Some Results about the Quality of Planar Mesh Elements

Philippe P. Pébay

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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Philippe P. Pébay

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Abstract: In this report are presented some new results about both triangle and planar quadrangle quality measures. First, in the context of triangular mesh optimization based on control over edge lenghts, the relations between the edge ratio and the usual shape quality measure ζ are fully examined. Second, a modification of the Frobenius norm is proposed, in order to comply with the case where the reference element is no longer equilateral, but a right isosceles triangle. Third, generalizations of ζ and ι to planar quadrangle quality measure are proposed and justified, and their limitations, depending on the user expectations, are exhibited, justifying the use of a more complex quality measure. Finally, a natural use of the Frobenius norm for quadrangular elements is proposed.

Key-words: Triangular meshing, quadrangle meshing, mesh quality.

Unit de recherche INRIA Rocquencourt
Domaine de Voluceau, Rocquencourt, BP 105, 78153 Le Chesnay Cedex (France)
Tlphone : +33 1 39 63 55 11 ¯ Tlcopie : +33 1 39 63 53 30

Quelques résultats concernant la qualité d'éléments de maillages plans.

Résumé : Ce rapport présente de nouveaux résultats concernant la qualité d'un élément d'un maillage triangulaire ou quadrangulaire. Tout d'abord, dans le contexte de l'optimisation de maillage basée sur le contrôle de la longueur des arêtes de celui-ci, les relations liant le rapport d'arête et la qualité de triangle usuelle ζ sont examinées en détail. Ensuite, une modification de la norme de FROBENIUS est proposée, afin de l'adapter au cas où l'élément de référence souhaité n'est plus équilatéral, mais rectangle isocèle. De plus, une généralisation de ζ et ι aux quadrangles plans est proposée et justifiée et ses limitations, dépendant des applications, sont exhibées, ce qui justifie alors l'emploi d'une mesure de qualité plus complexe. Enfin, une utilisation naturelle de la norme de FROBENIUS pour les éléments quadrangulaires est proposée.

Mots-clés : Maillage simplicial, maillage quadrangulaire, qualité de maillage.

Introduction

In the context of finite element analysis, it is well known that evaluation of mesh quality is a major issue. In the case of simplicial meshes, a general *concensus* has emerged, based on both experimental and theoretical results, concerning which shape "good" elements should have. As a general solution, one can say without risk that an optimal simplicial element is an equilateral simplex, with respect to a specified size map. *E.g.*, in the particular case of finite element analysis of elliptic problems, [3] shows that accuracy of the approximate solution is directly related to the closeness of such optimal elements.

Several measures have been proposed [1, 6, 10], in order to computee the geometric quality of a given element. More recently, alternative quality measures have been suggested [2, 5, 7], in order to estimate the deviation of a given simplex from the reference optimal element. Unfortunately, too rare efforts have been made in order to compare and examine the respective merits of the various quality measures that have been proposed. For triangle quality measures, [8] and [9] propose an exhaustive study of this issue, while [10] provide some results for tetrahederal meshes. One approach in mesh optimization is to control mesh elements *via* their edge lengths. In other words, the determinant factor becomes the edge ratio which does not behave as usual quality measures. The first results that are presented hereafter are to express the most widely used quality measure, ζ as denoted in [8], in terms of τ, the edge ratio, and to provide quality bounds and *optimum* for a given edge ratio.

Concerning planar quadrangles, few general results are available. [11] proposes to estimate such elements by the means of several criterions, such as *aspect ratio*, *skewness* and *stretching factor*. Albeit natural for some geometries, such measures are not as well defined in a general context. Alternate measures have been proposed by [6], without providing comparisons between them. Therefore, this report examines the extension of some triangle quality measures to quadrangles, and in particular provides a full analyis of the extension of ζ. It is shown that this quality measures satisfies the desired extremal and, depending on the context, asymptotic properties. When, these asymptotic properties do not comply with the requirements of the application, the reasons why a quality measure introduced in [6] ashould be prefered are examined. Concerning κ_2, the nice properties of triangles, in particular the fact that, when they are non-degenerate, their edge vectors are linearly independent, do not extend to quadrangles. For this reason, quality measure based on matrix norms cannot be directly extended to such elements. However, since a quadrangle can be decomposed in two different pairs of triangles, one idea is to estimate the quality of these triangles. Hence, this report also examines the efficiency of this idea and proposes an adaptation of matrix norms, when the reference element is no longer an equilateral, but a right isosceles triangle. This allows to, finally, propose and examine another quality measure for planar quadrangle.

1 About triangle quality

1.1 Preliminaries

In this section, we consider a non-degenerate triangle $t = ABC$ with area $\mathcal{A}$, half-perimeter p, edges of lengths $a = BC$, $b = AC$ and $c = AB$, and we denote the angle at vertex A (resp. B, C) as α (resp. β, γ) and the radius of the inscribed (resp. circumscribed) circle of t as r (resp. R). In addition, the vertices A, B and C are defined respectively by the position vectors $\mathbf{v}_0$, $\mathbf{v}_1$ and $\mathbf{v}_2$ in an arbitrary orthonormal affine reference frame. For simplicity, we choose a frame of reference parallel to the plane of the triangle t, in which case the coordinates of the position vectors $\mathbf{v}_0$, $\mathbf{v}_1$ and $\mathbf{v}_2$ are respectively denoted as (x_0, y_0), (x_1, y_1) and (x_2, y_2).
The following standard norm-like notations are also used:

$$\begin{aligned} |t|_0 &= \min(a, b, c) \\ |t|_2 &= \sqrt{a^2 + b^2 + c^2} \\ |t|_\infty &= \max(a, b, c) \\ \theta_0 &= \min(\alpha, \beta, \gamma) \\ \theta_\infty &= \max(\alpha, \beta, \gamma). \end{aligned} \tag{1}$$

Some results from elementary geometry are assumed without proof (see for example [4] for proofs and details). In particular, the following well-known relations will be used:

$$2R = \frac{abc}{2\mathcal{A}} = \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}, \tag{2}$$

where $\mathcal{A}$ is given by

$$\mathcal{A} = rp, \tag{3}$$

as well as by HERON's formula:

$$\mathcal{A} = \sqrt{p(p-a)(p-b)(p-c)}. \tag{4}$$

Finally, it is recalled that the edge ratio is defined as:

$$\tau = \frac{|t|_\infty}{|t|_0}, \tag{5}$$

see [9] for a study of the behavior of τ, with respect to the extremal angles of t.

1.2 Shape dependency on edge ratio

An usual measure of the shape quality of a triangle is:

$$\zeta = \frac{p}{r} \tag{6}$$

whose normalization coefficient is $\frac{1}{3\sqrt{3}}$ and which can be expressed as a function $\tilde{h}$ of only on two variables, given by the ratios of two edges to the third one:

$$\frac{1}{\zeta^2} = \tilde{h}\left(\frac{b}{a}, \frac{c}{a}\right) = \frac{\left(\frac{b}{a} + \frac{c}{a} - 1\right)\left(1 + \frac{b}{a} - \frac{c}{a}\right)\left(1 + \frac{c}{a} - \frac{b}{a}\right)}{\left(1 + \frac{b}{a} + \frac{c}{a}\right)^3}, \tag{7}$$

see [9] for details. In particular, if x denotes the ratio of the middle edge length to the smallest,

$$\frac{1}{\zeta^2} = \tilde{h}(x, \tau) = \frac{(x + \tau - 1)(x + 1 - \tau)(\tau + 1 - x)}{(x + \tau + 1)^3}. \tag{8}$$

The following constraints must be satisfied for x: on the one hand, $1 \leqslant x \leqslant \tau$, on the other hand $1 + x > \tau$, according to the triangular inequation[1]. These conditions are summarized as:

- if $\tau < 2$, then $x \in [1, \tau]$;
- if $\tau \geqslant 2$, then $x \in]\tau - 1, \tau]$.

For the sake of clarity, the less strict but more concise condition $x \in \Omega_\tau = [\max(1, \tau - 1), \tau]$ is used, keeping in mind that the case $x = \tau - 1$ when $\tau \geqslant 2$ corresponds to a degenerate triangle. We therefore study the variations of

$$\begin{array}{cccc} f: & \Omega_\tau & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \dfrac{(x + \tau - 1)(x + 1 - \tau)(\tau + 1 - x)}{(x + \tau + 1)^3} \end{array} \tag{9}$$

[1]Strict, since the triangle is supposed to be non-degenerate

The first-order derivative of this $\mathcal{C}^\infty(\Omega_\tau)$ is:

$$f'(x) = \frac{4(1-\tau-\tau^2+\tau^3+2\tau x-(1+\tau)x^2)}{(1+x+\tau)^4} \tag{10}$$

$$= \frac{4}{(1+x+\tau)^4}P(x) \tag{11}$$

where P denotes the following polynomial in $\mathbb{R}[X]$:

$$P = -(1+\tau)X^2 + 2\tau X + \tau^3 - \tau^2 - \tau + 1 \tag{12}$$

whose discriminant is $\Delta_P(\tau) = 4(1-\tau^2+\tau^4)$. It is straigthforward to show that Δ_P is a strictly increasing bijection of τ from $[1,+\infty[$ onto $[4,+\infty[$. In particular, for any τ, $\Delta_P(\tau) > 0$ and therefore P has the two following distinct real roots:

$$\begin{aligned} x_1 &= \frac{\tau+\sqrt{\tau^4-\tau^2+1}}{\tau+1} \\ x_2 &= \frac{\tau-\sqrt{\tau^4-\tau^2+1}}{\tau+1} \end{aligned} \tag{13}$$

Now, given the bounds for x, the relative positions of 1, $\tau-1$, x_1, x_2 and τ must be precised. First, $\tau \geqslant 1$ implies $\tau^4-\tau^2+1 \geqslant 1$ thus $x_2 < 1$ and $x_1 \geqslant 1$, with equality if and only if $\tau = 1$. Hence, x_2 is always outside of the definition range of x. Concerning x_1 and τ,

$$\tau - x_1 = \frac{\tau^2-\sqrt{\tau^4-\tau^2+1}}{\tau+1} = \frac{\tau^2-1}{(\tau+1)(\tau^2+\sqrt{\Delta_P(\tau)})} \geqslant 0 \tag{14}$$

with equality if and only if $\tau = 1$. In other words, $1 \leqslant x_1 \leqslant \tau$, with equality if and only if $\tau = 1$. Finally, it has to be checked whether or not $x_1 \geqslant \tau - 1$. In fact, the condition $x_1 \geqslant \tau - 1$ is equivalent to the following one:

$$\sqrt{\tau^4-\tau^2+1} \geqslant \tau^2-\tau-1 \tag{15}$$

and it is straightforward to see that $X^2 - X - 1$ has two distinct real roots, $\frac{1-\sqrt{5}}{2}$, negative, and $\frac{1+\sqrt{5}}{2}$, which belongs to $]1,2[$. Hence, when $1 \leqslant \tau < \frac{1+\sqrt{5}}{2}$, $\tau^2-\tau-1 < 0$ and (15) is obviously satisfied, since a square root is necessarily positive. If $\tau \geqslant \frac{1+\sqrt{5}}{2}$, then both sides of (15) are positive, thus

$$\sqrt{\tau^4-\tau^2+1} \geqslant \tau^2-\tau-1 \Leftrightarrow \tau^4-\tau^2+1 \geqslant (\tau^2-\tau-1)^2 = \tau^4-2\tau^3-\tau^2+2\tau+1. \tag{16}$$

In other words, $x_1 \geqslant \tau - 1$ is equivalent to $\tau(\tau^2+1) \geqslant 0$ which is, obviously, always true since $\tau \geqslant 1$.

As a consequence, P, thus f', has the sign of $1+\tau$, *i.e.*, is positive, between x_2 and x_1. In addition, $x_2 \leqslant \max(1, \tau-1) \leqslant x_1 \leqslant \tau$ and f is defined over $[\max(1,\tau-1),\tau]$. Hence, f is strictly increasing (resp. decreasing) over $[\max(1,\tau-1), x_1]$ (resp. $[x_1,\tau]$); moreover, since it is a continuous function, f reaches its unique and global (resp. two and local) strict *maximum* (resp. *minima*) for $x = x_1$ (resp. for $x = \max(1,\tau-1)$ and $x = \tau$). More precisely,

$$f(1) \geqslant 0 \quad \Leftrightarrow \quad \max(1, \tau-1) = 1 \tag{17}$$

with equalities on both sides of the equivalency if and only if $\tau = 2$. Therefore, the two local *minima* of f can be summarized as follows:

$$f(\max(1,\tau-1)) = \max\left(\frac{\tau^2(2-\tau)}{(2+\tau)^3}, 0\right) \tag{18}$$

$$f(\tau) = \frac{2\tau-1}{(2\tau+1)^3}. \tag{19}$$

These results provide a triangle construction strategy[2]: given an edge ratio τ, the best possible quality in the sense of ζ is achieved when the length of the middle edge is $\frac{\tau+\sqrt{\tau^4-\tau^2+1}}{\tau+1}$ times the length of the shortest edge; this unique *minimum* of ζ corresponds to the unique *maximum* of f, which is:

$$f(x_1) = \frac{2\tau(-1+\tau+\tau^3-\tau^4+\sqrt{1-\tau^2+\tau^4}+\tau^2(2+\sqrt{1-\tau^2+\tau^4}))}{(1+3\tau+\tau^2+\sqrt{1-\tau^2+\tau^4})^3} \tag{20}$$

Concerning the worst possible ζ quality, it is attained either when the shortest and the middle edges have the same length (case $\tau \leqslant 2$) or when the triangle tends towards being flat (case $\tau > 2$). In the lattest case, obviously, $\zeta \to +\infty$. Finally, since

$$\frac{\tau+\sqrt{\tau^4-\tau^2+1}}{\tau+1} \underset{\tau\to+\infty}{\sim} \tau, \tag{21}$$

it can be added that as edge ratio increases, the optimal middle edge length, in the sense of ζ, tends towards the largest one. These results are displayed in Figure 1, for values of τ below 2. When τ exceeds 2, ζ is no longer bounded above, since t can get as flattened as desired and, therefore, f tends towards 0. This corresponds to the limiting case $x = \tau - 1$. See Figure 2 for such values of τ.

[2] in particular, in the context of mesh optimization based on edge lengths.

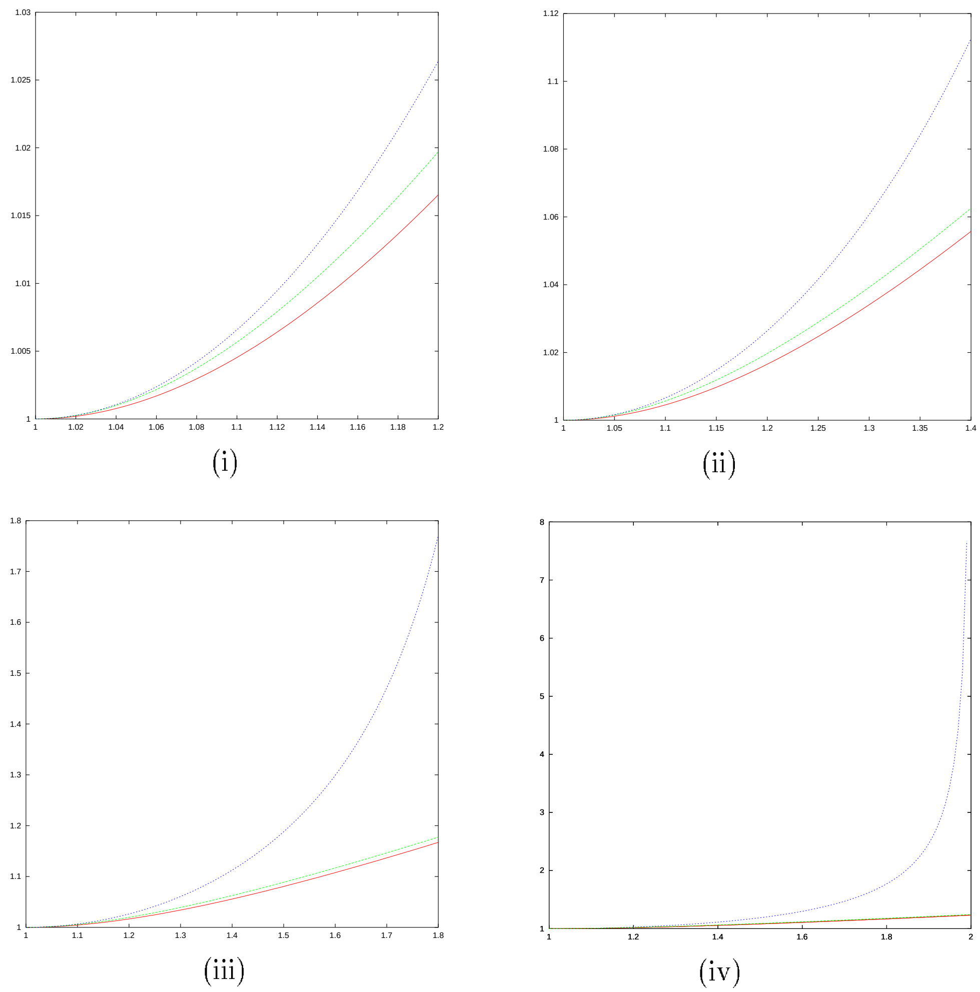

Figure 1: Best and worst possible $\frac{\zeta}{3\sqrt{3}}$ quality as a function of τ: (i): $\tau < 1.2$, (ii): $\tau < 1.4$, (iii): $\tau < 1.8$, (iv): $\tau < 2$. Intermediate curves represent $\frac{\zeta}{3\sqrt{3}}$ when $x = \tau$.

1.3 Adaptation of κ_2 to right isosceles triangles

An interesting approach to estimate triangle quality has been proposed by various authors (*cf.* [2, 5, 7]), based on the singular values of a matrix which expresses the affine transformation between the mesh element and a given reference element. More precisely, these works have focused on the case where the reference element is

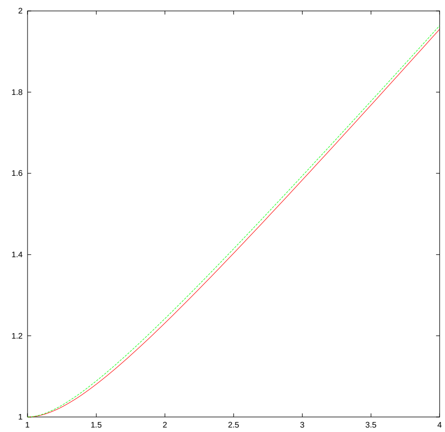

Figure 2: Best $\frac{\zeta}{3\sqrt{3}}$ quality as a function of τ (below), and values of $\frac{\zeta}{3\sqrt{3}}$ when $x = \tau$ (above), when $\tau < 4$.

a regular simplex, since this element is generally supposed to be the best possible for isotropic simplicial meshes. An in-depth examination of the variations of such quality measure, as well as a comparison with other quality measures has been made in [8] and [9].

To our knowledge, there is not yet any analysis of the case where the reference element is a right isosceles triangle; one reason is that such elements do not really correspond to the kind of triangles that are generally wished in the context of finite element analysis. However, in the goal of extending this measure to quadrangular meshes, such elements become naturally the desired ones.

As described in [2], we define an *edge-matrix* of t by:

$$T_0 = \begin{pmatrix} x_1 - x_0 & x_2 - x_0 \\ y_1 - y_0 & y_2 - y_0 \end{pmatrix} \tag{22}$$

and let W be the edge-matrix of a reference isosceles right triangle, for example

$$W = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{23}$$

meaning that W is, simply, the identity matrix of $\mathbb{R}^2$. Hence, $T_0 W^{-1} = T_0$ is the matrix that maps the reference element into T_0 and let define matrix-norms based on the singular values σ of T_0. Obviously, the symmetry which arises when W is an equilateral triangle vanishes with this new reference element. In particular, t is considered as being optimal only if the right angle is in A. This property allows a strict control, not only over the shape of t, but also on the vertex at which the right angle should be. The singular values are given by the positive square-roots of the eigenvalues of the positive definite matrix $T_0^T T_0$. Now,

$$T_0^T T_0 = \begin{pmatrix} u & w \\ w & v \end{pmatrix}, \tag{24}$$

where

$$\begin{aligned} u &= |\mathbf{v}_1 - \mathbf{v}_0|^2, & (25)\\ v &= |\mathbf{v}_2 - \mathbf{v}_0|^2, & (26)\\ w &= (\mathbf{v}_2 - \mathbf{v}_0) \cdot (\mathbf{v}_1 - \mathbf{v}_0). & (27) \end{aligned}$$

In the above expressions, $\cdot$ denotes the usual scalar product. The singular values σ of T_0 are thus obtained from the characteristic equation of $T_0^T T_0$ as

$$\begin{aligned} \sigma^4 &- \left(|\mathbf{v}_1 - \mathbf{v}_0|^2 + |\mathbf{v}_2 - \mathbf{v}_0|^2\right) \sigma^2 \\ &+ |\mathbf{v}_1 - \mathbf{v}_0|^2 |\mathbf{v}_2 - \mathbf{v}_0|^2 - \left((\mathbf{v}_2 - \mathbf{v}_0) \cdot (\mathbf{v}_1 - \mathbf{v}_0)\right)^2 = 0. \end{aligned} \tag{28}$$

Alternatively, this equation can be written as

$$\sigma^4 - 2(b^2 + c^2)\sigma^2 + 4\mathcal{A}^2 = 0. \tag{29}$$

Hence,

$$\sigma_1^2 + \sigma_2^2 = b^2 + c^2 \tag{30}$$

and $\sigma_1 \sigma_2 = 2\mathcal{A}$ where σ_1^2 and σ_2^2 ($0 < \sigma_1 \leq \sigma_2$) are the two roots of (29). A quality measure can be constructed from the condition number of any unitarily invariant norm of the matrix T_0 (*cf.* [2]). One such family is derived from the SCHATTEN p-norms defined by:

$$N_p(T_0) = (\sigma_1^p + \sigma_2^p)^{1/p} \quad , \quad p \in [1, +\infty[. \tag{31}$$

The case $p = 2$ is the FROBENIUS norm, the limiting case $p \to \infty$ is the spectral norm and the case $p = 1$ is the trace norm. A non-normalized quality measure is given by the condition number $\kappa_p(T_0)$ which is defined as

$$\kappa_p(T_0) = \left[(\sigma_1^p + \sigma_2^p) \left(\sigma_1^{-p} + \sigma_2^{-p} \right) \right]^{1/p}. \tag{32}$$

For the particular case $p = 2$, and using (2), it follows that

$$\kappa_2(T_0) = \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1 \sigma_2} = \frac{b^2 + c^2}{2\mathcal{A}} = \frac{b^2 + c^2}{bc \sin \alpha} \tag{33}$$

and, assuming that $\xi = \frac{b}{c}$, which is allowed since t is non degenerate and thus $c \neq 0$, it then follows that

$$\kappa_2(T_0) = \frac{\xi^2 + 1}{\xi \sin \alpha} = \left(x + \frac{1}{x} \right) \frac{1}{\sin \alpha} \tag{34}$$

Therefore, let consider the mapping

$$\begin{array}{rccc} g: & \mathbb{R}_+^* \times]\,0, \pi\,[& \longrightarrow & \mathbb{R}_+^* \\ & (\xi, \alpha) & \longmapsto & \left(x + \frac{1}{x} \right) \frac{1}{\sin \alpha} \end{array} \tag{35}$$

which is $\mathcal{C}^\infty$ over the open domain $\mathbb{R}_+^* \times]\,0, \pi\,[$; hence, any local *extremum* of g is attained at a stationary point. The first order derivatives are:

$$\frac{\partial g}{\partial \xi}(\xi, \alpha) = \left(1 - \frac{1}{\xi^2} \right) \frac{1}{\sin \alpha} \tag{36}$$

$$\frac{\partial g}{\partial \alpha}(\xi, \alpha) = -\left(\xi + \frac{1}{\xi} \right) \frac{\cos \alpha}{\sin^2 \alpha} \tag{37}$$

and, given the definition domain, the only stationary point is $(1, \frac{\pi}{2})$. In order to check whether this case corresponds, as expected, to a *minimum*, one has to make sure that the hessian matrix is positive definite. The second-order derivatives are given by:

$$\frac{\partial^2 g}{\partial \xi^2}(\xi, \alpha) = \frac{2}{\xi^3 \sin \alpha} \tag{38}$$

$$\frac{\partial^2 g}{\partial \alpha^2}(\xi, \alpha) = \left(\xi + \frac{1}{\xi} \right) \frac{\sin^2 \alpha + 2 \cos^2 \alpha}{\sin^3 \alpha} \tag{39}$$

$$\frac{\partial^2 g}{\partial \xi \partial \alpha}(\xi, \alpha) = -\left(1 - \frac{1}{\xi^2} \right) \frac{\cos \alpha}{\sin^2 \alpha} \tag{40}$$

which gives, when $(\xi, \alpha) = (1, \frac{\pi}{2})$,

$$\frac{\partial^2 g}{\partial \xi^2} \left(1, \frac{\pi}{2} \right) = 2 \tag{41}$$

$$\frac{\partial^2 g}{\partial \alpha^2} \left(1, \frac{\pi}{2} \right) = 2 \tag{42}$$

$$\frac{\partial^2 g}{\partial \xi \partial \alpha} \left(1, \frac{\pi}{2} \right) = 0. \tag{43}$$

Thus, the hessian determinant is equal to $4 > 0$ and the first diagonal entry is $2 > 0$. Hence, the hessian matrix is locally positive definite around the critical point, which therefore corresponds to a strict local *minimum* of g. Since g is $\mathcal{C}^\infty$ over its open and connected definition domain, the unicity of the critical point ensures that this *minimum* is, also, absolute. In other words, $\kappa_2(T_0)$ is[3] minimal only for right ($\alpha = \frac{\pi}{2}$) isosceles ($\xi = 1 \Leftrightarrow b = c$) triangles. In this case, the value of $\kappa_2(T_0)$ is, obviously 2, which provides the normalization coefficient.

Remark 1.1 *It is interesting to examine the case of some usual configurations:*

- *if t is equilateral, then $\kappa_2(T_0) = \frac{4}{\sqrt{3}}$;*
- *if t is right in A, then $\kappa_2(T_0) = \frac{b^2+c^2}{bc}$, which equals 2 if and only if $b = c$, and tends to $+\infty$ as either $\frac{b}{c}$ or $\frac{c}{b}$ does;*
- *if t is right in B, then $\kappa_2(T_0) = \frac{a^2+2c^2}{ac}$, which at best equals $2\sqrt{2}$, when $a = c\sqrt{2}$, and tends to $+\infty$ as either $\frac{a}{c}$ or $\frac{c}{a}$ does. In particular, if t is also isosceles, then $\kappa_2(T_0) = 3$.*

2 Deriving quadrangle quality from triangle quality

2.1 Preliminaries

In this section, $q = ABCD$ is supposed to be a non-degenerate convex planar quadrangle, with area $\mathcal{A}$, half-perimeter p, edges of lengths $a = AB$, $b = BC$, $c = CD$ and $d = DA$ and denote the angle at vertex A (resp. B, C, D) as α (resp. β, γ, δ) and θ the arithmetic mean of α and γ. In addition, the vertices A, B, C and D are defined respectively by the position vectors $\mathbf{v}_0$, $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ in an arbitrary orthonormal affine reference frame. For simplicity, we choose a frame of reference parallel to the plane of quadrangle q, in which case the coordinates of the position vectors $\mathbf{v}_0$, $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are respectively denoted as (x_0, y_0), (x_1, y_1),(x_2, y_2) and (x_3, y_3). In the particular case of a rectangle, we denote as λ the *stretching factor*, *i.e.* the ratio of the shortest edge to the longest.

[3] with the specified reference element.

We shall also use the following standard norm-like notations:

$$\begin{aligned} |q|_0 &= \min(a,b,c,d) \\ |q|_2 &= \sqrt{a^2+b^2+c^2+d^2} \\ |q|_\infty &= \max(a,b,c,d) \\ \theta_0 &= \min(\alpha,\beta,\gamma,\delta) \\ \theta_\infty &= \max(\alpha,\beta,\gamma,\delta). \end{aligned} \tag{44}$$

Most of the useful metric equalities of triangles do not extend to quadrangles, and this is the first obstacle to the generalization of results such as those presented in [8, 9] in the case of triangles. Nevertheless, HERON's formula can be generalized for quadrangles:

$$\mathcal{A} = \sqrt{(p-a)(p-b)(p-c)(p-d) - abcd\cos^2\theta}. \tag{45}$$

This gives an opportunity to discuss the choice of α and γ in the definition of θ. It is well known that the sum of the four angles of a convex quadrangle is equal to 2π; in other words, $\alpha+\gamma$ and $\beta+\delta$ are supplementary thus have opposite cosines, hence equal squared cosines. Therefore, whatever pair of opposite angles is picked in order to define θ, formula (45) returns the same result, *i.e.* it is symmetrical.

2.2 Quadrangles and triangles

A natural approach to mesure the quality of any given non-denerate convex quadrangle consists in seeing it as a pair of two non-degenerate triangles sharing one common edge, which is also a diagonal of the quadrangle. Hence, an apparently good idea would be to examine the qualities of these two triangles, but which quality ? Generally speaking (see [8] or [9] for a notable exception), the quality of a triangle is considered to be optimal[4] only for equilateral triangles. Unfortunately, the following example shows that using such triangle quality measurements for quadrangles is not straightforward.

Figure 3 illustrates the case where q is a rhombus, such as one of its diagonals has the same length as its edges. Hence, if q can be decomposed either in two equilateral triangles or in two obtuse isosceles triangles. In the general sense of triangle quality, the former case is considered as optimal, while the latter is far from this. In other words, the choice of the particular partition of q in two triangles has an effect over the resulting quadrangle quality measurement; which one shall be chosen ?

[4] more precisely, reaches its strict and unique *minimunm*, 1.

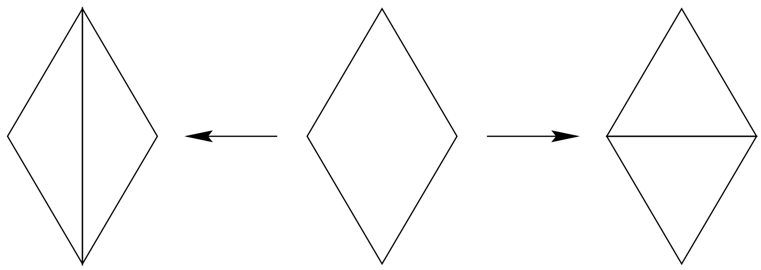

Figure 3: The two triangular covering-ups of the same rhombus.

The only certainty at this point is that either both triangular decompositions of q must be taken into account, or another approach of quadrangle quality, independent from the underlying triangles, must be used.

2.3 The quadrangle space

Prior to discussing about the space of quadrilaterals, it is useful to recall some results concerning triangles. First, equality up to homotopy is an equivalency, in the strict mathematical sense: reflexive, symmetrical and transitive. In addition, quality measures in the general sense (*cf.* [8, 9]) are invariant through homotopy. In other words, for quality mesure purposes, it is sufficient to consider triangles which have one side with unitary length. In this context, there is a canonic bijection between $\mathbb{R}^2$ and the set of such triangles[5].

Therefore, it is straightforward to define a natural bijection between the set of quadrangles, up to homotopy, and $\mathbb{R}^4$: any quadrangle $ABCD$ is homotopic to a single quadrangle $A'B'C'D'$ such as $A'C' = 1$, thus to a couple of triangles which have one side with unit length, hence to $\mathbb{R}^2 \times \mathbb{R}^2$, which is canonically isomorphic to $\mathbb{R}^4$. Hence, it is natural to see the set of quadrangles, up to homotopy, as isomorphic to $\mathbb{R}^4$.

Meanwhile, the set of convex planar quadrangles can thus be seen as a subset of $\mathbb{R}^4$. In particular, if x, y and z denote the ratio between three edge lengths to the fourth one, these variables must satisfy some bound conditions, so that they

[5] this allows in particular to transport the algebraic structure of $\mathbb{R}^2$ to this set of triangles: *e.g.* it is possible to add such triangles.

effectively correspond to an actual[6] non-degenerate quadrangle q. First, as q is assumed to be non-degenerate, it is clear that any edge length is strictly smaller than the sum of the three other ones; hence,

$$\frac{p}{a} = \frac{a+b+c+d}{2a} = \frac{1}{2} + \frac{b+c+d}{2a} < \frac{1}{2} + \frac{a}{2a} = 1 \tag{46}$$

whence $x < 1$ and, for the same reasons, $y < 1$ and $z < 1$. In addition,

$$x + y + z = \frac{a+b+c}{p} = \frac{2(a+b+c)}{a+b+c+d} < 2. \tag{47}$$

2.4 Edge ratio

It seems natural to extend τ, the edge ratio which has been defined for triangles, to quadrangles. In this case, we have:

$$\tau = \frac{|q|_\infty}{|q|_0} \tag{48}$$

2.4.1 *Extremum*

By definition, $\tau \geqslant 1$, with equality if and only if $|q|_\infty = |q|_0$. In other words, τ has a unique *minimum*, 1, which is strict, and reached for, and only for, rhombii. In particular, even a very flattened rhombus, "close" to a degenerate element, is considered as optimal by τ.

2.4.2 Asymptotic behavior

As the quadrangle-space is four-dimensional, it is not as natural to examine the asymptotic behavior, as in the case of triangles. However, it is possible to examine some usual configurations.

First, if q is rectangle, then the edge ratio $\tau_\perp$ becomes equal to λ, the stretching-coefficient, and it is obvious that, as the rectangle stretches, this ratio grows unbounded. In other words, τ does not consider right angles as being a sufficient condition for a good element.

To the contrary, for any element whose edges have close lengths, τ is close to 1. Flattening this element does not change the value of τ, even when one angle tends to 0.

Given both extremal and asymptotic properties of τ, it appears that this extension of a triangle quality measure is, at best, very specific, at worst, unsuitable.

[6] up to homotopy.

2.5 Edge to inradius

Among triangle qualities that extend naturally to quadraliterals is the comparizon of edge lengths with inradius. Of course, any convex quadrangle does not have, in general, an inscribed circle[7] and, hence, it might seem paradoxical to intend to extend such edge to inradius comparizons to quadrangles. However, in the case of a non-degenerate triangle t, it follows from (3) that:

$$\frac{p_t}{r_t} = \frac{p_t^2}{\mathcal{A}_t} \tag{49}$$

where $\mathcal{A}_t$, r_t and p_t respectively denote the area, inradius and halfperimeter of t. Therefore, this quality measurement can be extended directly to q:

$$\zeta = \frac{p^2}{\mathcal{A}}. \tag{50}$$

Similarily, the *aspect-ratio* can be extended to q:

$$\iota = \frac{p|q|_\infty}{\mathcal{A}}, \tag{51}$$

and these two measures are related *via* the following inequality:

$$\zeta = \frac{p(a+b+c+d)}{2\mathcal{A}} \leqslant \frac{4p|q|_\infty}{2\mathcal{A}} = 2\,\iota \tag{52}$$

with equality if an only if $p = 2|q|_\infty$, *i.e.* if and only if q is a rhombus.

2.5.1 *Extremum*

Combining (45) and (3) gives:

$$\zeta = \sqrt{\frac{p^4}{(p-a)(p-b)(p-c)(p-d) - abcd\cos^2\theta}} \tag{53}$$

and, as for triangles, it is much more convenient[8] to try to minimize $h = \frac{1}{\zeta^2}$ rather than to maximize ζ. h is, of course, a function of only the five variables a, b, c, p

[7] In fact, such an incircle exists if an only if the sums of opposite edge lenghts are equal to each other.

[8] and, obviously, equivalent.

and θ, and can be expressed as follows:

$$h(a,b,c,p,\theta) = \frac{(p-a)(p-b)(p-c)(a+b+c-p) - abc(2p-a-b-c)\cos^2\theta}{p^4} \tag{54}$$

and it is clear that, for any $a \in \mathbb{R}_+^*$, $h(\frac{a}{p}, \frac{b}{p}, \frac{c}{p}, 1, \theta) = h(a,b,c,p,\theta)$. This means that h only depends on four variables, given by the ratios of three edges lengths to the half-perimeter, plus the "torsion" angle θ. In other words, h is invariant through homotopy, as expected, since ζ is non-dimensional. We therefore study the variations of

$$\begin{array}{rccc} \tilde{h}: & \mathcal{Q} & \longrightarrow & \mathbb{R} \\ & (x,y,z,\theta) & \longmapsto & (1-x)(1-y)(1-z)(x+y+z-1)+xyz(x+y+z-2)\cos^2\theta \end{array} \tag{55}$$

whose first-order derivatives are:

$$\frac{\partial \tilde{h}}{\partial x}(x,y,z,\theta) = (2x+y+z-2)((1-y)(1-z)+yz\cos^2\theta) \tag{56}$$

$$\frac{\partial \tilde{h}}{\partial y}(x,y,z,\theta) = (x+2y+z-2)((1-x)(1-z)+xz\cos^2\theta) \tag{57}$$

$$\frac{\partial \tilde{h}}{\partial z}(x,y,z,\theta) = (x+y+2z-2)((1-x)(1-y)+xy\cos^2\theta) \tag{58}$$

$$\frac{\partial \tilde{h}}{\partial \theta}(x,y,z,\theta) = xyz(x+y+z-2)\sin 2t \tag{59}$$

Since, by assumption, $x+y+z<2$ and none of x, y nor z are null, the stationary-point condition implies that $2\theta \in \pi\mathbb{Z}$ thus, since $2\theta \in]\,0,\pi[$, necessarily $\theta = \frac{\pi}{2}$. Hence, $(x,y,z,\frac{\pi}{2})$ is a stationary-point if and only if

$$\left\{ \begin{array}{rcl} (2x+y+z-2)(1-y)(1-z) & = & 0 \\ (x+2y+z-2)(1-x)(1-z) & = & 0 \\ (x+y+2z-2)(1-x)(1-y) & = & 0 \end{array} \right. \tag{60}$$

which is equivalent, since neither x nor y nor z is equal to 1,

$$\left\{ \begin{array}{rcl} 2x+y+z & = & 2 \\ x+2y+z & = & 2 \\ x+y+2z & = & 2 \end{array} \right. \tag{61}$$

whose only solution is, clearly, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Therefore, $\tilde{h}$ has a unique stationary point, when the quadrangle is a square. Now,

$$\frac{\partial^2 \tilde{h}}{\partial x^2}(x, y, z, \theta) = 2(y + z - yz - 1 + yz\cos^2\theta) \tag{62}$$

$$\frac{\partial^2 \tilde{h}}{\partial y^2}(x, y, z, \theta) = 2(x + z - xz - 1 + xz\cos^2\theta) \tag{63}$$

$$\frac{\partial^2 \tilde{h}}{\partial z^2}(x, y, z, \theta) = 2(x + y - xy - 1 + yz\cos^2\theta) \tag{64}$$

$$\frac{\partial^2 \tilde{h}}{\partial \theta^2}(x, y, z, \theta) = 2xyz(2 - x - y - z)\cos 2t \tag{65}$$

$$\frac{\partial^2 \tilde{h}}{\partial x \partial y}(x, y, z, \theta) = (1 - z)(2x + 2y + z - 3) + z(2x + 2y + z - 2)\cos^2\theta \tag{66}$$

$$\frac{\partial^2 \tilde{h}}{\partial x \partial z}(x, y, z, \theta) = (1 - y)(2x + y + 2z - 3) + y(2x + y + 2z - 2)\cos^2\theta \tag{67}$$

$$\frac{\partial^2 \tilde{h}}{\partial y \partial z}(x, y, z, \theta) = (1 - x)(x + 2y + 2z - 3) + x(x + 2y + 2z - 2)\cos^2\theta \tag{68}$$

$$\frac{\partial^2 \tilde{h}}{\partial x \partial \theta}(x, y, z, \theta) = yz(2 - 2x - y - z)\sin 2t \tag{69}$$

$$\frac{\partial^2 \tilde{h}}{\partial y \partial \theta}(x, y, z, \theta) = xz(2 - x - 2y - z)\sin 2t \tag{70}$$

$$\frac{\partial^2 \tilde{h}}{\partial z \partial \theta}(x, y, z, \theta) = xy(2 - x - y - 2z)\sin 2t \tag{71}$$

Hence, the hessian matrix of $\tilde{h}$ in $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{\pi}{2})$ is given by:

$$H_{\tilde{h}_\square} = -\frac{1}{8}\begin{pmatrix} 4 & 2 & 2 & 0 \\ 2 & 4 & 2 & 0 \\ 2 & 2 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{72}$$

and is clearly negative-definite; more precisely, the characteristic polynomial of matrix

$$\begin{pmatrix} 4 & 2 & 2 & 0 \\ 2 & 4 & 2 & 0 \\ 2 & 2 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{73}$$

is $(X-1)(X-2)^2(X-8)$, thus the eigenvalues of $H_{\tilde{h}_\square}$ are $-\frac{1}{8}$, $-\frac{1}{4}$ (double) and -1. Hence, the stationary point is a *maximum*, meaning that ζ reaches its only *minimum* for squares; in addition, this *minimum*, denoted as $\zeta_\square$, is strict.

From these extremal property of ζ can be deduced another one for ι: firstly, since (52) is an equality only for rhombii, it is the case for squares, thus $\zeta_\square = 2\iota_\square$. Now, combining the minimization of ζ with (52), leads to:

$$\iota \geqslant \frac{\zeta}{2} \geqslant \frac{\zeta_\square}{2} = \iota_\square, \tag{74}$$

showing that, as ζ, ι is minimal for squares.

2.5.2 Asymptotic behavior

In the case where q is a rectangle, ζ becomes:

$$\zeta = \frac{(|q|_0 + |q|_\infty)^2}{4|q|_0|q|_\infty} = \frac{1}{4}\left(\lambda + \frac{1}{\lambda} + 2\right) \tag{75}$$

and is a function of λ, denoted as $\zeta_\perp$. It is clear that $\zeta_\perp$ is strictly increasing over $[1, +\infty[$ (the definition domain of λ) and that $\lim_{\lambda\to+\infty} \zeta_\perp = +\infty$. Moreover,

$$\zeta_\perp(\lambda) \underset{\tau\to+\infty}{\sim} \lambda = \tau_\perp(\lambda) \tag{76}$$

meaning that ζ behaves asymptotically as τ for rectangles.
If q is a rhombus, then its area can be expressed as:

$$\mathcal{A} = a^2 \sin\alpha \tag{77}$$

thus ζ becomes:

$$\zeta_\diamond(\alpha) = \frac{a^2}{4a^2\sin\alpha} = \frac{1}{4\sin\alpha} \tag{78}$$

which diverges to $+\infty$ at the bounds of $]0, \pi[$, the domain definition of α.

Another interesting asymptotic case occurs when two consecutive edges of q tend towards being aligned, in other words when q gets close to being a triangle t, as shown Figure 4. In this case, called *triangular degeneracy*, it is obvious that q and t share both perimeter and area. Hence, $\zeta(q)$, in the sense of quadrangle quality, tends towards $\zeta(t)$, in the sense of triangle quality.

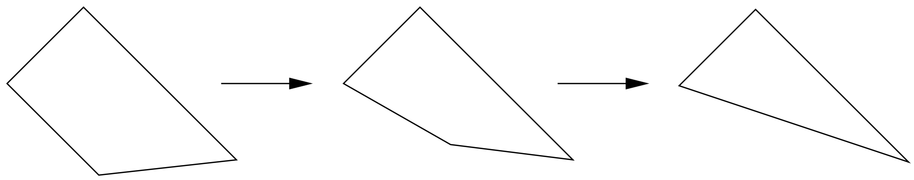

Figure 4: Triangle degeneracy.

From both extremal properties and asymptotic behavior that have been demonstrated, one can conclude that ζ can be extended as a generic quality measure, suitable for both triangles and quadrangles; in addition, a continuous transition between these two kinds of elements is ensured. However, this particular property might be, in some respects, a major drawback since, for some applications, one might wish to avoid triangular degeneracy of quadrangles. In this case, the quality measure should diverge to infinity, rather than tend towards the quality of the limiting triangular element.

2.6 Avoiding triangle degeneracy

[6] propose the following quadrangle quality measure:

$$\mathcal{Q} = \frac{|q|_2\, h_{\max}}{\min_i \mathcal{A}_i} \tag{79}$$

where $\mathcal{A}_i$ denotes the area of the triangle whose edges are those of q adjacent to vertex i and the diagonal opposed to this vertex, and $h_{\max}$ is the *maximum* among $|q|_\infty$ and the two diagonal lengths of q. Obviously, $|q|_\infty \leqslant h_{\max}$.

First, it is useful to remark that computing each of the $\mathcal{A}_i$ allows to detect, on the fly, whether or not q is convex, non-convex, self-intersected, degenerate (*cf.* [6] for details). Computationally speaking, this is of the greatest interest, since both topological consistency checking and geometrical quality measurement can be done at the same time.

In addition, according to CAUCHY-SCHWARZ inequality,

$$(\forall (u_1, ..., u_n) \in \mathbb{R}_+^n) \quad \sum_{k=1}^{k=n} u_k \leq \sqrt{n \sum_{k=1}^{k=n} u_k^2}, \tag{80}$$

thus $p \leqslant |q|_2$, with equality if and only if q is a rhombus. Last,

$$\min_i \mathcal{A}_i \leqslant \frac{\mathcal{A}}{2} \tag{81}$$

with equality if and only if q is a square. Hence,

$$\frac{\zeta}{2} \leqslant \iota = \frac{p|q|_\infty}{\mathcal{A}} \leqslant \frac{|q|_2\, h_{\max}}{2\min_i \mathcal{A}_i} = \frac{\mathcal{Q}}{2}. \tag{82}$$

and, in particular, $\zeta \leqslant \mathcal{Q}$.

An example of how $\mathcal{Q}$ distinguishes elements that neither ζ nor ι would is provided by Figure 5. More precisely, in this case, ABC is a unitary equilateral triangle, while ACD is iscosceles in D, with $AD = CD = x$. Obviously, x must belong to $]\,\frac{1}{2}, +\infty\,[$

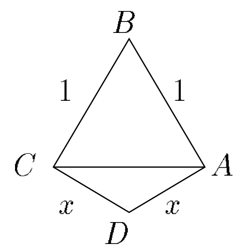

Figure 5: Kite element.

but, since the aim is here to examine the case of a triangle degeneracy, the interval is limited to $]\,\frac{1}{2}, 1\,[$. It is straightforward to determine ζ, ι and $\mathcal{Q}$ as function of x for $q = ABCD$. In fact,

$$\min_i \mathcal{A}_i(x) = \frac{1}{4}\sqrt{x^2 - \frac{1}{4}} \tag{83}$$

$$p(x) = 1 + x \tag{84}$$

$$|q|_2(x) = \sqrt{2(1+x^2)} \tag{85}$$

$$|q|_\infty(x) = 1 \tag{86}$$

$$\mathcal{A}(x) = \frac{\sqrt{3}}{4} + \frac{1}{4}\sqrt{x^2 - \frac{1}{4}}. \tag{87}$$

Hence, when $x \to \frac{1}{2}$, both ζ and ι tend towards finite values (respectively, $3\sqrt{3}$ and $2\sqrt{3}$), while $\mathcal{Q} \to +\infty$.

As a matter of fact, $\mathcal{Q}$, as introduced by [6], has all the desired properties for a quadrangle quality measure: extremal and asymptotic. In addition to ζ and ι, it handles triangle degneracy; in particular, this means that $\mathcal{Q}$ is not continuous with any underlying triangle quality measure.

2.7 Adaptation of κ_2 to quadrangles

It has been shown in Subsection 1.3 how κ_2, as studied in [8] for equilateral triangle, can be adapted when the reference element is a right isosceles triangle with, in addition, a specific control over which edge is the hypotenuse. The main motivation of this modification was to allow, in a second step, to be able to adapt κ_2 to quadrangles.

Considering the generic planar quadrangle q, four different triangles might be evaluated by the means of κ_2: ABD, BCA, CDB and DCA, with respective edge-matrices T_0, T_1, T_2 and T_3. Now, it follows that:

$$T_0 + T_1 + T_2 + T_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{88}$$

i.e.,

$$T_3 = -T_0 - T_1 - T_2. \tag{89}$$

In other words, it is unnecessary to evaluate the four edge-matrices at each vertex of the quadrangle, since any of them is a linear combination of the three other ones. This simply means that, given three vertex angles and edge ratios, the quadrangle is fully determined, up to homotopy.

Now, considering κ_2, as it has been previously modified for right isosceles triangles, the qualities of each of these four triangles are, respectively,

$$\kappa_2(T_0) = \frac{a^2+d^2}{ad\sin\alpha}, \quad \kappa_2(T_1) = \frac{a^2+b^2}{ab\sin\beta}, \tag{90}$$

$$\kappa_2(T_2) = \frac{b^2+c^2}{bc\sin\gamma}, \quad \kappa_2(T_3) = \frac{c^2+d^2}{cd\sin\delta}. \tag{91}$$

According to (88), $\kappa_2(T_3)$ can be directly derived from $\kappa_2(T_0)$, $\kappa_2(T_1)$ and $\kappa_2(T_2)$. However, this dependency is no longer linear, since singular values and, hence, polynomial equations, are involved. Therefore, although it might appear as more elegant to design a quadrangle quality measure, depending only on three of the underlying triangle qualities, it is certainly much more costly. For this reason, a more realistic and certainly more efficient idea is to take into account the four qualities. In this context, a natural approach is to consider their arithmetic mean, and to define the FROBENIUS norm of the quadrangle as follows:

$$\kappa_2(q) = \frac{\kappa_2(T_0) + \kappa_2(T_1) + \kappa_2(T_2) + \kappa_2(T_3)}{4}. \tag{92}$$

Remark 2.1 *The choice of the arithmetic mean is an* a priori *without any further justification. One might of course prefer to use the euclidean norm instead. Nevertheless, it would not be a good idea to use a the max norm, as illustrated by Figure 6:* $\max_i \kappa(T_i)$ *cannot detect the fact than one quadrangle is "less" distorted than the other.*

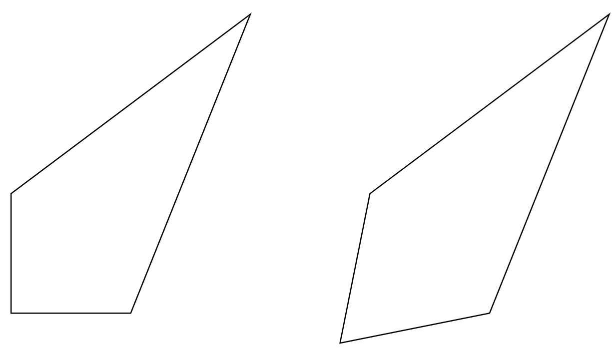

Figure 6: Both quadrangles share the same $\max_i \kappa(T_i)$.

The previously demonstrated extremal properties of κ_2 for right isosceles triangles show that

$$(\forall i \in \{0,1,2,3\}) \quad \kappa(T_i) \geqslant 2 \tag{93}$$

with equality if and only if T_i is a right isosceles triangle. Hence, $\kappa_2(q) \geqslant 2$, with equality if and only if each T_i is a right isosceles triangle, *i.e.* q is a square. Hence, $\kappa_2(q)$ complies with the desired *optimum* property for planar quadrangle quality measures. In addition, in the case where q is a rhombus,

$$(\forall i \in \{0,1,2,3\}) \quad \kappa(T_i) = \frac{a^2+b^2}{\mathcal{A}} \tag{94}$$

from which it follows that

$$\kappa_2(q) = \frac{|t|_2^2}{2\mathcal{A}} \tag{95}$$

which is very close of the result that has been exhibited in [8] for κ_2, when the reference element is an equilateral triangle: in this case, the following identity arises:

$$\kappa_2 = \frac{|t|_2^2}{2\sqrt{3}\mathcal{A}_t}. \tag{96}$$

It is very satisfactory to obtain the same result, up to a constant factor, for both triangles[9] and quadrangles.

Conclusion

The direct relations between edge-based quality and the shape of a triangle which are provided in the first part of this report might be useful in the context of triangular mesh adaptation or optimization, based on edge refinement. In particular, the fact that, given an edge ratio τ, the best possible quality in the sense of ζ is achieved when the length of the middle edge is $\frac{\tau+\sqrt{\tau^4-\tau^2+1}}{\tau+1}$ times the length of the shortest edge provides a triangle construction strategy.

Concerning quadrangle quality measures, the results demonstrated in this report can be summarized as in table 1. Column "~ 1" indicates which particular element optimizes the normalized quality; column "rectangle λ" provides the asymptotic behavior of the normalized quality when the element is a rectangle with stretching factor λ; column "triangle degeneracy" indicates whether or not the quality measure tends towards infinity in the case of a triangle degeneracy of the quadrangle.

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`http://www.mscsoftware.com/`

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[9] in the usual case, when the reference element is equilateral.

	~ 1	rectangle λ	triangle degeneracy
τ	rhombus	$\sim \lambda$	no
$\frac{1}{4}\zeta$	square	$\sim \frac{\lambda}{4}$	no
$\frac{1}{2}\iota$	square	$\sim \frac{\lambda}{2}$	no
$\frac{1}{4\sqrt{2}}\mathcal{Q}$	square	$\sim \frac{\lambda}{2}$	yes
$\frac{1}{2}\kappa_2$	square	$\sim \frac{\lambda}{2}$	no

Table 1: Summary of quadrangle quality measures: stat. refers to the existence of a stationary value when the quality is optimal. The kind of "good" triangle is indicated as either right or equilateral.

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Unit de recherche INRIA Rocquencourt
Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)

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