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*Approximation of the Curvature Measures of a
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Approximation of the Curvature Measures of a Smooth Surface endowed with a Mesh

David Cohen-Steiner , Jean-Marie Morvan

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Abstract: This report deals with the approximation of a smooth surface M by a triangulated mesh T . We give an explicit bound on the difference of the curvature measures of M and the curvature measures of T , when T is close to M . The result is obtained by applying the theory of the normal cycle.

Key-words: Normal cycle, approximation, surface, curvature, mesh, triangulation

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Approximation des mesures de courbure d'une surface lisse munie d'un maillage

Résumé : Ce rapport étudie l'approximation d'une surface lisse M par une triangulation proche T . Nous donnons une borne explicite de la différence des mesures de courbure définies sur M et les mesures de courbures définies sur T . Ce résultat est obtenu en utilisant la théorie du cycle normal.

Mots-clés : Cycle normal, approximation, surface, courbure, maillage, triangulation

1 Introduction

This report gives applications to a theoretical result appeared in [7], concerning an approximation of curvatures measures of a smooth hypersurface. We deal with the case where the smooth hypersurface is approximated by a mesh.

In [7] we proved the following result; (we use the same notations - \mathcal{F} denotes the flat norm on the space of currents, \mathbf{M} denotes the mass of currents, $N(A)$ denotes the normal cycle of a geometric object A , \mathcal{M}_k^A denotes the k^{th} -mean curvature measure of A):

Theorem 1 *Let M^{n-1} be a smooth closed (oriented) hypersurface of \mathbb{E}^n bounding a compact subset K and \mathcal{C} be a geometric compact subset of \mathbb{E}^n whose boundary $\mathcal{B} = \partial\mathcal{C}$ is strongly close to M^{n-1} . Let B be any regular Borel subset of \mathbb{E}^n included in \mathcal{B} . Then,*

$$\mathcal{F}(N(\mathcal{C})|_{(B \times \mathbb{E}^n)} - N(K)|_{(\text{pr}(B) \times \mathbb{E}^n)}) \leq \max(\delta_B, \alpha_B) \left(\frac{\sup_B(1, \|h_B\|)}{1 - \delta_B \|h_B\|} \right)^{n-1} (\mathbf{M}(N(\mathcal{C})|_{(B \times \mathbb{E}^n)}) + \mathbf{M}(\partial N(\mathcal{C})|_{(B \times \mathbb{E}^n)})),$$

where

- $\delta_B = \delta(B, \text{pr}(B))$,
- $\|h_B\|$ denotes the maximum of the norm of the second fundamental form h of M^{n-1} restricted to $\text{pr}(B)$,
- $\alpha_B = \sup_{(p,n) \in \text{spt}(N(\mathcal{C})|_{B \times \mathbb{E}^n})} |\angle(\xi_{\text{pr}(p)}, n)|$.

Using the same notations, we get the following

Corollary 1 *Let B be any regular Borel subset of \mathcal{B} . Then, for every $k, 0 \leq k \leq n-1$,*

$$|\mathcal{M}_k^{\mathcal{C}}(B) - \mathcal{M}_k^K(\text{pr}(B))| \leq C(n, k) \max(\delta_B, \alpha_B) \left(\frac{\sup_B(1, \|h_B\|)}{1 - \delta_B \|h_B\|} \right)^{n-1} (\mathbf{M}(N(\mathcal{C})|_{(B \times \mathbb{E}^n)}) + \mathbf{M}(\partial(N(\mathcal{C})|_{B \times \mathbb{E}^n}))),$$

where $\delta_B = \delta(B, \text{pr}(B))$ is the Hausdorff distance between B and $\text{pr}(B)$, $\|h_B\|$ is the maximum of the norm of the second fundamental form of M^{n-1} restricted to $\text{pr}(B)$, and $C(n, k)$ is a constant depending on the dimension of the ambient space.

In the following of this article, we shall apply these results to surfaces in \mathbb{E}^3 , approached by triangulated meshes. Remark that [17] gives a bound on the difference the area $\mathcal{A}(M)$ of a smooth surface M and a the area $\mathcal{A}(T)$ of a triangulation closely inscribed to it, in term of the area of T , the angular deviation α of T with respect to M , and is the relative curvature $\omega_M(T)$ of M with respect to T , (see the definitions below):

$$\|\mathcal{A}(M) - \mathcal{A}(T)\| \leq \mathcal{A}(T) \frac{(1 + \omega_M(T))^2 - \cos \alpha}{(1 - \omega_M(T))^2},$$

In particular,

$$|\mathcal{A}(M) - \mathcal{A}(T)| = O(\alpha^2 + \delta),$$

where δ is the Hausdorff distance between M and T . The goal of this paper is to get similar result concerning the curvature measures of T and M .

To simplify the notations, we put $\mathcal{M}_A^K = \mathcal{M}_0^K$, $\mathcal{M}_H^K = \mathcal{M}_1^K$, $\mathcal{M}_G^K = \mathcal{M}_2^K$. Remark that if K bounds M , then $\mathcal{M}_A^K = \mathcal{A}(M)$. Our main result is the following:

Theorem 2 *Let K be a compact subset of \mathbb{E}^3 whose boundary M is a smooth (closed oriented embedded) hypersurface of \mathbb{E}^3 . Let C be a compact subset of \mathbb{E}^3 whose boundary is a triangulated mesh T strongly close to M . Let B be the interior of a union of triangle of T . Then,*

$$\begin{aligned} & |\mathcal{M}_H^C(B) - \mathcal{M}_H^K(pr(B))| \leq \\ & 2 \max(\delta_B, \alpha_B) \left(\frac{\sup_B(1, \|h_B\|)}{1 - \delta_B \|h_B\|} \right)^2 (\mathcal{A}(B) + 2\alpha_B(l_B + \mathbf{n}^{v\circ}) + 4\pi \sin^2 \frac{\alpha_B}{2} n_e + l_{\partial B}); \\ & |\mathcal{M}_G^C(B) - \mathcal{M}_G^K(pr(B))| \leq \\ & \max(\delta_B, \alpha_B) \left(\frac{\sup_B(1, \|h_B\|)}{1 - \delta_B \|h_B\|} \right)^2 (\mathcal{A}(B) + 2\alpha_B(l_B + \mathbf{n}^{v\circ}) + 4\pi \sin^2 \frac{\alpha_B}{2} n_e + l_{\partial B}); \end{aligned}$$

where $\delta_B = \delta(B, pr(B))$ is the Hausdorff distance between B and $pr(B)$, $\|h_B\|$ is the maximum of the norm of the second fundamental form of M restricted to $pr(B)$, l_B , (resp. $l_{\partial B}$) denotes the sum of the lengths of the edges of T lying in B , (resp. the length of ∂B), n_e denotes the number of edges of T lying in B , and $\mathbf{n}^{v\circ}$ denotes the number of vertices belonging to the boundary of B .

This theorem can be interpreted as follows: Suppose that one deals with a mesh T in \mathbb{E}^3 . This mesh can be considered of an approximation of an infinity of smooth surfaces M . Of course, it is impossible to evaluate the local geometry of any M , without other assumptions. However, roughly speaking, Theorem 2 claims that every surface M in which T is closely inscribed, and

- whose normal vector field is close to the normal of the faces,
- whose second fundamental form is not too big,

has a local geometry close to the one of T . Moreover, the error between the mutual curvatures is bounded by an explicit constant depending on T ¹ and the two *a priori* previous assumptions. Together with the results of [2], this shows in particular that all surfaces that are ε -sampled (see a precise definition below) by a set of points P with ε sufficiently small have close curvature measures. Indeed, if one takes as T the result of the surface reconstruction algorithm described in [2] applied to a set of points P , and as M any surface ε -sampled by P , it appears that M and T satisfy the assumptions of theorem 2, and have

¹Interestingly enough, this constant depends only on the intrinsic geometry of T .

thus close curvature measures. To a certain extent, this shows that it makes sense to talk of the curvature of a point cloud, provided it is an ε -sample of at least one surface for ε sufficiently small.

One can give a coarser but more readable corollary of theorem 2 :

Corollary 2 *Under the same assumptions as theorem 2 :*

$$|\mathcal{M}_H^C(B) - \mathcal{M}_H^K(pr(B))| \leq \mathcal{K}\varepsilon,$$

$$|\mathcal{M}_G^C(B) - \mathcal{M}_G^K(pr(B))| \leq \mathcal{K}\varepsilon,$$

with

$$\mathcal{K} = O\left(\sum_{t \in T, t \subset \bar{B}} r(t)^2\right) + O\left(\sum_{t \in \partial T, t \subset \bar{B}, t \cap \partial B \neq \emptyset} r(t)\right),$$

where $r(t)$ denotes the circumradius of the triangle t . In particular, if the sampling is locally uniform, then

$$\mathcal{K} = O(\text{area}(B) + \text{length}(\partial B)).$$

When the vertices of T are an ε -sample of M , and T is the Delaunay triangulation of its vertices restricted to M , one can give an even simpler formulation of theorem 2 :

Corollary 3 *Let \mathcal{P} be an locally uniform [15] ε -sample of M , with $\varepsilon < 0.06$. Let B be the interior of a union of triangle of T . Then,*

$$|\mathcal{M}_H^C(B) - \mathcal{M}_H^K(pr(B))| = O(\varepsilon),$$

$$|\mathcal{M}_G^C(B) - \mathcal{M}_G^K(pr(B))| = O(\varepsilon).$$

2 Smooth submanifolds

2.1 Geometry of smooth surfaces of \mathbb{E}^3

In the following, a smooth surface M means a \mathcal{C}^2 surface which is regular, oriented, with or without boundary ∂M , embedded in the (oriented) euclidean space \mathbb{E}^3 . Let denote the boundary of M . M is endowed with the Riemannian structure induced by the standard scalar product of \mathbb{E}^3 . We denote by da the area form on M , $\mathcal{A}(B)$ the area of a Borel subset B of M , and by ds the canonical orientation of ∂M . Let ξ be the unitary normal vector field (compatible with the orientation of M) and h be the second fundamental form of M associated with ξ . Its determinant at a point p of M is the Gauss curvature $G(p)$, its trace is the mean curvature $H(p)$. The maximal curvature of M at p is $\|h_p\| = \max(|\lambda^1(p)|, |\lambda^2(p)|)$, where $\lambda_1(p)$ and $\lambda_2(p)$ are the eigenvalues of the second fundamental form at p . Finally, the

maximal curvature of M is $\|h_M\| = \sup_{p \in M} \|h_p\|$.

From now on, we shall assume that M is compact. It is well known that there exists an open set U_S of \mathbb{E}^3 containing M and a continuous map pr from U_S onto M satisfying the following: if p belongs to U_S , then there exists a unique point $pr(p)$ realizing the distance from p to M , (pr is nothing but the orthogonal projection onto M); (a proof of this proposition can be found in [8]). We shall also need the notion of *reach of a surface*, introduced in [8].

Definition 1 *The reach r_M of a surface M is the largest $r > 0$ for which pr is defined on the (open) tubular neighborhood $U_r(M)$ of radius r of M .*

This implies that the reach r_M of M is smaller than the minimal radius of curvature of M . Thus, we have:

$$\|h_M\| r_M \leq 1.$$

The *medial axis* of a smooth surface M of \mathbb{E}^3 is the closure of the set of points in \mathbb{E}^3 with more than one nearest neighbor on M . The *local feature size* $LFS(p)$ at a point p of M is the euclidean distance from p to (the nearest point of) the medial axis. Thus we have:

$$r_M = \min_{p \in M} LFS(p).$$

2.2 Curvature measures on smooth surfaces

The basic tool of our approach is the *normal cycle* of a geometric compact subset of \mathbb{E}^3 , introduced in [19], (see also [20], [11]). The normal cycle of a compact body whose boundary is a smooth submanifold M of \mathbb{E}^3 is nothing but its unit normal bundle, considered as a (closed) $(n - 1)$ -current of $\mathbb{E}^3 \times \mathbb{S}^2$.

Let $\omega_A, \omega_G, \omega_H$ be the three canonical invariant forms on $\mathbb{E}^3 \times \mathbb{S}^2$, (see [10] for instance). It is classical that, if B is a Borel subset of \mathbb{E}^3 , then

$$\mathcal{M}_A^K(B) = \int_{B \times \mathbb{E}^n} \omega_A = \int_B da = \mathcal{A}(B);$$

$$\mathcal{M}_G^K(B) = \int_{B \times \mathbb{E}^n} \omega_G = \int_B G da;$$

$$\mathcal{M}_H^K(B) = \int_{B \times \mathbb{E}^n} \omega_H = \int_B H da.$$

3 Triangulated meshes

A *triangulated mesh* (or a *triangulation*) T is a topological surface which is a (finite and connected) union of triangles of \mathbb{E}^3 , such that the intersection of two triangles is either empty, or equal to a vertex, or equal to an edge. In our context, T is the boundary of a compact subset \mathcal{C} of \mathbb{E}^3 .

3.1 Normal cycle of a triangulated mesh

Let \mathcal{C} be a simplex of \mathbb{E}^3 , the boundary of which is a triangulated mesh T . The normal cycle $N(\mathcal{C})$ of \mathcal{C} is the integral closed current of $\mathbb{E}^3 \times \mathbb{S}^2$ defined by the (smooth closed oriented) surface S of $\mathbb{E}^3 \times \mathbb{S}^2$ given by

$$S = \{(m, \xi) \in \mathbb{E}^3 \times \mathbb{S}^2 \text{ such that } \langle m\vec{n}, \xi \rangle \geq 0, \forall n \in T\}.$$

In general, \mathcal{C} can be considered as an union of simplices. Its normal cycle is obtained by computing the normal cycle of each simplex, and by using the fundamental additivity property:

$$N(A \cup B) = N(A) + N(B) - N(A \cap B).$$

The result does not depend on the decomposition. Consequently, the normal cycle $N(\mathcal{C})$ of a convex body \mathcal{C} whose boundary is a triangulated mesh T can be decomposed as a sum of three currents: $N^f(\mathcal{C})$ above the faces, $N^e(\mathcal{C})$ above the edges, and $N^v(\mathcal{C})$ above the vertices of T .

1. Above a face f of T , the support of $N^f(\mathcal{C})$ is the set of points

$$\{(m, \xi), m \in f, \xi \text{ unit vector normal to } T \text{ at } m\},$$

and $N^f(\mathcal{C})$ restricted to f can be identified with the current defined by integration above this portion of 2-plane, (endowed with the canonical orientation).

2. Above an edge e , the support of $N^e(\mathcal{C})$ is the portion of cylinder

$$\{e \times C \in \mathbb{E}^3 \times \mathbb{S}^2\},$$

where C is the arc of circle of \mathbb{S}^2 defined as the unit vectors delimited by the normals of the two faces incident to e . If e is convex (with respect to \mathcal{C}), then $N^e(\mathcal{C})$ above e can be identified with the current defined by integration over this portion of cylinder, (endowed with the canonical orientation), and with multiplicity $+1$. If e is concave, (with respect to \mathcal{C}), the same description holds, but with multiplicity -1 .

3. Above a vertex v , the support of $N^v(\mathcal{C})$ is a portion of the unit sphere $v \times \mathbb{S}^2$, and $N^v(\mathcal{C})$ above v can be identified with a linear combinaison of currents defined by integration over portions of the unit sphere over v , endowed with correct multiplicity.

To be precise, the multiplicity of each point h of the unit sphere over v is the index of v with respect to h , that is the integer

$$\mu(v, h) = \chi(St^+(v, h)),$$

where $\chi(St^+(v, h))$ is the Euler characteristic of the union of relative interiors of cells of \mathcal{C} incident to v and lying in the half plane $\{x : \langle \vec{v}\vec{x}, h \rangle \geq 0\}$, (see [3] for a general study, we take the convention that the relative interior of a point is the point itself).

3.2 The mass of the normal cycle of a triangulated mesh

Let B be the interior of an union of triangles of T . We shall evaluate the mass of $N(\mathcal{C})|_{B \times \mathbb{E}^n}$. We decompose it into three terms:

- The mass over the faces: $\mathbf{M}(N^f(\mathcal{C})|_{B \times \mathbb{E}^3})$ is nothing but the area $\mathcal{A}(B)$ of B ;
- The mass over the edges: $\mathbf{M}(N^e(\mathcal{C})|_{B \times \mathbb{E}^3})$ is the sum over all edges e of B of $l(e)|\angle(e)|$, where $l(e)$ denotes the length of the edge e , and $|\angle(e)|$ denotes the absolute value of the angle between the normals of the incident faces to e lying in B .
- The mass over the vertices: The mass $\mathbf{M}(N^v(\mathcal{C})|_{v \times \mathbb{E}^3})$ of the normal cycle over each vertex v can be computed as follows: we decompose the sphere $v \times \mathbb{S}^2$ over v as a disjoint union U_i of (Borel) subsets on which the multiplicity $\mu(v, h)$ is constant when h varies on U_i . Denoting by μ_i this constant, we have:

$$\mathbf{M}(N^v(\mathcal{C})|_{v \times \mathbb{E}^3}) = \sum_i |\mu_i| \mathcal{A}(U_i),$$

and

$$\mathbf{M}(N^v(\mathcal{C})|_{B \times \mathbb{E}^3}) = \sum_{v \text{ vertex in } B} \mathbf{M}(N^v(\mathcal{C})|_{v \times \mathbb{E}^3}).$$

3.3 Curvature measures on a triangulated mesh

Let B be the interior of an union of triangles of T . We shall evaluate the invariant forms $\omega_A, \omega_G, \omega_H$ on $B \times \mathbb{E}^3$. Looking at the expression of these forms, it appears clearly that ω_A is null over every edge and every vertex of B , ω_G is null over every triangle and every edge of B , ω_H is null over every vertex and every triangle of B . Consequently, one gets:

- $\mathcal{M}_A^{\mathcal{C}}(B) = \langle N(\mathcal{C})|_{B \times \mathbb{E}^3}, \omega_A \rangle = \mathcal{A}(B)$;
- $\mathcal{M}_G^{\mathcal{C}}(B) = \langle N(\mathcal{C})|_{B \times \mathbb{E}^3}, \omega_G \rangle = \sum_{v \text{ vertex in } B} \sum_i \alpha_i(v)$;
that is, the sum (over all vertices p lying in B) of the angle defect at v ;
- $\mathcal{M}_H^{\mathcal{C}}(B) = \langle N(\mathcal{C})|_{B \times \mathbb{E}^3}, \omega_H \rangle = \sum_e \text{edge in } B l(e)\angle(e)$;
where $\angle(e)$ denotes the angle between the two triangles incident to e , endowed with the suitable sign as defined in 3.1.

4 Triangulated mesh closely inscribed in a smooth surface

Following [13], we say that a triangulated mesh of \mathbb{E}^3 is *inscribed* in a smooth surface M if all its vertices belong to M . We say that a triangulated mesh T is *closely inscribed* in a smooth surface M if:

1. the triangulation T lies in $U_r(M)$, where r is the reach of M ,
2. all vertices of T belong to M ,
3. all vertices of ∂T belong to ∂M ,
4. the restriction of ξ to T is one to one.

The main invariant involved in the study of the couple M and T is the angular deviation between the normals. We give now a precise definition:

Definition 2 *Let M be a smooth surface and T a triangulated mesh closely inscribed in M . Let B be the interior of an union of triangles of T . The **angular deviation** between B and $pr(B)$ is the maximal angle α_B between the normals of the faces of B and the corresponding normals to $pr(B)$ on M .*

Let T be a triangulated mesh closely inscribed in a smooth surface M . Given a bound on the angular deviation of the corresponding normals, we shall deduce a bound on the mass of the normal cycle of T . We need the following

Proposition 1 *Let M be a smooth surface and T a triangulated mesh closely inscribed in M . Let B be the interior of an union of triangles of T . Let α_B denote the angular deviation between B and $pr(B)$, l_B denote the sum of the length of the edges of T lying in B , \mathbf{n}^e denote the maximum number of edges having a common vertex in B , n_v denote the number of vertices of T lying in B , $l_{\partial B}$ denote the length of ∂B , (that is the sum of the length of the edges forming ∂B), and $\mathbf{n}^{v\partial}$ denote the number of vertices of ∂B . Then,*

1. $\mathbf{M}(N^e(\mathcal{C})|_{B \times \mathbb{E}^3}) \leq 2\alpha_B l_B$;
2. $\mathbf{M}(N^v(\mathcal{C})|_{B \times \mathbb{E}^3}) \leq 4\pi \sin^2 \frac{\alpha_B}{2} n_v \mathbf{n}^e$;
3. $\mathbf{M}(N(\mathcal{C})|_{B \times \mathbb{E}^3}) \leq \mathcal{A}(B) + 2\alpha_B l_B + 4\pi \sin^2 \frac{\alpha_B}{2} n_v \mathbf{n}^e$;
4. $\mathbf{M}(\partial(N(\mathcal{C})|_{B \times \mathbb{E}^3})) \leq l_{\partial B} + 2\alpha_B \mathbf{n}^{v\partial}$;

Sketch of Proof of Proposition 1:

1. Over an edge e , the support of the normal cycle is reduced to a portion of cylinder, which can be identified with the product of e by an arc of circle \mathbf{c} of \mathbb{S}^2 . The angle spanned by any point of \mathbf{c} with the normal ξ of the surface at any vertex of e being smaller than α_B , the result follows by summing over all edges of B .
2. Over a vertex v , the support of the normal cycle lies in $v \times \mathbb{S}^2$. We can bound its mass as follows: Consider two adjacent faces f_1 and f_2 belonging to the one-ring of v . The normal of these two faces span with ξ_v a geodesic triangle in \mathbb{S}^2 whose area is smaller than $4\pi \sin^2 \frac{\alpha_B}{2}$. On the other hand, the mass of the normal cycle over v is smaller than the sum of the areas of these geodesic triangles. If the one ring of v contains $n^e(v) \leq n^e$ edges, then the mass of the normal cycle over v is smaller than $n^e \alpha_B^2$. Consequently, if B contains n^v vertices, then

$$\mathbf{M}(N^v(\mathcal{C})|_{B \times \mathbb{E}^3}) \leq 4\pi \sin^2 \frac{\alpha_B}{2} n^e n_v.$$

3. is clear by summing the terms over the faces, the edges and the vertices, and the boundary term.
4. The boundary of $(N(\mathcal{C})|_{B \times \mathbb{E}^3})$ is composed of edges corresponding to edges of ∂B , and arcs of circle above the vertices belonging to ∂B . The result follows.

As an immediate consequence, using the same notations, and remarking that $n_v n^e \leq n_e$, we get Theorem 2.

Remark that the previous theorem deals with the cardinals of the vertices and the edges, and with the topology of the triangulation. It does not involve the geometry of the faces. Consequently, it is "only" an approximation result, which cannot imply any convergence result without any other assumptions. In [18], the authors compare the behaviour of the normal vector field of a smooth surface M with the normal of each face of a triangulation which is closely inscribed in M . To summarize these results, we introduce the following notations and definitions: If T is a triangulation, t denotes a generic face of T , and \mathcal{T}_T denotes the set of faces of T .

- $\eta(t)$ denotes the *length of the longest edge* of t , and $\mathcal{A}(t)$ its area. The *height* of the triangulation T is:

$$\eta(T) = \max_{t \in \mathcal{T}_T} \eta(t).$$

- $r(t)$ denotes the *circumradius* of a triangle t . The *circumradius* $r(T)$ of T is the maximal circumradius of the triangles t of T .
- $\theta(t)$ denotes the *fatness* of the triangle t , that is the minimal angle of t .

- Following [?], we introduce the notion of *straightness of a triangle* t . It is the real number

$$str(t) = \max_{p \text{ vertex of } t} |\sin(\theta_p)|,$$

where θ_p is the angle at p of t . Remark in particular that if β is any of the three angles of a triangle t , we have:

$$2\theta(\Delta) \leq |\sin \beta| \leq str(\Delta).$$

The *straightness* of T is:

$$str(T) = \min_{t \in \mathcal{T}_T} str(t).$$

- The *fatness* of a triangle t is the minimal angle of ; the *fatness* of T is the minimal angle of all triangles t of T , (see [13] for instance).

Remark that the *straightness* is a weaker condition than the *fatness*.

Suppose now that T be a triangulated mesh closely inscribed in a smooth surface M .

- The *relative curvature of T with respect to M* is the real number defined by:

$$\omega_S(T) = \sup_{m \in T \setminus \partial T} \|pr(m) - m\| \|h_{\xi(m)}\|.$$

- The *relative height of T with respect to M* is the real number defined by:

$$\pi_S(T) = \sup_{t \in \mathcal{T}_T} \sup_{m \in t} \eta(t) \|h_{pr(m)}\|.$$

These geometric tools are related by the following inequalities, (see [18] for details):

Lemma 1 *Let M be a smooth surface of \mathbb{E}^3 and T a triangulated mesh closely inscribed in M . Then the Hausdorff distance δ_t between a triangle t and its projection $pr(t)$ on M satisfies:*

$$\delta_t \leq \eta(t) \leq 2r(t).$$

Proof of Lemma 1:

- Let p be a point of t . Let v be a vertex of t . Since t is inscribed in M , then, $d(p, pr(t)) \leq d(p, v) \leq \eta(t)$. Conversely, if m is a point of $pr(t)$, consider the intersection point p of the normal line to M at m with t . Since t is closely inscribed to M , m is the (unique) orthogonal projection of p onto M , and $d(m, t) = d(m, p) \leq d(p, v) \leq \eta$. Consequently, $\delta_t \leq \eta(t)$.
- The second inequality is trivially true in any triangle.

Lemma 2 *Let M be a smooth surface of \mathbb{E}^3 and T a triangulated mesh closely inscribed in M . Then, if $\pi_S(T) \leq \frac{1}{2}$, the angular deviation α_t satisfies*

$$\sin \alpha_t \leq 2 \|h_{pr(t)}\| r(t).$$

In particular,

$$\alpha_T = O(r(T)).$$

The Proof of Lemma 2 can be found in [18].

From now on, for simplicity, we put

$$s(B) = \sum_{t \in T, t \subset \bar{B}} r(t)^2,$$

$$sd(B) = \sum_{t \in \partial T, t \subset \bar{B}, t \cap \partial B \neq \emptyset} r(t).$$

Theorem 1 and Corollary 1 and the previous Lemmas implies, one gets:

Proposition 2 *With the notations of Theorem 2, one has*

1. $\mathbf{M}(N^f(\mathcal{C})|_{B \times \mathbb{E}^3}) = O(s(B))$,
2. $\mathbf{M}(N^e(\mathcal{C})|_{B \times \mathbb{E}^3}) = O(B)$,
3. $\mathbf{M}(N^v(\mathcal{C})|_{B \times \mathbb{E}^3}) = O(s(B))$,
4. $\mathbf{M}(\partial(N(\mathcal{C})|_{B \times \mathbb{E}^3})) = O(sd(B))$.

Proof of Proposition 2:

1. The support of the normal cycle over a triangle t can be identified with t itself, Consequently, its mass is the area of t , which is obviously bounded by the area of its circumcircle. Summing over all triangles of B , one find:

$$\mathbf{M}(N^f(\mathcal{C})|_{B \times \mathbb{E}^3}) = O\left(\sum_{t \in T, t \subset \bar{B}} r(t)^2\right) = O(s(B)).$$

2. The support of the normal cycle over an edge e can be identified with the portion of cylinder whose axis is e , and the angle is the dihedral angle between the two adjacent triangles t and t' . Now, the maximum angle α_t (*resp.* $\alpha_{t'}$ between t (*resp.* t') and its projection on M is $O(r(t))$ by Lemma 2. Consequently, summing over all edges lying in B , one find

$$\mathbf{M}(N^e(\mathcal{C})|_{B \times \mathbb{E}^3}) = O(s(B)).$$

3. Let v be a vertex of T . Let $v_1 \dots v_n$ be the vertices of T which have a common edge with v . The support of the normal cycle of T over v is included in the spherical triangles spanned by the normal n to M at v , the normal n_i to the triangle $vv_i v_{i+1}$ and the normal n_{i+1} to the triangle $vv_{i+1} v_{i+2}$, $1 \leq i \leq n$. By Lemma 2, the area of any such triangle is $O(r(vv_i v_{i+1}) + r(vv_{i+1} v_{i+2}))^2$. Summing over all v in B , we get:

$$\mathbf{M}(N^v(\mathcal{C})|_{B \times \mathbb{E}^3}) = O(s(B)).$$

4. $\partial(N(\mathcal{C})|_{B \times \mathbb{E}^3})$ decomposes into two parts : a union of line segments corresponding to the edges of ∂B , and a union of circle arcs corresponding to the edges of B meeting ∂B . The mass of the former is obviously $O(sd(B))$ and the one of the latter also by lemma 2.

5 ϵ -sample on a surface

In this paragraph, we shall deal with classical particular triangulations: the ones which are spanned by a sample on the smooth surface M . Let us give the basic definitions: a finite set \mathcal{S} on M is called a *sample* of M . We denote by $\text{Del}(\mathcal{S})$ the Delaunay triangulation associated to \mathcal{S} . Moreover, we denote by $\text{Del}_M(\mathcal{S})$ the restriction of $\text{Del}(\mathcal{S})$ to M , that is, the set of faces of $\text{Del}(\mathcal{S})$ whose dual Voronoi faces cut M . In the following, we say that \mathcal{S} is *good* if $\text{Del}_M(\mathcal{S})$ is homeomorphic to M and stay homeomorphic to M if we add sample points. An ϵ -sample on M is a sample such that for every point m of M , the ball $B(m, \epsilon \text{LFS}(m))$ encloses at least one point of \mathcal{S} . It is well known that if $\epsilon < 0.1$, any ϵ -sample of M is a good sample, [1]. If the sample is good, any dual edge of $\text{Del}_M(\mathcal{S})$ cuts M in a unique point.

In [1] and [2], N. Amenta and al. evaluate the Hausdorff distance between $\text{Del}_M(\mathcal{S})$ and M in terms of ϵ , and prove in particular that if $\epsilon < 0.06$, then, $\text{Del}_M(\mathcal{S})$ is closely inscribed in M . On the other hand, one has

Lemma 3 *Let \mathcal{S} be an ϵ -sample on M , with $\epsilon < 0.06$. If B be any Borel subset of $\text{Del}_M(\mathcal{S})$, then the Hausdorff distance $\delta_B = \delta(B, pr(B))$ satisfies:*

$$\delta_B \leq \epsilon \sup_{m \in pr(B)} \text{LFS}(m),$$

Moreover,

$$r(B) \leq \epsilon \sup_{m \in pr(B)} \text{LFS}(m).$$

Proof of Lemma 3: Let t be a triangle of $\text{Del}_M(\mathcal{S})$. Then, the Delaunay ball centered in M and being circumscribed to t has a radius greater than $\epsilon \sup_{m \in pr(t)} \text{LFS}(m)$. We deduce immediately that

$$\delta_t \leq \epsilon \sup_{m \in pr(t)} \text{LFS}(m),$$

$$r(t) \leq \epsilon \sup_{m \in pr(t)} LFS(m);$$

We deduce immediately Lemma 3.

Proposition 3 *Let S be an ϵ -sample on M , with $\epsilon < 0.06$. Then, the angle deviation satisfies:*

$$\alpha_{Del_M(S)} = O(\epsilon).$$

Proof of Proposition 3: This is an obvious consequence of Lemma 3. Indeed, one has:

$$\alpha_{Del_M(S)} = O(r(T)) \text{ and } r(T) = O(\epsilon).$$

Consequently, one has proved Corollary 2.

Remark that in general, the difference between the curvatures of B and $pr(B)$ is not $O(\epsilon)$. To get a better result, we need more restrictive assumptions on the ϵ -sample. If the ϵ -sample on M is *locally uniform* in the sense of [15], it is known that the triangles of its restricted Delaunay triangulation are well-shaped, so that

$$s(B) = O(\text{area}(B)), \quad sd(B) = O(\text{length}(\partial B)).$$

We deduce immediately that $\mathcal{K} = O(\text{area}(B) + \text{length}(\partial B))$, and Corollary 3 is proved.

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Contents

1	Introduction	3
2	Smooth submanifolds	5
2.1	Geometry of smooth surfaces of \mathbb{E}^3	5
2.2	Curvature measures on smooth surfaces	6
3	Triangulated meshes	7
3.1	Normal cycle of a triangulated mesh	7
3.2	The mass of the normal cycle of a triangulated mesh	8
3.3	Curvature measures on a triangulated mesh	8
4	Triangulated mesh closely inscribed in a smooth surface	9
5	ϵ-sample on a surface	13



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