{Y})]\langle\bar{\Delta}\rangle,$$

where $\cdot$ is the product with an element of $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]$.

Let $\gamma = \{\gamma_{ij}\}_{i,j \in \mathbb{N}_m}$ be the elements in $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]\langle\bar{\Delta}\rangle$ defined as

$$\gamma_{ij} = \bar{\delta}_i \circ \bar{\delta}_j - \bar{\delta}_j \circ \bar{\delta}_i - [\bar{\delta}_i, \bar{\delta}_j].$$

Mapping $\bar{\delta}_i$ on δ_i, the elements of $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]\langle\bar{\Delta}\rangle$ can be considered as operators on $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]$. The actions of the γ_{ij} for instance is given by:

$$\begin{array}{rccl} \gamma_{ij} : & \mathcal{K}[\mathcal{W}_m(\mathcal{Y})] & \to & \mathcal{K}[\mathcal{W}_m(\mathcal{Y})] \\ & p & \mapsto & \delta_i\left(\delta_j(p)\right) - \delta_j\left(\delta_i(p)\right) - \sum_{l=1}^m c_{ijl}\delta_l(p) \end{array}$$

The γ_{ij} are derivations on $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]$ i.e. $\gamma_{ij}(p+q) = \gamma_{ij}(p) + \gamma_{ij}(q)$ and $\gamma_{ij}(p\,q) = \gamma_{ij}(p)\,q + p\,\gamma_{ij}(q)$. We shall refer to those derivations as the *commutation rules* of the derivations $\{\delta_1, \ldots, \delta_m\}$.

In the following we shall simply write δ_i for $\bar{\delta}_i$ and the multiplications $\circ$ and $\cdot$ shall be omitted.

A ranking on $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]$ is *compatible* with the commutation rules if for all $y_I \in \mathcal{W}_m(\mathcal{Y})$ and $i,j \in \mathbb{N}_m$ we have $[\delta_i, \delta_j](y_I) \prec y_{i.j.I}$. Note that it is not enough to check that $[\delta_i, \delta_j](y) \prec y_{j.i}$

Example 4.1 Assume $c_{211} = y_{2.1.1}\,y_2 = -c_{121}$ and $c_{212} = y_{2.1.1}\,y_1 = -c_{122}$. Then $[\delta_2, \delta_1](y) = 0$ but $[\delta_2, \delta_1](y_1) = y_{2.1.1}\,(y_2\,y_{1.1} - y_1\,y_{2.1})$

Let Ω be the set of all the generated *commutation relationships* induced by the γ_{ij}. That is

$$\Omega = \left\{\delta^J \gamma_{ij}(p) \mid i,j \in \mathbb{N}_m,\ J \in \mathcal{W}_m,\ p \in \mathcal{K}[\mathcal{W}_m(\mathcal{Y})]\right\}$$

We shall show that under some natural conditions on the coefficients $\{c_{ijl}\}_{i,j,l\in\mathbb{N}_m}$ of the commutation rules we have an isomorphism:

$$\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]/(\Omega) \cong \mathcal{K}[\mathcal{T}_m(\mathcal{Y})].$$

4.2 Normalization

In this section we exhibit a subset Γ of Ω such that $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]/(\Gamma)$ is isomorphic to $\mathcal{K}[\mathcal{T}_m(\mathcal{Y})]$.

Let

$$\Gamma = \{\delta^J \gamma_{ji}(y_I) \mid i,j \in \mathbb{N}_m, I, J \in \mathcal{W}_m \text{ s.t. } j > i,\, i.I \in \mathcal{M}_m\}.$$

Γ is stable under the actions of $\delta_1, \ldots, \delta_m$ and thus so is the generated ideal (Γ).

LEMMA 4.2 *Assume $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]$ is endowed with a ranking that is compatible with the commutation rules. Γ is a triangular set. Its set of leaders is the set of all the non monotone formal derivatives.*

PROOF: For a compatible ranking the leader of $\delta^J \gamma_{ji}(y_I)$ is $y_{J.j.i.I}$ when $j > i$. If $K = (k_1, \ldots, k_p) \in \mathcal{W}_m$ is not monotone then y_K is the leader of a single element of Γ. Taking

- t to be the smallest integer s.t. $(k_t, k_{t+1}, \ldots, k_p)$ is monotone. Note that $2 \leq t \leq p$ as K is not monotone.
- $j = k_{t-1}$ and $i = k_t$. Note that $j > i$.
- $I = ()$ if $t = p$ or $I = (k_{t+1}, \ldots, k_p)$ otherwise. Note that $i.I$ is monotone.
- $J = ()$ if $t = 2$ or $J = (k_1, \ldots, k_{t-2})$ otherwise

y_K is the leader of $\delta^J \gamma_{ji}(y_I)$. □

LEMMA 4.3 *Assume $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]$ is endowed with a ranking that is compatible with the commutation rules. Any element of $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]$ is equal modulo (Γ) to a polynomial of $\mathcal{K}[\mathcal{M}_m(\mathcal{Y})]$.*

PROOF: We proceed by contradiction. Among the polynomials of $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]$ the class of which modulo (Γ) does not include a polynomial in $\mathcal{K}[\mathcal{M}_m(\mathcal{Y})]$, take one, p, for which the highest ranking non monotone derivative y_K is minimal. This non monotone derivative is the leader of an element γ of Γ. Since γ is of degree one in y_K with initial 1, p can be rewritten modulo γ into a q s.t. the only non monotone derivatives that q contain rank lower than y_K. By the choice of p and y_K, q must be equal modulo (Γ) to a polynomial in monotone derivatives only. So must p then. □

We can write down an algorithm to rewrite any polynomial in $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]$ into a polynomial of $\mathcal{K}[\mathcal{M}_m(\mathcal{Y})]$ modulo (Γ). It consists at each step in rewriting the highest ranking non monotone derivative in the polynomial with the only element of Γ having this derivative as leader. As rankings are well orders (Proposition 3.2), the process terminates. We shall call that algorithm normal_Γ. For any $p \in \mathcal{K}[\mathcal{W}_m(\mathcal{Y})]$, $\mathsf{normal}_\Gamma(p) - p \in (\Gamma)$. We shall see in next lemma that it is indeed a normal form.

LEMMA 4.4 *Assume $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]$ is endowed with a ranking that is compatible with the commutation rules. (Γ) is a prime ideal and the set of monotone derivatives, $\mathcal{M}_m(\mathcal{Y})$, forms a maximally independent set modulo (Γ).*

PROOF: Consider $\Gamma' = \{\mathsf{lead}(\gamma) - \mathsf{normal}_\Gamma(\gamma - \mathsf{lead}(\gamma)) \mid \gamma \in \Gamma\}$. Evidently $(\Gamma) = (\Gamma')$. The only non monotone derivative in an element of Γ' is its leader and it appears with degree one and initial 1.

If A is a finite subset of Γ', A is a regular chain. The set of leaders of A is exactly the set of the non monotone derivatives appearing in A. There is a finite number of monotone derivatives appearing in A. By linearity of the elements of A in their leaders and the properties of regular chains, $(A) = (A):I_A^\infty$ is prime and that finite number of monotone derivatives is a transcendence basis for it [8].

If p belongs to (Γ'), there is a finite subset A of Γ' s.t. $p \in (A)$. From what precedes, p cannot involve monotone derivatives only. As (A) is prime there is no $p_1, p_2 \notin (A)$ s.t. $p = p_1 p_2$. There is thus no $p_1, p_2 \notin (\Gamma')$ s.t. $p = p_1 p_2$. Consequently $(\Gamma') = (\Gamma)$ is prime and the set of monotone derivatives is algebraically independent modulo (Γ). As any other derivative is algebraically dependent over those modulo (Γ') the conclusion follows. □

PROPOSITION 4.5 *Assume $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]$ is endowed with a ranking that is compatible with the commutation rules. The algorithm normal_Γ expresses a morphism from $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]$ to $\mathcal{K}[\mathcal{M}_m(\mathcal{Y})]$ the kernel of which is (Γ). $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]/(\Gamma)$ is isomorphic to $\mathcal{K}[\mathcal{T}_m(\mathcal{Y})]$.*

PROOF: By the previous two lemmas, for any $p \in \mathcal{K}[\mathcal{W}_m(\mathcal{Y})]$, there is a unique element in the class of p modulo (Γ) that belongs to $\mathcal{K}[\mathcal{M}_m(\mathcal{Y})]$. That element is $\mathsf{normal}_\Gamma(p)$. It follows that normal_Γ is a ring epimorphism from $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]$ to $\mathcal{K}[\mathcal{M}_m(\mathcal{Y})]$. Its kernel is (Γ).

Recall that $\mathcal{K}[\mathcal{M}_m(\mathcal{Y})]$ is isomorphic, through λ to $\mathcal{K}[\mathcal{T}_m(\mathcal{Y})]$. Thus $\lambda \circ \mathsf{normal}_\Gamma : \mathcal{K}[\mathcal{W}_m(\mathcal{Y})] \to \mathcal{K}[\mathcal{T}_m(\mathcal{Y})]$ is a ring epimorphism with (Γ) as kernel. It follows that $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]/(\Gamma)$ is isomorphic to $\mathcal{K}[\mathcal{T}_m(\mathcal{Y})]$. □

For visualisation we have the following commuting diagram.

$$\begin{array}{ccc}
\mathcal{K}[\mathcal{W}_m(\mathcal{Y})] & \xrightarrow{\mathsf{normal}_\Gamma} & \mathcal{K}[\mathcal{M}_m(\mathcal{Y})] \\
\downarrow & & \downarrow \lambda \\
\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]/(\Gamma) & \xrightarrow{\sim} & \mathcal{K}[\mathcal{T}_m(\mathcal{Y})]
\end{array}$$

4.3 Quotient ring

We shall prove in this section that given some natural conditions on the commutation rule the ideal $(\Gamma) = (\Omega)$, i.e. that (Γ) contains all the commutation relationships on $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]$.

PROPOSITION 4.6 *If the commutation rules satisfy:*

1. $[\delta_i, \delta_j] = -[\delta_j, \delta_i]$ *for all* $i, j \in \mathbb{N}_m$
2. $[[\delta_i, \delta_j], \delta_k] + [[\delta_j, \delta_k], \delta_i] + [[\delta_k, \delta_i], \delta_j] = 0$ *for all* $i, j, k \in \mathbb{N}_m$.

and if $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]$ *can be endowed with a semi-orderly ranking that is compatible with the commutation rules then* $\delta^I \gamma_{ij}(p) \in (\Gamma)$, *for all* $p \in \mathcal{K}[\mathcal{W}_m(\mathcal{Y})]$, $i, j \in \mathbb{N}_m$ *and* $I \in \mathcal{W}_m$.

Before proceeding with the proof let us make two comments on the hypotheses of the theorem to see that those conditions are quite natural and not really restrictive.

The two first condition are conditions on the coefficients $\{c_{ijl}\}_{i,j,l \in \mathbb{N}_m}$. The first is $c_{ijl} = -c_{jil}$ for all $i, j, l \in \mathbb{N}_m$. Note that the condition implies, for a compatible ranking, that $[\delta_i, \delta_j](y_I)$ ranks lower than $y_{j.i.I}$ as well as than $y_{i.j.I}$. As a consequence, for $y_I \in \mathcal{W}_m(\mathcal{Y})$, $\mathsf{normal}_\Gamma(y_I) = y_{\hat{I}} + p$, where $p \prec y_{\hat{I}}$.

The second condition is

$$\sum_{\mu=1}^{m} c_{ij\mu} c_{\mu kl} + c_{jk\mu} c_{\mu il} + c_{ki\mu} c_{\mu jl} = \delta_k(c_{ijl}) + \delta_i(c_{jkl}) + \delta_j(c_{kil}), \quad \forall i, j, k, l \in \mathbb{N}_m.$$

If any of those conditions is not satisfied, there is a polynomial of $\mathcal{K}[\mathcal{M}_m(\mathcal{Y})]$ that belongs to (Ω). For instance if the conditions $c_{ijl} + c_{jil} = 0$ are not all satisfied then $\gamma_{ij}(y) + \gamma_{ji}(y) = \sum_{l=1}^{m} (c_{ijl} - c_{jil}) y_l \in \Omega$. Assuming that the c_{ijl} are constants, it implies that there is a $\mathcal{K}$-linear dependency of the first order derivatives implied by the commutation relationship. Similarly, if the second condition is not satisfied, we shall find that some polynomial of $\mathcal{K}[\mathcal{M}_m(\mathcal{Y})]$ belong to (Ω). For example, if all the c_{ijk} belong to $\mathcal{K}$ and do not satisfy the condition $\sum_{\mu=1}^{m} c_{ij\mu} c_{\mu kl} + c_{jk\mu} c_{\mu il} + c_{ki\mu} c_{\mu jl} = 0$ then $y_i \in (\Omega)$ for all $i \in \mathbb{N}_m$. We thus understand that those conditions are highly desirable.

A sufficient condition for the existence of a semi-orderly ranking that is compatible with the commutation rules is that all the c_{ijl} are of order one or less: $\{c_{ijl}\}_{i,j,l} \subset \mathcal{K}[y_I \mid y \in \mathcal{Y}, |I| \leq 1]$.

PROOF: As (Γ) is stable under the actions of the derivations, it is enough to prove that $\gamma_{ij}(p) \in (\Gamma)$. Furthermore, as the γ_{ij} are derivations on $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]$, it is enough to prove that $\gamma_{ij}(y_I) \in (\Gamma)$ for all $i, j \in \mathbb{N}_m$ and all $y_I \in \mathcal{W}_m(\mathcal{Y})$.

As $[.,.]$ is assumed to be anti-symmetric we need only to consider the case $i > j$. The proof is by induction.

The result is true for any $y \in \mathcal{Y}$ and $I = ()$ as $\gamma_{ij}(y) \in \Gamma$ by construction.

Take $(i,j) \in \mathbb{N}_m \times \mathbb{N}_m$ such that $i > j$, $y \in \mathcal{Y}$ and $I \in \mathcal{W}_m$ such that $I \neq ()$. Assume that for any $(a,b,J,z) \in \mathbb{N}_m \times \mathbb{N}_m \times \mathcal{W}_m \times \mathcal{Y}$ such that $z_{a.b.J} \prec y_{i.j.I}$ we have $\gamma_{ab}(z_J) \in (\Gamma)$. We shall show that $\gamma_{ij}(y_I) \in (\Gamma)$.

As the ranking is compatible with the commutation rules and $[.,.]$ is anti-symmetric, we can write

$$y_I \equiv y_{\hat{I}} + \text{terms ranking lower than } y_{\hat{I}} \mod (\Gamma).$$

By induction hypothesis and because (Γ) is stable by derivation we have

$$\gamma_{ij}(y_I) \equiv \gamma_{ij}(y_{\hat{I}}) \mod (\Gamma)$$

We have to consider two cases. Either $j \preceq_{\text{lex}} \hat{I}$ or the contrary. In the first case $\gamma_{ij}(y_{\hat{I}}) \in \Gamma$ by construction.

In the second case, there is $k \in \mathbb{N}_m$ and $K \in \mathcal{W}_m$ s.t. $\hat{I} = k.K$ with $j > k$.

A straightforward calculation leads us to observe that

$$\begin{array}{rl} & \gamma_{ij}\,\delta_k - \delta_k\gamma_{ij} + \gamma_{jk}\,\delta_i - \delta_i\,\gamma_{jk} + \gamma_{ki}\,\delta_j - \delta_j\,\gamma_{ki} \\ = & [[\delta_i,\delta_j],\delta_k] + [[\delta_j,\delta_k],\delta_i] + [[\delta_k,\delta_i],\delta_j] + \sum_{l=1}^{m} c_{ijl}\gamma_{lk} + c_{jkl}\gamma_{li} + c_{kil}\gamma_{lj} \end{array}$$

Using the identity above and the Jacobi identity of the second condition we can write

$$\begin{array}{rcl} \gamma_{ij}(y_{\hat{I}}) & = & \gamma_{ij}(y_{k.K}) = \gamma_{ij}\,\delta_k(y_K) \\ & = & \delta_k\,\gamma_{ij}(y_K) + \delta_i\,\gamma_{jk}(y_K) + \delta_j\,\gamma_{ki}(y_K) \\ & & -\gamma_{jk}\,\delta_i(y_K) - \gamma_{ki}\,\delta_j(y_K) \\ & & +\sum_{l=1}^{m} c_{ijl}\gamma_{lk}(y_K) + c_{jkl}\gamma_{li}(y_K) + c_{kil}\gamma_{lj}(y_K) \\ & = & \delta_k\,\gamma_{ij}(y_K) + \delta_i\,\gamma_{jk}(y_K) + \delta_j\,\gamma_{ki}(y_K) \\ & & -\gamma_{jk}\,(y_{i.K}) - \gamma_{ki}\,(y_{j.K}) \\ & & +\sum_{l=1}^{m} c_{ijl}\gamma_{lk}(y_K) + c_{jkl}\gamma_{li}(y_K) + c_{kil}\gamma_{lj}(y_K) \end{array}$$

As $i > j > k$, $j.k.i.K \prec_{\text{lex}} i.j.k.K$ and $k.i.j.K \prec_{\text{lex}} i.j.k.K$. Thus $y_{j.k.i.K} \prec y_{i.j.\hat{I}} \preceq y_{i.j.I}$ and $y_{k.i.j.K} \prec y_{i.j.I}$. By induction hypothesis $\gamma_{jk}\,(y_{i.K})$ and $\gamma_{ki}\,(y_{j.K})$ belong to (Γ).

As $|i.j.K|$, $|j.k.K|$, $|k.i.K|$, on the one hand, and $|l.k.K|$, $|l.i.K|$, $|l.j.K|$, on the other one, are smaller than $|i.j.k.K| = |i.j.I|$, the associated derivatives $y_{i.j.K}$, $y_{j.k.K}$, $y_{k.i.K}$, $y_{l.k.K}$, $y_{l.i.K}$, $y_{l.j.K}$ rank lower than $y_{i.j.I}$ since the ranking is assumed to be semi-orderly. By induction hypothesis we can conclude that $\gamma_{ij}(y_K)$, $\gamma_{jk}(y_K)$, $\gamma_{ki}(y_K)$, $\gamma_{lk}(y_K)$, $\gamma_{li}(y_K)$, $\gamma_{lj}(y_K)$ belong to (Γ). As (Γ) is stable under the action of the elements of Δ, $\delta_k\,\gamma_{ij}(y_K)$, $\delta_i\,\gamma_{jk}(y_K)$ and $\delta_j\,\gamma_{ki}(y_K)$ belong to (Γ) and the conclusion follows. □

4.4 Derivations on the quotient ring

After summarizing what we obtained so far we shall endow $\mathcal{K}[\mathcal{T}_m(\mathcal{Y})]$ with explicit derivations that satisfy the commutation rules.

From now we assume that the commutation rules satisfy:

1. $[\delta_i, \delta_j] = -[\delta_j, \delta_i]$ for all $i, j \in \mathbb{N}_m$
2. $[[\delta_i, \delta_j], \delta_k] + [[\delta_j, \delta_k], \delta_i] + [[\delta_k, \delta_i], \delta_j] = 0$ for all $i, j, k \in \mathbb{N}_m$.

and that $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]$ is endowed with a semi-orderly ranking compatible with the commutation rules. Recall that it means that

$$[\delta_i, \delta_j](y_J) < y_{i.j.J} \quad \forall i, j \in \mathbb{N}_m,\ \forall J \in \mathcal{W}_m,\ y \in \mathcal{Y},$$

and that for any $y \in \mathcal{Y}$ and $I, J \in \mathcal{W}_m$ we have $|I| < |J| \Rightarrow y_I < y_J$.

We considered the set

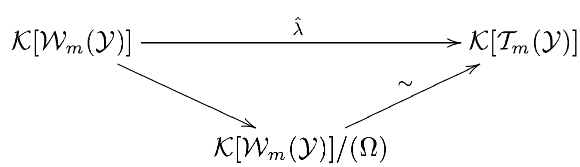
$$\Omega = \left\{\delta^J \gamma_{ij}(p) \mid i, j \in \mathbb{N}_m,\ J \in \mathcal{W}_m,\ p \in \mathcal{K}[\mathcal{W}_m(\mathcal{Y})]\right\}$$

and its subset

$$\Gamma = \{\delta^J \gamma_{ji}(y_I) \mid i, j \in \mathbb{N}_m, I, J \in \mathcal{W}_m \text{ s.t. } j > i,\ i.I \in \mathcal{M}_m\}.$$

to which is associated the algorithm normal_Γ that is an epimorphism from $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]$ to $\mathcal{K}[\mathcal{M}_m(\mathcal{Y})]$. In particular normal_Γ is the identity on $\mathcal{M}_m(\mathcal{Y})$. As $[.,.]$ is anti-symmetric, $\mathsf{normal}_\Gamma(y_I) = y_{\hat{I}} + p$ where $\hat{I}$ is the monotone word that is obtained by permutation of the components of I and p is a polynomial ranking strictly lower than $y_{\hat{I}}$.

In the above described situation, we proved that $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]/(\Gamma) \cong \mathcal{K}[\mathcal{T}_m(\mathcal{Y})]$ and that $(\Gamma) = (\Omega)$. We thus have $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]/(\Omega) \cong \mathcal{K}[\mathcal{T}_m(\mathcal{Y})]$ with the following commuting diagram:



where $\hat{\lambda} = \lambda \circ \mathsf{normal}_\Gamma$.

As (Ω) is stable under the actions of $\Delta = \{\delta_1, \ldots, \delta_m\}$, Δ also defines derivations on $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]/(\Omega)$ and we use the ring isomorphism between $\mathcal{K}[\mathcal{W}_m(\mathcal{Y})]/(\Omega)$ and $\mathcal{K}[\mathcal{T}_m(\mathcal{Y})]$ ito transport the differential structure.

Let $\hat{c}_{ijl} = \hat{\lambda}(c_{ijl})$. On $\mathcal{K}[\mathcal{T}_m(\mathcal{Y})]$ consider the set of derivations $\hat{\Delta} = \{\hat{\delta}_1, \ldots, \hat{\delta}_m\}$ so that $\hat{\delta}_i|_{\mathcal{K}} = 0$ and defined on the $y_\alpha \in \mathcal{T}_m(\mathcal{Y})$ recursively as follow:

$$\hat{\delta}_i\left(y_\alpha\right) = \begin{cases} y_{\alpha+\epsilon_i} & \text{if } \alpha_1 = \ldots = \alpha_{i-1} = 0 \\ \hat{\delta}_j \hat{\delta}_i\left(y_{\alpha-\epsilon_j}\right) + \sum_{l=1}^{m} \hat{c}_{ijl} \hat{\delta}_l\left(y_{\alpha-\epsilon_j}\right) & \begin{array}{l}\text{where } j < i \text{ is s.t. } \alpha_j > 0 \\ \text{while } \alpha_1 = \ldots = \alpha_{j-1} = 0\end{array} \end{cases}$$

where ϵ_i is the element of $\mathbb{N}^m$ having only 0 as components except at the i^{th} position where there is a 1. That definition is in fact adapted from [23].

PROPOSITION 4.7 *For all* $i \in \mathbb{N}_m$, $\hat{\lambda} \circ \delta_i = \hat{\delta}_i \circ \hat{\lambda}$.

PROOF: We only need to prove that $\hat{\lambda}\left(\delta_i y_I\right) = \hat{\delta}_i \hat{\lambda}(y_I)$ for all the $y_I \in \mathcal{W}_m(\mathcal{Y})$. The proof is by induction. The base cases are immediate: for any $y \in \mathcal{Y}$ and $i \in \mathbb{N}_m$, $\hat{\lambda}\left(\delta_i y\right) = \hat{\lambda}(y_i) = y_{\epsilon_i} = \hat{\delta}_i \hat{\lambda}(y)$.

Assume now that $\hat{\lambda}(\delta_k z_K) = \hat{\delta}_k \hat{\lambda}(z_K)$ for all $(k, K, z) \in \mathbb{N}_m \times \mathcal{W}_m \times \mathcal{Y}$ such that $z_{k.K} < y_{i.I}$ for some $(i, I, y) \in \mathbb{N}_m \times \mathcal{W}_m \times \mathcal{Y}$. We shall show the result is also true for (i, I, y).

Consider first the case where I is monotone. Let $\alpha \in \mathcal{T}_m$ be s.t. $\hat{\lambda}(y_I) = \lambda(y_I) = y_\alpha$ and $j \in \mathbb{N}_m$ s.t. $I = j.J$ for some $J \in \mathcal{W}_m$. If $i \leq j$ then $i.I$ is monotone and $\alpha_1 = \ldots = \alpha_{i-1} = 0$ so that

$$\hat{\lambda}(\delta_i y_I) = \hat{\lambda}(y_{i.I}) = y_{\alpha+\epsilon_i} = \hat{\delta}_i y_\alpha = \hat{\delta}_i \hat{\lambda}(y_I).$$

In the other case, when $i > j$, we have $\alpha_1 = \ldots = \alpha_{j-1} = 0$. We can write

$$\delta_i y_I = \delta_i \delta_j y_J \equiv \delta_j \delta_i y_J + \sum_{l=1}^{m} c_{ijl} \delta_l(y_J) \mod \Gamma$$

so that

$$\hat{\lambda}(\delta_i y_I) = \hat{\lambda}(\delta_j y_{i.J}) + \sum_{l=1}^{m} \hat{c}_{ijl} \hat{\lambda}(\delta_l y_J).$$

Since $j.i.J \prec_{\text{lex}} i.j.J = i.I$ we have $y_{j.i.J} \prec y_{i.I}$ so that by induction hypothesis $\hat{\lambda}(\delta_j y_{i.J}) = \hat{\delta}_j \hat{\lambda}(\delta_i y_J)$. As the ranking is semi-orderly $y_{i.J}$ and $y_{l.J}$ rank lower than $y_{i.I}$ since $|i.J| = |l.J| = |i.I| - 1$. By induction hypothesis we can thus write

$$\hat{\lambda}(\delta_i y_I) = \hat{\delta}_j \hat{\delta}_i \hat{\lambda}(y_J) + \sum_{l=1}^{m} \hat{c}_{ijl} \hat{\delta}_l \hat{\lambda}(y_J).$$

As $\hat{\lambda}(y_J) = y_{\alpha-\epsilon_j}$ we recognize that $\hat{\lambda}(\delta_i y_I) = \hat{\delta}_i \hat{\lambda}(y_I)$.

The case where I is not monotone is easily disposed off by induction since $y_I \equiv y_{\hat{I}} + p \mod (\Gamma)$ where p is a polynomial involving monotone derivatives ranking lower than $y_{\hat{I}}$. By the definition of ranking $y_{\hat{I}} \prec y_I$ and the conclusion follows easily. □

In next section we shall start working on $\mathcal{K}[\mathcal{T}_m(\mathcal{Y})]$ with the derivations $\hat{\Delta}$. The following property is then essential in adapting classical differential algebra to this new setting.

PROPOSITION 4.8 *Let $y \in \mathcal{Y}$, $\alpha, \beta \in \mathcal{T}_m$ and $J \in \mathcal{W}_m$ such that $\lambda_0(J) = \beta$. $\hat{\delta}^J(y_\alpha) = y_{\alpha+\beta} + r$ where $r \in \mathcal{K}[\mathcal{T}_m(\mathcal{Y})]$ involves only derivatives that are lower than $y_{\alpha+\beta}$.*

PROOF: As the ranking is compatible witht the commutation rules and $[\cdot,\cdot]$ is antisymmetric, for any $I \in \mathcal{W}_m$, $\mathsf{normal}_\Gamma(y_I) = y_{\hat{I}} + r$ where r involves monotone derivatives that are strictly lower than $y_{\hat{I}}$.

Let I be the monotone word s.t. $\lambda(I) = \alpha$. Then

$$\hat{\delta}^J(y_\alpha) = \hat{\delta}^J\left(\hat{\lambda}(y_I)\right) = \hat{\lambda}\left(\delta^J(y_I)\right) = \hat{\lambda}\left(y_{J.I}\right) = \lambda\left(y_{\widehat{J.I}} + r\right)$$

where r involves only monotone derivatives that are strictly lower than $y_{\widehat{J.I}}$. We thus have $\hat{\delta}^J(y_\alpha) = y_{\alpha+\beta} + \lambda(r)$ and $\lambda(r)$ involves only derivatives that are lower than $y_{\alpha+\beta}$. □

5 Constructive differential algebra

In last section we saw that if we can endow $\mathcal{K}[\mathcal{T}_m(\mathcal{Y})]$ with a semi-orderly ranking that is compatible with the commutation rules we can endow $\mathcal{K}[\mathcal{T}_m(\mathcal{Y})]$ with explicit derivations that provide the desired differential structure. We generalize the constructions of classical differential algebra that lead to the fundamental theorems and to effective algorithms. After exhibiting the fundamental basic properties, the definitions and proofs are not really different than in the classical case [12, 9]. We shall prove some key facts to see the tricks we have to introduce but for details we shall refer to [12] for fundamental theoretical results and to [9] for recent developments and algorithms.

5.1 Differential polynomial rings

Let $\mathcal{Y} = \{y_1, \ldots, y_n\}$ be a set of differential indeterminates. We consider the polynomial ring in infinitely many variables $\mathcal{T}_m(\mathcal{Y}) = \{y_\alpha \mid y \in \mathcal{Y},\ \alpha \in \mathbb{N}^m\}$, called the derivatives, with coefficient in a field $\mathcal{K}$ of characteristic zero. We endow $\mathcal{K}[\mathcal{T}_m(\mathcal{Y})]$ with a set of m derivations $\Delta = \{\delta_1, \ldots, \delta_m\}$ for which $\mathcal{K}$ is a field of constants and defined recursively on $\mathcal{T}_m(\mathcal{Y})$ by

$$\delta_i(y_\alpha) = \begin{cases} y_{\alpha+\epsilon_i} & \text{if } \alpha_1 = \ldots = \alpha_{i-1} = 0 \\ \delta_j \delta_i\left(y_{\alpha-\epsilon_j}\right) + \sum_{l=1}^{m} c_{ijl}\, \delta_l\left(y_{\alpha-\epsilon_j}\right) & \begin{array}{l}\text{where } j < i \text{ is s.t. } \alpha_j > 0 \\ \text{while } \alpha_1 = \ldots = \alpha_{j-1} = 0\end{array} \end{cases}$$

where the family $\{c_{ijl}\}_{i,j,l\in\mathbb{N}_m}$ of elements of $\mathcal{K}[\mathcal{T}_m(\mathcal{Y})]$ is such that for all $i,j,k,l \in \mathbb{N}_m$

- $c_{ijl} = -c_{jil}$
- $\sum_{\mu=1}^{m} c_{ij\mu}c_{\mu kl} + c_{jk\mu}c_{\mu il} + c_{ki\mu}c_{\mu jl} = \delta_k(c_{ijl}) + \delta_i(c_{jkl}) + \delta_j(c_{kil})$

For the sake of simplicity we introduce the following definition instead of the phrase *compatible semi-orderly ranking.*

Definition 5.1 *An* admissible ranking *on $\mathcal{K}[\mathcal{T}_m(\mathcal{Y})]$ is a total order $\prec$ on $\mathcal{T}_m(\mathcal{Y})$ s.t.*

- $|\alpha| < |\beta| \;\Rightarrow\; y_\alpha \prec y_\beta,\ \forall \alpha,\beta \in \mathcal{T}_m,\ \forall y \in \mathcal{Y}$,
- $y_\alpha \prec z_\beta \;\Rightarrow\; y_{\alpha+\gamma} \prec z_{\beta+\gamma},\ \forall \alpha,\beta,\gamma \in \mathcal{T}_m,\ \forall y,z \in \mathcal{Y}$.
- $\sum_{l\in\mathbb{N}_m} c_{ijl}\delta_l(y_\alpha)$ *ranks lower than* $y_{\alpha+\epsilon_i+\epsilon_j}$ *for all* $y_\alpha \in \mathcal{T}_m(\mathcal{Y})$ *and all* $i,j \in \mathbb{N}_m$.

Note that the first condition implies the usual condition $y_\alpha \prec y_{\alpha+\gamma},\ \forall y \in \mathcal{Y},\ \alpha,\gamma \in \mathcal{T}_m$.

If the c_{ijl} are differential polynomials that involve derivative of order one or less than any orderly ranking is admissible. Note nonetheless that for the motivating example it was possible to choose an elimination ranking. It is also possible to have the c_{ijl} involve derivatives of higher order.

Example 5.2 Assume $m = 1, n = 1$ and $c_{211} = y_{(2,0)}y_{(0,1)}$ and $c_{212} = -y_{(2,0)}y_{(1,0)}$. Then $c_{211}\delta_1(y) + c_{212}\delta_2(y) = 0$ and for all derivatives y_α with $|\alpha| \geq 1$ the property is clear.

If $\mathcal{K}[\mathcal{T}_m(\mathcal{Y})]$ can be endowed with an admissible ranking then, by the previous section,

$$\delta_i\delta_j(p) - \delta_j\delta_i(p) = \sum_{l=1}^{m} c_{ijl}\delta_l(p), \quad \forall p \in \mathcal{K}[\mathcal{T}_m(\mathcal{Y})].$$

In this case we shall say that $\mathcal{K}[\mathcal{T}_m(\mathcal{Y})]$ is a *differential polynomial ring with non trivial commutation rules for the derivations* $\Delta = \{\delta_1,\ldots,\delta_m\}$ and we write $\mathcal{K}[\![\mathcal{Y}]\!]$.

5.2 Differential ideals

A *differential ideal* in $\mathcal{K}[\![\mathcal{Y}]\!]$ is an ideal that is stable under the action of $\Delta = \{\delta_1,\ldots,\delta_m\}$. We shall note $[\Phi]$ the *differential ideal generated* by a non empty subset Φ of $\mathcal{K}[\![\mathcal{Y}]\!]$. $[\Phi]$ is defined as the intersection of all differential ideals containing Φ. The ideal generated by $\mathcal{W}_m(\Phi) = \{\delta^I\phi \mid I \in \mathcal{W}_m, \phi \in \Phi\}$ is a differential ideal. It is furthermore contained in all the differential ideals containing Φ. Thus $[\Phi] = (\mathcal{W}_m(\Phi))$. One shall show below that it is enough to consider derivation according to monotone words.

A differential ideal is *radical* if whenever a positive power of an element belongs to the differential ideal the element itself belongs to the differential ideal. $[\![\Phi]\!]$, the radical differential ideal generated by Φ, is defined as the intersection of all the radical differentials containing Φ. As in the classical setting, one shows that

$$[\![\Phi]\!] = \left\{ p \in \mathcal{K}[\![\mathcal{Y}]\!] \mid \exists k \in \mathbb{N} \setminus \{0\},\ p^k \in [\Phi] \right\}.$$

PROPOSITION 5.3 *For all $I \in \mathcal{W}_m$ with $|I| \geq 2$ there is a family $\{a_L\}_{L \in \mathcal{M}_m}$ of polynomial functions in $\left\{\delta^K(c_{ijl}) \mid |K| \leq |I| - 2\right\}$ with coefficients in $\mathbb{Z}$ such that*

$$\delta^I = \delta^{\hat{I}} + \sum_{\substack{L \in \mathcal{M}_m \\ |L| < |I|}} a_L\, \delta^L.$$

PROOF: For all $I \in \mathcal{W}_m$ there is a permutation σ_I that takes I to $\hat{I}$, a monotone word. Any permutation can be written as the composition of a finite number of transpositions of neighboring elements. We call length the smallest number of such transpositions needed to decompose a permutation.

The proof is by induction on $|I|$ and the length of σ_I. The base cases are trivial: the result is true if $|I| = 2$ since $\delta_i\delta_j = \delta_j\delta_i + \sum_{l=1}^m c_{ijl}\, \delta_l$ and if the length of σ_I is 0, meaning that I is monotone.

Take $I \in \mathcal{W}_m$, $|I| > 2$, and assume the result is true for all J s.t. either $|J| < |I|$ or $|J| = |I|$ but the length of σ_J is lower than the length of σ_I. We proceed to prove that the result is then true for I.

Take ρ to be the first transposition of neighboring elements in a minimal decomposition of σ_I and let J be the image of I by ρ. Then $\sigma_I = \sigma_J \circ \rho$ and the length of σ_J is strictly lower than the length of σ_I. There exist $L, K \in \mathcal{W}_m$ and $i, j \in \mathbb{N}_m$ s.t. $I = L.j.i.K$ and $J = L.i.j.K$. Thus

$$\begin{aligned} \delta^I &= \delta^L(\delta_i\delta_j + \sum_{l=1}^m c_{jil}\, \delta_l)\delta^K \\ &= \delta^J + \sum_{l=1}^m \sum_{M \star N = L} \delta^M(c_{ijl})\, \delta^N \delta_l \delta^K \end{aligned}$$

where $M \star N = L$ means that M is a subword of L and N is its complement in L.

On the one hand $|N.l.K| < |I|$ so that we can apply the induction hypothesis to $\delta^{N.l.K} = \delta^N\delta_l\delta^K$. On the other hand σ_J has a length strictly smaller than σ_I so that we can also apply the induction hypothesis on δ^J. As $|M| \leq |I| - 2$, the conclusion follows for δ^I. □

For $I = (i_1, i_2, \ldots, i_p) \in \mathcal{W}_m$ we used the notation $\delta^J = \delta^{i_1}\delta^{i_2}\ldots\delta^{i_p}$. For $\alpha \in \mathcal{T}_m$ we take the convention[4] that δ^α is δ^I where I is the only monotone word that corresponds to the term α.

[4]Note that this convention allows to minimize the recursion steps in the definition of derivations.

Corollary 5.4 *Let Φ be a non empty subset of $\mathcal{K}[\![\mathcal{Y}]\!]$. The differential ideal $[\Phi]$ is the ideal generated by $\mathcal{T}_m(\Phi) = \{\delta^\alpha \phi \mid \phi \in \Phi, \alpha \in \mathcal{T}_m\}$.*

That means that any element p of $[\Phi]$ can be written $p = \sum_{\phi \in \Phi, \alpha \in \mathcal{T}_m} a_{\alpha,\phi}\, \delta^\alpha \phi$ where the $a_{\alpha,\phi}$ form a family of $\mathcal{K}[\![\mathcal{Y}]\!]$ with finite support.

Note that $\delta^\alpha \delta^\beta \neq \delta^{\alpha+\beta}$. Instead Proposition 5.3 immediately implies the following.

Corollary 5.5 *For all pair $\alpha, \beta \in \mathcal{T}_m$ there is a family $\{a_\gamma\}_{\gamma \in \mathcal{T}_m}$ of polynomial functions in $\{\delta^\mu(c_{ijl}) \mid |\mu| \leq |\alpha+\beta| - 2\}$ with coefficients in $\mathbb{Z}$ such that*

$$\delta^\alpha \delta^\beta = \delta^{\alpha+\beta} + \sum_{|\gamma| < |\alpha+\beta|} a_\gamma\, \delta^\gamma.$$

Derivation of products have nonetheless that same nice formulas.

Proposition 5.6 *Let $p, q \in \mathcal{K}[\![\mathcal{Y}]\!]$ and $\alpha \in \mathcal{T}_m$. We have*

$$\delta^\alpha(p\,q) = \sum_{\mu+\nu=\alpha} \delta^\mu(p)\, \delta^\nu(q).$$

Lemma 5.7 *Let $p, q \in \mathcal{K}[\![\mathcal{Y}]\!]$ and $\alpha \in \mathcal{T}_m$. We have*

$$p^{|\alpha|+1} \delta^\alpha(q) \equiv p^{|\alpha|} \delta^\alpha(p\,q) \mod (\, \delta^\gamma(p\,q) \mid |\gamma| < |\alpha|\,).$$

proof: The proof is by induction on $|\alpha|$. The result is true if $|\alpha| = 0$. Take α with $|\alpha| > 0$ and assume the result is true for all β with $|\beta| < |\alpha|$. Consider $i \in \mathbb{N}_m$ the smallest index with $\alpha_i \neq 0$. Take $\beta = \alpha - \epsilon_i$ so that $|\beta| = |\alpha| - 1$ and $\delta_i \delta^\beta = \delta^\alpha$.

By induction hypothesis $p^{|\alpha|} \delta^\beta(q) \equiv p^{|\beta|} \delta^\beta(p\,q) \mod (\, \delta^\gamma(p\,q) \mid |\gamma| < |\beta|\,)$. So, we have on the one hand

$$p\, \delta_i \left(p^{|\alpha|}\, \delta^\beta(q) \right) \equiv |\beta|\, \delta_i(p) p^{|\beta|}\, \delta^\beta(p\,q) + p^{|\alpha|}\, \delta^\alpha(p\,q) \mod (\, \delta^\gamma(p\,q), \delta_i \delta^\gamma(p\,q) \mid |\gamma| < |\beta|\,).$$

and on the other hand

$$p\, \delta_i \left(p^{|\alpha|}\, \delta^\beta(q) \right) = |\alpha|\, \delta_i(p)\, p^{|\alpha|}\, \delta^\beta(q) + p^{|\alpha|+1}\, \delta^\alpha(q).$$

As the induction hypothesis implies $p^{|\alpha|}\, \delta^\beta(q) \in (\, \delta^\gamma(p\,q) \mid |\gamma| < |\alpha|\,)$ and Corollary 5.5 implies $\delta_i \delta^\gamma(p\,q) \equiv \delta^{\epsilon_i+\gamma}(p\,q) \mod (\, \delta^\mu(p\,q) \mid |\mu| < |\gamma|\,)$ we can conclude that

$$p^{|\alpha|+1}\, \delta^\alpha(p\,q) \equiv p\, \delta_i \left(p^{|\alpha|}\, \delta^\beta(q) \right) \equiv p^{|\alpha|}\, \delta^\alpha(p\,q) \mod (\, \delta^\gamma(p\,q) \mid |\gamma| < |\alpha|\,).$$

□

If H is a subset of $\mathcal{K}[\![\mathcal{Y}]\!]$ we denote by H^∞ the monoid of elements that divide a power product of elements of H. Let I be a differential ideal in $\mathcal{K}[\![\mathcal{Y}]\!]$. The saturation of I by H is defined as

$$I:H^\infty = \{\, p \in \mathcal{K}[\![\mathcal{Y}]\!] \mid \exists h \in H^\infty, h\,p \in I\,\}.$$

From the previous lemma it is easily shown that $I:H^\infty$ is a differential ideal.

5.3 Differential reduction and triangular sets

We assume that an admissible ranking is given on $\mathcal{K}[\![\mathcal{Y}]\!]$. Proposition 4.8 induces the following property that is the basis of differential reduction.

Proposition 5.8 *Assume that for $p \in \mathcal{K}[\![Y]\!]$ we have lead$(p) = y_\alpha$. For any $\beta \in \mathcal{T}_m$ with $|\beta| > 0$, rank $\left(\delta^\beta(p)\right) = y_{\alpha+\beta}$ and init $\left(\delta^\beta(p)\right) =$ sep(p).*

Let p, q be elements of $\mathcal{K}[\![\mathcal{Y}]\!]$, $q \notin \mathcal{K}$. and $y_\alpha = \mathsf{lead}(q)$. The differential polynomial p is *partially reduced* w.r.t. q if it involves no derivative of the type $y_{\alpha+\gamma}$ where $|\gamma| > 0$. It is *reduced* if additionnally the degree in y_α of p is lower than the one of q.

Thanks to Proposition 5.8 there is an algorithm pd-red based on derivation and pseudo-division that, given p, q as before, returns a differential polynomial that is partially reduced w.r.t. q with the property that

$$\mathsf{pd\text{-}red}(p,q) \equiv s\,p \mod [q]$$

where s is a power product of factors of $\mathsf{sep}(p)$. Similarly d-red computes a differential polynomial that is reduced w.r.t. q with the property that

$$\mathsf{d\text{-}red}(p,q) \equiv h\,p \mod [q]$$

where h is a power product of factors of $\mathsf{sep}(p)$ and $\mathsf{init}(p)$.

Indeed, if p is not partially reduced w.r.t. q there is a $y_{\alpha+\gamma}$, $|\gamma| > 0$ present in p. Take the highest ranking such derivative. As $\mathsf{rank}(\delta^\gamma q) = y_{\alpha+\gamma}$, the remainder of the (sparse) pseudo-division of p by $\delta^\gamma q$ w.r.t. $y_{\alpha+\gamma}$ involves no derivatives $y_{\alpha+\beta}$ with $y_{\alpha+\gamma} \preceq y_{\alpha+\beta}$. Proceeding inductively on the remainders we obtain a differential polynomial with the required properties for $\mathsf{pd\text{-}red}(p,q)$. Adding a pseudo-division by q we obtain the algorithm d-red.

We can thus define *(weak) differential triangular sets*, *differential chains* and *autoreduced sets* just as in [9, Section 3.2]. They are finite. If A is a differential triangular set we note respectively I_A and S_A the set of the initials and separants of its elements. Also $H_A = S_A \cup I_A$.

It is trivial to translate differential reduction by a weak differential triangular set from [9, Algorithm 3.12 and 3.13] as we did for reduction by a single differential polynomial. For a differential polynomial $p \in \mathcal{K}[\![\mathcal{Y}]\!]$ we can thus compute $\mathsf{d\text{-}red}(p, A)$ a differential polynomial that is reduced w.r.t. all the elements of A so that

$$\exists h \in H_A^\infty \text{ s.t. } h\,p \equiv \mathsf{d\text{-}red}(p, A) \mod [A]$$

and similarly for $\mathsf{pd\text{-}red}$ with $h \in S_A^\infty$.

5.4 Coherence

The keystone of characteristic decomposition algorithms in the classical case is coherence and the related Rosenfeld lemma. We proceed to show the result in our new setting. There is no essential difference with the classical case. We shall use nonetheless the fact that the ranking is semi-orderly in the finite test for coherence.

For $y_\alpha \in \mathcal{T}_m(\mathcal{Y})$ we note $\mathcal{T}_m(\mathcal{Y})_{<y_\alpha}$ the set of derivatives that rank lower than y_α. Let A be a differential triangular set. $\mathcal{T}_m(A)$ denotes the set $\{\delta^\alpha a \mid \alpha \in \mathcal{T}_m,\, a \in A\}$. We note $\mathcal{T}_m(A)_{<y_\alpha} = \mathcal{T}_m(A) \cap \mathcal{T}_m(\mathcal{Y})_{<y_alpha}$.

DEFINITION 5.9 *Let A be a d-triangular set in $\mathcal{F}[\![\mathcal{Y}]\!]$ and H a subset of $\mathcal{F}[\![\mathcal{Y}]\!]$. It is implicitly assumed that $S_A \subset H$ or otherwise we consider $H \cup S_A$ instead of H. A is said to be* coherent *away from H (or H-coherent for short) if whenever $a, b \in A$ are such that $\mathit{lead}(a) = y_\alpha$ and $\mathit{lead}(b) = y_\beta$ for some $y \in \mathcal{Y}$ and $\alpha, \beta \in \mathcal{T}_m$ then for any $\gamma \in (\alpha + \mathcal{T}_m) \cap (\beta + \mathcal{T}_m)$ we have*

$$\mathit{sep}(b)\,\delta^{\gamma-\alpha}(a) - \mathit{sep}(a)\,\delta^{\gamma-\beta}(b) \in (\mathcal{T}_m(A)_{<y_\gamma}) : H^\infty.$$

THEOREM 5.10 *Let A be a d-triangular set and $H \supset S_A$ a set of differential polynomials partially reduced w.r.t. A. If A is H-coherent then any differential polynomial of $[A] : H^\infty$ that is partially reduced w.r.t. A belongs to $(A) : H^\infty$.*

PROOF: For $a \in A$ we note u_a and s_a respectively the leader and the separant of a. Let us consider $p \in [A] : H^\infty$. There thus exists a finite subset D of $\mathcal{T}_m^+ \times A$ s.t. for some $h \in H^\infty$ we can write

$$h\,p = \sum_{(\alpha,a) \in D \subset \mathcal{T}_m{}^+ \times A} p_{\alpha,a}\,\delta^\alpha(a) + \sum_{a \in A} p_a\,a \tag{1}$$

for some $p_a, p_{\alpha,a} \in \mathcal{K}[\![\mathcal{Y}]\!]$. For each equation of type (1) we consider v to be the highest ranking derivative of $\mathsf{lead}\left(\mathcal{T}_m{}^+(A)\right)$ that appears effectively in the right hand side.

Assume that p is partially reduced w.r.t. A. If the set D is empty then $p \in (A) : H^\infty$. Assume, for contradiction, that there is no relation of type (1) with an empty D for p. Among all the possible relationships (1) that can be written, we consider one for which v is minimal.

Consider $E = \{(\alpha, a) \in D \mid \mathsf{lead}(\delta^\alpha(a)) = v\}$ and single out any $(\bar\alpha, \bar a)$ of E. As A is H-coherent, for all (α, a) of E we have $s_{\bar a}\, \delta^\alpha(a) \equiv s_a\, \delta^{\bar\alpha}(\bar a) \mod (\mathcal{T}_m(A)_{<v}) : H^\infty$. Thus

$$s_{\bar a}\, h\, p \equiv \left(\sum_{(\alpha,a)\in E} s_a\, p_{\alpha,a} \right) \delta^{\bar\alpha}(\bar a) + \sum_{(\alpha,a)\in D\setminus E} s_{\bar a} p_{\alpha,a}\, \delta^\alpha a + \sum_{a\in A} s_{\bar a} p_a\, a \mod (\mathcal{T}_m(A)_{<v}) : H^\infty \quad (2)$$

so that we can find $k \in H^\infty$ s.t.

$$k\, p = q_{\bar\alpha,\bar a}\, \delta^{\bar\alpha}(\bar a) + \sum_{\substack{(\alpha,a)\in \mathcal{T}_m^+ \times A, \\ \mathsf{lead}(\delta^\alpha(a)) < v}} q_{\alpha,a}\, \delta^\alpha a + \sum_{a\in A} q_a\, a \quad (3)$$

for some $q_a, q_{\alpha,a} \in \mathcal{F}[\![\mathcal{Y}]\!]$.

We proceed now to eliminate v from the coefficients $q_{\alpha,a}$ and q_a. We make use of the fact that $s_{\bar a}\, v = \delta^{\bar\alpha}(\bar a) - \mathsf{tail}(\delta^{\bar\alpha}(\bar a))$. Recall that $\mathsf{tail}(\delta^{\bar\alpha}(\bar a))$ contains only derivatives lower than v. Multiplying both sides of (3) by $s_{\bar a}^d$, where d is the degree of v in the right hand side, and replacing $s_{\bar a}\, v$ by $\delta^{\bar\alpha}(\bar a) - \mathsf{tail}(\delta^{\bar\alpha}(\bar a))$ we can rewrite the relationship obtained as

$$s_{\bar a}{}^d k\, p = r_d\, \delta^{\bar\alpha}(\bar a)^d + \ldots + r_1\, \delta^{\bar\alpha}(\bar a) + r_0 \quad (4)$$

where $r_0, r_1, \ldots, r_d$ no longer contain v and $r_0 \in (\mathcal{T}_m(A)_{<v})$. The only occurrences of v in that right hand side is through $\delta^{\bar\alpha}(\bar a)$. Because p and the elements of H are partially reduced w.r.t. A, v does not appear in the left hand side. The coefficients r_i, for $1 \le i \le d$ must be zero. We have thus exhibited a relationship like (1) with a v lower than what we started from. This contradicts our hypotheses. □

We proceed to give a finite test for coherence, after a preliminary lemma.

LEMMA 5.11 *Let A be a d-triangular set and H a finite subset of $\mathcal{K}[\![\mathcal{Y}]\!]$ and $y_\alpha \in \mathcal{T}_m(\mathcal{Y})$. If $p \in \left(\mathcal{T}_m(A)_{<y_\alpha}\right) : H^\infty$ then $\delta^\beta(p) \in \left(\mathcal{T}_m(A)_{<y_{\alpha+\beta}}\right) : H^\infty$*

PROOF: From Proposition 5.6 and Proposition 5.8 one easily deduces that if $q \in \left(\mathcal{T}_m(A)_{<y_\alpha}\right)$ then $\delta^\beta(q) \in \left(\mathcal{T}_m(A)_{<y_{\alpha+\beta}}\right)$.

As $p \in \left(\mathcal{T}_m(A)_{<y_\alpha}\right) : H^\infty$ there exists $h \in H^\infty$ s.t. $h\, p \in \left(\mathcal{T}_m(A)_{<y_\alpha}\right)$. By Lemma 5.7

$$h^{|\beta|+1} \delta^\beta(p) \equiv h^{|\beta|} \delta^\beta(h\, p) \mod (\delta^\gamma(h\, p) \mid |\gamma| < |\beta|)\,.$$

As the ranking is semi-orderly $|\gamma| < |\beta|$ implies that $y_{\alpha+\gamma} \prec y_{\alpha+\beta}$. So by induction on $|\beta|$ we prove that $h^{|\beta|+1}\, \delta^\beta(p) \in \left(\mathcal{T}_m(A)_{<y_{\alpha+\beta}}\right)$. The conclusion follows. □

For $\alpha, \beta \in \mathcal{T}_m$, we denote $\alpha \diamond \beta$ the element of $\mathcal{T}_m$ having $\max(\alpha_i, \beta_i)$ for i^{th} component. Any element of $(\alpha + \mathcal{T}_m) \cap (\beta + \mathcal{T}_m)$ can be written $\alpha \diamond \beta + \mu$ for some $\mu \in \mathcal{T}_m$.

Proposition 5.12 *Let A be a weak d-triangular set and H a subset of $\mathcal{K}[\![\mathcal{Y}]\!]$, $S_A \subset H$. If for all $a, b \in A$ s.t. $\mathit{lead}(a) = y_\alpha$ and $\mathit{lead}(b) = y_\beta$ for some $y \in \mathcal{Y}$ and $\alpha, \beta \in \mathcal{T}_m$ we have*

$$\Delta(a,b) = \mathit{sep}(b)\, \delta^{\alpha\diamond\beta-\alpha}(a) - \mathit{sep}(a)\, \delta^{\alpha\diamond\beta-\beta}(b) \in (\mathcal{T}_m(A)_{<y_{\alpha\diamond\beta}}) : H^\infty$$

then A is H-coherent.

PROOF: We deduce from Proposition 5.6 that for $h, p \in \mathcal{K}[\![\mathcal{Y}]\!]$ and $\gamma \in \mathcal{T}_m$, $\delta^\gamma(hp) \equiv h\delta^\gamma(p)$ mod $(\delta^\mu(p) \mid |\mu| < |\gamma|)$. Take $\gamma = \alpha\diamond\beta$ and $\gamma + \nu$ an element of $(\alpha + \mathcal{T}_m) \cap (\beta + \mathcal{T}_m)$. Then $\Delta(a,b) = \mathsf{sep}(b)\, \delta^{\gamma-\alpha}(a) - \mathsf{sep}(a)\, \delta^{\gamma-\beta}(b)$ and by the first remark

$$\begin{aligned}\delta^\mu\left(\Delta(a,b)\right) \quad \equiv \quad & \mathsf{sep}(b)\, \delta^\mu\delta^{\gamma-\alpha}(a) - \mathsf{sep}(a)\, \delta^\mu\delta^{\gamma-\beta}(b) \\ & \mod (\delta^\nu\delta^{\gamma-\alpha}(a), \delta^\nu\delta^{\gamma-\beta}(b) \mid |\nu| < |\mu|).\end{aligned}$$

According to the hypotheses and Lemma 5.11, $\delta^\mu\left(\Delta(a,b)\right)$ belongs to $(\Theta A_{<y_{\gamma+\mu}}) : H^\infty$. Now $\delta^\mu\delta^{\gamma-\alpha}(a) \equiv \delta^{\mu+\gamma-\alpha}(a) \mod (\delta^\nu(a) \mid |\nu| < |\mu+\gamma-\alpha|)$ by Corollary 5.5 and $\mathsf{lead}(\delta^\nu(a)) = y_{\alpha+\nu} < y_{\gamma+\mu}$ for $|\nu| < |\mu+\gamma-\alpha|$ by Proposition 5.8 and the fact that the ranking is semi-orderly. Similarly for $\delta^\mu\delta^{\gamma-\beta}(b)$. The conclusion follows. □

5.5 Characteristic decomposition

The definitions and first properties of differential characteristic sets, characterizable differential ideals and characteristic decomposition are exposed in details in [9, Section 3.3 and 5]. This is all based on the grounds of differential reduction. We saw that that goes through without difficulty. We just sketch the content of those sections in our new setting.

We define *characteristic sets* of differential ideals as differential chains of minimal rank in those differential ideals. Any differential ideal admits a characteristic set. If C is a characteristic set of a differential ideal I then $C \subset I \subset [C] : H_C^\infty$ and $\forall p \in I, \mathsf{d\text{-}red}(p, C) = 0$.

We can define as in [7, 9] *characterizable* differential ideals. They are the differential ideals $[C] : H_C^\infty$ where C is a characteristic set of $[C] : H_C^\infty$. Consequently $p \in [C] : H_C^\infty \Leftrightarrow \mathsf{d\text{-}red}(p, C) = 0$. As expanded upon in [9, Section 5.1] C is a characteristic set of $[C] : H_C^\infty$ iff C is a *regular differential chain*. In that case $[C] : H_C^\infty = [C] : S_C^\infty$. Note that prime differential ideals are characterizable for any admissible ranking. Some radical differential ideals though can be characterizable for some ranking but not for another one.

Rosenfeld's lemma is the keystone of the characteristic decomposition algorithms [12, 3, 4] in the classical case. For the algorithms of [7, 9] we need another major ingredient: the fact that any radical differential ideal is the intersection of finitely many prime ideals. That fact is an easy consequence of the basis theorem that asserts that any radical differential ideal is finitely generated (as a radical differential ideal). The proof of the basis theorem goes through into our new setting. In the appendix we revisit the one in [12] in our new setting together with the result on the decomposition into prime differential ideals. As we proved a version of Rosenfeld's lemma in our new setting, all the algorithms go through. We thus only state the existence of those algorithms in our new setting.

Assume $\mathcal{K}[\![\mathcal{Y}]\!]$ is endowed with an admissible ranking. Consider Φ be a finite set of differential polynomials in $\mathcal{K}[\![\mathcal{Y}]\!]$. Algorithms [12, 3, 7, 4, 9] are easily translated to compute a finite set $\mathcal{C}$ of regular differential chains such that

$$[\![\Phi]\!] = \bigcap_{C \in \mathcal{C}} [C] : S_C^\infty.$$

for any finite subset Φ of $\mathcal{K}[\![\mathcal{Y}]\!]$.

Those characteristic decomposition algorithm allow to test membership and answer differential elimination questions. For examples of use of characteristic decomposition, in the classical case, see [9, Section 8]

Using the recursive formulation of derivation, the software *diffalg* [2] that implements an algorithm derived from [3, 7, 9] can be adapted to this new and more general situation at a small cost.

A The basis theorem

We adapt here the proof of the basis theorem as dissiminated over several chapters in [12]. Beside rewriting it in our new setting we make the simplifications owing to the the fact that we are dealing with characteristic zero only. We also give the result about the decomposition of radical differential ideals into prime ideals that is an easy consequence of the basis theorem.

PROPOSITION A.1 *For all subset $\Phi, \Psi \subset \mathcal{K}[\![\mathcal{Y}]\!]$ we have $[\![\Phi]\!] \cap [\![\Psi]\!] = [\![\Psi \cdot \Phi]\!]$, where $\Psi \cdot \Phi$ denotes the set of all the products of the pairs in $\Psi \times \Phi$.*

PROOF: Trivially $\Psi \cdot \Phi$ is a subset of $[\![\Phi]\!]$ and $[\![\Psi]\!]$ and therefore of their intersection. Conversely, take $p \in [\![\Phi]\!] \cap [\![\Psi]\!]$. There exist $r, s \in \mathbb{N} \setminus \{0\}$ s.t.

$$\begin{array}{rcl} p^r & = & \sum_{\phi \in \Phi, \alpha \in \mathcal{T}_m} a_{\phi,\alpha} \delta^\alpha(\phi) \\ p^s & = & \sum_{\psi \in \Psi, \beta \in \mathcal{T}_m} b_{\psi,\beta} \delta^\beta(\psi) \end{array}$$

so that

$$p^{r+s} \quad = \quad \sum \; \phi \in \Phi, \psi \in \Phi, \alpha, \beta \in \mathcal{T}_m \quad a_{\phi,\alpha}\, b_{\psi,\beta} \delta^\alpha(\phi)\, \delta^\beta(\psi)$$

From Lemma 5.7 we easily deduce that $\delta^\alpha(\phi)\, \delta^\beta(\psi) \in [\![\phi\, \psi]\!]$ so that $p \in [\![\Psi \cdot \Phi]\!]$. □

LEMMA A.2 *Let $\mathcal{F}$ be the set of radical differential ideals in $\mathcal{K}[\![\mathcal{Y}]\!]$ that are not finitely generated. $\mathcal{F}$ has a maximal element (for the partial order of inclusion) and all maximal elements are prime.*

PROOF: The first part is by Zorn's lemma that says: if every chain in $\mathcal{F}$ has an upper bound then $\mathcal{F}$ has a maximal element. Consider $\{J_i\}_{i \in \mathbb{N}}$ a family in $\mathcal{F}$ forming a chain, i.e. s.t. $J_i \subsetneq J_{i+1}$. $J = \bigcup_{i=1}^{\infty} J_i$ is also a radical differential ideal. If there existed a finite subset Φ s.t. $J = [\![\Phi]\!]$ then Φ would be contained in some J_k for some $k \in \mathbb{N}$ and we would have $J_k = J_{k+1} = \ldots = J = [\![\Phi]\!]$. Thus J belongs to $\mathcal{F}$.

Assume now that P is a maximal element of $\mathcal{F}$ and take $a, b \notin P$. As P is maximal in $\mathcal{F}$, there exist Φ, Ψ finite subsets of $\mathcal{K}[\![\mathcal{Y}]\!]$ s.t. $[\![P \cup \{a\}]\!] = [\![\Phi]\!]$ and $[\![P \cup \{b\}]\!] = [\![\Psi]\!]$. As

$$[\![P \cup \{a\, b\}]\!] = [\![P \cup \{a\}]\!] \cap [\![P \cup \{b\}]\!] = [\![\Phi]\!] \cap [\![\Psi]\!] = [\![\Phi \cdot \Psi]\!]$$

$a\, b$ cannot belong to P since otherwise it would contradict the hypothesis that P is not finitely generated. Thus P is rime. □

THEOREM A.3 (THE BASIS THEOREM). *For any radical differential ideal J in $\mathcal{K}[\![\mathcal{Y}]\!]$ there exists a finite subset Φ of $\mathcal{K}[\![\mathcal{Y}]\!]$ such that $J = [\![\Phi]\!]$.*

PROOF: Assume for contradiction that the set $\mathcal{F}$ of radical differential ideals that are not finitely generated is non empty. By Lemma A.2, we can consider P a maximal element in

$\mathcal{F}$. P is prime. Let C be a characteristic set for P and h be the product of the initials and separants of C. Then $P = [C]:H_C^\infty = [\![C]\!]:h$.

As $h \notin P$ and P is maximal in $\mathcal{F}$ there exists a finite subset Φ of $\mathcal{K}[\![\mathcal{Y}]\!]$ s.t. $[\![P \cup \{h\}]\!] = [\![\Phi]\!]$. Any element of Φ can be written as a linear combination of h and its derivatives and elements of P. Let Ψ be the finite set of elements of P coming into those linear combinations. Then $[\![P \cup \{h\}]\!] = [\![\Phi]\!] = [\![\{h\} \cup \Psi]\!]$.

As $P = P:h \cap [\![P \cup \{h\}]\!] = P \cap [\![\{h\} \cup \Psi]\!]$, by Corollary A.1, $P = [\![P \cdot (\{h\} \cup \Psi)]\!]$ and as $P \cdot \Psi \subset [\![\Psi]\!]$ we have $P = [\![(P \cdot \{h\}) \cup \Psi]\!]$. Now $P = [\![C]\!]:h$ so that $P \cdot \{h\} \subset [\![C]\!] \subset P$. Thus $P \subset [\![C \cup \Psi]\!] \subset P$. It follows that $P = [\![C \cup \Psi]\!]$ is finitely generated. □

THEOREM A.4 *In $\mathcal{K}[\![\mathcal{Y}]\!]$ any radical differential ideal is the intersection of a finite number of prime differential ideals. The set of prime differential ideals coming into such a decomposition with no superfluous component is unique.*

PROOF: As any radical differential ideal of $\mathcal{K}[\![\mathcal{Y}]\!]$ is finitely generated any ascending chain of radical differential ideal is finite. Thus among any set of radical differential ideal there is one that is maximal with respect to inclusion.

Let p be an element that does not belong to J. We show that there is a prime differential ideal that contains J and not p. Take P to be a radical differential ideal that contains J and not p that is maximal w.r.t. that property. Take $a, b \in \mathcal{K}[\![\mathcal{Y}]\!]$ s.t. $ab \in P$ so that $P = [\![P \cup \{a\}]\!] \cap [\![P \cup \{b\}]\!]$. If neither a nor b belonged to P then p would belong to $[\![P \cup \{a\}]\!]$ and $[\![P \cup \{b\}]\!]$ by the maximality hypothesis on P. This cannot be the case since p does not belong to their intersection. P is prime. Thus J is the intersection of all the prime differential ideals that contain it.

Assume the set of radical differential ideals that cannot be written as a finite intersection of prime differential ideal is not empty and consider J a maximal element in that set. Obviously J is not prime. There thus exist $a, b \notin J$ s.t. $ab \in J$. Then $J = [\![J \cup \{a\}]\!] \cap [\![J \cup \{b\}]\!]$. Since $J \subsetneq [\![J \cup \{a\}]\!]$ it must be that $[\![J \cup \{a\}]\!]$ is an intersection of a finite number of prime differential ideals and similarly for $[\![J \cup \{b\}]\!]$. Thus J has to be an intersection of a finite number of prime differential ideals.

Assume that $J = \bigcap_{P \in \mathcal{P}} P = \bigcap_{Q \in \mathcal{Q}} Q$ where $\mathcal{P}$ and $\mathcal{Q}$ are finite set of prime differential ideals such that no element of it contains another one. Then for any $P \in \mathcal{P}$, $\bigcap_{Q \in \mathcal{Q}} Q \subset P$ so that there exists Q such that $Q \subset P$. Similarly there is a $P' \in \mathcal{P}$ s.t. $P' \subset Q$. It must be that $P = Q = P'$. Thus $\mathcal{P} = \mathcal{Q}$. □

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