Choosability of bipartite graphs with maximum degree *Delta*

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Abstract: Let $G = (V(G), E(G))$ be a graph. A *list assignment* is an assignment of a set $L(v)$ of integers to every vertex v of G. An L-colouring is an application C from $V(G)$ into the set of integers such that $C(v) \in L(v)$ for all $v \in V(G)$ and $C(u) \neq C(v)$ if u and v are joined by an edge. A (k, k')-*list assignment* of a bipartite graph G with bipartition (A, B) is a list assignment L such that $|L(v)| = k$ if $v \in A$ and $|L(v)| = k'$ if $v \in B$. A bipartite graph is (k, k')-*choosable* if it admits an L-colouring for every $(k.k')$-list assignment L. In this paper, we study the (k, k')-choosability of graphs. Alon and Tarsi [2] proved in an algebraic and non-constructive way, that every bipartite graph with maximum degree Δ is $(\lceil \Delta/2 \rceil + 1, \lfloor \Delta/2 \rfloor + 1)$-choosable. In this paper, we give an alternative and constructive proof to this result. We conjecture that this result is sharp (i.e. there is a bipartite graph with maximum degree Δ that is not $(\lceil \Delta/2 \rceil, \lfloor \Delta/2 \rfloor + 1)$-choosable) and prove it for $\Delta \leq 5$. Moreover, for a fixed $\Delta \in \{4, 5\}$, we show that given a bipartite graph with maximum degree Δ and a $(\lceil \Delta/2 \rceil, \lfloor \Delta/2) \rfloor + 1)$-list assignment L, it is NP-complete to decide if G is L-colourable. At last, we give upper bounds for the minimum size $n_3(\Delta)$ of a non $(3, 3)$-choosable bipartite graph with maximum degree Δ: $n_3(5) \leq 846$ and $n_3(6) \leq 128$.

Key-words: list colouring, choosability, bipartite graph, NP-complete

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Choisissabilité des graphes bipartis de degré maximum Δ

Résumé : Soit $G = (V(G), E(G))$ un graphe. Une *affectation de listes* est l'affectation à chaque sommet v de G d'un ensemble d'entiers $L(v)$. Une L-*coloration* est une application C de $V(G)$ dans l'ensemble des entiers telle que $C(v) \in L(v)$ pour tout $v \in V(G)$ et $C(u) \neq C(v)$ si u et v sont reliés par une arête. Une (k, k')-*affectation de listes* d'un graphe biparti G de bipartition (A, B) est une affectation de listes l telle que $|L(v)| = k$ si $v \in A$ et $|L(v)| = k'$ si $v \in B$. Un graphe biparti est (k, k')-*choisissable* s'il admet une L-coloration pour toute (k, k')-affectation de listes L. Dans ce rapport, nous étudions la (k, k')-choisissabilité des graphes. Alon et Tarsi ont prouvé, de manière algébrique et non constructive, que tout graphe biparti de degré maximum Δ est $(\lceil \Delta/2 \rceil + 1, \lfloor \Delta/2 \rfloor + 1)$-choisissable. Nous donnons une preuve constructive de ce résultat. Nous conjecturons que ce résultat est le meilleur possible (i.e. il y a des graphes bipartis de degré maximum Δ qui ne sont pas $(\lceil \Delta/2 \rceil, \lfloor \Delta/2 \rfloor + 1)$-choisissables). Nous montrons ceci pour $\Delta \leq 5$. De plus, pour $\Delta \in \{4, 5\}$ fixé, nous montrons qu'étant donnés un graphe biparti de degré maximum Δ et une $(\lceil \Delta/2 \rceil, \lfloor \Delta/2) \rfloor + 1)$-affectation de listes L, il est NP-complet de décider si G est L-colorable. Enfin, nous donnons des bornes supérieures du nombre minimum de sommets $n_3(\Delta)$ d'un graphe biparti de degré maximum Δ non $(3, 3)$-choisissable: $n_3(5) \leq 846$ et $n_3(6) \leq 128$.

Mots-clés : coloration sur liste, choisissabilité, graphe biparti, NP-complet

The list colouring problem is a variation and generalization of the well-known problem of colouring the vertices of a graph with as few colours as possible so that adjacent vertices get distinct colours. The additionnal requirement in this concept is that every vertex v has to be coloured with a colour from a set $L(v)$ of allowed colours which is assigned to every vertex of the graph.

List colouring is well motivated by various practical or theoretical problems. It provides a natural interpretation for scheduling problems [3, 4], extendability of partial Latin squares [1] and frequency assignment problems [11, 16]. Other interesting problems which lead to list colourings may be found in [15].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *degree* $d(v)$ of a vertex v is the number of edges incident to it. The *maximum degree* of G is $\Delta(G) := \max\{d(v)|v \in V(G)\}$. A *list assignment* L is an assignment of a set $L(v)$, called the *list of* v, of integers to every vertex $v \in V(G)$. If all lists $L(v)$ have the same number k of elements then L is called a *k-list assignment* and k the *length* of L.

Let L be a list assignment. An *L-colouring* is a application C from $V(G)$ into the integers set such that $C(v) \in L(v)$ for all v in V(G) and $C(u) \neq C(v)$ if u and v are joined by an edge. A graph is *L-colourable* if it admits an L-colouring. Notice that these definitions lead to the problem of k-colourability of G if all lists are identical with $L(v) = \{1, 2, \ldots, k\}$.

A graph is *k-choosable* if it is L-colourable for all k-list assignment L. Of course, it is interesting to ask about the shortest length k of list assignment such that the graph is always list colourable. The *choice number* $ch(G)$ of G is the smallest integer such that G is k-choosable. Obviously, the choice number is at least as big as the *chromatic number* $\chi(G)$, that is the smallest integer such that G is k-colourable. Furthermore, the ratio $ch(G)/\chi(G)$ may be arbitrarily large. In particular, Erdös, Rubin and Taylor [6] established that for arbitrary k there are *bipartite* (i. e. 2-colourable) graphs that are not k-choosable. Moreover, Gravier [8] proved that every non-k-choosable graphs may be constructed from non-k-choosable bipartite graphs using three operations. This theorem is an analogue of Hajós's Theorem on colourings [10] stating that every non-k-colourable graphs can be obtained from the complete graph K_{k+1} by three operations (almost identical to those of Gravier). Hence, bipartite graphs play a special role in investigations of list colourings and choosability, more or less the same as complete graphs do in colourability. And since they have a simple structure in relation to ordinary vertex colourings, it seems to be reasonable to consider first the choosability of bipartite graphs. Rubin[6] characterized all 2-choosable graphs. Mahadev, Roberts and Santhanakrishnan [13] started to characterize complete bipartite 3-choosable graphs and Shende and Tesman [17] and O'Donnell [14] completed this characterization. In particular, $K_{7,7}$ is not 3-choosable.

In this paper, we investigate the choosability of bipartite graphs with maximum degree Δ and the complexity of the corresponding problem. More precisely, we study the (k, k')-choosability of bipartite graphs. A bipartite graph with bipartition (A, B) is *(k, k')-choosable* if it is L-colourable for every *(k, k')-list assignment* L, i. e. such that $|L(v)| = k$ if $v \in A$ and $|L(v)| = k'$ if $v \in B$.

Alon and Tarsi [2] proved the following :

Theorem 1 (Alon et Tarsi [2]) *Every bipartite graph with maximum degree* Δ *is* $(\lceil \Delta/2 \rceil + 1, \lfloor \Delta/2 \rfloor + 1)$*-choosable.*

However, the proof given by Alon and Tarsi is not constructive. In the first section, we give a constructive proof of this result. We conjecture that the lower bound $(\lceil \Delta/2 \rceil + 1, \lfloor \Delta/2 \rfloor + 1)$ of Theorem 1 is sharp.

Conjecture 1 *For any* Δ*, there is a bipartite graph with maximum degree* Δ *that is not* $(\lceil \Delta/2 \rceil, \lfloor \Delta/2 \rfloor + 1)$*-choosable.*

We also consider the complexity of the corresponding list colouring problem:
Δ Bipartite graph List Colouring Problem (Δ-BLCP) :
Instance: A bipartite graph G with maximum degree Δ and a $(\lceil \Delta/2 \rceil, \lfloor \Delta/2 \rfloor + 1)$-list assignment.
Question: Is G L-colourable?

For $\Delta \leq 3$, Conjecture 1 holds and Δ-BLCP is polynomial since the 2-List Colouring problem is easy to solve as observed in both early papers [18] and [6]. In Section 2 and 3, we prove that Conjecture 1 holds and that Δ-BLCP is NP-complete when Δ is 4 and 5, respectively.

In particular, we exhibit a non 3-choosable bipartite graph G_5 with maximum degree 5 which has a thousand of vertices. A natural question is to ask for the minimum number of vertices $n_3(\Delta)$ of a non-3-choosable bipartite graphs with maximum degree at most Δ. Our example give us $n_3(5) \leq 846$. The complete bipartite $K_{7,7}$ is not 3-choosable. And there is no non-3-choosable bipartite graph with less than 14 vertices (see [12]). Hence, $n_3(\Delta) = 14$ if $\Delta \geq 7$. In the last section, we show that $n_3(6) \leq 128$.

1 Constructive proof of Theorem 1.

The aim of this section is to give a constructive proof of the following theorem due to Alon and Tarsi.

Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$. The *outneighbourhood* of a vertex v is the set $N^+(v) := \{x | (v, x) \in A(D)\}$. Its cardinality, called the *outdegree* of v is denoted by $d^+(v)$. If $y \in N^+(v)$, we say that v *dominates* y. A *stable set* S of D is a set of vertices such that there is no arc (u, v) with $u \in S$ and $v \in S$. A *kernel* of a digraph D is a stable set S such that every vertex is either in S or dominated by a vertex of S.

Lemma 1 (Bondy, Boppana, Siegel [7]) *Let* D *be a digraph. Let* L *be a list assignment such that* $d^+(v) + 1 = |L(v)|$ *for each vertex* v*. If every induced subdigraph of* D *has a kernel then* D *is* L*-colourable.*

Moreover the proof of this lemma is constructive and give the following algorithm to find an L-colouring of D.

Let $\bigcup_{v\in V(D)} L(v) = \{c_1, c_2, \dots, c_p\}$.
For $i = 1$ to n do
Find a kernel K_i of the graph induced on G by the vertices whose list contains c_i.
Assign the colour c_i to every vertex of K_i.
Set $G = G - K_i$.

Lemma 2 *Every bipartite digraph has a kernel.*

Proof. Let D be a bipartite digraph. One can find one of its kernel K with following algorithm :

Step 0: Initialize K to the empty set.
Step 1: If there is a vertex x with no inneighbour then,
add x to K, set $V(D) := V(D) \setminus (N^+(x) \cup \{x\})$ and go to Step 1.
Step 2: Add one of the vertex classes of the bipartition of D to K.

Let us prove that the set K is a kernel. Let x and y be two vertices of K. Suppose for contradiction that (x, y) is an arc of D. Then x and y have not been put in K by Step 2. If x has been put in K before y then $y \notin N^+(x)$, which contradicts that (x, y) is an arc. And y has been put in K before x, then x is not an inneighbour of y which is again a contradiction. Hence K is a stable. Now let v be a vertex of $V(D) \setminus K$. If it has been deleted from $V(D)$ at Step 1 and it belongs to $N^+(x)$ for some $x \in K$. If not, it was in D at Step 2 thus it has an inneighbour y that is in the other vertex class of the bipartition. And $y \in K$ since $x \notin K$. So, every vertex of $V(D) \setminus K$ is dominated by a vertex of K. Hence K is a kernel of D. ■

Proof. *(of Theorem 1)* By König's Theorem, the edges of G may be partitionned in Δ matchings $M_1, M_2, \dots, M_\Delta$. Let D be the orientation of G such that every edge is oriented from A to B if and only if it is in $\bigcup_{i=1}^{\lceil \Delta/2 \rceil} M_i$. Then every vertex of A has outdegree at most $\lceil \Delta/2 \rceil$ and every vertex of B has outdegree at most $\lfloor \Delta/2 \rfloor$. Then Lemmas 1 and 2 yield the result. ■

2 Bipartite graphs with maximum degree 4

Let H and $L_H[1]$ be the graph and its list assignment depicted Figure 1. For any integer i, let $L_H[i]$ be the list assignment obtained from $L_H[1]$ by applying a permutation σ on the integers which satisfies $\sigma(1) = i$.

Proposition 1 *For any i, H is not $L_H[i]$-colourable.*

Proof. It suffices to prove it for $i = 1$. Suppose that there is a $L_H[1]$-colouring C. Then $C(u) = 4$ thus d, e and f are not coloured 4. Since the complete bipartite $K_{3,3}$ is not list colourable with lists $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$ on each vertex class, we get a contradiction. ■

Corollary 1 *There is a bipartite graph with maximum degree 4 that is not $(3, 2)$-choosable.*

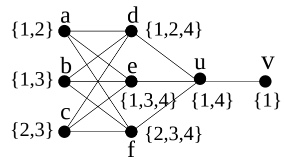

Figure 1: The graph H and its list assignment $L_H[1]$

Proof. Let G be the graph consisting of three copies H_i, $1 \leq i \leq 3$ of H whose vertices v are identified. Let L be the list assignment such that $L(v) = \{1,2,3\}$ and that L coincides with $L_H[i]$ on $V(H_i) \setminus v$ for $1 \leq i \leq 3$. Then G is not L-colourable. Indeed if v is coloured $i \in \{1,2,3\}$ then Proposition 1 give a contradiction in H_i. ■

To prove the NP-completeness of 4-BLCP, we need to prove the NP-completeness of the following problem.

Auxiliary Bipartite List Colouring Problem (ABLCP)
Instance: A bipartite graph G with bipartition (A, B) such that each vertex has degree 2 or 3 and a list assignment L such that $|L(v)| = 2$ if $v \in A$ and $|L(v)| = d(v)$ if $v \in B$.
Question: Is G L-colourable?

Theorem 2 (Gravier [9]) *ABLCP is NP-complete.*

Proof. Given a Boolean formula Φ in conjunctive normal form, with a set C of clauses over the set X of variables. Let us define the graph G_Φ as follows: Its vertex set is $C \cup \{(c,x) | x \in c \in C\} \cup \{(c,\bar{x}) | x \in c \in C\}$. For all $x \in c \in C$, the vertex (c,x) is joined to the vertex c and $(c,\bar{x})$. For each $x \in X$, the subgraph induced by the vertices (c,x) and $(c,\bar{x})$ for $x \in c$ is a cycle.

The symbols x and $\bar{x}$ will be taken for the colours, and the list assignment L is defined as follows:

$$L(c) := \{\bar{x} | x \in c\} \cup \{x | \neg x \in c\} \quad \forall c \in C$$

and

$$L((c,x)) = L((c,\bar{x})) := \{x, \bar{x}\} \quad \forall x \in c \in C \ .$$

With $A = \{(c,x) | x \in c \in C\}$ and $B = C \cup \{(c,\bar{x}) | x \in c \in C\}$, (A, B) is a bipartition of G_Φ such that $|L(v)| = 2$ if $v \in A$ and $|L(v)| = d(v)$ if $v \in B$.

In an L-colouring, for each $x \in X$, the vertices of $\{(c,x) | x \in c \in C\}$ must get the same colour because the subgraph induced by the vertices (c,x) and $(c,\bar{x})$ for $x \in c$ is a cycle. It can be seen that there is a one-to-one correspondance between the satisfying truth assignments of Φ and the L-colourings. Then the NP-completeness of 3-SAT yield the result. ■

Theorem 3 *4-BLCP is NP-complete.*

Proof.

Let (G, L) be an instance of the ABLCP. An equivalent instance (G', L') may be obtained for the 4-BLCP: For each vertex $x \in A$ whose list $L(x)$ has cardinality 2, set $L'(x) = L(x) \cup \{e\}$ for some $e \notin L(v)$, identify x to the vertex v of a copy H_x of H and set $L'(u) = L_H[e](u)$ for $u \in V(H_x) \setminus x$. It is easy to check that G' has degree at most four. ■

3 Non 3-choosable bipartite graph with maximum degree 5

In this section, we construct a non-3-choosable bipartite graph with maximum degree 5. For that, we need four intermediary graphs.

Let M and $L_M[4,5]$ be the graph and its list assignment depicted Figure 2. Note that M is bipartite and that the vertices a and b lie in different parts of the bipartition. For any two integers $i \neq j$, let $L_M[i,j]$ be the list assignment obtained from $L_M[4,5]$ by applying a permutation σ on the integers which satisfies $\sigma(4) = i$ and $\sigma(5) = j$.

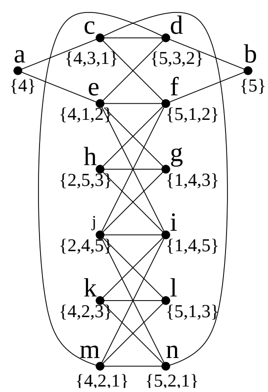

Figure 2: The graph M and its list assignment $L_M[4,5]$.

Proposition 2 *For any $i \neq j$, M is not $L_M[i,j]$-colourable.*

Proof. Without loss of generality, it suffices to prove it for $L_M[4,5]$. Suppose that M admits a $L_M[4,5]$-colouring C: colour 4 is forbidden on c and e,and 5 is forbidden on d and f. If $C(f) = 1$, we force $C(c) = 3$ and $C(e) = 2$, and then d may not be coloured. So, we have $C(f) = 2$ and, consequently, $C(e) = 1$. The same reasonnement applied on vertices g, h, i and j shows that $C(i) = 4$ and $C(j) = 5$, and on the vertices k, l, m and n, it gives that $C(m) = 2$ and $C(n) = 1$. Hence colours 4 and 1 are forbidden on c and colours 5 and 2 are forbidden on d, and we must have $C(c) = C(d) = 3$, which is impossible. ■

Let N and $L_N[4,5]$ be the graph and its list-assignment depicted Figure 3. The vertices a_1 and b_1, and a_2 and b_2 are identified with the vertices a and b of the two copies of M, M_1 and M_2 respectively. And $L_N[4,5]$ coincides with $L_M[5,1]$ (resp. $L_M[5,2]$) on $V(M_1) \setminus \{a_1, b_1\}$ (resp. $V(M_2) \setminus \{a_2, b_2\}$). Note that the graphs N is bipartite and that s and w lie in the same part of the bipartition. For any two integers $i \neq j$, let $L_N[i,j]$ be the list assignment obtained from $L_N[4,5]$ by applying a permutation σ on the colours $\{1, \cdots, 5\}$ which satisfies $\sigma(4) = i$ and $\sigma(5) = j$.

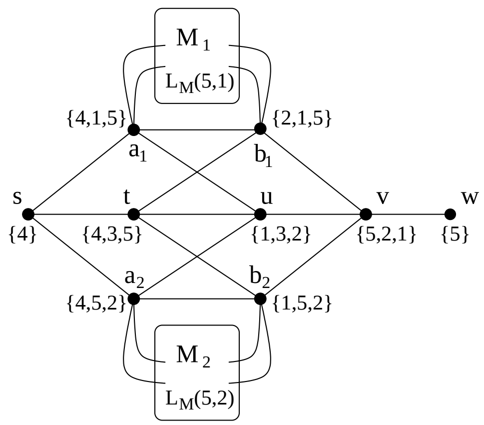

Figure 3: The graph N and its list assignment $L_N[4,5]$.

Proposition 3 *For any $i \neq j$, N is not $L_N[i,j]$-colourable.*

Proof. Suppose that N admits a $L_N[4,5]$-colouring C. Colour 4 is forbidden on vertices a_1, a_2 and t and colour 5 is forbidden on vertex v. If $C(v) = 1$, then $C(b_2) \neq 1$ and $C(u) \neq 1$. Moreover, $C(b_2)$ can not be 5, otherwise $C(t) = 3$ and $C(a_2) = 2$ and u can not be coloured. Then $C(b_2) = 2$ and $C(a_2) = 5$ but this contradicts Proposition 2. Consequently, v may not be coloured 1. Analogously, by symmetry, v can not be coloured 2. So, N is not $L_N[4,5]$-colourable. ■

Let P and $L_P[4,5]$ be the graph and its list assignment depicted Figure 4. The vertices a_3 and b_3 are identified with the vertices a and b of a copy of M, M_3. And $L_P[4,5]$ coincides with $L_M[2,1]$ (resp. $L_N[3,1]$) on $V(M_3) \setminus \{a_3, b_3\}$ (resp. $V(N) \setminus \{s, w\}$). The graph P is bipartite and s and w lie in different part of the bipartition. For $i \neq j$, let $L_P[i,j]$ be the list assignment obtained from $L_P(4,5)$ by applying a permutation σ on the integers which satisfies $\sigma(4) = i$ and $\sigma(5) = j$.

Proposition 4 *For any $i \neq j$, P is not $L_P[i,j]$-colourable.*

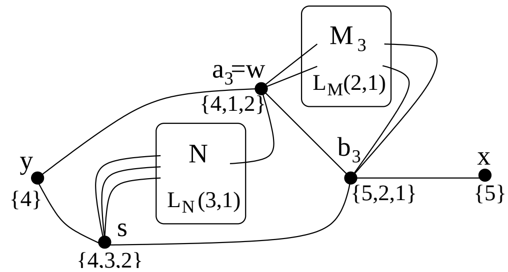

Figure 4: The graph P and its list assignment $L_P[4,5]$.

Proof. Let C be an $L_P[4,5]$-colouring of P. Then $C(s) \neq 4$, $C(a_3) \neq 4$ and $C(b_3) \neq 5$. Furthermore $C(b_3) = 2$. Otherwise, $C(b_3) = 1$ so $C(a_3) = 2$ which contradicts Proposition 2. Hence $C(w) = 1$ and $C(s) = 3$. This contradicts Proposition 2. So P is not $L_P[4,5]$-colourable. ■

Let Q be the graph obtained from two copies P_1 and P_2 by identifying the vertices coresponding to x and y and adding an edge between them. See Figure 5. And let $L_Q[4]$ be the list assignment coinciding with $L_P[1,4]$ (resp. $L_P[2,4]$) on $V(P_1) \setminus y$ (resp. $V(P_2) \setminus y$) and $L_Q[4](y) = \{1,2,4\}$.

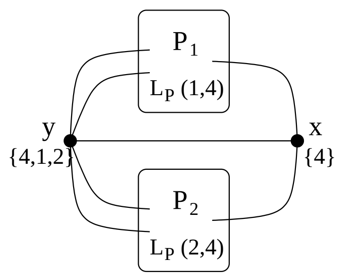

Figure 5: The graph Q and its list assignment $L_Q[4]$.

Proposition 5 *Q is not $L_Q[4]$-colourable.*

Proof. Clearly, Q is not $L_Q[4]$-colourable. Indeed y may not be coloured because of the edge $\{y,x\}$ and by Proposition 4 it can not be coloured 1 or 2 because of P_1 and P_2. ■

Let G_5 and L_5 be the graph depicted Figure 6: Each Q_i, $1 \leq i \leq 10$, is a copy of Q and L coincides with $L_Q(4)$ on each $Q_i \setminus x_i$. Obviously, G_5 is bipartite and has maximum degree 5.

Proposition 6 *G is not L_5-colourable, so it is not 5-choosable.*

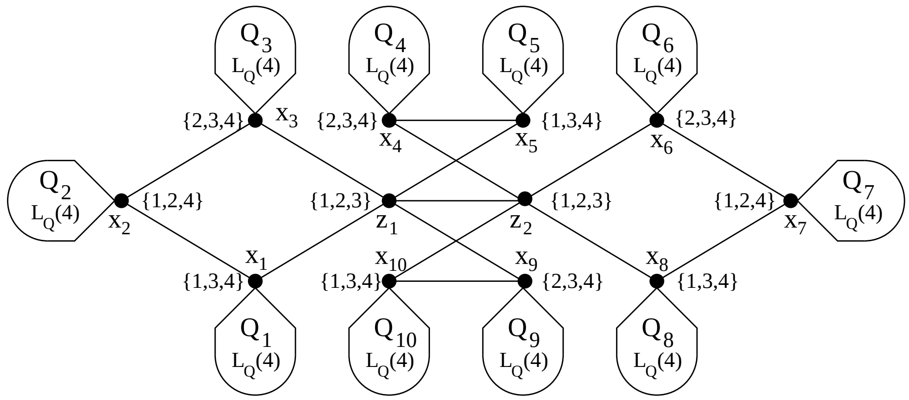

Figure 6: The graph G_5 and its list assignment L_5.

Proof. Suppose that G_5 admits a L_5-colouring C. Every x_i, $1 \leq i \leq 10$, is not coloured 4 by Proposition 5. If $C(z_1) = 3$, then $C(x_1) = 3$ and $C(x_3) = 2$ and we can not colour x_2. Analogously $C(z_2) = 3$ leads to a contradiction. If $C(z_1) = 1$ then $C(z_2) = 2$, so $C(x_4) = C(x_5) = 3$ which is a contradiction. And if $C(z_1) = 1$ then $C(z_2) = 2$ thus $C(x_9) = C(x_{10}) = 3$ which is a contradiction. ■

Remark 3.1 *In the proof of Proposition 6, the non colourable list assignment uses five different colours. This is the minimum number of colour for a counterexample. Indeed if a list assignment L uses only four colour then it is always L-colourable as observed in [5].*

Theorem 4 *5-BLCP is NP-complete.*

Proof. Let R be the subgraph of G_5 induced by the vertices of $V(G_5) \setminus (V(Q_1) \setminus \{x_1\})$ and let $L_R[1]$ be the list assignment defined by $L_R[4](v) = L_5(v)$ if $v \in V(R)$. For any integer i, let $L_R[i]$ be the list assignment obtained from $L_R[4]$ by applying a permutation σ on the integers which satisfies $\sigma(4) = i$. From the proof of Proposition 6, we deduce that every $L_R[4]$-colouring C of R satisfies $C(x_1) = 4$. Thus for any integer i, every $L_R[i]$-colouring C of R satisfies $C(x_1) = i$. Let (G, L) be an instance of the ABLCP. An equivalent instance (G'', L'') may be obtained for the 5-BLCP: For each vertex $x \in V(G)$ whose list $L(x)$ has cardinality 2, set $L''(x) = L(x) \cup \{e\}$ for some $e \notin L(v)$, join x to the vertex v of a copy R_x of R and set $L''(u) = L_R[e](u)$ for $u \in V(R_x)$. It is easy to check that G'' has degree at most five. ■

4 Upper bounds for $n_3(\Delta)$

The graph G_5 has 942 vertices, so $n_3(5) \leq 942$. However, one can get a slightly better bound:

Proposition 7 $n_3(5) \leq 846$.

Proof. Let G be the graph obtained from G_5 by doing the following : For $4 \leq i \leq 6$, (resp. $i = 3$) identify the vertices h, g, k and l of each subgraph M of Q_i with the vertices h, g, k and l of the identical M in Q_{14-i} (resp. Q_1). It is easy to check that G is bipartite, because identified vertices lied in the same part of the bipartition, and G has maximal degree 5. Moreover, for two identified vertices u and v, $L_5(u) = L_5(v)$. Hence, L_5 is still well-defined and G is not L_5-colourable. The graph G has 846 vertices (24 less per pair of Q_i). Hence $n_3(5) \leq 846$. ■

Let us now construct a small non 3-choosable bipartite graph with maximum degree 6. As in the previous section, we need some intermediary graphs.

Proposition 8 *Let* $(\{b, c, d\}, \{b', c', d'\})$ *be the bipartition of* $K_{3,3}$. *And let* L *be the following list assignment :* $L(b) = \{1, 2\} = L(b')$, $L(d) = L(c') = \{1, 3\}$, $L(c) = \{2, 3, 4\}$ *and* $L(d') = \{2, 3\}$. *Then any* L*-colouring* C *of* $K_{3,3}$ *(of the two) satisfies* $C(c) = 4$, $C(b) = C(d) = 1$, $C(b') = 2$ *and* $C(c') = 3$.

Proof. Let C be an L-colouring of $K_{3,3}$. Since $K_{3,3}$ is not list colourable with lists $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$ on each vertex class, $C(c) = 4$. Moreover $C(b') = 2$ otherwise $C(b) = 2$ and $C(d) = 3$ and d' can not be coloured. In the same way, $C(c') = 3$. It follows that $C(b) = C(d) = 1$. ■

Proposition 9 *Let* F *be the graph and* L_F *the list assignment depicted Figure 7. Then* F *is not* L_F*-colourable.*

Proof. Suppose that F admits a L_F-colouring C. Then $C(v) = 4$, thus $C(v')$ is 2 or 5.

Suppose that it is 5. Then b', c' and d' may not be coloured 5. Moreover b and d may not be coloured 4. Thus by Proposition 8, $C(c) = 4$, $C(b) = 1$ and $C(b') = 2$ and $C(c') = 3$. Hence $C(a') = 5 = C(a)$ which is a contradiction.

Analogously, if $C(v') = 2$, we get the contradiction $C(g') = 2 = C(g)$ ■

Let G_6 and L_6 be the graph and its list-assignment depicted Figure 8. The vertex v_i, $1 \leq i \leq 9$, b_1, is identified with the vertex v of the copy F_i of F. And L_6 coincides with L_F on $F_i \setminus v_i$ for $1 \leq i \leq 9$.

Proposition 10 G_6 *is not* L_6*-colourable. So it is not 3-choosable.*

Proof.

Let us show that G_6 is not L-colourable. By Proposition 9, none of the v_i may be coloured with 4. Suppose now that x is coloured with 1 then v_1 must be coloured 2 and v_3 coloured 3 then v_2 may not be coloured which is a contradiction. Analogously, if x is coloured 2 (resp. 3), we obtain a contradiction along the 4-cycle (x, v_4, v_5, v_6) (resp. (x, v_7, v_8, v_9)). ■

The graph G_6 has 145 vertices, so $n_3(6) \leq 145$. Once again, one can get a better bound:

Proposition 11 $n_3(6) \leq 128$.

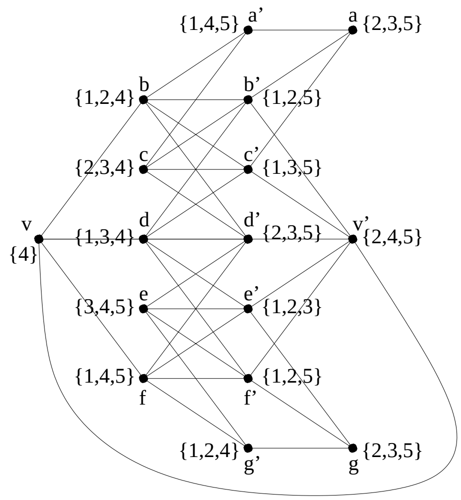

Figure 7: The graph F and the list assignment L_F

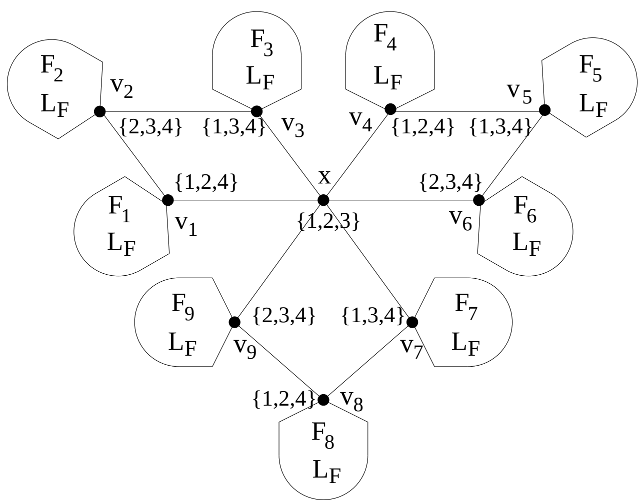

Figure 8: The graph G_6

Proof. Let G be the graph obtained from G_6 by doing the following : For $i \in \{1,4,7\}$ (resp. $i=2$), identify the vertices a', a, g' and g of F_i with the vertices a', a, g' and g of F_{i+2} (resp. F_5). And identify the vertex a of F_8 with the vertex g of F_8. It is easy to check that G is bipartite, because identified vertices lied in the same part of the bipartition, and G has maximal degree 6. Moreover, for two identified vertices u and v, $L_6(u) = L_6(v)$. Hence, G is not L_6-colourable and has 128 vertices. Hence $n_3(6) \leq 128$. ■

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