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Integrability of nilpotent sub-Riemannian structures

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Abstract: We consider the action of a Lie group that leaves invariant the sub-Riemannian structures associated to Goursat systems and Euclidean metrics. The low dimensional cases contain the well-known Heisenberg group, as well as the nilpotent groups associated to the Martinet and Engel systems. For arbitrary dimensions, we establish the complete integrability of the associated adjoint system by means of the explicit calculation of Casimir functions of the associated nilpotent Poisson algebra. We define a new algebraic curve in total space and compute two examples of sub-Riemannian extremals in cotangent bundle. Our approach fits certain non-holonomic classical problems, particularly that of the physical model for a plasma considered as a gas of non interacting charged particles under the influence of a static non homogeneous magnetic field.

Key-words: Sub-Riemannian structures, Heisenberg, Martinet and Engel systems, Casimir functions, nilpotent Poisson algebra., exact integrability, non-holonomic classical problems

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Intégrabilité des structures sous-Riemanniennes nilpotentes

Résumé : On considère l'action d'un groupe de Lie qui laisse invariantes les structures sous-Riemanniennes associées aux systèmes (distributions) de Goursat et aux métriques Euclidiennes. Parmi les cas de petite dimension : le fameux groupe de Heisenberg, ainsi que les groupes nilpotents associés aux systèmes de Martinet et de Engel.

On prouve, par un calcul explicite des fonctions de Casimir de l'algèbre de Poisson, l'intégrabilité exacte du système adjoint associé, et ce en dimension quelconque. On définit une nouvelle courbe algébrique dans l'espace total, et on calcule deux exemples d'extrémales sous-Riemanniennes dans le fibré cotangent. L'approche proposée s'applique à un certain nombre de problèmes non holonômes classiques, et en particulier au modèle physique d'un plasma considéré comme un gaz de charges électriques soumises à un champ magnétique statique inhomogène, mais n'interagissant pas entre elles.

Mots-clés : Structures sous-Riemanniennes, groupe de Heisenberg, systèmes de Martinet et de Engel, fonctions de Casimir, algèbre de Poisson, l'intégrabilité exacte, problèmes non holonômes classiques

Contents

1 Introduction

There is a great variety of problems with non-holonomic constraints that can be formulated in the framework of what now is known as sub-Riemannian geometry. The general idea is to associate to the set of constraints written as a Pfaffian system, a sub-bundle of the configuration space of the system; such procedure allows to define the distance by means of horizontal curves.

A prototypical problem illustrating this approach is the case of the model for a weak density plasma, considered as a gas composed of non-interacting charged particles under the influence of a static non homogeneous magnetic field $\vec{B}$ produced by external sources or generated by the electric currents associated to the movement of the particles, see for instance [1]. An electric field $\vec{E}$ can also be taken into account, in such a way that the macroscopic behavior of the gas is determined by the statistical average of all possible trajectories of the equation

$$m\frac{d\vec{v}}{dt} = q(\vec{E} + \vec{v} \times \vec{B}). \tag{1}$$

For the case of $\vec{E} = 0$, the movement is determined by the so-called Lorentz force, which is perpendicular to both $\vec{v}$ and $\vec{B}$. The trajectories are helices gyrating around $\vec{\Omega} = -\frac{q}{m}\vec{B}$ and parallel translating in the direction of $\vec{\Omega}$.

Furthermore, if we take $\vec{B}$ to be parallel to the z axis, then equation (1) is equivalent to the Lagrange equations in coordinates (x, y, z), corresponding to the Lagrangian

$$\mathcal{L} = \lambda_0[\frac{m}{2}(\dot{x} + \dot{y})] + \lambda_1[\dot{z} - (y\dot{x} - x\dot{y})],$$

which includes a non-holonomic constraint that can be written as

$$dz - (ydx - xdy) = 0.$$

This constraint determines the sub-bundle of $T\mathbb{R}^3$ generated by the vector fields $X_1 = \partial x + y\partial z$ and $X_2 = \partial y - x\partial z$, which provide the basis for the sub-Riemannian geometry of $\mathbb{R}^3$ introduced by R.W. Brockett in [2] and discussed by some other authors.

In conclusion, the non-holonomic nature of the problem forces to extend the analysis of the trajectories to higher dimensional spaces, composed in this case by the positions and the swept areas, situation that leads naturally to the study of distributions underlying certain sub-Riemannian structures, for details we refer the reader to the recently appeared book by R. Montgomery [3].

In this paper we study rank two distributions equivalent to the Goursat distribution and a dynamical system associated to it. We follow as a model, the physical situation described above, where an electric charge q evolves in the Euclidean plane M subject to a magnetic field perpendicular M. As already mentioned, the motion of q is better described when immersing M into a higher dimensional manifold P. For simplicity, we consider only magnetic fields, which are polynomial of degree N in a single variable.

Our main result exhibits explicitly the Lie group which describes the symmetries of the sub-Riemannian structure, given in terms of a rank two distribution of vector fields $\Delta \subset TP$ and a flat Riemannian metric $\langle \cdot, \cdot \rangle$. We shall show that the methods based on a detailed study of this geometry provide an additional insight into the behavior of the trajectories. We shall also exhibit the corresponding Lie algebras, describing the symmetries of the trajectories in total space.

Nilpotent Lie groups are essential in our solutions and contain as the simplest nilpotent subgroup the well-known Heisenberg group. The intrinsic geometry defined by this group on $\mathbb{R}^3$ provides a model for the higher dimensional cases, in this sense all them can be though as deformations of it. Also the so called Martinet case considered by R. Montgomery, [4] and by B. Bonnard and coworkers, [5] is naturally contained as a particular example corresponding to linear magnetic fields.

Some general results concerning non-homogeneous magnetic fields were first mentioned by R. Montgomery in the aforementioned reference, in contrast with that paper, we present here explicit and detailed results. The 1-forms associated to the non homogeneous magnetic fields, lead us to study Goursat distributions and the extension of the standard two dimensional Lagrangian in base space to higher dimensions associated to certain constants of motion. In the final sections of this work, we define a new algebraic curve in the total $N+3$-dimensional space. We show that the sub-Riemannian geodesics are defined by this curve and by the Pfaffian system of constraints.

2 Sub-Riemannian structures

Let P be a $N+3$ dimensional smooth manifold, having in mind the immersion in P of the two dimensional Euclidean plane, we denote the coordinates as $q = (x, y, u_1, \cdots, u_{N+1})$, we consider the following non-holonomic constraints

$$\dot{q} = \dot{x} Z_1(q) + \dot{y} Z_2(q), \tag{2}$$

given in terms of the rank two distribution $\Delta = \{Z_1, Z_2\}$.

We assume that the distribution Δ is bracket generating, which means that there is an integer n such that $\Delta_p^n = T_pP$ for each $p \in P$. For each i, we take $\Delta^{i+1} = [\Delta, \Delta^i]$, which leads to the flag

$$\Delta_p \subset \Delta_p^2 \subset \cdots \subset T_pP.$$

An important quantity here is the so-called growth vector n_p of Δ at the point p, defined as $n_p = (n_{1p}, n_{2p}, \cdots, n_{np})$, where $n_{ip} = dim\, \Delta_p^i$. Here, $n_{1p} = 2$, and generically we shall have $n_{np} = N+3$, but this last number can be smaller under some circumstances.

In this work we assume that the distribution Δ generates the nilpotent Lie algebra $\mathfrak{g}$ with the following non-zero brackets

$$[Z_1, Z_2] =: Z_3, \cdots, [Z_1, Z_{N+2}] =: Z_{N+3}. \tag{3}$$

Observe that that Z_{N+3} is central and that the subalgebras generated by the fields $\{Z_k,\ k \neq 1\}$ are Abelian. The subalgebras $\mathfrak{g}_i$ generated by $\{Z_1, Z_{i+1}, Z_{i+2}, \ldots, Z_{N+3}\}$ for $i \geq 1$ lead to

$$\mathfrak{g} = \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots \supset \mathfrak{g}_{N+3},$$

where $\mathfrak{g}_{i+1}$ is an ideal in $\mathfrak{g}_i$. The algebra $\mathfrak{g}$ is solvable and we have the flag of Abelian ideals

$$\mathfrak{g} = \widetilde{\mathfrak{g}}_1 \supset \widetilde{\mathfrak{g}}_2 \supset \cdots \supset \widetilde{\mathfrak{g}}_{N+3},$$

where $\widetilde{\mathfrak{g}}_i$ is the ideal generated by $\{Z_i, Z_{i+1}, \ldots, Z_{N+3}\}$ for $i > 1$.

The pairing $(\omega, Z) = \sum f_i g_i$, written in a particular basis where $\omega = \sum f_i e_i^*$ and $Z = \sum g_i e_i$, such that $(e_i^*, e_j) = \delta_{ij}$, leads to the splitting

$$T_pP = Ver_p \oplus Hor_p,$$

where Ver_p is called the vertical subspace at p and Hor_p the horizontal subspace at p of T_pP. Here we have $Hor = \Delta$, and $Ver = \mathfrak{g}_o$ where $\mathfrak{g}_0$ is the Abelian algebra generated by $\{Z_3, \ldots, Z_{N+3}\}$.

Definition 2.1 *A sub-Riemannian geometry is a triple $\{P, \Delta \langle , \rangle_p\}$, where $\langle , \rangle_p$ is an smoothly varying inner product in $\Delta_p \subset T_pP$ for each $p \in P$. The set $\{\Delta, \langle , \rangle\}$ is called a sub-Riemannian structure.*

We define the inner product on Δ by considering $\{Z_1, Z_2\}$ to be an orthonormal set in such a way that $\langle \dot{q}, \dot{q} \rangle = (\dot{x}^2 + \dot{y}^2)/2$. We consider then the extremum of the action

$$S = \int_0^T \langle \dot{q}, \dot{q} \rangle dt, \tag{4}$$

subject to the non-holonomic constraints given by (2). The length of the curve $q : [0, T] \to \mathbb{R}^{N+3}$ is given by

$$L = \int_0^T \langle \dot{q}, \dot{q} \rangle^{1/2} dt.$$

The infimum of the lengths of all curves $q(t)$ joining two points q_i and q_f, with $\dot{q}(t) \in \Delta(q(t))$ for almost each t, is called the sub-Riemannian distance between the points q_i and q_f.

We consider the Euclidean space $M \subset P$ as the base space with coordinates (x, y). The relation (2) defines the horizontal lifts Z_1 and Z_2 of tangent vectors on TM. The horizontal lift gives a smoothly varying family of linear maps $\sigma_p : T_xM \to T_pP$, where $x = \pi(p) \in M$ is the projection of p by the submersion $\pi : P \to M$. The subspaces $\pi^{-1}(x) \subset T_pP$ are the fibers of π.

A connection is defined by the horizontal distribution Hor, formed by the set of horizontal subspaces. The horizontal lift satisfies $d\pi_p \circ \sigma_p =$ identity mapping on T_xM and $\sigma(gp) = g\sigma(p)$, with $g \in G_0$, where G_0 is the Lie group associated to the Abelian subalgebra $\mathfrak{g}_0$.

Altogether, the natural setting for this problem is the principal bundle (P, M, π, G_0), with total space P, base space M and with Abelian structure group G_0. The family of left invariant vector fields with respect to the group law of G, the Lie group of $\mathfrak{g}$, does not in general coincide with the original set of vector fields determined by the non-holonomic constraints (2). When this occurs we obtain the following result.

Theorem 2.2 *The dynamical system defined by the metric $(\dot{x}^2+\dot{y}^2)/2$ and the non-holonomic constraints $\dot{q} = \dot{x}Z_1(q) + \dot{y}Z_2(q)$ is invariant under the left action of G on P.*

Proof. The result follows directly from the definition of left invariant vector fields.

□

3 Goursat Distribution

A usual Pfaffian system on P is the so called Goursat system which is defined by the kernel of the following Pfaffian system

$$
\begin{aligned}
\nu_1 &= du_1 + y dx, \\
\nu_2 &= du_2 + u_1 dx, \\
&\vdots \\
\nu_{N+1} &= du_{N+1} + u_N dx.
\end{aligned}
$$

These forms have a well defined degree. Setting $deg(x) = deg(y) = 1$, we get $deg(u_i) = i+1$. Particular cases for which some of these forms are absent could be of interest, in such cases the dimension of the manifold would be correspondingly lower.

These one-forms define what is also known as a chained system. By adding, as usual, the 1-forms dx and dy one completes a linearly independent set which encodes the constraints of (2) and yields a rank two distribution $\{Y_1, Y_2\}$, with

$$Y_1 = \partial_x - \sum_{i=1}^{N+1} u_{i-1}\partial_{u_i}, \quad Y_2 = \partial_y,$$

here for notational reasons we take $u_0 = y$. As before, $Y_{i+2} := [Y_1, Y_{i+1}] = \partial_{u_i}$ for $i = 1, \cdots N+1$, and the Y_i's satisfy the same algebra as the Z_i's.

The vector fields Y_i are left invariant with respect to left translations associated to the Lie group G_1 with product law

$$\alpha\,\alpha' = (\alpha_1 + \alpha'_1, \alpha_2 + \alpha'_2, \alpha_3 + \alpha'_3 - \alpha'_1\alpha_2, \cdots, \alpha_{N+3} + \alpha'_{N+3} - \alpha'_1\alpha_{N+2}),$$

thus we have the following result.

Theorem 3.1 *The dynamical system defined by the metric* $(\dot{x}^2+\dot{y}^2)/2$ *and the Kernel of the* 1*-forms* $\nu_i,\ i=1,\cdots,N+1$ *is invariant under the left action of* G_1.

Proof. The result follows once more from the definition of left invariant vector fields. □

4 An alternative distribution

In this section we give an alternative distribution which is equivalent to the Goursat distribution defined in the last section. An analytic map between the involved variables in both problems can be also given. The geometrical problem consists now in the calculation of the extremum of the action (4), subject to non-holonomic constraints determined by the Pfaffian system given by the following 1-forms

$$\omega_k = d\theta_k - \frac{1}{k!}x^k dy, \quad k=1,\cdots,N+1, \tag{5}$$

here $q=(x,y,\theta_1,\cdots,\theta_{N+1})$ are the coordinates of the total space P and the non-holonomic constraints (2) are given in terms of the vector fields

$$X_1=\partial_x, \quad X_2=\partial_y+\sum_{i=1}^{N+1}\frac{1}{i!}x^i\partial_{\theta_i}. \tag{6}$$

The Lie brackets yield in our problem to the same nilpotent Lie algebra $\mathfrak{g}$ defined above with

$$X_{j+2}=\sum_{i=j}^{N+1}\frac{1}{(i-j)!}x^{i-j}\partial_{\theta_i}, \quad j=1,\cdots,N+1.$$

The above vector fields are left invariant with respect to left translations associated to the Lie group G_2 with product law

$$\alpha\cdot\alpha'=(\alpha_1+\alpha_1',\alpha_2+\alpha_2',\alpha_3'',\ldots,\alpha_{N+3}''),$$

with

$$\begin{aligned}
\alpha_3'' &= \alpha_3+\alpha_3'+\alpha_1\alpha_2',\\
&\vdots\\
\alpha_k'' &= \alpha_k+\alpha_k'+\sum_{j=2}^{k-1}\frac{1}{(k-j)!}\alpha_1^{k-j}\alpha_j',\\
&\vdots
\end{aligned}$$

$$\alpha''_{N+3} = \alpha_{N+3} + \alpha'_{N+3} + \frac{1}{(N+1)!}\alpha_1^{N+1}\alpha'_2 + \frac{1}{N!}\alpha_1^N\alpha'_3 + \cdots + \alpha_1\alpha'_{N+2}.$$

Thus, we have the following result.

Theorem 4.1 *The dynamical system defined by the metric* $(\dot{x}^2 + \dot{y}^2)/2$ *and the Kernel of the* 1*-forms* $\omega_i = d\theta_i - x^i dy/i!, \quad i = 1, \ldots, N+1$ *is invariant under the left action of* G_2.

Proof. The result follows once more from the definition of left invariant vector fields.

□

5 Equivalence of the distributions

The vector fields $\{X_i\}$ and $\{Y_i\}$ are not left invariant with respect to the group law of the Lie group obtained directly by the exponential mapping. Moreover, both the kernel of the Goursat system and that of the alternative Pfaffian system, encode indistinctly the non-holonomic constraints (2). In this sense, both systems under consideration are equivalent, more precisely:

Theorem 5.1 $Ker\{\nu_j\}$ *and* $Ker\{\omega_j\}$ *are equivalent under the following coordinate transformation*

$$u_j = \frac{(-)^j}{j!}x^j y + \sum_{i=0}^{j-1}\frac{(-)^i}{i!}x^i\theta_{j-i}, \quad \theta_j = \frac{1}{j!}x^j y + \sum_{i=0}^{j-1}\frac{1}{i!}x^i u_{j-i}.$$

Proof. By direct substitution.

□

A different control problem which is also related with the Goursat distribution is the so-called N-trailer problem. However, the proof of the equivalence of both distributions is by far more involved, see for instance [6]. The study of the equivalence of Pfaffian systems goes back to E. Cartan [7], and involves fine theoretical issues that go beyond the purposes of this paper, see for instance [8].

6 Trajectories

Consider now a curve $C(t)$ on M, as $C : I \subset \mathbb{R} \mapsto M$ by $t \mapsto C(t)$, passing through the point $x_0 = C(0)$ and with p_0 in the fiber $\pi^{-1}(x)$ at x_0. Then the parallel transport of p_0 along C is given by the curve defined by $d\hat{C}(t)/dt = \sigma_p dC(t)/dt$, with $\hat{C}(0) = p_0$ and $\hat{C}(t) = p$. The curve $\hat{C}$ projects by π onto the curve C. We select here $x_0 = 0$.

If the underlying manifold is connected and the distribution is bracket generating, the Chow-Rashevsky's Theorem, see [9] and [4] implies that any two points can be connected by a smooth horizontal path $\hat{C}$.

In this section we shall show how to integrate the equations of motion for the present problem. We shall consider the trajectories only for the non-holonomic constraints (2) and the metric $(\dot{x}^2+\dot{y}^2)/2$. In this case the Lagrangian is given by

$$\mathcal{L} = \lambda_0 \frac{1}{2}(\dot{x}^2+\dot{y}^2) + \lambda\cdot(\dot{q} - \dot{x}Z_1(q) - \dot{y}Z_2(q)),$$

with $q=(x,y,\theta_1,\dots,\theta_{N+1})$, $\lambda=(0,0,\lambda_1,\dots,\lambda_{N+1})$, and the Z_i's acting on q coordinate by coordinate.

The λ_k, $k=1,\dots,N+1$, are Lagrange parameters, in general time dependent and the number λ_0 can take the values 0 or 1. Solutions for $\lambda_0=1$ are called normal extremals and those for $\lambda_0=0$ are abnormal extremals. An extremal is called strictly abnormal if it is not the projection of a normal horizontal curve, see [5]. For the particular Pfaffian system (5), we have

$$\mathcal{L} = \lambda_0 \frac{1}{2}(\dot{x}^2+\dot{y}^2) + \sum_{k=1}^{N+1} \lambda_k(\dot{\theta}_k - \frac{1}{k!}x^k\dot{y}).$$

Let us now consider first the normal extremals, the canonical momenta are given by

$$\begin{aligned} p_x &= \frac{\partial}{\partial \dot{x}}\mathcal{L} = \dot{x}, \\ p_y &= \frac{\partial}{\partial \dot{y}}\mathcal{L} = \dot{y} - \sum_k \frac{\lambda_k}{k!}x^k, \\ p_j &= \frac{\partial}{\partial \dot{\theta}_j}\mathcal{L} = \lambda_j, \quad j=1,\dots,N+1. \end{aligned} \tag{7}$$

The equations of motion for this problem are thus

$$\ddot{x} = eB_z(x)\dot{y}, \quad \ddot{y} = -eB_z(x)\dot{x}, \quad \dot{\lambda}_i = 0, \tag{8}$$

with

$$eB_z(x) = -\sum_{i=1}^{N+1} \frac{\lambda_i}{(i-1)!}x^{i-1}.$$

All Lagrange parameters λ_i are constant, and for simplicity, we shall assume $\lambda_{N+1}\neq 0$.

The first two equations can be interpreted as the equations of motion of an electric charge e of unit mass in the plane $x-y$ subject to the action of an non homogeneous magnetic field B_z in the perpendicular direction to the plane. The horizontal trajectory $\hat{C}(t)$ is given by the solutions $x(t)$ and $y(t)$ and by the solutions of the Pfaffian system $\dot{\theta}_i - x^i\dot{y}/i! = 0$, for $i=1,\dots,N+1$. For the Goursat system the equations for x and y remain the same.

From the equations of motion we obtain as usual the statement of energy conservation, that is, the following quantity is constant along the trajectories

$$\mathcal{E} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2). \tag{9}$$

For simplicity we take $\mathcal{E} = 1/2$, which implies $\dot{x}^2 + \dot{y}^2 = 1$ which is tantamount to measuring trajectories by its arc length.

Remark 6.1 *Some other selections of constraints leading to the same equations of motion in the base space can be interpreted as equivalent gauges for the magnetic vector potential 1-form* $A = \lambda \cdot (dq - Z_1(q)dx - Z_2(q)dy)$ *with* $dA = B_z\, dx \wedge dy = d(A + d\phi)$, *for an arbitrary gauge scalar function* ϕ. *These arbitrary gauge choices are related to the group of automorphisms of the Lie algebra* $\mathfrak{g}$.

We integrate first

$$\dot{y} = \dot{y}_0 + r(x) = \dot{y}_0 - \int_o^x eB_z(v)dv, \tag{10}$$

where $r(x)$ is a $N+1$-degree polynomial.

Theorem 6.2 *The* x *variable satisfies the equation*

$$\xi^2 = p_{2N+2}(x), \tag{11}$$

with $\xi = \dot{x}$ *and the polynomial* $p_{2N+2} = a_0 + a_1 x + \cdots + a_{2N+2}x^{2N+2}$ *of degree* $2N+2$ *in* x *with known real coefficients. There are at most* $N+1$ *regions of the* x*-axis where the motion is allowed. The boundary points of these regions are the real zeros of* $p_{2N+2}(x)$.

Proof. The assertion follows from the substitution of equation (10) into equation (9), obtaining so

$$p_{2N+2}(x) = 1 - (\dot{y}_0 + r(x))^2. \tag{12}$$

Now, observe that $x, \dot{x}, y$ and $\dot{y}$ must be real in our problem, hence we need $p_{2N+2} \geq 0$. But since p_{2N+2} is a polynomial, we infer that the x-axis has allowed regions only for x such that $p_{2N+2} > 0$ and forbidden regions for x such that $p_{2N+2} < 0$, with a set of boundary points given by the real zeros of p_{2N+2}. We need at least two real roots in a nontrivial problem, and obtain a set of at most $N+1$ allowed regions such that $p_{2N+2}(x) \geq 0$, since the leading coefficient of p_{2N+2} is negative and equal to $-(\lambda_{N+1}/(N+1)!)^2$.

□

Remark 6.3 *It is clear the importance of knowing the number of real roots of* $p_{2N+2}(x)$, *this leads us in a natural way to see the holomorphic extension of equation (11). For* $x, \xi \in \mathbb{C}$ *the associated algebraic curve is hyperelliptic of genus* N. *The coefficients* a_i *can lead to singular algebraic curves for certain initial conditions associated with multiple zeros of* ξ^2.

Now, since $\dot{y}$ is a polynomial of degree $N+1$ in x, then equation (9) can be interpreted as the equation for conservation of energy for the one dimensional motion of a point particle of unit mass with total energy $\mathcal{E} = T + V(x)$, kinetic energy $T = \dot{x}^2/2$ and *effective potential energy*

$$V(x) = \frac{1}{2}\dot{y}^2 = \frac{1}{2}\left(\dot{y}_0 + \sum_{i=0}^{N} \frac{\lambda_{i+1}}{(i+1)!} x^{i+1}\right)^2 . \tag{13}$$

Similarly, equation (8) for $\ddot{x}$ can be interpreted as the corresponding equation of motion. This interpretation makes clear the existence of allowed and forbidden gaps for motion, according to whether the energy $\mathcal{E}$ is larger or smaller than the effective potential energy $V(x)$, respectively.

In terms of the polynomial defined in (11), we have $p_{2N+2} = 1 - 2V$. In consequence, the relation $1 \geq p_{2N+2} \geq 0$, which corresponds to an allowed gap, is equivalent to $1 \geq 2V(x) \geq 0$. Additionally, we note here that the maxima of $V(x)$ are points of unstable equilibrium of this mechanical analogy. These points are associated with singular algebraic curves as we shall see.

Corollary 6.4 *The time dependence of the coordinate $x(t)$ is given by the inverse function of the hyper-elliptic integral*

$$t(x) = \int_0^{x(t)} \frac{d\zeta}{\sqrt{p_{2N+2}(\zeta)}}. \tag{14}$$

Here, it is necessary that $p_{2N+2}(\zeta) > 0$ for $0 < \zeta < x(t)$, with at most simple zeros at $\zeta = 0$ and/or $\zeta = x(t)$.

Proof. The result follows integrating equation (11). To make sense of the integrand, t must be real, so that $p_{2N+2} \geq 0$. Further, if p_{2N+2} has a double zero at x_ρ, then as x approaches x_ρ the integrand behaves locally as a constant times $1/(\zeta - x_\rho)$ and the integral grows to infinity.

□

Corollary 6.5 *If there exist $x(t_\rho) = x_\rho$ such that $B_z(x_\rho) = 0$ and $\dot{x}(t_\rho) = 0$, the polynomial $p_{2N+2}(x)$ has at least a double root at $x = x_\rho$.*

Proof. From equation (12) it follows that

$$\frac{d}{dx} p_{2N+2}(x) = -2r'(x)(\dot{y}_0 + r(x)).$$

Therefore, the derivative is also zero at x_ρ, since $r'(x) = -eB_z(x)$. Now, since $\dot{x}(t_\rho) = 0$ means that $p_{2N+2}(x_\rho) = 0$, this polynomial has at least a double zero at x_ρ.

□

For example, for $\dot{x}_0 = 0$ and $\lambda_1 = 0$, i.e. $a_0 = 0 = a_1$, the curve has at least a double point for $N > 0$ at $x = 0$. Note that since $eB_z(x) = -\lambda_1 - \lambda_2 x + \cdots$, the example corresponds to a trajectory along a line (the y-axis) for which the magnetic field is zero.

Since for this last conclusion, the remaining parallel lines on which the magnetic field is zero can be at any other values of x, the same applies *for all* straight lines on which $\vec{B}$ is zero.

Theorem 6.6 *If there exist at least a real zero of $eB_z(x)$, the abnormal extremals are not strictly abnormal and correspond to straight lines. Otherwise, there no abnormal extremals.*

Proof. Abnormal extremals correspond to $\lambda_0 = 0$, for the electric charge system the corresponding Lagrangian is

$$\mathcal{L}^{abnormal} = \sum_{i=1}^{N+1} \lambda_i (\dot{\theta}_i - \frac{1}{i!} x^i \dot{y}).$$

The Euler-Lagrange equations reduce to

$$\begin{aligned} eB_z(x)\dot{y} &= 0, \\ eB_z(x)\dot{x} &= 0, \\ \lambda_j &= constant. \end{aligned}$$

The solutions are either the set of points for which $\dot{x} = \dot{y} = 0$, which are not allowed for non zero total energy, or the set of real points for which $B_z(x) = 0$, which correspond to straight lines $\dot{x} = 0$. In that case $\dot{y}$ is constant and $\ddot{x} = \ddot{y} = 0$, in the base space.

Therefore, either the abnormal extremals are not strictly abnormal or there no abnormal extremals at all.

□

Note also, that the case $eB_z(x_\rho) = 0$, $x_\rho = 0$ and $N = 1$ corresponds to the abnormal trajectories for the Martinet distribution discussed in [5].

Theorem 6.7 *The solutions of the singular algebraic curves, resulting for zero discriminant of $p_{2N+2}(x)$, are associated to the abnormal extremals as well as to trajectories approaching them asymptotically.*

Proof. The last corollary implies that the polynomial $p_{2N+2}(x)$ has double roots along the abnormals and therefore, the algebraic curve $\xi^2 = p_{2N+2}(x)$ is singular.

Now, since $\xi = \dot{x}$ is always zero on these straight lines, we deal with trivial solutions of the algebraic equation, which are located precisely on the unstable points. For non-trivial solutions of the same singular algebraic equation we obtain trajectories which tend asymptotically towards the abnormal extremals on the base space as can be seen from equation (14), for x approaching a double root of p_{2N+2}.

□

Corollary 6.8 *The admissible trajectories with* $\dot{x}_0 = \cos(\phi_0)$*, which reach* $x \neq 0$ *are determined by the wedge-like sets* $-1 - r(x) < \sin(\phi_0) < 1$*, for* $-2 \leq r(x) \leq 0$ *and* $-1 < \sin(\phi_0) < 1 - r(x)$*, for* $2 \geq r(x) \geq 0$*. At the boundary values* $\sin(\phi_0) = -1 - r(x)$ *for* $r(x) \leq 0$ *and* $\sin(\phi_0) = 1 - r(x)$ *for* $2 \geq r(x) \geq 0$*, and if further* $eB_z(x) = 0$*, then the trajectories are non-abnormal and correspond to singular algebraic curves, otherwise they correspond also to non-singular curves.*

Proof. Set $\dot{y}(x) = \sin(\phi)$, then from equation (10) it follows that $\sin(\phi) = \sin(\phi_0) + r(x)$. Therefore $-1 < \sin(\phi_0) + r(x) < 1$, and from this relation follow both inequalities. The limit values correspond to $\dot{y}_\rho = \pm 1$, which lead to singular curves if additionally $eB_z(x) = 0$.

□

7 Momentum and Casimir functions.

Let us now introduce the so-called momentum functions π_X associated to a vector field X on a manifold M, as functions on the cotangent bundle, see for instance [10].

$$\pi_X(x,p) = p(X(x)), \quad p \in T_x^*M, \quad p : T_xM \to \mathbb{R}.$$

In coordinates $p = \sum p_i dq_i$, $X = \sum X_i(x,p)\partial_{q_i}$ and $\pi_X(x,p) = \sum X_i(x,p)p_i$.

Under this correspondence the Lie brackets are associated to the negative of the corresponding Poisson brackets, in particular $[\partial_x, x] = 1$ yields $\{p_x, x\} = -1$. To simplify the notation we shall write $\pi_i := \pi_{X_i}$, then the Poisson bracket realization of the algebra is given by the non-zero brackets $\{\pi_1, \pi_i\} = -\pi_{i+1}$ for $i = 2, \ldots, N+2$.

We calculate now, in a coordinate free fashion, Casimir functions of the universal enveloping algebra of the above nilpotent Poisson algebra associated to the nilpotent Lie algebra given by relations (3). Clearly π_{N+3} is a Casimir function.

Theorem 7.1 *The generators satisfy*

$$\pi_{N+3}^i \pi_{N+2-i} = \sum_{j=-1}^{i} \frac{\alpha_{j+1}}{(i-j)!}\, \pi_{N+2}^{i-j}, \quad i = 0, \ldots, N, \tag{15}$$

where the α_k *are arbitrary constants,* $\alpha_0 = 1$ *and* $\alpha_1 = 0$.

Proof. It is sufficient to verify all Poisson brackets with π_1.

□

Theorem 7.2 *The following* N *quantities are Casimir elements of the algebra*

$$c_j = \frac{(-)^j}{j!}\, \pi_{N+2}^j + \sum_{k=0}^{j-1} \frac{(-)^{j-k-1}}{(j-k-1)!}\, \pi_{N+3}^k\, \pi_{N+2-k}\, \pi_{N+2}^{j-k-1}, \quad j = 2, \ldots, N+1. \tag{16}$$

Proof. All Poisson brackets can be easily shown to vanish. It follows then that

$$c_2 = \pi_{N+3}\pi_{N+1} - \pi_{N+2}^2/2 \quad \text{for} \quad N \geq 1$$

□

Corollary 7.3 *The Casimir elements c_j satisfy the following recursion formula*

$$c_{i+1} = \pi_{N+3}^i \pi_{N+2-i} - \sum_{j=-1}^{i-1} \frac{c_{j+1}}{(i-j)!} \pi_{N+2}^{i-j}, \quad i = 1, \ldots, N, \quad c_0 = 1, \ c_1 = 0. \tag{17}$$

Proof. The formula follows immediately from the expression (15) for the generators, identifying the arbitrary constants α_i with the c_i's.

□

Theorem 7.4 *The normal extremals in cotangent space are given by the intersection of the cylinders with directrices*

$$c_{N+1} - \pi_{N+3}^N \pi_2 + \sum_{j=-1}^{N-1} \frac{c_{j+1}}{(N-j)!} \pi_{N+2}^{N-j} = 0, \quad c_0 = 1, \ c_1 = 0, \tag{18}$$

and $\pi_1^2 + \pi_2^2 - 1 = 0$, located on the planes (π_2, π_{N+2}) and (π_1, π_2) respectively in the $\{\pi_1, \pi_2, \pi_{N+2}\}$-subspace, with $\pi_{N+3} = constant$ and π_i, $i = 3, \ldots, N+1$ given by (17).

Proof. The surfaces follow from equation (17) for $i = N$ and from (9) with $\mathcal{E} = 1/2$.

□

The figure 1 illustrates a simple non-degenerated example for $N = 2$, where(18) is a cubic cylinder $\pi_4^3/6 + c_2\pi_4 + c_3 - \pi_5^2\pi_2 = 0$.

In particular, for the basis of equations (6), the fields $(\partial_x, \partial_y, \partial_{\theta_i})$ for $i > 2$ are mapped to the canonical momenta (p_x, p_y, p_{θ_i}), and with equation (7)

$$\begin{aligned} \pi_1 &= \dot{x}, \\ \pi_2 &= \dot{y}, \\ \pi_{j+2} &= \sum_{i=j}^{N+1} \frac{\lambda_i}{(i-j)!} x^{i-j}, \quad j = 1, \ldots, N+1, \end{aligned}$$

where we recognize $\pi_3 = -eB_z(x)$, $\pi_{N+2} = \lambda_N + \lambda_{N+1}x$, $\pi_{N+3} = \lambda_{N+1}$, and the initial conditions $\pi_{1,0} = \dot{x}_0$, $\pi_{2,0} = \dot{y}_0$ and $\pi_{i+2,0} = \lambda_i$ for $i = 1, \ldots, N+1$. The following simple facts follow.

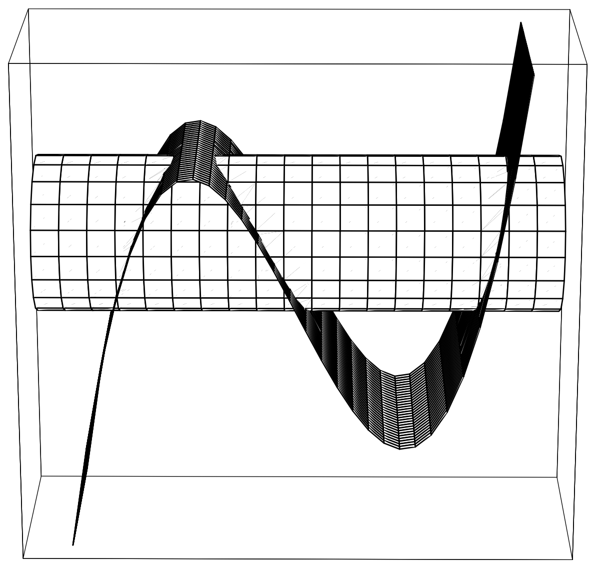

Figure 1: Trajectory in cotangent space for $N = 2$, as intersection of two cylinders.

Corollary 7.5 *In the basis (6), the Casimir functions are given by (16) with the momenta* π_{i+2} *for* $i > 0$ *replaced by the constant parameters* λ_i *and* π_2 *by* $\dot{y}_0$.

Proof. The assertions follow from the equalities

$$\pi_{j+2} = -\frac{d^{j-1}}{dx^{j-1}} eB_z(x), \quad \lambda_j = -\frac{d^{j-1}}{dx^{j-1}} eB_z(x)|_{x=0}, \quad j = 1, \dots, N+1.$$

□

Corollary 7.6 *The trajectory in cotangent space is given in terms of the algebraic curve* $f(a, b) = 0$, *of genus* N, *with*

$$f(a,b) = c^{2N}a^2 - c^{2N} + (c_{N+1} + \sum_{j=-1}^{N-1} \frac{c_{j+1}}{(N-j)!} b^{N-j})^2, \tag{19}$$

where $a = \pi_1$, $b = \pi_{N+2}$, $c = -\pi_{N+3}$ *and* $\{a, b\} = c$.

Proof. It follows directly from equation (18).

□

The algebraic curve can be solved by quadratures for b since $\dot{b} = -ac$. For $N = 0$, the curve is rational, for $N = 1$, elliptic, and for $N > 1$, hyperelliptic.

The figure 2 shows the real curve of genus two associated with figure 1. The closed curves are the allowed values for (b, a).

In the figure 3, the corresponding potential energy (13) is shown. The horizontal line corresponds to the total energy.

In the figure 4 and figure 5, a degenerated genus two case is exemplified. The double point in the left closed curve of figure 4 corresponds to an abnormal trajectory.

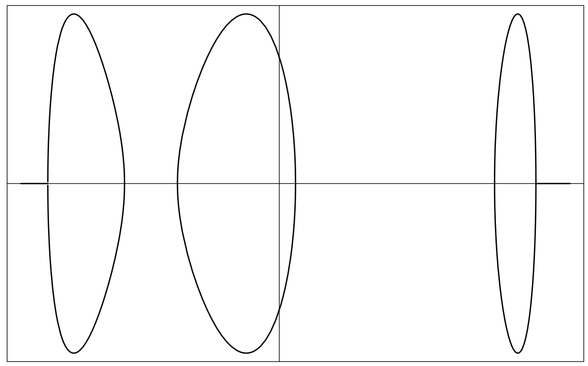

Figure 2: Real part of the genus two curve corresponding to Figure 1

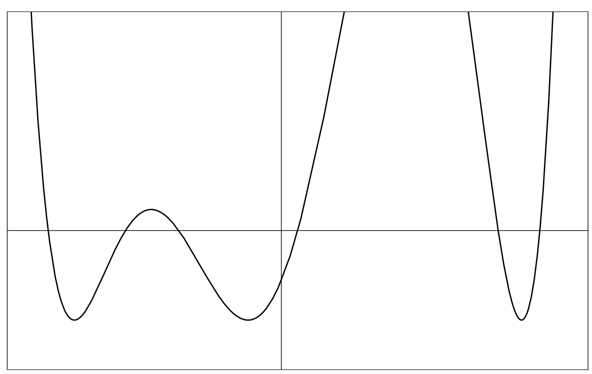

Figure 3: The potential energy associated with Figure 2

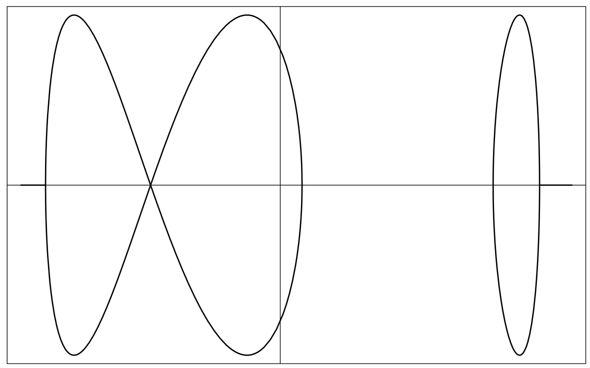

Figure 4: The real part of a degenerated genus two curve in cotangent space. The double point of the left closed curve corresponds to an abnormal trajectory.

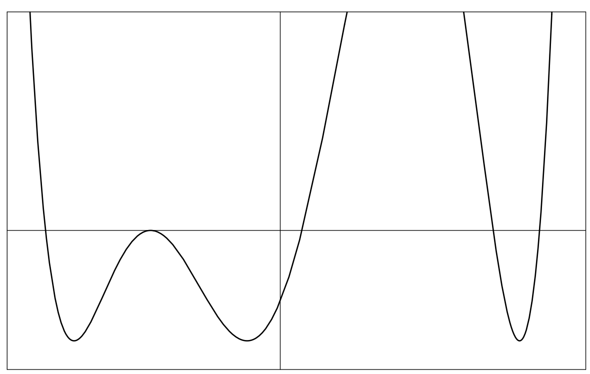

Figure 5: The potential energy for Figure 4

8 Sub-Riemannian geodesics in total space

Here we consider the trajectories in terms only of the coordinates of total space. For simplicity we assume that all trajectories start at $x(0) = y(0) = \theta_1(0) = \cdots = \theta_N = 0$.

Lemma 8.1 *The trajectories lie on the integral surface given by the cylinder*

$$(\dot{x}_0 \, - \sum_{i=1}^{N+1} \lambda_i \theta_{i-1})^2 \, + \, 2V(x) - 1 = 0. \tag{20}$$

On the planes (x, θ_{i-1}), $i = 1, \ldots, N+1$, the cylinder has cross sections given by the real hyperelliptic curves

$$(\lambda_i \theta_{i-1} - \dot{x}_0)^2 \, + \, 2V(x) - 1 = 0, \quad 1 \geq 2V(x). \tag{21}$$

Proof. Rewrite the equation for $\ddot{x}$ as $\ddot{x} = -\sum_{i=1}^{N+1} \lambda_i \dot{\theta}_{i-1}$ and integrate it to obtain

$$\dot{x} = \dot{x}_0 \, - \sum_{i=1}^{N+1} \lambda_i \theta_{i-1}, \tag{22}$$

with $\theta_i(0) = 0$ and $\theta_0(t) = y(t)$. Thus, equation (20) results, which contains only the coordinates of total space and no time derivatives of them. It is an algebraic curve in total space.

□

Therefore, in total space coordinates $\{x, y, \theta_i\}$ the trajectory is given by energy conservation and the Pfaffian system $d\theta_i - x^i \, dy/i! = 0$ for $i = 1, \ldots, N+1$.

Theorem 8.2 *In terms of (Abelian) differentials ϖ_i we have*

$$dy = \dot{y}_0 \, \varpi_0 \, + \sum_{i=1}^{N+1} \frac{\lambda_i}{i!} \varpi_i, \tag{23}$$

with

$$\varpi_i = \frac{x^i}{\xi} \, dx, \tag{24}$$

where $\xi = \sqrt{p_{2N+2}(x)}$.

Proof.

From the Pfaffian system, we obtain for the normal extremals

$$dy = \frac{\dot{y}_0 \, + \sum_{i=1}^{N+1} \lambda_i x^i / i!}{\sqrt{1 - 2V(x)}} \, dx. \tag{25}$$

With the definition of the ϖ_i the result follows.

□

The differentials $\{\varpi_i,\ i = 0, \ldots, 2N\}$ form a basis [11]. All differentials ϖ_k with $k \in \mathbb{N}$ can be given recursively in terms of the basis using the following result.

Proposition 8.3 *The differentials ϖ_i satisfy*

$$(s+N+1)a_{2N+2}\varpi_{s+2N+1} = d(x^s\,\xi) - \sum_{i=0}^{2N+1}(s+\frac{i}{2})a_i\varpi_{i+s-1},\ s = 0, 1, \ldots \tag{26}$$

Proof.

The result follows after the computation of $d(x^s\,\xi)$ and using the expression for ξ^2 in terms of x. For $s \geq 0$, all ϖ_k with $k > 2N$ are obtained. For $s \leq -1$, all ϖ_j with $j \leq -1$ are given in terms of the set ϖ_n with $n = -1, 0, 1, \ldots, 2N$. But a simple argument allows to find a relation between these last differentials and therefore, the set of differentials without the ϖ_{-1} is enough to obtain all ϖ_i.

□

The differentials ϖ_j for $j = 0, 1, \cdots, N-1$ are holomorphic for x complex. The differentials ϖ_k, with $k = N, \cdots, 2N$ have two poles of order $k - N + 1$. The following Abelian integrals are important here. Let

$$I_i(x) = \int_0^x \varpi_i. \tag{27}$$

Proposition 8.4 *We have*

$$y(t) = \dot{y}_0\,t + \sum_{i=1}^{N+1}\frac{\lambda_i}{i!}I_i(x), \tag{28}$$

where $p_{2N+2}(x) \geq 0$ for $x \geq 0$, and initial condition $y_0 = 0$.

Proof.

The result follows from equations (14) and (23), since $t = I_0(x)$. Time plays the role of a local uniformizing parameter.

□

Lemma 8.5 *The differentials of the Pfaffian system can be expressed in terms of the basis of Abelian differentials by means of*

$$d\theta_i = \frac{1}{i!}\dot{y}_0\varpi_i + \sum_{j=1}^{N+1}\frac{\lambda_j}{i!j!}\varpi_{j+i}, \quad i = 0, 1, \ldots, N+1. \tag{29}$$

Proof. This relations result from the original Pfaffian system and the expression of dy in terms of Abelian differentials.

□

Theorem 8.6 *The sub-Riemannian geodesics in total space are given by hyperelliptic integrals as*

$$\theta_i = \frac{1}{i!}\dot{y}_0 I_i(x) + \sum_{j=1}^{N+1} \frac{\lambda_j}{i!j!} I_{j+i}(x), \quad i = 0, 1, \ldots, N+1, \tag{30}$$

with $y = \theta_0$ *and* $x(t)$ *given by the inverse function of (14).*

Proof. The integration of equation (29) yields the result. □

9 Conclusions

A physical problem with non-holonomic constraints has been presented, which can be transformed into a variational problem with a Goursat system. The present new geometrical approach seems to allow future generalizations. We have integrated by quadratures the equations of motion corresponding to Goursat systems and a flat metric. The trajectories in total space are given in terms of hyperelliptic integrals. A set of algebraic curves in cotangent space determine the normal trajectories. Abnormal trajectories are shown to be non-strictly abnormal and their relation with singular algebraic curves has been considered. The Casimir elements of an associated Poisson algebra have been computed explicitly.

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