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Abstract: In this note, we show that if two surfaces in are homeomorphic, then a simple and purely topological condition is sufficient to ensure the existence of an isotopy between them. When the surfaces are connected, the condition is merely that one surface is contained in some topological thickening of the other and separates the two boundary components of that thickening. The proof of this result is based on basic 3-manifold topology.

Key-words: isotopy, 3-manifold topology

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Une condition assurant l’isotopie entre deux surfaces

Résumé : Dans cette note, nous montrons que quand deux surfaces sont homéomorphes, une condition simple et purement topologique suffit à montrer l’existence d’une isotopie entre elles. Quand les surfaces sont connexes, la condition est simplement qu’une des surfaces est contenues dans un épaississement topologique de l’autre et sépare les deux composantes du bord de cet épaississement. La preuve de ce résultat est basée sur la théorie des 3-variétés.

Mots-clés : isotopie, topologie des 3-variétés

The goal of this report is to prove the following proposition, which can be useful for guaranteeing the topological correctness of the approximation of a surface. The background on 3-manifold topology required for the proof of this proposition can be found in [1]. In the following, all maps and manifolds considered are $\mathscr{C}^\infty$.

Proposition 1 *Let $\hat{S}$ be a orientable compact connected surface without boundary and let S be a surface such that*

- *$\hat{S}$ is homeomorphic to S,*
- *S is embedded in $V = \hat{S} \times [0,1]$,*
- *$S \cap (\hat{S} \times \{0\}) = \emptyset$ and $S \cap (\hat{S} \times \{1\}) = \emptyset$,*
- *$V \setminus S$ has two connected components, one containing $\hat{S} \times \{0\}$ and the other one containing $\hat{S} \times \{1\}$.*

Then S is isotopic to $\hat{S}$ in V.

For technical reasons our proof of proposition 1 does not work when $\hat{S}$ is a sphere. Fortunately, isotopy always holds in this case, since there is no smooth knotted 2-sphere in $\mathbb{R}^3$ (this follows from Schoenflies theorem, see [2] p.34). From now on, we assume that $\hat{S}$ is not a sphere. The proof of proposition 1 is based upon the following theorem (see [1] p.16 for a proof).

Theorem 2 *Let $\tilde{V}$ be a connected compact irreducible Seifert-fibered manifold. Then any essential surface S in $\tilde{V}$ is isotopic to a surface which is either vertical, i.e. a union of regular fibers, or horizontal, i.e. transverse to all fibers.*

Let us explain the various terms involved in this theorem. A 3-manifold N is said *irreducible* if any 2-sphere embedded in N bounds a 3-ball embedded in N. A *Seifert-fibered manifold* is a 3-manifold that decomposes into a union of topological circles, the *fibers*, satisfying certain properties. In particular, the cartesian product of a surface S and a circle S^1 is a Seifert-fibered 3-manifold, with fibers the circles $\{x\} \times S^1$ for $x \in S$. We will not explain what a *regular* fiber is, but in the previous case, which will be ours, all fibers are regular. An orientable surface without boundary S embedded in a 3-manifold N is said *incompressible* if none of its components is homeomorphic to a sphere and if for any (topological) disk $D \subset N$ whose boundary is included in S, there is a disk $D' \subset S$ such that $\partial D = \partial D'$. Any disk D for which there is no D' is called a *compressing disk* for S (see figure 1 for an example of compressing disk). The notion of *essential surface* of a 3-manifold is similar to the one of incompressible surface, but more restrictive. However, when the 3-manifold has no boundary, both notions coincide.

In our setting, $\tilde{V}$ is the trivial Seifert-fibered manifold $\hat{S} \times S^1$, which we obtain by identifying the two boundary components of $V = \hat{S} \times [0,1]$. We will still denote by S the surface corresponding to S in $\tilde{V}$. We first prove that $\tilde{V}$ and S fulfill the hypothesis of theorem 2 and then deduce that S is isotopic to $\hat{S}$. Because we assume that $\hat{S}$ is not a 2-dimensional sphere, $\tilde{V} = \hat{S} \times S^1$ is irreducible ([1] prop 1.12 p.18). We now prove the following

Lemma 3 *S is an essential surface in $\tilde{V}$.*

Proof. Since $\tilde{V}$ has no boundary it is sufficient to prove that S is incompressible. Let $x \in S^1$ be the point corresponding to the endpoints of $[0,1]$ and denote by $\hat{S}$ the section $\hat{S} \times \{x\}$ in $\tilde{V}$. Suppose S is compressible. So one can find a simple curve γ on S which does not bound a disk in S and which bounds an embedded disk D in $\tilde{V}$. Do the following surgery: cut S along γ and glue a disk homotopic to D along each of the two boundary components of $S \setminus \gamma$ (see figure 1). By doing so, one obtains a new surface with Euler characteristic greater than $\chi(S) = \chi(\hat{S})$. The previous surgery does not change the homology class: the new surface is homologous to S. Also, the surface S (with well chosen orientation) is homologous to $\hat{S}$, as $\hat{S}$ and S form the boundary of an open subset in $\tilde{V}$. On the other hand, it follows from Künneth formula ([G] p.198) that the homology class of $\hat{S}$ in $\tilde{V} = \hat{S} \times S^1$ is not zero. So one of the connected components S' of the new surface has a non zero homology class in $\tilde{V}$. Moreover, S' has a smaller genus than the one of S. Indeed, suppose it is not the case. As the new surface has a larger Euler characteristic than $\chi(S)$ and has at most two connected components, the only possibility is that this surface is the disjoint union of S' and a sphere. Considering the complement of the compressing disk in the sphere component shows that ∂D bounds a disk in S, which is a contradiction.

Note that it is possible to choose D such that $D \cap \hat{S} = \emptyset$: among all the embedded disks with γ as boundary that meet $\hat{S}$ in a finite number n of simple closed curves, take as D the one such that n is minimum. Suppose that n is

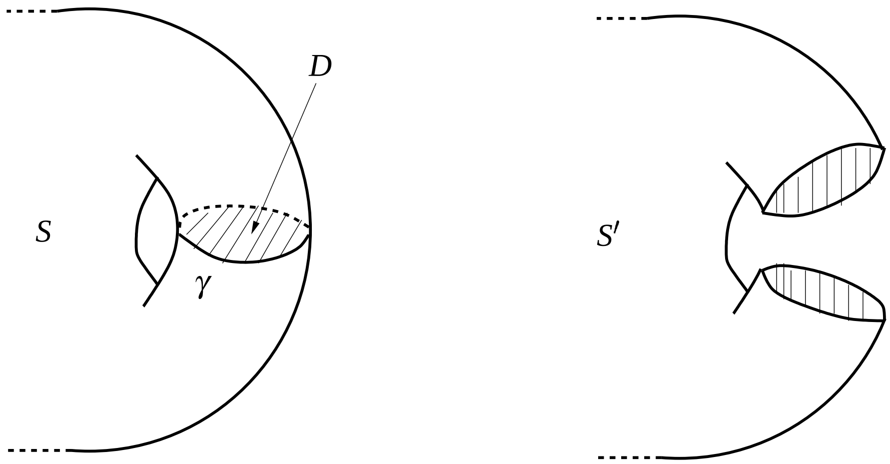

Figure 1: Surgery along a compressing disk.

not zero. Among all these intersection curves, there is at least one curve α bounding a disk in $D \setminus (\hat{S} \cap D)$ (when the curves are nested, consider any innermost curve on D, see fig. 2 on the right). Also, the surface $\hat{S}$ is incompressible, since the injection of $\hat{S}$ in $\tilde{V}$ induces an injection between corresponding fundamental groups (see [1] p. 10). As a consequence, α bounds a disk in $\hat{S}$ and one can then make an isotopy to obtain a disk D' such that $D' \cap \hat{S} = (D \cap \hat{S}) \setminus \alpha$. This contradicts the minimality of n (see fig. 2).

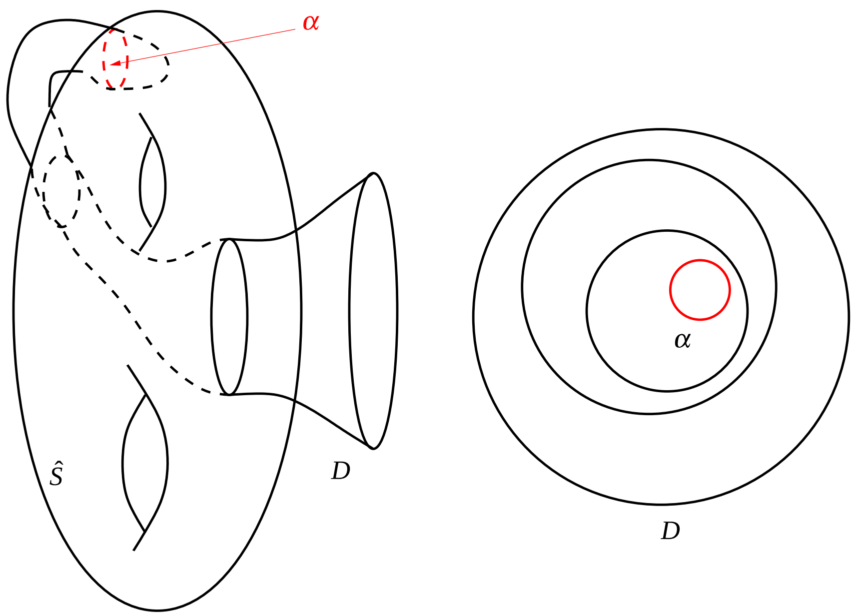

Figure 2: Decreasing the number of components of $D \cap \hat{S}$.

The previous surgery cannot be iterated an infinite number of times, since the genus of S' decreases each time. Upon termination, one obtains a surface, called S' again, which is incompressible or the sphere S^2, and which does not intersect the surface $\hat{S}$ because we chose compressing disks that do not meet $\hat{S}$. If S' is a 2-sphere, it does not bound a 3-ball because its homology class in $H_2(\tilde{V})$ is not zero. This implies that $\tilde{V}$ is not irreducible: contradiction. So S' is an incompressible surface. Applying theorem 2, one deduces that S' is isotopic to either a horizontal or a vertical surface.

Claim: *S' is not isotopic to a vertical surface.*

Proof: Suppose it is. Then there exists a surface S'' which is an union of fibers of $\tilde{V}$ and which is isotopic to S'. Choose

one fiber ϕ included in S''. Its intersection number with $\hat{S}$ is equal to 1 and has to remain constant during the isotopy. So S' contains a simple closed curve whose intersection number with $\hat{S}$ is equal to 1, namely the image of ϕ under the isotopy. But S' does not intersect $\hat{S}$: contradiction.
Hence S' is isotopic to a horizontal surface, which is a covering of $\hat{S}$ under the canonical projection of $\tilde{V}$. But this is not possible since $genus(S') < genus(S)$. So, S is incompressible, which concludes the proof of lemma 3. □

Now, it follows from theorem 2 that S is isotopic to either a horizontal or a vertical surface. S does not intersect $\hat{S}$, so it cannot be isotopic to a vertical surface, by the same argument as above. So S is isotopic to a horizontal surface. This surface is a covering of $\hat{S}$ under the canonical projection of $\tilde{V}$. Because $\tilde{V} \setminus S$ is connected, it follows from [1] p.17-18 that the covering is trivial. Hence, S is isotopic to a horizontal surface which meets each fiber in one point. It is now a classical fact that this horizontal surface can be "pushed along the fibers" to construct an isotopy to $\hat{S}$ (see Fig.). Note that, using the same argument as the one used previously to prove that one can construct S' such that it does not intersect $\hat{S}$, the isotopy $f_t, t \in [0,1]$ between $\hat{S}$ and S can be chosen so that $f_t(\hat{S}), t \in]0,1]$ never intersects $\hat{S}$. So S is isotopic to $\hat{S}$ in V.

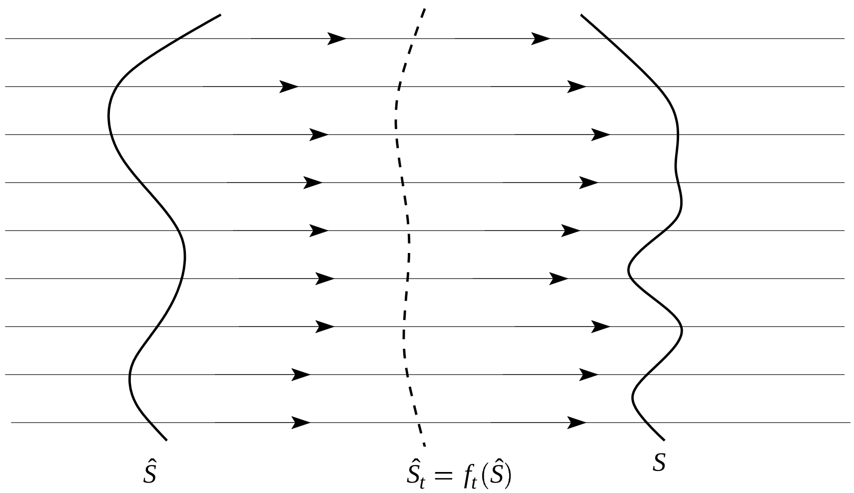

Figure 3: Pushing S to $\hat{S}$ along the fibers of $\tilde{V}$.

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