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Abstract: In this paper, we compare the normal vector field of a compact (oriented) smooth surface S with the normals of a triangulated mesh T whose vertices belong to S. As a corollary, we deduce an approximation of the area of S by the area of T. We apply this result to the restricted Delaunay triangulation obtained with a sample of S. Using Chew's algorithm, we build sequences of triangulations inscribed on S, whose curvature measures tend to the curvature measures of S.

Key-words: Surface, area, curvature, Hausdorff distance, mesh, Delaunay triangulation, sample.

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Sur l'approximation du champ de vecteurs normal d'une surface lisse

Résumé : Dans cet article, nous comparons le champ de vecteurs normal d'une surface lisse S compacte (orientée) aux normales des faces d'une triangulation T dont les sommets appartiennent à cette surface. Comme corollaire, nous déduisons une approximation de l'aire de S par l'aire de T. Nous appliquons ce résultat à la triangulation de Delaunay restreinte obtenue à partir d'un échantillonage de S. En utilisant l'algorithme de Chew, nous construisons des suites de triangulation inscrites sur S, dont les mesures de courbure tendent vers les mesures de courbure de S.

Mots-clés : Surface, aire, courbure, distance de Hausdorff, maillage, triangulation de Delaunay, échantillonage.

1 Introduction

In this paper, we are interested in the relationship between a smooth surface S and a triangulated mesh T inscribed in it (*i.e.* whose vertices belong to the smooth surface). In particular we wonder whether we can approximate the normal vector field of an (oriented) smooth surface with the normals of a triangulated mesh.

Roughly speaking, we evaluate this approximation in terms of the geometry of the triangulated mesh, the local curvature of the smooth surface and the local Hausdorff distance between the two surfaces. This result is very general and must be compared with [12]. In the context of surface reconstruction from scaterred sample points, N. Amenta et al. (c.f. [1] and [2]) also get approximations of the normals of a smooth surface. In [9], J. Fu has results of convergence of the curvature measures of a sequence of triangulated meshes to the curvature measures of a smooth surface.

This paper is organized as follows. Section 3 states our main theorem. In Section 4 is an application: using *Chew*'s algorithm, we produce a sequence of triangulations which are close to S, and whose area tends to the area of S. Section 5 sketches the proofs.

The background on differential geometry can be found in [5], [7], [11], [14].

2 Definitions

We recall here some classical definitions which concern smooth surfaces, triangulated meshes and the relative position of two surfaces. For more details on triangulated meshes, one may refer to [9].

2.1 Smooth surfaces

In the following, a smooth surface means a $\mathcal{C}^2$ surface which is regular, oriented, with or without boundary. Let S be a smooth surface of the (oriented) Euclidean space $\mathbb{E}^3$. Let ∂S denote the boundary of S. S is endowed with the Riemannian structure induced by the standard scalar product of $\mathbb{E}^3$. We denote by da the area form on S and by ds the canonical orientation of ∂S. Let ν be the unitary normal vector field (compatible with the orientation of S) and h be the second fundamental form of S associated with ν. Its determinant at a point p of S is *the Gauss curvature* $G(p)$, its trace is *the mean curvature* $H(p)$. *The maximal curvature of* S *at* p is $\rho_p = max(|\lambda^1(p)|, |\lambda^2(p)|)$, where $\lambda_1(p)$ and $\lambda_2(p)$ are the eigenvalues of the second fundamental form at p. Finally, *the maximal curvature of* S is

$$\rho_S = \sup_{p \in S} \rho_p.$$

We need the following

Proposition 1 *Let* S *be a smooth compact (oriented) surface of* $\mathbb{E}^3$. *Then there exists an open set* U_S *of* $\mathbb{E}^3$ *containing* S *and a continuous map* ξ *from* U_S *onto* S *satisfying the* *following property: if p belongs to U_S, then there exists a unique point $\xi(p)$ realizing the distance from p to S (ξ is nothing but the orthogonal projection onto S).*

A proof of this proposition can be found in [8].

The open set U_S depends locally and globally on the smooth surface S. Locally, the normals of S do not intersect in U_S. Globally, U_S depends on points which can be far from one another on the surface, but close in $\mathbb{E}^3$.

We shall also need the notion of *reach of a surface*, introduced in [8].

Definition 1 *The reach of a surface S is the largest $r > 0$ for which ξ is defined on the (open) tubular neighborhood $U_r(S)$ of radius r of S.*

This implies that the reach r_S of S is smaller than the minimal radius of curvature of S (see [11]). Thus, we have:

$$\rho_S \, r_S \leq 1.$$

The *reach* is linked to the notion of *medial axis* and *local feature size (LFS)* (see [1] for instance). The *medial axis* of a smooth surface S of $\mathbb{E}^3$ is the closure of the set of points in $\mathbb{E}^3$ with more than one nearest neighbor on S. The *local feature size* $LFS(p)$ at a point p of S is the Euclidean distance from p to (the nearest point of) the medial axis.

2.2 Triangulated meshes

2.2.1 Generalities

A triangulated mesh T is a topological surface which is a (finite and connected) union of triangles of $\mathbb{E}^3$, such that the intersection of two triangles is either empty, or equal to a vertex, or equal to an edge.

We denote by $\mathcal{T}_T$ the set of triangles of T and by Δ a generic triangle of T.

- $\eta(\Delta)$ denotes the length of the longest edge of Δ, and $\mathcal{A}(\Delta)$ its area.
- The fatness of Δ is the real number

$$\theta(\Delta) = \frac{\mathcal{A}(\Delta)}{\eta(\Delta)^2}.$$

- We introduce the notion of *straightness of a triangle* Δ. It is the real number

$$\mathrm{str}(\Delta) = \sup_{p \text{ vertex of } \Delta} |\sin(\theta_p)|,$$

  where θ_p is the angle at p of Δ.

Globally,

- The area $\mathcal{A}(T)$ is the sum of the areas of all the triangles of T.
- The height of T is:
$$\eta(T) = \sup_{\Delta \in \mathcal{T}_T} \eta(\Delta).$$
- The fatness of T is:
$$\theta(T) = \min_{\Delta \in \mathcal{T}_T} \theta(\Delta).$$
- The *straightness* of T is:
$$\mathrm{str}(T) = \min_{\Delta \in \mathcal{T}_T} \mathrm{str}(\Delta).$$

Remark that the *straightness* is a weaker condition than the *fatness*. The *straightness* of a triangle is large if there exists at least one angle whose sinus is large. The *fatness* of a triangle is large if the sinus of every angle is large.

2.2.2 Triangulated mesh closely inscribed in a smooth surface

- We say that a triangulated mesh of $\mathbb{E}^3$ is inscribed in a smooth surface S if all its vertices belong to S.
- A triangulated mesh T is closely inscribed in a smooth surface S if:
  1. the triangulation T lies in $U_r(S)$, where r is the reach of S,
  2. all vertices of T belong to S,
  3. all vertices of ∂T belong to ∂S,
  4. the restriction of ξ to T is one to one.
- Let T be a triangulated mesh closely inscribed in a smooth surface S. Let m be a point lying in the interior of a triangle Δ of T. Let N^Δ be the normal line through m to Δ. We put
$$\alpha_m = (\widehat{N^\Delta, \nu^S_{\xi(m)}}) \in \left[0, \frac{\pi}{2}\right].$$
The real number α_m is defined almost everywhere on T. (If m is a point on an edge or a vertex, one can define α_m by taking the supremum of the angles between the triangles which contain m and the normal $\nu^S_{\xi(m)}$).
We can define the real number
$$\alpha = \sup_{m \in T} \alpha_m.$$
α is called *the maximal angle between the normals of S and T*.
- Let T be a triangulated mesh closely inscribed in a smooth surface S. *The relative curvature of T with respect to S* is the real number defined by:
$$\omega_S(T) = \sup_{m \in T \setminus \partial T} ||\xi(m) - m|| \rho_{\xi(m)}.$$

- The *relative height* of T with respect to S is the real number defined by:

$$\pi_S(T) = \sup_{\Delta \in \mathcal{T}} \sup_{m \in \Delta} \eta(\Delta)\rho_{\xi(m)}.$$

Remark A triangulated mesh T closely inscribed in a smooth surface S satisfies:

- $\omega_S(T) \leq \pi_S(T)$,
- $\omega_S(T) \leq 1$.

3 An Approximation on the Normal Vector Field

In this paragraph, we compare the behaviour of the normal vector field of a smooth surface S with the normal of each face of a triangulation which is closely inscribed in S. We express the approximation in terms of the geometric invariants that we introduced in the previous paragraph.

Theorem 1 *Let S be a smooth surface and T a triangulated mesh closely inscribed in S. Then the maximal angle $\alpha_{\max}$ between the normals of S and T satisfies*

$$\sin \alpha_{\max} \leq \left(\frac{\sqrt{10}}{2\ str(T)\ (1 - \omega_S(T))} + \frac{1}{1 - \omega_S(T)} \right) \pi_S(T).$$

Corollary 1 *Let S be a smooth surface and T a triangulated mesh closely inscribed in S.*

$$\textit{If} \quad \pi_S(T) \leq \frac{1}{2},$$

then the maximal angle $\alpha_{\max}$ between the normals of S and T satisfies

$$\sin \alpha_{\max} \leq \left(\frac{4}{str(T)} + 2 \right) \pi_S(T).$$

In a previous paper, ([13]), we got an approximation of the area of a smooth surface by the area of an almost-differentiable surface close to it, in terms of the angles between the corresponding normals. Using this result we obtain immediately the following

Corollary 2 *Let S be a (compact orientable) C^2-surface in $\mathbb{E}^3$ and T a triangulated mesh closely inscribed in S. If*

$$\left(\frac{4}{str(T)} + 2 \right) \pi_S(T) \leq 1,$$

then the area of S satisfies:

$$\frac{\sqrt{1 - \left(\frac{4}{str(T)} + 2 \right)^2 \pi_S(T)^2}}{(1 + \omega_S(T))^2} \mathcal{A}(T) \leq \mathcal{A}(S) \leq \frac{1}{(1 - \omega_S(T))^2} \mathcal{A}(T).$$

As an obvious consequence, we get the following convergence result:

Corollary 3 *Let S be a (compact orientable) C^2 surface in $\mathbb{E}^3$. Let T_n be a sequence of triangulations closely inscribed in S. If*

1. *the length of the edges of T_n tends to zero when n goes to infinity,*
2. *the straigthness of T_n is (uniformly) bounded from below by a positive constant,*

then,

$$\lim_{n\to\infty} \mathcal{A}(T_n) = \mathcal{A}(S).$$

Since S is compact, the second condition may be weakened by asking that $\pi_S(T_n)$ tends to zero when n goes to infinity, *(in some sense, the length of the edges may be "large" when the curvature is "small"...).*

4 On *Chew*'s algorithm

We build a sequence of *"nice"* triangulations based on *Chew*'s algorithm, (c.f. [6]) and we apply corollary 1 and corollary 3. Let us remind the context.

4.1 Sample on a smooth surface and Delaunay triangulation

A finite set $\mathcal{S}$ on S is called a *sample* of S. An ϵ-sample on S is a sample such that for every point m of S, the ball $B(m, \epsilon \mathrm{LFS}(m))$ encloses at least one point of $\mathcal{S}$.
We denote by $\mathrm{Del}(\mathcal{S})$ the Delaunay triangulation associated to $\mathcal{S}$. Moreover, we denote by $\mathrm{Del}_S(\mathcal{S})$ the restriction of $\mathrm{Del}(\mathcal{S})$ to S, that is the set of faces of $\mathrm{Del}(\mathcal{S})$ whose dual Voronoi faces cut S.

In the following, we say that $\mathcal{S}$ is *good* if $\mathrm{Del}_S(\mathcal{S})$ is homeomorphic to S and stay homeomorphic to S if we add sample points. It is well known that if $\epsilon < 0.1$, any ϵ-sample of S is a good sample (c.f. [1]). If the sample is good, any dual edge of $\mathrm{Del}_S(\mathcal{S})$ cuts S in a unique point.

4.2 An application using *Chew*'s algorithm

Let S be a compact surface, $\mathcal{S}$ be a $\tilde{\epsilon}$-sample of S, ($\tilde{\epsilon} < 0.1$), $\mathrm{Del}_S(\mathcal{S})$ be the restricted Delaunay triangulation associated to the sample $\mathcal{S}$ of S. From $\mathrm{Del}_S(\mathcal{S})$, *Chew*'s algorithm builds a triangulation which is finer than $\mathrm{Del}_S(\mathcal{S})$, and whose fatness is $> 30°$ if S is closed and $> 20.7°$ if S has a boundary.

Now, to get a triangulation which is close to S (for the Hausdorff distance), we work as follows: We consider a (good) $\tilde{\epsilon}$-sample, $\tilde{\epsilon} < 0.1$. We fix $\epsilon > 0$. To each face σ of $\mathrm{Del}(\mathcal{S})$, we associate the (unique) sphere whose center c is on S and which contains the vertices of σ.

Let $r_{\mathcal{S}}$ be the maximal circumradius of the faces of $\mathrm{Del}_S(\mathcal{S})$. If $r_{\mathcal{S}} > \epsilon$, we consider the all faces of $\mathrm{Del}(\mathcal{S})$ whose circumradius is larger than ϵ. We add the corresponding centers $c's$ to $\mathcal{S}$, build $\mathrm{Del}(\mathcal{S} \cup \{c's\})$ and $\mathrm{Del}_S(\mathcal{S} \cup \{c's\})$. Iterating this process, we obtain a triangulation $T(\epsilon)$ such that

- the length of every edge is $< \epsilon$,
- the fatness is bounded by below by an *universal* constant (independent of ϵ).

In [3], J.D. Boissonnat proved that this algorithm terminates. Moreover, it is easy to see that the Hausdorff distance between S and the final triangulation is less than ϵ, and that for ϵ small enough, $T(\epsilon)$ is closely inscribed in S. In particular, using Corollary 1 we get the following

Corollary 4 *Let S be a smooth surface and $\mathcal{S}$ be a (good) ϵ-sample of S. Let $\epsilon > 0$. Then the triangulation $T(\epsilon)$ obtained by the previous algorithm is such that the maximal angle $\alpha_{\max}$ between the normals of S and $T(\epsilon)$ satisfies:*

$$\sin \alpha_{\max} \leq C(\rho_S)\epsilon,$$

where $C(\rho_S)$ denotes a constant depending only on the maximal curvature ρ_S of S.

Consider now the sequence $\epsilon_n = \frac{1}{n}$. By the previous process, we associate a triangulation T_n satisfying the following properties:

- each T_n is closely inscribed in S;
- the Hausdorff distance between T_n and S tends to 0 when n tends to infinity;
- the fatness of T_n is uniformly bounded below by a positive constant.

This implies in particular (by using corollary 3),

$$\lim_{n \to \infty} \mathcal{A}(T_n) = \mathcal{A}(S).$$

Moreover, we can apply classical results on geometric measure theory, and deduce immediately that the curvature measures of T_n converge to the curvature measures of S for the flat topology, and in particular for the weak topology, (see [8], [9] for the background and the details on *curvature measures*).

5 Proof of Theorem 1

We need technical lemmas.

5.1 A purely geometric result

The following lemma is a purely geometric result in $\mathbb{E}^3$:

Lemma 1 *Let Δ be a triangle whose vertices are p, p_1 and p_2. If $\alpha_p \in \left[0, \frac{\pi}{2}\right]$ denotes the angle between a normal to the triangle and the axis (O, z), then*

1.
$$cos^2(\alpha_p) = \frac{\cos^2\theta_1 \cos^2\theta_2 - \sin^2\theta_1 \sin^2\theta_2 - \cos^2\gamma + 2\cos\gamma \sin\theta_1 \sin\theta_2}{\sin^2(\theta_2 - \theta_1) + 2\sin\theta_1 \sin\theta_2 \cos(\theta_2 - \theta_1) + \cos^2\theta_1 \cos^2\theta_2 - \sin^2\theta_1 \sin^2\theta_2 - \cos^2\gamma},$$

   where $\theta_i \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is the angle between $\vec{pp_i}$ and the orthogonal projection of $\vec{pp_i}$ onto the plane orthogonal to (O, z) which contains p ($\theta_i \geq 0$ iff the third composante of $\vec{pp_i}$ is positive) and $\gamma \in]0, \pi[$ is the angle of Δ at p.

2. *In particular, if $|\sin\theta_1| \leq \epsilon$ and $|\sin\theta_2| \leq \epsilon$, then*

$$\sin(\alpha_p) \leq \frac{\sqrt{10}\epsilon}{\sin\gamma},$$

   where $\gamma \in]0, \pi[$ is the angle of Δ at p.

Proof of Lemma 1

1. We put $\eta_1 = pp_1$, $\eta_2 = pp_2$ and $N = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. We can assume that $p = 0$. In spherical coordinates, we get:

$$p = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad p_1 = \begin{pmatrix} \eta_1 \cos\phi_1 \cos\theta_1 \\ \eta_1 \sin\phi_1 \cos\theta_1 \\ \eta_1 \sin\theta_1 \end{pmatrix}, \quad p_2 = \begin{pmatrix} \eta_2 \cos\phi_2 \cos\theta_2 \\ \eta_2 \sin\phi_2 \cos\theta_2 \\ \eta_2 \sin\theta_2 \end{pmatrix},$$

$$\text{with} \left\{ \begin{array}{l} \theta_i \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \\ \phi_i \in [0, 2\pi]. \end{array} \right.$$

   We put

$$\vec{pp_1} \wedge \vec{pp_2} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

   Thus

$$cos(\alpha_p) = \left| < N, \frac{\vec{pp_1} \wedge \vec{pp_2}}{\|\vec{pp_1} \wedge \vec{pp_2}\|} > \right| = \sqrt{\frac{c^2}{a^2 + b^2 + c^2}}.$$

   We have:

$$\begin{array}{ll} a & = \eta_1\eta_2 \ (\cos\theta_1 \sin\phi_1 \sin\theta_2 - \cos\theta_2 \sin\phi_2 \sin\theta_1), \\ b & = \eta_1\eta_2 \ (\cos\theta_2 \cos\phi_2 \sin\theta_1 - \cos\theta_1 \cos\phi_1 \sin\theta_2), \\ c & = \eta_1\eta_2 \ \cos\theta_1 \cos\theta_2 \ (\cos\phi_1 \sin\phi_2 - \sin\phi_1 \cos\phi_2) \\ & = \eta_1\eta_2 \ \cos\theta_1 \cos\theta_2 \ \sin(\phi_2 - \phi_1). \end{array}$$

This implies that

$$\begin{aligned}\frac{a^2+b^2}{\eta_1^2\eta_2^2} &= \cos^2\theta_1\sin^2\theta_2 \;+\; \cos^2\theta_2\sin^2\theta_1\\ &\quad -2\cos\theta_1\cos\theta_2\sin\theta_1\sin\theta_2\cos(\phi_2-\phi_1)\\ &= \sin^2(\theta_2-\theta_1)+2\cos\theta_1\cos\theta_2\sin\theta_1\sin\theta_2(1-\cos(\phi_2-\phi_1)).\end{aligned}$$

On the other hand, we get:

$$\begin{aligned}\cos\gamma &= \frac{<\vec{pp_1},\vec{pp_2}>}{\eta_1\eta_2}\\ &= \cos\theta_1\cos\theta_2(\cos\phi_1\cos\phi_2+\sin\phi_1\sin\phi_2)+\sin\theta_1\sin\theta_2\\ &= \cos\theta_1\cos\theta_2\cos(\phi_2-\phi_1)+\sin\theta_1\sin\theta_2.\end{aligned}$$

Suppose that $\cos\theta_1\neq 0$ and $\cos\theta_2\neq 0$. Then

$$\cos(\phi_2-\phi_1)=\frac{\cos\gamma-\sin\theta_1\sin\theta_2}{\cos\theta_1\cos\theta_2}.$$

Therefore

$$\begin{aligned}\frac{a^2+b^2}{\eta_1^2\eta_2^2} &= \sin^2(\theta_2-\theta_1)+2\cos\theta_1\cos\theta_2\sin\theta_1\sin\theta_2\left(1-\frac{\cos\gamma-\sin\theta_1\sin\theta_2}{\cos\theta_1\cos\theta_2}\right)\\ &= \sin^2(\theta_2-\theta_1)+2\sin\theta_1\sin\theta_2\cos(\theta_2-\theta_1)-2\cos\gamma\sin\theta_1\sin\theta_2,\\ \frac{c^2}{\eta_1^2\eta_2^2} &= \cos^2\theta_1\cos^2\theta_2\;\left(1-\left(\frac{\cos\gamma-\sin\theta_1\sin\theta_2}{\cos\theta_1\cos\theta_2}\right)^2\right)\\ &= \cos^2\theta_1\cos^2\theta_2-\sin^2\theta_1\sin^2\theta_2-\cos^2\gamma+2\cos\gamma\sin\theta_1\sin\theta_2,\\ \frac{a^2+b^2+c^2}{\eta_1^2\eta_2^2} &= \sin^2(\theta_2-\theta_1)+2\sin\theta_1\sin\theta_2\cos(\theta_2-\theta_1)\\ &\quad +\cos^2\theta_1\cos^2\theta_2-\sin^2\theta_1\sin^2\theta_2-cos^2\gamma.\end{aligned}$$

Suppose now that $\cos\theta_1=0$. Thus the angle α_p is equal to $\frac{\pi}{2}$ or $-\frac{\pi}{2}$ and $\cos^2\alpha_p=0$. Since $\cos\gamma=\sin\theta_2$, the result is still true.

2. We have

$$|\sin\theta_i|\leq\epsilon \quad\text{and}\quad |\cos\theta_i|\geq\sqrt{1-\epsilon^2}.$$

Thus

$$\begin{aligned}&\cos^2\theta_1\cos^2\theta_2-\sin^2\theta_1\sin^2\theta_2-\cos^2\gamma+2\cos\gamma\sin\theta_1\sin\theta_2\\ \geq\;& (1-\epsilon^2)-\epsilon^4-\cos^2\gamma-2\epsilon^2\\ =\;& \sin^2\gamma-4\epsilon^2,\end{aligned}$$

and

$$\begin{aligned}&\sin^2(\theta_2-\theta_1)+2\sin\theta_1\sin\theta_2\cos(\theta_2-\theta_1)\\ &+\cos^2\theta_1\cos^2\theta_2-\sin^2\theta_1\sin^2\theta_2-cos^2\gamma\\ \leq\;& 4\epsilon^2+2\epsilon^2+1-cos^2\gamma\\ =\;& \sin^2\gamma+6\epsilon^2.\end{aligned}$$

Then

$$\cos^2(\alpha_p)\geq\frac{\sin^2\gamma-4\epsilon^2}{\sin^2\gamma+6\epsilon^2}.$$

Suppose now that $\frac{6\epsilon^2}{\sin^2\gamma} < 1$. Then, we get

$$\cos^2(\alpha_p) \geq \left(1 - \frac{4\epsilon^2}{\sin^2\gamma}\right)\left(1 - \frac{6\epsilon^2}{\sin^2\gamma}\right) \geq 1 - \frac{10\epsilon^2}{\sin^2\gamma},$$

and

$$\sin(\alpha_p) \leq \frac{\sqrt{10}\epsilon}{\sin\gamma}.$$

If $\frac{6\epsilon^2}{\sin^2\gamma} \geq 1$, the result is still true.

5.2 Comparing the length of a geodesic and its chord

We need the following result (compare with [10], which deals with the convex case) whose proof can be found in [13]:

Proposition 2 *Let S be a smooth surface of $\mathbb{E}^3$ without boundary, U_S an open subset of $\mathbb{E}^3$ where the map $\xi : U_S \to S$ is well defined. Then,*

1. *the map ξ is C^1 in U_S and satisfies for every $m \in U_S$:*

$$D\xi(m)(Z_m) = 0, \forall Z_m \text{ orthogonal to } T_{\xi(m)}S,$$

$$D\xi(m)(X_m) = (Id + \delta_m \epsilon_m A_{\xi(m)})^{-1}(X_m), \forall X_m \text{ parallel to } T_{\xi(m)}S,$$

where $\epsilon_m =< \nu_{\xi(m)}, \frac{\xi(m)-m}{||\xi(m)-m||} > \in \{-1, +1\}$.

2. *In particular, the matrix of $D\xi(m) : \mathbb{E}^3 \to T_{\xi(m)}S$ (in local orthonormal frames $(e^1_{\xi(m)}, e^2_{\xi(m)}, \nu^S_{\xi(m)})$ and $(e^1_{\xi(m)}, e^2_{\xi(m)})$) is given by:*

$$\begin{pmatrix} \frac{1}{1+\delta_m\epsilon_m\lambda^1_{\xi(m)}} & 0 & 0 \\ 0 & \frac{1}{1+\delta_m\epsilon_m\lambda^2_{\xi(m)}} & 0 \end{pmatrix}.$$

Corollary 5 *Let S be a smooth compact surface of $\mathbb{E}^3$, U_S a neighborhood of S where the map $\xi : U_S \to S$ is well defined, p and q two points on S such that $[p,q] \subset U_S$ and $\xi(]p,q[) \subset S \setminus \partial S$. Then the distance l_{pq} between p and q on S satisfies:*

$$l_{pq} \leq \frac{1}{1-\omega}\, pq,$$

where $\omega = \sup_{m \in]p,q[} \|\xi(m) - m\| \, \rho_{\xi(m)}$.

Proof of Lemma 5 Since $\xi([p,q])$ is a curve on S, its length is larger than the length l_{pq} of the geodesic whose ends are p and q. Therefore:

$$l_{pq} \leq l(\xi([p,q])) \leq \sup_{m\in]p,q[} |D\xi(m)|\ pq.$$

On the other hand, for every $m \in]p,q[$, we have

$$\|\xi(m)-m\|\ \rho_{\xi(m)} < 1.$$

Since $\xi(p)=p$ and $\xi(q)=q$, $\omega < 1$. Therefore Proposition 2 implies that:

$$|D\xi(m)| \leq \frac{1}{1-\|\xi(m)-m\|\ \rho_{\xi(m)}} \leq \frac{1}{1-\omega},$$

and corollary 5 is proved.

5.3 Comparing the normals at a vertex

We shall prove the following

Proposition 3 *Let S be a smooth surface, Δ a triangle closely inscribed in S and p a vertex of Δ. Then the angle $\alpha_p \in \left[0,\frac{\pi}{2}\right]$ between the normals of S and Δ at p satisfies:*

$$\sin(\alpha_p) \leq \frac{\sqrt{10}\pi_S(\Delta)}{2\ \sin\gamma_p\ (1-\omega_S(\Delta))},$$

where γ_p is the angle of Δ at p.

This proposition is a consequence of the following

Lemma 2 *Let S be smooth surface, Δ a triangle closely inscribed in S, p and q two vertices of Δ. Then the angle $\theta \in \left[0,\frac{\pi}{2}\right]$ between $\vec{pq}$ and the orthogonal projection of $\vec{pq}$ onto T_pS satisfies:*

$$\sin\theta \leq \frac{\rho_S\ l}{2},$$

where l is the distance on S between p and q.

Proof of Lemma 2 Let $\mathcal{C}$ denote a geodesic of S linking p and q. $\mathcal{C}$ is parametrized by arc length by:

$$\gamma : [0,l] \rightarrow S,$$

with $\gamma(0)=p$ and $\gamma(l)=q$. A simple calculation gives:

$$\begin{aligned} \gamma(l)-\gamma(0) &= l\gamma'(0)+\textstyle\int_0^l (l-t)\gamma''(t)dt, \\ \text{thus}\quad \gamma'(0) &= \tfrac{\vec{pq}}{l} - \tfrac{1}{l}\textstyle\int_0^l (l-t)\gamma''(t)dt. \end{aligned}$$

Let $\vec{v} = \frac{1}{l}\int_0^l (l-t)\gamma''(t)dt$ and $\vec{e} = \frac{\vec{pq}}{pq}$. We have:

$$\gamma'(0) = \frac{pq}{l}\vec{e} - \vec{v} \text{ with } \begin{cases} \|\vec{e}\| = 1, \\ \|\vec{v}\| \leq \frac{l\rho_{\xi_S}}{2}, \end{cases}$$

and

$$\sin\theta = \inf_{\substack{\vec{u}\in T_pS \\ \|\vec{u}\|=1}} \|\vec{u}\wedge\vec{e}\| \leq \frac{l\rho_S}{2}.$$

Proof of Proposition 3 Denote by l_1 the distance on S between p and p_1, by l_2 the distance on S between p and p_2. Since T is closely inscribed in S, thanks to corollary 5 we get:

$$l_1 \leq \frac{pp_1}{1-\omega_S(\Delta)} \leq \frac{\eta_\Delta}{1-\omega_\Delta} \quad \text{and} \quad l_2 \leq \frac{\eta_\Delta}{1-\omega_\Delta}.$$

Therefore, lemma 2 implies that

$$\sin\theta_1 \leq \frac{\rho_{\xi(\Delta)} l_1}{2} \leq \frac{\pi_S(\Delta)}{2(1-\omega_S(\Delta))} \quad \text{and} \quad \sin\theta_2 \leq \frac{\pi_S(\Delta)}{2(1-\omega_S(\Delta))}.$$

Then lemma 1 implies

$$\sin(\alpha_p) \leq \frac{\sqrt{10}}{\sin\gamma_p}\ \frac{\pi_S(\Delta)}{2(1-\omega_S(\Delta))} = \frac{\sqrt{10}\ \pi_S(\Delta)}{2\ \sin\gamma_p\ (1-\omega_S(\Delta))}.$$

5.4 Comparing the normals of a smooth surface

Proposition 4 *Let S be a smooth compact oriented surface of $\mathbb{E}^3$, Δ a triangle closely inscribed in S, p and s two points on Δ. Then the angle $\alpha_{sp} \in \left[0, \frac{\pi}{2}\right]$ between two normals $\nu^S_{\xi(p)}$ and $\nu^S_{\xi(s)}$ at $\xi(p)$ and $\xi(s)$ satisfies:*

$$\sin(\alpha_{sp}) \leq \frac{\pi_S(\Delta)}{1-\omega_\Delta},$$

where η_Δ is the height of Δ and ω_Δ is the relative curvature of Δ with respect to $\xi(\Delta)$.

This proposition is the consequence of the following lemma, which is a direct application of the mean value theorem:

Lemma 3 *Let S be a smooth compact oriented surface of $\mathbb{E}^3$, a and b two points of S. The angle $\alpha_{ab} \in \left[0, \frac{\pi}{2}\right]$ between two normals ν^S_a and ν^S_b at a and b satisfies:*

$$\sin(\alpha_{ab}) \ \leq \rho_S\ L_S(a,b),$$

where ρ_S the maximal curvature of S and $L_S(a,b)$ the distance on S between a and b.

The proof of Proposition 4 is an abvious consequence of Lemma 3 and Proposition 2.

Theorem 1 can be immediately deduced from Propositions 3 and 4, since

$$\sin(\alpha_s) \leq \sin(\alpha_p) + \sin(\alpha_{sp}).$$

6 Conclusion

In this paper, we give a very general result relating the normal vector field of a smooth surface and the normal vector field of a triangulated mesh. We hope that this kind of result can be usefull to get a general theoretical approach to the approximation of the geometrical properties of a smooth surface and a triangulated mesh close to it.

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