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On the controllability of linear juggling mechanical systems

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Abstract: This paper deals with the controllability of a class of nonsmooth complementarity mechanical systems. Due to their particular structure they can be decomposed into an "object" and a "robot", consequently they are named *juggling* systems. It is shown that the accessibility of the "object" can be characterized by nonlinear constrained equations, or generalized equations. Examples are presented, including a simple model of backlash. The main focus of the work is about linear jugglers, but extensions towards more complicated models are considered.

Key-words: unilateral constraints, nonsmooth mechanics, impacts, controllability, complementarity problems, accessibility.

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Commandabilité des jongleurs mécaniques linéaires

Résumé : Ce papier concerne la caractérisation de la commandabilité d'une classe de systèmes mécaniques non-réguliers. Ces systèmes peuvent être décomposés en un "objet" et un "robot", et sont donc appelés *jongleurs*. On montre que l'accessibilité de l'objet peut être caractérisée par des équations non-linéaires contraintes. Des exemples sont présentés, tels que le modèle simple de jeu dynamique. Ce travail est centré essentiellement sur les jongleurs linéaires, mais on considère des extensions vers des systèmes plus complexes.

Mots-clés : Conraintes unilatérales, mécanique non-régulière, impacts, commandabilité, problèmes de complémentarité, accessibilité.

1 Introduction

Manipulating objects by pushing and hitting (also called nonprehensile manipulation) is an important robotic task, see [2] [15] [10] [22] and references therein. It is easily recast in the setting of so-called juggling systems [8] [16] [25] (a class that encompasses systems with dynamic backlash [18], manipulators with dynamic passive environments, controlled structures, hopping machines, tethered sattelites [12] etc). Studies on the controllability of such nonlinear nonsmooth dynamical systems require the development of specific analysis tools, due to their very particular features [6] [15] [24] [16]. The paper [16] contains a very nice study of a juggling system and its reachable subspaces, and proposes a general method for the design of feedback control in order to stabilize specific trajectories. In particular the studied juggler is not small-time locally controllable. Global criteria for accessibility may be needed.

This paper is dedicated to investigate a way to characterize the controllability properties of a subclass of juggling systems, which we choose to name *linear jugglers*. It appears that despite the fact this class of jugglers may represent the simplest juggling systems, their controllability is not easy to establish in general since they anyway remain highly nonlinear dynamical systems. This work is a step forward in the study of the controllability properties of jugglers as introduced in [6], since it presents some tools which allow one to characterize in a general way whether the considered system possesses the required accessibility properties, or not. The paper is organized as follows: in section 2 we introduce the dynamics of jugglers; in section 3 the controllability framework is developed; section 4 is devoted to illustrating the theoretical setting by an example (dynamic backlash). Section 5 shows how the tools can be extended to study more complex systems. Conclusions end the paper. Some definitions and calculations are provided in appendices. A preliminary version of this work can be found in [7].

2 System's dynamics

Let us consider the following class of complementarity dynamical systems [9]

$$\left\{\begin{array}{l} \dot{z}_1 = f_1(z_1, t, \lambda) \\ \\ \dot{z}_2 = f_2(z_2, t, u, \lambda) \\ \\ 0 \leq h(z_1, z_2) \perp \lambda \geq 0 \\ \\ \text{Collision mapping} \end{array}\right. \tag{1}$$

which has been named *juggling systems* in [6], where $z_1 \in R^{n_1}$, $z_2 \in R^{n_2}$, $h(\cdot,\cdot)$ and $\lambda \in R^m$, and $u \in R^{n_u}$. The z_1-dynamics represents the dynamics of the "object" (which may be a real object like a puck, or the center of gravity dynamics of a flying system [5] §8.7), while the z_2-dynamics is that of the "robot". The signal λ in (1) is a vector of Lagrange multipliers which represents the contact force between the two parts of the system, if the system is a mechanical system. When the "distance" function $h(z_1, z_2) > 0$ then the interaction is $\lambda = 0$, and the force is allowed to be $\lambda > 0$ only if $h(z_1, z_2) = 0$. At times of impact, λ is no longer a function but is a Dirac measure so that the dynamics becomes algebraic [5]. Obviously the free-motion dynamics ($\lambda \equiv 0$) is not controllable. In this work we focus on a subclass of dynamics as in (1) that we may call *linear jugglers*

$$\begin{cases} M_1\ddot{q}_1 = A_1^T\lambda \\ M_2\ddot{q}_2 = Eu + A_2^T\lambda \\ 0 \leq Aq + B \perp \lambda \geq 0 \\ \dot{q}(t_k^+) = \mathrm{prox}_M[\dot{q}(t_k^-), V(q(t_k))] \end{cases} \tag{2}$$

In (2) $q_1 \in I\!R^{n_1/2\times 1}$, $q_2 \in I\!R^{n_2/2\times 1}$, $q^T = (q_1^T, q_2^T)$ is a $\frac{n_1+n_2}{2}$-dimensional vector of generalized coordinates, $A = (A_1, A_2) \in I\!R^{m\times(n_1/2+n_2/2)}$, i.e. A_1 is made of the first $\frac{n_1}{2}$ columns of A whereas A_2 is made of the last $\frac{n_2}{2}$ columns of A. Also $E \in I\!R^{n_2/2\times n_u}$, $M_1 \in I\!R^{\frac{n_1}{2}\times\frac{n_1}{2}}$, $M_2 \in I\!R^{\frac{n_2}{2}\times\frac{n_2}{2}}$, and $B \in I\!R^m$ are all constant, $\lambda \in I\!R^m$. Clearly both n_1 and n_2 are even integers. The "prox_M" denotes the proximation in the kinetic metric, i.e. $\dot{q}(t_k^+)$ is the closest vector to $\dot{q}(t_k^-)$, inside the set $V(q(t_k))$, and with the distance deduced from the scalar product x^TMy, x and y two vectors of appropriate dimension. The times t_k generically denote impact times. At $t = t_k$ the multiplier $\lambda = p_k\delta_{t_k}$ is a Dirac measure, $p_k \geq 0$, and the dynamics in (2) can be rewritten as

$$\begin{cases} M_1(\dot{q}_1(t_k^+) - \dot{q}_1(t_k^-)) = A_1^T p_k \\ M_2(\dot{q}_2(t_k^+) - \dot{q}_2(t_k^-)) = A_2^T p_k \\ \dot{q}(t_k^+) = \mathrm{prox}_M[\dot{q}(t_k^-), V(q(t_k))] \end{cases} \tag{3}$$

The tangent cone to the domain $Aq+B \geq 0$ is defined as $V(q(t)) = \{x \in I\!R^{\frac{n_1}{2}} \times I\!R^{\frac{n_2}{2}} \mid Ax = A_1x_1 + A_2x_2 \geq 0\}$. Let us choose Moreau's collision rule with restitution e. Let us denote A^i the ith row of the matrix A, $V_i(q(t)) = \{x \in I\!R^{\frac{n_1}{2}} \times I\!R^{\frac{n_2}{2}} \mid A^ix = A_1^ix_1 + A_2^ix_2 \geq 0\}$, $1 \leq i \leq m$, and let M =blockdiag(M_1, M_2). We will also use $(\cdot)_+$ to denote the positive part of $(\cdot)$ and $(\cdot)_-$ the negative part. Notice that $\mathrm{prox}_M(y, V_i(q(t_k))) = y - \langle y, n_i\rangle_+ n_i$ with $n_i = \frac{-1}{\sqrt{A^iM^{-1}(A^i)^T}} \begin{pmatrix} M_1^{-1}(A_1^i)^T \\ M_2^{-1}(A_2^i)^T \end{pmatrix} = \frac{-1}{\sqrt{A^iM^{-1}(A^i)^T}} \begin{pmatrix} N_1^i \\ N_2^i \end{pmatrix}$, $i \in \{1, ..., m\}$, and the scalar product $\langle\cdot,\cdot\rangle$ is in the kinetic metric. Consequently we can rewrite the impact law with restitution $e \in [0,1]$ as [17]

$$\dot{q}(t_k^+) = -e\dot{q}(t_k^-) + (1+e)\left[\dot{q}(t_k^-) - \frac{1}{A^i M^{-1}(A^i)^T}\left[A_1^i \dot{q}_1(t_k^-) + A_2^i \dot{q}_2(t_k^-)\right]_- \begin{pmatrix} N_1^i \\ \\ N_2^i \end{pmatrix}\right] \quad (4)$$

The choice of the model, especially the impact rule, will not be discussed here. See [5] [23] for more informations, and [24] [22] for a control study based on the Routh's 2-dimensional model of impact with friction. Notice that there could be a different e_i for each constraint. For the sake of simplicity of the subsequent presentation, we assume that $e_i = e$, $1 \le i \le m$. One should however be aware of the fact that such rigid body models have been proved to be quite useful for numerical applications and to provide very good motion prediction for virtual prototyping of complex nonsmooth mechanical systems, see e.g. [1]. In [6] a criterion to characterize the controllability properties of the z_1-dynamics has been proposed. It bases on the derivation of a partial impact Poincaré map

$$\begin{aligned} P_\Sigma : \quad & \Sigma \to \Sigma \\ & \dot{q}_1(k) \mapsto P_\Sigma(\dot{q}_1(k), \dot{q}_2(k), k) = \dot{q}_1(k+1) \end{aligned} \quad (5)$$

where $\dot{q}_i(k) \triangleq \dot{q}_i(t_k^-)$, $i = 1, 2$. If P_Σ together with state-input inequality constraints is controllable (state $\dot{q}_1(k)$, control input $\dot{q}_2(k)$), then the z_1-dynamics in (1) is said *velocity controllable through the impacts* (VCTI). Obviously other properties (like accessibility, or reachability through the impacts) can be studied. The VCTI therefore focuses on velocities only, which may be sufficient in certain tasks. Actually it has been introduced because it is not possible to derive a discrete-time system with state $z_1(k)$ and input $z_2(k)$ because the integration of the z_1-dynamics on (t_k, t_{k+1}) necessarily involves $q_2(k+1)$, see remark 1 below. The study in [24] characterizes reachable subsets of velocities achievable via a single impact. We shall come back on the relationships between the VCTI and the work in this note in remark 3.

3 A controllability criterion

In view of this let us investigate another path to characterize the controllability properties of linear juggling systems as in (2).

3.1 Controllability through the impacts

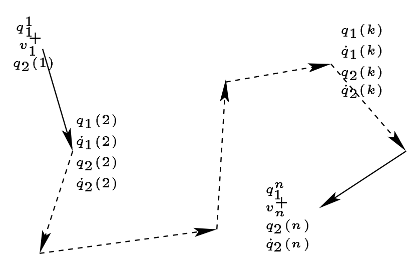
In this study we choose to control the object's dynamics through impacts with the robot's dynamics. Therefore phases of persistent contact between these two parts of the system, are excluded from the following developments. Similarly as in [6] we make the following

Assumption 1 There exists u such that $\dot{q}_2(k)$ can be given arbitrary values at arbitrary positions on the object's orbits. Moreover a unique arbitrary constraint can be striked at times t_k.

Assumption 2 The matrix A_1 satisfies $\text{rank}(A_1) \geq 2$.

It is clear that assumption 2 implies $m \geq 2$, and excludes also some systems like the impacting pair [13] which models dynamic backlash and belongs to the class of systems as in (2). We will see that it can be relaxed. Also we deal only with simple impacts, i.e. impacts with a constraint $A^i q + B^i = 0$ for some $1 \leq i \leq m$. Multiple impacts can be incorporated in the proposed framework, by suitably modifying the reinitialization rule in (4), following for instance Moreau [20] [5]. Assumption 1 allows one to decouple the control problem into two parts: the control of the object's dynamics using λ as the input (i.e. using impacts), then the control of the overall system with u. Here we focus on the first part only and we suppose throughout the paper that assumption 1 holds. Let us notice that the algebraic form of the dynamics at the impact times, allows one to express λ as a function of positions and velocities. This is used to calculate the mappingP_Σ in (5). Therefore "using λ as the input" can be understood as using some of these variables as the input, in particular $\dot{q}_2(k)$ and $q_2(k)$ (which obviously is linked to $q_1(k)$, see (2)). Let us formulate the following controllability problem.

Problem 1 Given (q_1^1, v_1^+) and (q_1^n, v_n^+) with $v_1^+ \neq 0$, find a path $\{q_1(k)\}_{2 \leq k \leq n}$, $\{\dot{q}_1(k)\}_{2 \leq k \leq n}$, $\{q_2(k)\}_{1 \leq k \leq n}$ and $\{\dot{q}_2(k)\}_{2 \leq k \leq n}$, such that $(q_1(t_1), \dot{q}_1(t_1^+)) = (q_1^1, v_1^+)$ and $(q_1(t_n), \dot{q}_1(t_n^+)) = (q_1^n, v_n^+)$.



Notice that $n \geq 2$ by construction. The constraint on v_1^+ will be made clear later. Problem 1 is formulated such that an impact has occured at t_1 and one starts looking at the system just after this impact. However $v_n^+ = 0$ is a possible choice. This is why $\dot{q}_2(1)$ is not an unknown of the problem. One sees that if problem 1 has a solution, then the second stage of the controllability study is to find out a controller $u(\cdot)$ which drives the robot state towards the values given by the sequences $\{q_2(k)\}_{1 \leq k \leq n}$ and $\{\dot{q}_2(k)\}_{2 \leq k \leq n}$.

Since our main goal is to characterize controllability, we will focus later on the characterization of reachable subspaces and the accessibility of the systems in (2) and (4).

Definition 1 The reachable subspaces are defined as $\mathcal{R}^n\left[(q_1^1, v_1^+)\right] = \{(q_1^n, v_n^+) \mid$ problem 1 possesses at least one solution$\}$.

Definition 2 Let us denote $\bar{\mathcal{R}}^n\left[(q_1^1,v_1^+)\right] = \cup_{2\leq k\leq n}\mathcal{R}^k\left[(q_1^1,v_1^+)\right]$. The object's dynamics is called accessible when $\bar{\mathcal{R}}^n\left[(q_1^1,v_1^+)\right]$ contains an open set for some $n \geq 2$, and accessible in $N-1$ impacts if $\bar{\mathcal{R}}^n\left[(q_1^1,v_1^+)\right]$ contains an open set for all $n \geq N$.

The reason for the $N-1$ instead of N comes from the fact that the first impact that counts in the analysis is at time t_2. Let us denote $\mathcal{R}^n_{V_1}\left[(q_1^1,v_1^+)\right]$ the reachable set from (q_1^1,v_1^+) in $n-1$ impacts, with object's trajectories $(q_1(k),\dot{q}_1(k))_{2\leq k\leq n}$ remaining in a neighborhood V_1 of (q_1^1,v_1^+).

Definition 3 The object's dynamics is called locally accessible if $\bar{\mathcal{R}}^n_{V_1}\left[(q_1^1,v_1^+)\right]$ contains an open set for any V_1 and any $n \geq 2$. When the object's dynamics is accessible but not locally accessible, it is said to be globally accessible.

As pointed out in the introduction, some jugglers are not locally accessible. In other words given (q_1^1,v_1^+) and (q_1^n,v_n^+) in any neighborhood V_1, one may need to consider object's trajectories which do not remain inside V_1 to reach (q_1^n,v_n^+). This is essentially due to the unilaterality of the constraints (i.e. the complementarity conditions). Notice that the trajectories are understood as the positions and velocities at the impact times only. However the complete trajectories may leave V_1 as the simplest one degree-of-freedom juggler under gravity (see [6]) shows.

Definition 4 (CTI) The object's dynamics in (2) is controllable in $n-1$ impacts (or controllable through the impacts in $n-1$ impacts –CTI$(n-1)$–) if problem 1 has a solution for all $(q_1^1,v_1^+)\in I\!R^{n_1}$ and all $(q_1^n,v_n^+)\in I\!R^{n_1}$.

Let us denote $x_1^T = (q_1^T(2),\cdots,q_1^T(n-1))$, $x_1\in I\!R^{(n-2)\frac{n_1}{2}\times 1}$, $x_2^T = (q_2^T(1),...,q_2^T(n))$, $x_2\in I\!R^{n\frac{n_2}{2}\times 1}$, $x_3^T = (\dot{q}_1^T(2),\cdots,\dot{q}_1^T(n-1))$, $x_3\in I\!R^{(n-2)\frac{n_1}{2}\times 1}$, $x_4^T = (\dot{q}_2^T(2),\cdots,\dot{q}_2^T(n))$, $x_4\in I\!R^{(n-1)\frac{n_2}{2}\times 1}$. The first aim of this note is to prove the following

Lemma 1 *Let assumption 2 hold. Problem 1 has a solution if and only if the constrained equation $H_j(x_1,x_2,x_4)=0$, $G_j(x_1,x_2,x_4)\geq 0$ has a solution, where $H_j(\cdot)$ and $G_j(\cdot)$ are the nonlinear functions given in (27) below.*

This result is not surprising due to the complementarity conditions in the dynamics. It is the basis for subsequent analysis.

Proof: The proof (i.e. the construction of the functions $H_j(\cdot)$ and $G_j(\cdot)$) is divided in five steps which correspond to the constraints that the unknowns in problem 1 have to satisfy:

- **Final velocity equality:** From (4) we have

$$\begin{aligned}\dot{q}_1(t_k^+) &= \dot{q}_1(k) - (1+e)\frac{1}{A^{i(k)}M^{-1}(A^{i(k)})^T}\left[A_1^{i(k)}\dot{q}_1(k) + A_2^{i(k)}\dot{q}_2(k)\right]_- N_1^{i(k)} \\ &= \dot{q}_1(k) - a_-^k N_1^{i(k)}\end{aligned} \tag{6}$$

with $a_-^k = \frac{1+e}{A^{i(k)}M^{-1}(A^{i(k)})^T}\left[A_1^{i(k)}\dot{q}_1(k) + A_2^{i(k)}\dot{q}_2(k)\right]_-$. One notes that a_-^k is linear in its arguments $\dot{q}_1(k)$ and $\dot{q}_2(k)$ provided $A_1^{i(k)}\dot{q}_1(k) + A_2^{i(k)}\dot{q}_2(k) \leq 0$. From (2) it follows that

$$\dot{q}_1(k) = \dot{q}_1(t_{k-1}^+) \tag{7}$$

The index $i(k)$ means that the constraint $i(k)$, corresponding to the row $i(k)$ of the matrix A, is striked at the time t_k. For instance if $m = 2$ one may have $i(k) = 1$ or $i(k) = 2$. Since problem 1 is concerned with $n-1$ impacts, let us denote $\mathcal{I}_j = \{i(2), ..., i(n)\}$, $j \in \{1, ..., m^{(n-1)}\}$, the possible sequences of successive simple impacts with the m constraints. From (6) and (7) one can write

$$\begin{cases} \dot{q}_1(t_k^+) = \dot{q}_1(t_{k-1}^+) - a_-^k N_1^{i(k)} \\ \dot{q}_1(t_{k-1}^+) = \dot{q}_1(t_{k-2}^+) - a_-^{k-1} N_1^{i(k-1)} \\ ... \\ ... \\ \dot{q}_1(t_2^+) = \dot{q}_1(t_1^+) - a_-^2 N_1^{i(2)} \end{cases} \tag{8}$$

Consequently there are $m^{(n-1)}$ possible sequences as (8) within the formulation of problem 1. We obtain

$$\dot{q}_1(t_k^+) = \dot{q}_1(t_1^+) - \left(\sum_{j=2}^{k} a_-^j N_1^{i(j)}\right) \tag{9}$$

From problem 1 we can rewrite (9) as

$$\dot{q}_1(t_k^+) = v_1^+ - \left(\sum_{j=2}^{k} a_-^j\left(\dot{q}_1(j), \dot{q}_2(j)\right) N_1^{i(j)}\right) \tag{10}$$

and obviously at t_n one obtains

$$v_n^+ = v_1^+ - \left(\sum_{j=2}^{n} a_-^j\left(\dot{q}_1(j), \dot{q}_2(j)\right) N_1^{i(j)}\right) \tag{11}$$

where v_n^+ and v_1^+ are data of the problem. Consider (6) and (7), the linearity of a_-^j; in particular $a_-^1 = \frac{1+e}{A^{i(1)}M^{-1}(A^{i(1)})^T}\left[A_1^{i(1)}v_1^+ + A_2^{i(1)}\dot{q}_2(1)\right]_-$ and using (6) and (8) one can express $\dot{q}_1(j)$ as a linear function of x_4. For instance one has

$$\dot{q}_1(j) = \dot{q}_1(j-1) - a_-^{j-1}\left(\dot{q}_1(j-2) - a_-^{j-2}N_1^{i(j-2)}, \dot{q}_2(j-1)\right) N_1^{i(j-1)} \tag{12}$$

We have $\dot{q}_1(2) = v_1^+$, $\dot{q}_1(3) = v_1^+ - a_-^2(v_1^+, \dot{q}_2(2))N_1^{i(2)}$, $\dot{q}_1(4) = v_1^+ - a_-^3 \left[v_1^+ - a_-^2(v_1^+, \dot{q}_2(2))N_1^{i(2)}, \dot{q}_2(3)\right] N_1^{i(3)} - a_-^2(v_1^+, \dot{q}_2(2))N_1^{i(2)}$, and so on. Then from (10) and since $A_1^{i(k)}\dot{q}_1(k) + A_2^{i(k)}\dot{q}_2(k)$ is a scalar, one gets

$$\dot{q}_1(t_k^+) = v_1^+ + \mathcal{F}_k(e, A, M_1, M_2)x_4 + \mathcal{G}_k(v_1^+, e, M_1, A, M_2) \tag{13}$$

where $\mathcal{F}_k(\cdot)$ and $\mathcal{G}_k(\cdot)$ are constant matrices. Taking $k = n$ one notes that the equality in (13) represents a constraint on x_4 given as

$$\mathcal{F}_{\mathcal{I}_j}(e, A, M_1, M_2)x_4 + \mathcal{G}_{\mathcal{I}_j}(v_1^+, e, M_1, A, M_2) + v_n^+ = 0 \tag{14}$$

where $\mathcal{F}_{\mathcal{I}_j} \in I\!R^{\frac{n_1}{2} \times \frac{n_2}{2}(n-1)}$ and $\mathcal{G}_{\mathcal{I}_j} \in I\!R^{\frac{n_1}{2} \times 1}$.

- **Negative pre-impact velocities:** From the object's dynamics one gets

$$q_1(k+1) - q_1(k) = \dot{q}_1(t_k^+)\Delta_k \tag{15}$$

with $\Delta_k = t_{k+1} - t_k$, $1 \leq k \leq n-1$, and from the constraints expression at t_{k+1}

$$A_1^{i(k+1)}q_1(k+1) + A_2^{i(k+1)}q_2(k+1) + B^{i(k+1)} = 0,\ 0 \leq k \leq n-1 \tag{16}$$

and inserting (15)

$$A_1^{i(k+1)}q_1(k) + A_2^{i(k+1)}q_2(k+1) + B^{i(k+1)} + A_1^{i(k+1)}\dot{q}_1(t_k^+)\Delta_k = 0 \tag{17}$$

Remark 1 It clearly appears from (15) and (17) that $q_2(k)$ cannot be used as an input in a mapping for the computation of $q_1(k)$. ■

Notice from the rank condition in assumption 2 that $A_1^{i(k+1)}\dot{q}_1(t_k^+) = A_1^{i(k+1)}\dot{q}_1(t_{k+1}^-) \neq 0$ since one can always choose the vector $\left(A_1^{i(k+1)}\right)^T$ not normal to the vector $\dot{q}_1(t_k^+)$. Also one has from the impact existence condition $A_1^{i(k)}\dot{q}_1(k) + A_2^{i(k)}\dot{q}_2(k) \leq 0$ and (7)

$$A_1^{i(k+1)}\dot{q}_1(t_k^+) + A_2^{i(k+1)}\dot{q}_2(k+1) \leq 0,\quad 1 \leq k \leq n-1. \tag{18}$$

Still using recursively (12) it follows that (18) can be rewritten compactly as

$$\mathcal{K}_{\mathcal{I}_j}(v_1^+, A, e, M_1, M_2) + \mathcal{L}_{\mathcal{I}_j}(e, M_1, A, M_2)x_4 \geq 0 \tag{19}$$

for some constant matrices $\mathcal{K}_{\mathcal{I}_j}(\cdot) \in I\!R^{(n-1)\times 1}$, $\mathcal{L}_{\mathcal{I}_j}(\cdot) \in I\!R^{(n-1)\times \frac{n_2}{2}(n-1)}$ and the ≥ 0 is componentwise.

- **Final position equality:** Now from (15) it easily follows that

$$q_1^n - q_1^1 = \sum_{k=1}^{n-1} \Delta_k \dot{q}_1(t_k^+) \tag{20}$$

Using (17) to express Δ_k, inserting into (20) and using (16), we obtain

$$\begin{aligned} q_1^n - q_1^1 &= \sum_{k=1}^{n-1} \frac{-\dot{q}_1(t_k^+)}{A_1^{i(k+1)}\dot{q}_1(t_k^+)} \left[A_1^{i(k+1)} q_1(k) + A_2^{i(k+1)} q_2(k+1) + B^{i(k+1)}\right] \\ &= \sum_{k=1}^{n-1} \frac{-\dot{q}_1(t_k^+)}{A_1^{i(k+1)}\dot{q}_1(t_k^+)} A_1^{i(k+1)} \left[q_1(k) - q_1(k+1)\right] \end{aligned} \tag{21}$$

where q_1^n and q_1^1 are data of the problem. Now using the fact that we can express $\dot{q}_1(t_k^+)$ as a function of v_1^+ and x_4, see (13), for all $k \in \{2, ..., n\}$, we obtain from the second equality in (21)

$$\mathcal{A}_{\mathcal{I}_j}(v_1^+, A, M_1, M_2, x_4)x_1 + \mathcal{B}_{\mathcal{I}_j}(v_1^+, A, M_1, M_2, q_1^1, x_4) + \mathcal{R}_{\mathcal{I}_j}(e, A, M_1, M_2, v_1^+, x_4)q_1^n = 0 \tag{22}$$

for some matrices $\mathcal{A}_{\mathcal{I}_j}(\cdot) \in I\!R^{\frac{n_1}{2} \times \frac{n_1}{2}(n-2)}$, $\mathcal{B}_{\mathcal{I}_j}(\cdot) \in I\!R^{\frac{n_1}{2} \times 1}$ and $\mathcal{R}_{\mathcal{I}_j}(\cdot) = I_{n_1/2} - \frac{\dot{q}_1(t_{n-1}^+)A_1^{i(n)}}{A_1^{i(n)}\dot{q}_1(t_{n-1}^+)} \in I\!R^{\frac{n_1}{2} \times \frac{n_1}{2}}$ whose entries may be singular in x_4, but with a denominator that is linear in x_4, see (13).

- **Positive flight-times:** From (15) (17) we get

$$A_1^{i(k+1)}\left[q_1(k+1) - q_1(k)\right] = A_1^{i(k+1)}\dot{q}_1(t_k^+)\Delta_k \tag{23}$$

Since $A_1^{i(k+1)}\dot{q}_1(t_k^+) \neq 0$, the constraints $\Delta_k \geq 0$, $1 \leq k \leq n-1$, may therefore be written as (see (13))

$$\mathcal{H}_{\mathcal{I}_j}(v_1^+, e, A, M_1, M_2, x_4)x_1 + \mathcal{J}_{\mathcal{I}_j}(q_1^1, q_1^n, v_1^+, e, A, M_1, M_2, x_4) \geq 0 \tag{24}$$

for some matrices $\mathcal{H}_{\mathcal{I}_j}(\cdot) \in I\!R^{(n-1)\times\frac{n_1}{2}(n-2)}$ and $\mathcal{J}_{\mathcal{I}_j}(\cdot) \in I\!R^{(n-1)\times 1}$ whose components may be singular in x_4. The vector $\mathcal{J}_{\mathcal{I}_j}$ has the following structure

$$\begin{pmatrix} -\frac{A_1^{i(2)}q_1^1}{A_1^{i(2)}v_1^+} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{A_1^{i(n)}q_1^n}{A_1^{i(n)}\left[v_1^+ - \sum_{j=2}^{n-1} a_-^j N_1^{i(j)}\right]} \end{pmatrix} \tag{25}$$

Notice that the form of the inequalities in (24) depends on the signum of $A_1^{i(k+1)}\dot{q}_1(t_k^+)$ for each $1 \leq k \leq n-1$.

Remark 2 The constraint that $v_1^+ \neq 0$ in problem 1 appears as a necessary condition for the construction of the matrices. It is in fact an artefact due to the way we have formulated the controllability in problem 1. A solution for removing this technical assumption is to strike the object initially and then continuing.

• **Unilateral constraints active at impact times:** From (16) one can construct the set of equalities

$$\mathcal{C}_{\mathcal{I}_j}(A)x_1 + \mathcal{D}_{\mathcal{I}_j}(A)x_2 + \mathcal{E}_{\mathcal{I}_j}(B,q_1^1) + \mathcal{Q}_{\mathcal{I}_j}(A)q_1^n = 0 \tag{26}$$

with $\mathcal{C}_{\mathcal{I}_j}(\cdot) \in I\!R^{n\times \frac{n_1}{2}(n-2)}$, $\mathcal{D}_{\mathcal{I}_j}(\cdot) \in I\!R^{n\times n\frac{n_2}{2}}$, $\mathcal{E}_{\mathcal{I}_j}(\cdot) \in I\!R^{n\times 1}$, and $\mathcal{Q}_{\mathcal{I}_j}(A)q_1^n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ A_1^{i(n)}q_1^n \end{pmatrix}$,

$\mathcal{Q}_{\mathcal{I}_j} \in I\!R^{n\times\frac{n_1}{2}}$.

The functions $H_j(\cdot)$, $G_j(\cdot)$ can be constructed from (22) (26) (14), and (24) (19) respectively. The controllability problem formulated as problem 1 yields nonlinear equations ([1]) in x under inequality constraints, as follows

$$(S_1)\left\{\begin{array}{ll} \mathcal{A}_{\mathcal{I}_j}(v_1^+,A,M_1,M_2,x_4)x_1 + \mathcal{B}_{\mathcal{I}_j}(v_1^+,A,M_1,M_2,q_1^1,x_4) & \\ +\mathcal{R}_{\mathcal{I}_j}(e,A,M_1,M_2,v_1^+,x_4)q_1^n = 0 & \textbf{(i)} \\ \mathcal{C}_{\mathcal{I}_j}(A)x_1 + \mathcal{D}_{\mathcal{I}_j}(A)x_2 + \mathcal{E}_{\mathcal{I}_j}(B) + \mathcal{Q}_{\mathcal{I}_j}(A)q_1^n = 0 & \textbf{(ii)} \\ \mathcal{F}_{\mathcal{I}_j}(e,A,M_1,M_2)x_4 + \mathcal{G}_{\mathcal{I}_j}(v_1^+,e,M_1,M_2,A) + v_n^+ = 0 & \textbf{(iii)} \\ \mathcal{H}_{\mathcal{I}_j}(v_1^+,e,A,M_1,M_2,x_4)x_1 + \mathcal{J}_{\mathcal{I}_j}(q_1^1,q_1^n,v_1^+,e,A,M_1,M_2,x_4) \geq 0 & \textbf{(iv)} \\ \mathcal{K}_{\mathcal{I}_j}(v_1^+,A,e,M_1,M_2) + \mathcal{L}_{\mathcal{I}_j}(e,M_1,M_2,A)x_4 \geq 0 & \textbf{(v)} \\ j \in \{1,...,m^{(n-1)}\} & \end{array}\right. \tag{27}$$

that is a set of $n_1 + n$ equalities and $2(n-1)$ inequalities for each j. When assumption 2 is not satisfied, then one may add to (27) an additional constraint on x_4, obtained from (13) and stating that $A_1^{i(k+1)}v_1^+ + A_1^{i(k+1)}\mathcal{F}_k x_4 + A_1^{i(k+1)}\mathcal{G}_k \neq 0$, $1 \leq k \leq n-1$. One may first solve (27) and then investigate how this inequality modifies the reachable subspaces. We have therefore shown that problem 1 has a solution only if the constrained equation (S_1) has a solution for at least one j. Since the converse is obviously true lemma 1 is proved.

One notices that the negativity condition on the terms a_-^j is dropped in (27) thanks to the inclusion of the second inequality constraint into the constrained equation. ■

3.2 Some properties of the constrained equation

The set of equalities/inequalities in (27) has the following useful properties:

[1]It is interesting to note that the controllability problem as formulated in [15] also results in a nonlinear programme with equality and inequality constraints.

- **An equivalent formulation**

  If we use the first equality in (21) we get

$$\mathcal{A}_{\mathcal{I}_j}(v_1^+, A, M_1, M_2, x_4)x_1 + \mathcal{B}_{\mathcal{I}_j}(v_1^+, A, M_1, M_2, x_4)x_2 + \mathcal{R}_{\mathcal{I}_j}(e, A, B, M_1, M_2, q_1^1, x_4) + q_1^n = 0 \quad (28)$$

  where we use the same notations than in (22) but the matrices are different. The first column of $\mathcal{B}_{\mathcal{I}_j}$ in (28) is made of zeros because $q_2(1)$ is not in the right-hand-side of the first line of (21). Let us denote as (S_2) the constrained equation made of (28) and **(ii) (iii) (iv) (v)** in (27). The constrained equation (S_2) is equivalent to (S_1) in (27) because (26) is included in the set of equalities.

- **Linearity in the final state**

  Let $x_n^T = ((q_1^n)^T, (v_n^+)^T)$. Then **(i) (ii) (iii)** and **(iv)** are linear in x_n, so that both (S_1) and (S_2) can be rewritten as

$$(S_3)\begin{cases} \mathcal{M}_{\mathcal{I}_j}(x_4, q_1^1, v_1^+)x_n + \mathcal{N}_{\mathcal{I}_j}(x_1, x_2, x_4, q_1^1, v_1^+) = 0 & \textbf{(i)(ii)(iii)} \\ \mathcal{P}_{\mathcal{I}_j}(x_4, q_1^1, v_1^+)q_n^1 + \mathcal{T}_{\mathcal{I}_j}(x_1, x_4, q_1^1, v_1^+) \geq 0 & \textbf{(iv)} \\ \textbf{(v)} & \end{cases} \quad (29)$$

  where $\mathcal{M}_{\mathcal{I}_j}(x_4) = \begin{pmatrix} \mathcal{R}_{\mathcal{I}_j}(x_4) & 0 \\ \mathcal{Q}_{\mathcal{I}_j} & 0 \\ 0 & I_{n_1/2} \end{pmatrix}$ for (S_1), $\mathcal{M}_{\mathcal{I}_j} = \begin{pmatrix} I_{\frac{n_1}{2}} & 0 \\ \mathcal{Q}_{\mathcal{I}_j} & 0 \\ 0 & I_{\frac{n_1}{2}} \end{pmatrix}$ for (S_2), $\mathcal{N}_{\mathcal{I}_j}(x_4) = \begin{pmatrix} \mathcal{A}_{\mathcal{I}_j}(x_4)x_1 + \mathcal{B}_{\mathcal{I}_j}(x_4) \\ \mathcal{C}_{\mathcal{I}_j}x_1 + \mathcal{D}_{\mathcal{I}_j}x_2 + \mathcal{E}_{\mathcal{I}_j} \\ \mathcal{F}_{\mathcal{I}_j}x_4 + \mathcal{G}_{\mathcal{I}_j} \end{pmatrix}$ for (S_1), $\mathcal{N}_{\mathcal{I}_j}(x_4) = \begin{pmatrix} \mathcal{A}_{\mathcal{I}_j}(x_4)x_1 + \mathcal{B}_{\mathcal{I}_j}(x_4) + \mathcal{R}_{\mathcal{I}_j}(x_4) \\ \mathcal{C}_{\mathcal{I}_j}x_1 + \mathcal{D}_{\mathcal{I}_j}x_2 + \mathcal{E}_{\mathcal{I}_j} \\ \mathcal{F}_{\mathcal{I}_j}x_4 + \mathcal{G}_{\mathcal{I}_j} \end{pmatrix}$ for (S_2), $\mathcal{P}_{\mathcal{I}_j} = \begin{pmatrix} 0_{(n-2)\times\frac{n_1}{2}} \\ \frac{A_1^{i(n)}}{A_1^{i(n)}\left[v_1^+ - \sum_{j=2}^{n-1} a_-^j N_1^{i(j)}\right]} \end{pmatrix}$, $\mathcal{T}_{\mathcal{I}_j} = \mathcal{H}_{\mathcal{I}_j}(x_4)x_1 + \begin{pmatrix} -\frac{A_1^{i(2)}q_1^1}{A_1^{i(2)}v_1^+} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

- It is worthy to recall that the equation

$$x_n - A^n x_0 = KU \quad (30)$$

  characterizes the controllability of a linear time invariant system $x_{k+1} = Ax_k + Bu_k$, $x_k \in I\!R^n$, where K is the Kalman controllability matrix and $U^T = [u_0, ..., u_{n-1}]$, x_n and x_0 are the target and the initial state respectively. One may compare (29) with (30) to measure how nonlinear systems as in (2) are.

- The two functions H_j and G_j in lemma 1 satisfy $H_j(x_1, x_2, x_4) = H_{j,x_4}(x_1, x_2)$ and $G_j(x_1, x_2, x_4) = G_{j,x_4}(x_1, x_2)$, where $H_{j,x_4}(\cdot)$ and $G_{j,x_4}(\cdot)$ are linear. Since x_4 and

x_2 play the role of the input, this means that the problem is linear in the state but nonlinear in the input.

- **A constrained equation formulation**

  One can formulate problem 1 and rewrite (27) as

$$\begin{cases} F_j(x_1,x_4)=0 & \textbf{(i)(iii)} \\ (x_1,x_4)\in\Omega_j\subset I\!R^{(n-2)\frac{n_1+n_2}{2}} & \textbf{(iv)(v)} \\ \textbf{(ii)} & \end{cases} \tag{31}$$

  for all $j\in\{1,...,m^{(n-1)}\}$, some function $F_j(\cdot)$ and some closed subset of $I\!R^{(n-2)\frac{n_1+n_2}{2}}$. The problem is then under the canonical form studied in [19] for the variables x_1 and x_4. However the applicability of the general conditions of existence of a solution to (31) in [19] is not straightforward.

- Let us denote $x^T=(x_1^T,x_2^T,x_4^T)$. Then (27) can be written as

$$\begin{cases} H_j(x_1,x_2,x_4)=M_j(x_4)x+N_j(x_4)=0 & \textbf{(i)(i)(iii)} \\ G_j(x_1,x_2,x_4)=L_j(x_4)x+K_j(x_4)\geq 0 & \textbf{(iv)(v)} \end{cases} \tag{32}$$

  where $M_j(x_4)=\begin{pmatrix} \mathcal{A}_{\mathcal{I}_j}(x_4) & 0 & 0 \\ \mathcal{C}_{\mathcal{I}_j} & \mathcal{D}_{\mathcal{I}_j} & 0 \\ 0 & 0 & \mathcal{F}_{\mathcal{I}_j} \end{pmatrix}$ and $N_j(x_4)=\begin{pmatrix} \mathcal{B}_{\mathcal{I}_j}(x_4) \\ \mathcal{E}_{\mathcal{I}_j} \\ \mathcal{G}_{\mathcal{I}_j} \end{pmatrix}+\mathcal{M}_{\mathcal{I}_j}(x_4)x_n$, $L_j(x_4)=\begin{pmatrix} \mathcal{H}_{\mathcal{I}_j}(x_4) & 0 & 0 \\ 0 & 0 & \mathcal{L}_{\mathcal{I}_j} \end{pmatrix}$, i.e. $\mathcal{N}_{\mathcal{I}_j}$ in (29) is $\mathcal{N}_{\mathcal{I}_j}=M_j(x_4)x+\begin{pmatrix} \mathcal{B}_{\mathcal{I}_j}(x_4) \\ \mathcal{E}_{\mathcal{I}_j} \\ \mathcal{G}_{\mathcal{I}_j} \end{pmatrix}$.
  In (32) $M_j\in I\!R^{(n_1+n)\times(\frac{n_1}{2}(n-2)+\frac{n_2}{2}(2n-1))}$, $N_j\in I\!R^{(n_1+n)\times 1}$, $L_j\in I\!R^{2(n-1)\times\left(\frac{n_1}{2}(n-2)+\frac{n_2}{2}(2n-1)\right)}$, $K_j\in I\!R^{2(n-1)\times 1}$.

- **Planar objects**

  If $n_1=2$ the second line of (21) becomes trivial, so one has to use the first line of (21) and modify the constrained equations in consequence (see (28)). Then **(i)** is modified to the equality

$$\mathcal{A}_{\mathcal{I}_j}(A)x_1+\mathcal{B}_{\mathcal{I}_j}(A)x_2+\mathcal{R}_{\mathcal{I}_j}(q_1^1,B)+q_1^n=0 \quad \textbf{(i')} \tag{33}$$

  where all matrices are constant (in (28) they may generally be function of x_4). One sees that **(i') (ii) (iii)** can be rewritten as $M_jx+\bar{N}_j+\begin{pmatrix} q_1^n \\ \mathcal{Q}_{\mathcal{I}_j}q_1^n \\ v_n^+ \end{pmatrix}=0$ where the matrices

$M_j = \begin{pmatrix} \mathcal{A}_{\mathcal{I}_j} & \mathcal{B}_{\mathcal{I}_j} & 0 \\ \mathcal{C}_{\mathcal{I}_j} & \mathcal{D}_{\mathcal{I}_j} & 0 \\ 0 & 0 & \mathcal{F}_{\mathcal{I}_j} \end{pmatrix}$ and $\bar{N}_j = \begin{pmatrix} \mathcal{R}_{\mathcal{I}_j} \\ \mathcal{E}_{\mathcal{I}_j} \\ \mathcal{G}_{\mathcal{I}_j} \end{pmatrix}$ are constant for given initial data. This is the case for the backlash model in figure 1 where $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ L \end{pmatrix}$, $m = 2$, $n_1 = n_2 = 2$. We have $A_1^1 = 1$, $A_1^2 = -1$, $A_2^1 = -1$, $A_2^2 = 1$. This example will be treated in more detail in section 4. ■

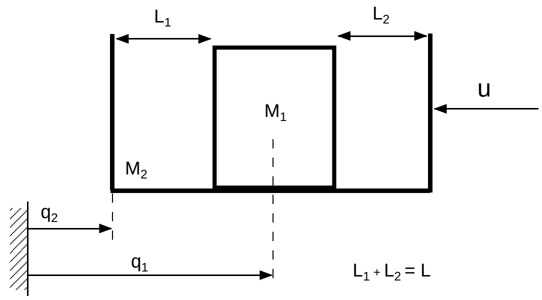


Figure 1: The impacting pair.

Remark 3 The system P_Σ in (5) can be written for (2) as

$$\dot{q}_1(k+1) = \dot{q}_1(k) - \frac{1+e}{A^{i(k)}M^{-1}(A^{i(k)})^T}\left[A_1^{i(k)}\dot{q}_1(k) + A_2^{i(k)}\dot{q}_2(k)\right]_- N_1^{i(k)} \qquad (34)$$

which is a discrete-time nonlinear system with state $\dot{q}_1(k)$ and input $\dot{q}_2(k)$. Provided the terms between brackets are negative (constraints (18)), the system in (34) is linear (noting that $x^T y.v$ for vectors x, y and v, can be rewritten as Ax with rows $A_i = v_i y^T$). So the VCTI property can be studied. The fact that the object velocities do not appear in (27) is just a consequence of this linearity.

3.3 Analytical study of accessibility

The number of impacts $n-1$ is an important parameter in the existence of a solution to problem 1. This combined with the fact that when $m \geq 2$ the index $i(k)$ may vary from one impact to the next, renders the controllability study a hard task in general. However attacking the problem of multiple impacts as a succession of simple impacts, inherently contains such cumbersome enumeration procedures [11] [6]. Notice that for problem 1 to make sense there must exist at least one feasible x^* for (27) and a nonempty open neighborhood of x_4^* in which the entries are non-singular . Since the denominators are linear in x_4 (consequently the singular set is closed), the matrices entries are even smooth in this neighborhood. Let us state the following:

Lemma 2 **(i)** *In (29) rank*($\mathcal{P}_{\mathcal{I}_j}$) = 1, **(ii)** *If* ($S_1$) *is used then one has rank*($\mathcal{M}_{\mathcal{I}_j}$) = $n_1 - 1$, **(iii)** *If* (S_2) *is used one has rank*($\mathcal{M}_{\mathcal{I}_j}$) = n_1.

Proof: **(i)** and **(iii)** follow by direct inspection of the matrices structures, in particular from (28) it follows that $\mathcal{M}_{\mathcal{I}_j} = \begin{pmatrix} I_{\frac{n_1}{2}} & 0 \\ \mathcal{Q}_{\mathcal{I}_j} & 0 \\ 0 & I_{\frac{n_1}{2}} \end{pmatrix}$; **(ii)** In (22) one has rank($\mathcal{R}_{\mathcal{I}_j}$) = $\frac{n_1}{2} - 1$. Indeed if v is an eigenvector of $\mathcal{R}_{\mathcal{I}_j}$ and λ the corresponding eigenvalue, one has $\mathcal{R}_{\mathcal{I}_j} v = \lambda v$. It follows that $\frac{\dot{q}_1(t_{n-1}^+) A_1^{i(n)}}{A_1^{i(n)} \dot{q}_1(t_{n-1}^+)} v = (1-\lambda) v$. The matrix $\frac{\dot{q}_1(t_{n-1}^+) A_1^{i(n)}}{A_1^{i(n)} \dot{q}_1(t_{n-1}^+)}$ has rank 1, and it is easy to see that all its eigenvalues are 0 except one eigenvalue equal to 1, whose eigenvector is $v = \dot{q}_1(t_{n-1}^+)$. Therefore $\mathcal{R}_{\mathcal{I}_j}$ has all eigenvalues equal to 1 except one that is 0. The result then follows from the structure of the matrix $\mathcal{Q}_{\mathcal{I}_j}$ in (26) and that of $\mathcal{M}_{\mathcal{I}_j}$. ■

Lemma 3 *Accessibility in the sense of definition 2 implies* $\frac{4n_1+n_2}{n_1+2n_2} \le n$.

Proof: The proof can be done by direct calculation and inspection of the structure of the matrices M_j and N_j in (32). One has rank($\mathcal{M}_{\mathcal{I}_j}$) $\le n_1$. In the general case of n impacts, one has $M_j \in I\!R^{(n_1+n)\times(\frac{n_1}{2}(n-2)+\frac{n_2}{2}(2n-1))}$. Accessibility implies that for a given (q_1^1, v_1^+), the equality in (29) be satisfied for x_n in an open set. Consequently it must hold that Im(M_j) $\supseteq$ Im($\mathcal{M}_{\mathcal{I}_j}$). The representations in (S_1) and (S_2) being equivalent one to each other, let us use (S_2). It follows that (rank (M_j) $\ge$ rank($\mathcal{M}_{\mathcal{I}_j}$)) $\Leftrightarrow \frac{n_1}{2}(n-2) + \frac{n_2}{2}(2n-1) \ge n_1 \Longleftrightarrow n \ge \frac{4n_1+n_2}{n_1+2n_2}$. ■

Clearly the lowerbounds in lemma 3 are conservative and should be refined. Let us propose the following, without proof:

Conjecture 1 *One has* $n_1 \le 2^n$.

3.3.1 The case $n = 2$

Controllability as in definition 4 holds if there exists n such that $\bar{\mathcal{R}}^n\left[(q_1^1, v_1^+)\right] = I\!R^{n_1}$ for any (q_1^1, v_1^+). One may examine the structure of $\mathcal{R}^2\left[(q_1^1, v_1^+)\right]$ in order to determine whether or not a covering of $I\!R^{n_1}$ is possible in a finite or infinite number of impacts (more exactly one should speak of the object configuration space $\times$ its tangent space). The reason may be that the study of the spaces $\mathcal{R}^2\left[(q_k^1, v_k^+)\right]$ may be more tractable, hence the characterization of $\mathcal{R}^2\left[(q_k^1, v_k^+)\right]$, $(q_k^1, v_k^+) \in \mathcal{R}^2\left[(q_{k-1}^1, v_{k-1}^+)\right]$, may be useful. Let us therefore better understand the constrained nonlinear equation (27) when $n = 2$. Notice first that x_1 no longer appears in the problem, so that $x^T = (x_2^T, x_4^T)$, $x \in I\!R^{\frac{3}{2}n_2}$.

Lemma 4 *If* $n = 2$ *in problem 1, then the matrices* M_j, N_j, L_j, K_j *in (32) are constant (i.e. they do not depend on* x_4*), and* **(iv)** *in (27) simplifies to an inequality* $\mathcal{J}_{\mathcal{I}_j}(q_1^1, q_1^2, v_1^+, A) \ge 0$. *Moreover rank*($M_j$) = 3 *and accessibility in one impact implies* $n_1 = 2$.

Proof: The proof is easy by inspection of the structure of the matrices in (27). In particular one has $M_j = \begin{pmatrix} 0 & \frac{v_1^+ A_2^{i(2)}}{A_1^{i(2)} v_1^+} & 0 \\ A_2^{i(1)} & 0 & 0 \\ 0 & A_2^{i(2)} & 0 \\ 0 & 0 & \frac{(1+e) N_1^{i(2)} A_2^{i(2)}}{A^{i(2)} M^{-1} (A^{i(2)})^T} \end{pmatrix} \in I\!R^{(n_1+2) \times \frac{3}{2} n_2}$. The rank upperbound follows. Using the formalism in (32) constructed from (28), one sees that $\text{rank}(\mathcal{M}_{\mathcal{I}_j}) = n_1$. For the system to be accessible, problem 1 has to have solutions for (q_1^2, v_2^+) evolving in a set containing an open set of $I\!R^{n_1}$. From (32) this implies that $\text{rank}(M_j) \geq \text{rank}(\mathcal{M}_{\mathcal{I}_j})$. Consequently $n_1 \leq 3$, i.e. $n_1 = 2$. ■

It is useful to think of problem 1 (i.e. of (27)) in terms of a convex quadratic program, which in turn can be posed as a LCP [21, §1.3.4]. Some definitions concerning LCPs are given in appendix A.

Corollary 1 *If $n = 2$, problem 1 has a solution only if the mixed linear complementarity problem (mLCP) in (37) is solvable.*

In the case $n = n_1 = n_2 = 2$ one may use LCPs to study accessibility and controllability, see section 4. The use of complementarity problems is important because it paves the way to theoretical and numerical studies.

Proof: If x^* is a solution of problem 1, then from (32) it is also a solution of the quadratic programme (QP)

$$\min \frac{1}{2}(M_j x + N_j)^T (M_j x + N_j),\ \mathcal{K}_{\mathcal{I}_j} + \mathcal{L}_{\mathcal{I}_j} x_4 \geq 0 \tag{35}$$

Hence the Karush-Kuhn-Tucker (KKT) necessary and sufficient conditions are satisfied, i.e. there exists a slack variable $\mu \in I\!R$ such that

$$\begin{cases} M_j^T M_j x^* + M_j^T N_j - L_j^T \mu = 0 \\ 0 \leq \mu \perp \mathcal{K}_{\mathcal{I}_j} + \mathcal{L}_{\mathcal{I}_j} x_4 \geq 0 \end{cases} \tag{36}$$

where $L_j = \begin{pmatrix} 0 & \mathcal{L}_{\mathcal{I}_j} \end{pmatrix} \in I\!R^{1 \times \frac{3}{2} n_2}$. If x^* is a KKT point of this programme [21, §9.3.1] and in addition $M_j x^* + N_j = 0$, then (q_1^2, v_2^+) is reachable from (q_1^1, v_1^+). If one can show that there is a set in $I\!R^{n_1} \ni (q_1^2, v_2^+)$ containing an open set, and such that both conditions are satisfied, then the system is reachable in one impact. If $M_j^T M_j \in I\!R^{\frac{3}{2} n_2 \times \frac{3}{2} n_2}$ is full-rank, one can easily transform (36) into a LCP(μ) with a positive definite matrix, see appendix B. If not, let us assume its rank is $4 \geq r \geq 1$ and denote $\bar{r} = \frac{3}{2} n_2 - r$. Let us define $E_r = (0\ I_r)$ and $E_{\bar{r}} = (I_{\bar{r}}\ 0)$ and W, $W^T W = I_{\frac{3}{2} n_2}$, such that $W M_j^T M_j W^T = \text{diag}(0_{\bar{r} \times \bar{r}}, D_j)$ where $D_j > 0$ is diagonal and $r \times r$. Then (36) can be rewritten as

$$\begin{cases} D_j E_r W x^* + E_r W M_j^T N_j - E_r W L_j^T \mu = 0 \\ E_{\bar{r}} W M_j^T N_j - E_{\bar{r}} W L_j^T \mu = 0 \\ 0 \leq \mu \perp \mathcal{K}_{\mathcal{I}_j} + \mathcal{L}_{\mathcal{I}_j} x_4^* \geq 0 \end{cases} \tag{37}$$

which is a mLCP. ■

One may solve the mLCP numerically, and then look for solutions which satisfy $M_j x^* + N_j = 0$.

Corollary 2 **(i)** $(q_1^2, v_2^+) \in \mathcal{R}^2[(q_1^1, v_1^+)] \Longrightarrow$ **(ii)** *$\mathcal{J}_{\mathcal{I}_j}(q_1^1, q_1^2, v_1^+) \geq 0$ and $x^*(0)$ is a KKT point of the QP in (35)* $\Longleftrightarrow$ **(iii)** *$\mathcal{J}_{\mathcal{I}_j}(q_1^1, q_1^2, v_1^+) \geq 0$ and the mLCP in (37) is solvable with $\mu = 0$. If Ker $(M_j^T) = \{0\}$ then* **(i)** $\Longleftrightarrow$ **(ii)**.

Proof: Simply notice that if (q_1^2, v_2^+) is reachable from (q_1^1, v_1^+) then $M_j x^* + N_j = 0$ so that $\mu = 0$, and $\mathcal{H}_{\mathcal{I}_j}$ in (24) vanishes. ■

3.3.2 The general case

A vector x is said admissible if the entries of the matrices in (27) are bounded at x. The set of admissible x is open.

Lemma 5 *Consider the constrained equation in (S_2). Then the system is accessible in $(n-1)$ impacts in the sense of definition 2 if and only if there exists a set V of admissible x and $j \in \{1, \ldots, m^{(n-1)}\}$ such that*

- **a)** $-[\mathcal{P}_{\mathcal{I}_j}(x_4), 0]\,(\mathcal{M}_{\mathcal{I}_j}^T \mathcal{M}_{\mathcal{I}_j})^{-1} \mathcal{M}_{\mathcal{I}_j}^T \mathcal{N}_{\mathcal{I}_j}(x) + \mathcal{T}_{\mathcal{I}_j}(x) \geq 0$ *for all* $x \in V$,
- **b)** $\mathcal{K}_{\mathcal{I}_j} + \mathcal{L}_{\mathcal{I}_j} x_4 \geq 0$ *for all* $x \in V$,
- **c)** $\mathcal{N}_{\mathcal{I}_j}(x) \in$ *Ker* $[I_{n_1} - \mathcal{M}_{\mathcal{I}_j}^T (\mathcal{M}_{\mathcal{I}_j}^T \mathcal{M}_{\mathcal{I}_j})^{-1} \mathcal{M}_{\mathcal{I}_j}]$ *for all* $x \in V$,
- **d)** $\bigcup_{j \in \{1, \ldots, m^{(n-1)}\}} \mathcal{M}_{\mathcal{I}_j}$ *span*$_{x \in V}$ $[\mathcal{N}_{\mathcal{I}_j}(x)]$ *contains an open set of* $\mathbb{R}^{n_1}$.

Proof: The proof is based on the following convex quadratic program in x_n, constructed from (S_2):

$$\begin{cases} \min \frac{1}{2} \left(\mathcal{M}_{\mathcal{I}_j} x_n + \mathcal{N}_{\mathcal{I}_j}(x)\right)^T \left(\mathcal{M}_{\mathcal{I}_j} x_n + \mathcal{N}_{\mathcal{I}_j}(x)\right) \\ \\ \mathcal{P}_{\mathcal{I}_j}(x_4) + \mathcal{T}_{\mathcal{I}_j}(x) \geq 0 \end{cases} \tag{38}$$

Notice that in (S_2), $\mathcal{M}_{\mathcal{I}_j}$ in (29) is constant of rank n_1 and the QP matrix $\mathcal{M}_{\mathcal{I}_j}^T \mathcal{M}_{\mathcal{I}_j} \in \mathbb{R}^{n_1 \times n_1}$ is positive definite. If x_n is reachable from (q_1^1, v_1^+) then necessarily x_n is a solution of the QP in (38). Equivalently [21, §1.3.3] x_n is a KKT point of the QP in (38), and the Lagrange multiplier of the KKT conditions is $\mu = 0$ since $\mathcal{M}_{\mathcal{I}_j} x_n + \mathcal{N}_{\mathcal{I}_j}(x) = 0$. It follows that items a) and b) are fulfilled and that $x_n = -(\mathcal{M}_{\mathcal{I}_j}^T \mathcal{M}_{\mathcal{I}_j})^{-1} \mathcal{M}_{\mathcal{I}_j}^T \mathcal{N}_{\mathcal{I}_j}(x)$. Accessibility implies that x_n varies in an open set of $\mathbb{R}^{n_1}$, hence item d) follows. Moreover the condition $\mathcal{M}_{\mathcal{I}_j} x_n + \mathcal{N}_{\mathcal{I}_j}(x) = 0$ implies item c). Reciprocally items **a) b) c)** and **d)** yield that the system is accessible in $(n-1)$ impacts.

If one uses (S_1), then similar conditions can be derived taking into account that rank$(\mathcal{M}_{\mathcal{I}_j}(x_4)) = n_1 - 1$. ■

4 Example: dynamic backlash model

As an illustration let us come back on the impacting pair in figure 1. The following is true

Lemma 6 *The dynamics of the impacting pair's object is accessible in 1 impact in the sense of definition 2.*

The proof is given in appendix B. It is an application of corollary 2, however we develop all the calculations and provide an accurate characterization of the reachable subspaces. It is a first step to show the following:

Lemma 7 *The dynamics of the impacting pair's object is controllable in 3 impacts in the sense of definition 4.*

The proof is given in appendix C. It relies on the characterization of the spaces $\mathcal{R}^2[(q_1^k, v_k^+)]$ which can be analytically computed. Both lemma 6 and 7 hold independently of $e \in [0,1]$. The shape of the reachable subspaces for the impacting pair that is found from the analysis in appendices B and C is intuitively sound (see figure 3).

5 Extensions

It is of interest to briefly investigate how much the complexity of the overall problem is increased when the dynamics in (2) is modified, and to indicate some manners to simplify the study.

5.1 Object subject to gravity

Let us assume that the object's dynamics is given by

$$M_1 \ddot{q}_1 = \mathbf{g} \tag{39}$$

for some constant $\mathbf{g}$. A systematic manner to test the VCTI for such systems has been proposed in [6, Lemma 3]. With respect to the framework developed here, one sees that (11) is replaced by

$$v_n^+ = v_1^+ + M_1^{-1}\mathbf{g}\left(\sum_{j=1}^{n-1} \Delta_j\right) - \sum_{j=2}^{n} a_-^j N_1^{i(j)} \tag{40}$$

Equation (17) becomes the equation of degree 2 in Δ_k

$$-A_1^{i(k+1)}\left[q_1(k+1) - q_1(k)\right] + A_1^{i(k+1)}\dot{q}_1(t_k^+)\Delta_k + \frac{1}{2}A_1^{i(k+1)}M_1^{-1}\mathbf{g}\Delta_k^2 = 0 \tag{41}$$

From (41) and (6) it follows that we can write for $1 \leq k \leq n-1$

$$\Delta_k = \Delta_k(\dot{q}_1(k), \dot{q}_2(k), q_1(k+1), q_1(k)) \geq 0 \tag{42}$$

Now (20) is replaced by

$$q_1^n - q_1^1 = \sum_{k=1}^{n-1}\left(\dot{q}_1(t_k^+)\Delta_k + \frac{1}{2}M_1^{-1}\mathbf{g}\Delta_k^2\right) \tag{43}$$

It appears that it is no longer possible to express $\dot{q}_1(k)$ as a linear function of x_4, as it was the case for (2). From (40) (42) (43) and (16) (18) the set of conditions in (27) is transformed into a nonlinear constrained equation

$$\begin{cases} \mathcal{A}_{\mathcal{I}_j}(x_1, x_2, x_3, x_4) = 0 \\ \mathcal{B}_{\mathcal{I}_j}(x_1, x_3, x_4) \geq 0 \\ j \in \{1, ..., m^{(n-1)}\},\ \mathcal{A}_{\mathcal{I}_j}(\cdot) \in I\!R^{n_1 \times 1},\ \mathcal{B}_{\mathcal{I}_j}(\cdot) \in I\!R^{2(n-1)\times 1} \end{cases} \tag{44}$$

As long as $A_1^{i(k+1)}M_1^{-1}\mathbf{g} \neq 0$, $\mathcal{A}_{\mathcal{I}_j}(\cdot,\cdot)$ and $\mathcal{B}_{\mathcal{I}_j}(\cdot,\cdot)$ are continuous in their arguments. In practice it may be interesting to seek particular solutions and then to numerically look for a set of reachable points. To that aim one may simplify (44) by looking for paths that satisfy $q_1(k) = q_1^1 = q_1^n$, $1 \leq k \leq n$ [2]. Then one is able to deduce that

$$\begin{cases} \dot{q}_1(k) = B^k \dot{q}_1(t_{k-1}^+),\ B^k = I_{n_1/2} - \frac{2M_1^{-1}\mathbf{g}A_1^{i(k)}}{A_1^{i(k)}M_1^{-1}\mathbf{g}} \\ \Delta_k = -2\frac{A_1^{i(k+1)}}{A_1^{i(k+1)}M_1^{-1}\mathbf{g}}\dot{q}_1(t_k^+) \end{cases} \tag{45}$$

where $I_{n_1/2}$ is the identity matrix, and that

$$\begin{cases} \dot{q}_1(k+1) = B^{k+1}\left[\prod_{j=2}^{k} B^j v_1^+ - \left(\sum_{j=2}^{k-1}\left\{\prod_{l=j+1}^{k} B^l a_-^j N_1^{i(j)}\right\} + a_-^k N_1^{i(k)}\right)\right] \quad if\ k \geq 3 \\ \dot{q}_1(2) = B^2 v_1^+ \\ \dot{q}_1(3) = B^3 B^2 v_1^+ - a_-^2 N_1^{i(2)} \end{cases} \tag{46}$$

[2] Such a path is meaningless for dynamics as in (2), see (15).

Thus one can express $\dot{q}_1(k)$ as a linear function of $\dot{q}_2(k)$. Then (44) takes the particular form ([3])

$$\left\{\begin{array}{l} \mathcal{A}_{\mathcal{I}_j}.\begin{pmatrix} x_2 \\ x_4 \end{pmatrix} + \mathcal{B}_{\mathcal{I}_j} = 0 \\ \\ \mathcal{C}_{\mathcal{I}_j}.x_4 + \mathcal{D}_{\mathcal{I}_j} \geq 0 \\ \\ \mathcal{E}_{\mathcal{I}_j}(x_4) + \mathcal{F}_{\mathcal{I}_j} x_4 + \mathcal{G}_{\mathcal{I}_j} = 0 \\ \\ \mathcal{H}_{\mathcal{I}_j} + \mathcal{J}_{\mathcal{I}_j} x_4 \geq 0 \\ \\ j \in \{1, ..., m^{(n-1)}\} \end{array}\right. \tag{47}$$

where the second inequality is deduced from (19). It is noticeable that $\mathcal{E}_{\mathcal{I}_j}(x_4)$ is quadratic in x_4. It has the generic form

$$\mathcal{E}_{\mathcal{I}_j}(x_4) = \left(\bar{x}_4^T \mathcal{Q}_{\mathcal{I}_j} \bar{x}_4\right) \mathcal{K}_{\mathcal{I}_j} \tag{48}$$

where $\bar{x}_4^T = \left(\dot{q}_2^T(2), ..., \dot{q}_2^T(n-1)\right)$, $\mathcal{K}_{\mathcal{I}_j} \in I\!R^{\frac{n-1}{2} \times 1}$. In the case $n = 3$ one obtains

$$\begin{array}{l} \mathcal{Q}_{\mathcal{I}_j} = \dfrac{\left(A_2^{i(2)}\right)^T A_2^{i(2)}}{\left[A^{i(2)} M^{-1} \left(A^{i(2)}\right)^T\right]^2} \in I\!R^{\frac{n-1}{2} \times \frac{n-1}{2}} \\ \\ \mathcal{K}_{\mathcal{I}_j} = \dfrac{2(1+e)^2 \left(A_1^{i(3)} N_1^{i(2)}\right)^2}{\left(A_1^{i(3)} M_1^{-1} \mathbf{g}\right)^2} \end{array} \tag{49}$$

For simplicity the other arguments (problem data) have been dropped in (47). One sees that the assumption $q_1(k) = q_1^1 = q_1^n$, $1 \leq k \leq n$ allows one to reduce the nonlinear constrained equation in (44) to a mixed linear/quadratic equation under linear inequality constraints in (47). It can be interpreted as a *local* configuration version of the controllability problem: investigate the accessibility as it is defined in problem 1, in a neighborhood of a position of impact q_1^1, and subsequently search for the reachable/controllable subspaces either analytically or numerically (continuation methods). The reachable sets are now defined as $\mathcal{R}_V^n\left[(q_1^1, v_1^+)\right] = \{(q_1^n, v_n^+) \mid$ problem 1 possesses at least one solution with $q_1(k) \in V, 1 \leq k \leq n, V$ a neighborhood of $q_1^1\}$. The constrained equation in (47) can also be obtained by expanding $\Delta_k(\delta_k)$ in (42) around $\delta_k = q_1(k+1) - q_1(k) = 0$ and neglecting all terms but the constant one (i.e. $\sqrt{1+\epsilon} \approx 1$).

5.2 Nonlinear unilateral constraints

Let us now examine the case when $h(z_1, z_2) = h(q_1, q_2)$ is a nonlinear function. The major discrepancy with the linear constraints case, is that a_-^k in (6) and N_1^i depend on $q(k)$,

[3] The notations do not mean that the terms in (27) or (44) or (47) are the same. Also the dot "." is used only to clarify the notations and avoid confusion between the arguments of the matrices and their multiplication by a vector.

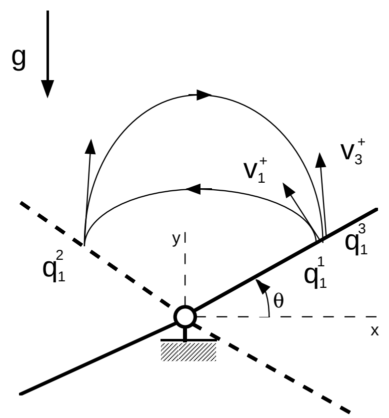


Figure 2: A particular orbit.

generally in a nonlinear fashion. As an example let us cite the so-called two-degree-of-freedom juggler [8] [6] (see figure 2) where

$$h(q_1, q_2) = y\cos(\theta) - x\sin(\theta) \tag{50}$$

with $q_1^T = (x, y)$ and $q_2 = \theta$. Let us notice that by doing the same assumption as in section 5.1, i.e. $q_1(k) = q_1^1 = q_1^n$, $1 \leq k \leq n$, it follows that $\nabla h(q(k))$ is a constant, provided the equation $h(q_1^1, q_2(k)) = 0$ has a solution $q_2(k) = q_2^1$. Then the framework of linear jugglers as in (2) and (27) is recovered. In addition one may add gravity in the object's dynamics and proceed as in section 5.1. The identification of particular solutions is sometimes easy, as for the two-degree-of-freedom juggler where vertical orbits obviously exist and satisfy $q_1^T(k) = (x, 0)$, $q_2(k) = 0$.

Remark 4 Once a particular solution has been identified, accessibility can be assessed by studying the local properties of the constrained equation around that solution. If the various functions appearing in the equalities and inequalities possess enough regularity, then the set $\bar{\mathcal{R}}_V^n$ may contain an open set and the system be accessible, where V is a neighborhood of the particular solution configuration. It is however important to notice that the local criterion may not be always applicable, and may also miss some reachable states. As an example let us consider the two-degree-of-freedom juggler in figure 2, which has been thoroughly studied in [16]. Let us assume that $\theta \in [-\theta_0, \pi + \theta_0]$. The points (q_1^1, v_1^+) and (q_1^3, v_3^+), where q_1^3 is in a neighborhood of q_1^1 and v_3^+ is in a neighborhood of v_1^+, can be joined only by a trajectory passing through $(q_1^2, \dot{q}_1(t_2^+))$ where obviously neither q_1^2 nor $\dot{q}_1(t_2^+)$ lie in the required neighborhoods. Hence this system cannot be locally accessible but may be globally accessible.

5.3 The effect of friction at impacts

The simplest fashion to incorporate friction at the impacts is to introduce an impact ratio in the collision rule [4]. A more sophisticated model is the Darboux-Keller's shock dynamics, which reduces to the Routh's model in the 2-dimensional case [5] [23]. For the sake of briefness of the paper we do not investigate here the influence of friction at impacts on the CTI. A velocity accessibility study can be found in [24] which incorporates friction via Routh's shock dynamics.

Remark 5 The yoyo dynamics [14] does not strictly pertain to (1) or (2), since u enters the constraint as $h(q_1, q_2, u(t))$, where $u(t)$ is the height at which one handles the yoyo and moves the string vertically. Interestingly enough, a simple network with a diode possesses quite similar features.

6 Conclusions

This note focuses on the controllability of a class of nonsmooth complementarity mechanical systems, named jugglers because of their particular dynamics. Potential and important applications can be found in nonprehensile manipulation, kinematic chains with dynamic backlash, controlled structures, manipulators with dynamic passive obstacle, hopping and running machines, vibro-impact processes etc. The simplest jugglers are examined, which anyway remain highly nonlinear systems. It is shown that the attainable subspaces of the object dynamics are characterized by constrained equations. These constrained equations are shown to possess a specific structure so that preliminary analytical results can be derived. Some paths which allow the designer to simplify them are indicated and an example is worked out.

A Complementarity Problems

Given $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^n$, the problem of finding $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ satisfying

$$y = Ax + B \geq 0,\ x \geq 0,\ x^T y = 0 \tag{51}$$

is called a Linear Complementarity Problem (LCP) [3]. It can be equivalently written as

$$0 \leq x \perp y = Ax + B \geq 0 \tag{52}$$

Roughly the LCP has a unique solution x^* whatever B if and only if A satisfies some positivity conditions, see [21]. Positive definiteness of A is sufficient.

Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times m}$, $C \in \mathbb{R}^{n\times m}$, $D \in \mathbb{R}^{m\times n}$, $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, the problem of finding $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ satisfying

$$a + Ax + Cy = 0,\ 0 \leq b + Dx + By \perp y \geq 0 \tag{53}$$

is called a mixed LCP (mLCP) [3].

B Accessibility in one impact

Let us consider the case $n_1 = n_2 = n = 2$. Then if problem 1 has a solution x^* and using notations in lemma 4, this must be the value that minimizes the quadratic function

$$\Psi(x) = \frac{1}{2}\left[M_j x + \bar{N}_j + \begin{pmatrix} q_1^2 \\ 0 \\ A_1^{i(2)} q_1^2 \\ v_2^+ \end{pmatrix}\right]^T \left[M_j x + \bar{N}_j + \begin{pmatrix} q_1^2 \\ 0 \\ A_1^{i(2)} q_1^2 \\ v_2^+ \end{pmatrix}\right]$$

subject to $L_j(x_4)x + K_j(x_4) \geq 0$, and $x = (q_2(1), q_2(2), \dot{q}_2(2))^T \in I\!R^{3\times 1}$. The matrices $M_j \in I\!R^{4\times 3}$ and $N_j \in I\!R^{4\times 1}$ are constant, see the last item in section 3.2. The inequality in (S_1) reduces to **(v)**, i.e. from (18): $-A_1^{i(2)} v_1^+ - A_2^{i(2)} x_4 \geq 0$, because **(iv)** disappears from the analysis in a first stage. The KKT necessary and sufficient conditions stipulate the existence of a Lagrange multiplier $\mu \in I\!R$ such that [21, §9.3.1]

$$\left\{ \begin{array}{l} M_j^T M_j x^* + M_j^T \bar{N}_j + M_j^T \begin{pmatrix} q_1^2 \\ 0 \\ A_1^{i(2)} q_1^2 \\ v_2^+ \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ A_2^{i(2)} \end{pmatrix} \mu = 0 \\ \\ \mu \geq 0,\ \mu^T (A_2^{i(2)} x_4^* + A_1^{i(2)} v_1^+) = 0,\ -A_2^{i(2)} x_4^* - A_1^{i(2)} v_1^+ \geq 0 \end{array} \right. \tag{54}$$

for a $j \in \{1, 2\}$ (in other words one can choose to strike with either constraint at time t_2 and to initialize the system with any constraint at time t_1). Let us assume that $\mathrm{rank}(M_j) = 3$. Then the second line in (54) is a LCP(μ) with LCP-matrix (a scalar in this case) $M_{LCP} = \begin{pmatrix} 0 & 0 & A_2^{i(2)} \end{pmatrix} (M_j^T M_j)^{-1} \begin{pmatrix} 0 \\ 0 \\ A_2^{i(2)} \end{pmatrix} > 0$. Therefore LCP($\mu$) always possesses a unique solution [21, theorem 3.13]. From the first line of (54) one has

$$x^*(\mu) = (M_j^T M_j)^{-1} \left[-M_j^T \bar{N}_j - M_j^T \begin{pmatrix} q_1^2 \\ 0 \\ A_1^{i(2)} q_1^2 \\ v_2^+ \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ A_2^{i(2)} \end{pmatrix} \mu \right] \tag{55}$$

Let us come back on the backlash model, where m_1 and m_2 are the masses of the two bodies in figure 1. If we choose to strike at t_1 the constraint 2 and at t_2 the constraint 1(i.e. $i(1) = 1$ and $i(2) = 2$), we have $A_1^{i(2)} = A_1^2 = -1$ and $A_2^{i(2)} = A_2^2 = 1$. Consider x_4^* calculated from (55) with $\mu = 0$, and let us denote it as $x_4^*(0)$. So if $x_4^*(0) - v_1^+ < 0$ the solution of LCP(μ) is $\mu = 0$. Injecting (55) into the left-hand-side of this inequality, one sees that the obtained scalar function is a linear function of (q_1^2, v_2^+) and since $\mathrm{rank}(M_j) = 3$, there is a half space in the (q_1^2, v_2^+)-plane such that $x_4^*(0) - v_1^+ < 0$. More precisely in this case one has

$M_j^T \begin{pmatrix} q_1^2 \\ 0 \\ A_1^{i(2)} q_1^2 \\ v_2^+ \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{-(1+e)m_2}{m_1+m_2} \end{pmatrix} \begin{pmatrix} q_1^2 \\ 0 \\ A_1^{i(2)} q_1^2 \\ v_2^+ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{-(1+e)m_2}{m_1+m_2} v_2^+ \end{pmatrix}$. Therefore $x_4^*(0)$ is a linear function of v_2^+ only. In addition to be a KKT point of the quadratic programme for the suitable values of v_2^+ in a half-plane, x^* in (55) and with $\mu = 0$ satisfies the equality $M_j x^* + N_j + \begin{pmatrix} q_1^2 \\ 0 \\ A_1^{i(2)} q_1^2 \\ v_2^+ \end{pmatrix} = 0$.

Let us examine now inequality **(iv)**. From (23) one finds $q_1^2 - q_1^1 = v_1^+ \Delta_1$. The constraint $\Delta_1 \geq 0$ therefore yields $q_1^2 \leq q_1^1$ if $v_1^+ \leq 0$, and $q_1^2 \geq q_1^1$ if $v_1^+ \geq 0$. We conclude about accessibility of the impacting pair in one impact since $\bar{\mathcal{R}}^2[(q_1^1, v_1^+)] = \mathcal{R}^2[(q_1^1, v_1^+)]$ is a quadrant in the (q_1^2, v_2^+)-plane, as intuitively expected (see figure 3).

C Controllability in three impacts

The proof relies on the following arguments:

- The goal is to show that for any initial data (q_1^1, v_1^+) one can reach any (q_1^4, v_4^+) after three impacts. In appendix B we have shown that $\mathcal{R}^2[(q_1^1, v_1^+)] = [q_1^1, +\infty) \times [-\infty, v_1^+)$ if $v_1^+ > 0$ and $\mathcal{R}^2[(q_1^1, v_1^+)] = (-\infty, q_1^1] \times [-\infty, v_1^+)$ if $v_1^+ < 0$.
- Let us choose in a second step a sequence of impacts, that comes after the second impact, that consists of a third impact at time t_3 and with the constraint number 1. In other words we restart the analysis and we choose $i(1) = 2$ and $i(2) = 1$. The new initial condition for this second sequence is (q_1^2, v_2^+), and the final condition is (q_1^3, v_3^+). This time calculations show that $\mathcal{R}^2[(q_1^2, v_2^+)] = (-\infty, q_1^2] \times [v_2^+, +\infty)$ if $v_2^+ < 0$, and $\mathcal{R}^2[(q_1^2, v_2^+)] = [q_1^2, +\infty) \times [v_2^+, +\infty)$ if $v_2^+ > 0$.
- The reachable sets which correspond to such a succession of impacts are depicted on figure 3. One can easily imagine what the sets look like if a third impact at t_4 is imposed, with the same sequence of attained surfaces as in the first case. The idea is therefore to prove controllability by looking for sets $\mathcal{R}^2[(q_1^k, v_k^+)]$ and $(q_1^k, v_k^+) \in \mathcal{R}^2[(q_1^{k-1}, v_{k-1}^+)]$.
- Inspection of the reachable sets in the four cases depicted on figure 3 shows that whatever the final state, one can always choose a combination of impacted surfaces in order to attain such a point of the object's state space. Consider for instance the point (a, b) on figure 3 **(b)**, with $|a| > |q_1^1|$ and $v_1^+ < 0$. It is not possible to reach this point in two impacts (figure 3 **(a)** and then 3 **(b)**). However choosing $v_2^+ > b$ one can reapply an initial sequence with positive velocity (figure 3 **(c)**) and reach the desired point.

- We conclude that it is possible to reach the whole plane after at most three impacts, i.e. $\bar{\mathcal{R}}^4[(q_1^1, v_1^+)] = \mathcal{R}^2[(q_1^1, v_1^+)] \cup \mathcal{R}^3[(q_1^1, v_1^+)] \cup \mathcal{R}^4[(q_1^1, v_1^+)] = I\!R^2$, which allows us to conclude about controllability in three impacts of the impacting pair's object. Possibly the system is controllable in less impacts than 3, because we have not checked all possible sequences between the set of indices $(1, 2)$ and the initial velocity signum (i.e. we have not constructed all the possible constrained equations in (27)).

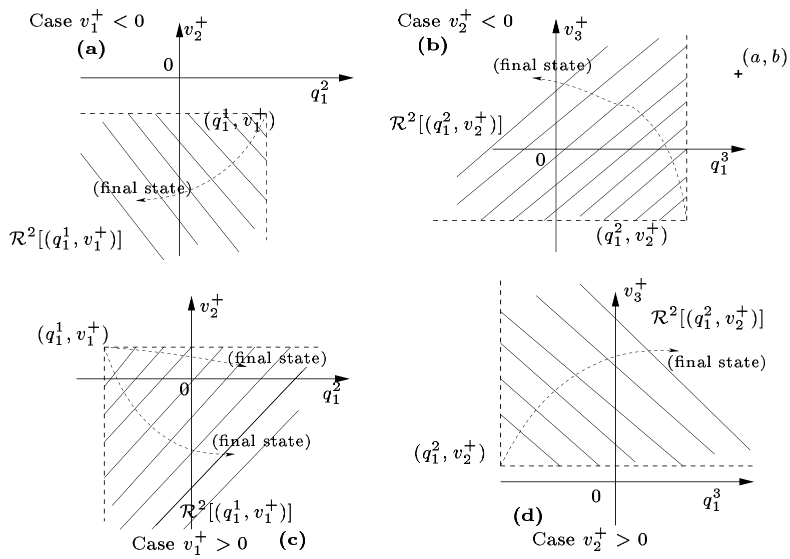


Figure 3: Reachable subspaces for the backlash model.

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