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Conforming Orthogonal Meshes

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Abstract: In this paper, we show how to compute an orthogonal mesh that strictly conforms to a given simplicial complex $\mathcal{SC}$ in $\mathbb{R}^d$. This result will be applied to improve the treatment of wells in oil reservoir simulation. The trend for this problem is to use hybrid meshes obtained by inserting radial well meshes in a structured reservoir grid. Our work can be used to generate a transition mesh connecting the reservoir mesh to well meshes and preserving strict conformity.

Key-words: Meshes, orthogonal meshes, conforming meshes power diagram, regular triangulations, hybrid meshes, transition meshes, oil reservoir simulation

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Maillages orthogonaux conformes

Résumé : Ce papier montre comment construire un maillage orthogonal strictement conforme à un complexe simplicial donné. Cette procédure peut s'appliquer au traitement des puits dans les simulations de réservoirs pétroliers. La tendance actuelle pour ce problème consiste à utiliser un maillage hybride obtenu en insérant un maillage radial représentant le puits dans un maillage héxaèdrique structuré décrivant le réservoir. Notre méthode peut être utilisée pour générer un maillage de transition qui raccorde le maillage du puits au maillage du réservoir en préservant la conformité du maillage final.

Mots-clés : Maillages, maillages orthogonaux, maillages conformes, diagrammes de puissances, triangulation régulières, maillages hybrides, maillages de transition, simulation de réservoirs pétroliers

1 Introduction

This work is motivated by a meshing problem arising in the field of oil reservoir simulation. Efficient tools can accurately determine the complex geological architecture of a reservoir. A spatial discretization of this structure is then required in order to simulate and forecast fluid flow processes through numerical simulations. Typically the reservoir field is described through a corner point geometry (CPG) grid, which is basically a structured hexahedral grid slightly distorded to fit the mesh to geological features. Around wells, flows are known to have a radial symetry. Thus, in those areas a structured radial mesh is suitable to increase the accuracy of flux computations. Finally, to get a global mesh, these different meshes have to be connected through unstructured transition meshes, leading to hybrid meshes. This paper deals with the generation of transition meshes.

The numerical schemes used during the simulation processes impose conditions on the transition meshes. A transition mesh is required to be a polyhedral mesh with convex cells and to preserve strict conformity, which means that any two adjacent cells share a unique facet. Moreover, finite volume computations require the mesh to be an orthogonal mesh, which means that there is an embedding of the dual complex such that each face is orthogonal to its dual face.

Treatment of wells is a well known problem in oil reservoir simulation [7, 3]. Different methods have been studied to introduce a well in a reservoir mesh. The first method consists in locally refining a coarse cartesian grid into a finer local grid around the well [12]. This widely used method holds for structured grids, leads to non conforming meshes and increases broadly the number of cells. Another way to integrate well meshes into a reservoir mesh is to generate a global unstructured mesh [10]. As we would like to keep a kind of structure as much a possible, this solution is not held. As a compromise, hybrid meshes have been studied and two main approaches have been proposed. The first one refines locally one or more cells of the cartesian grid with an arbitrarily fine curvilinear orthogonal mesh that defines a radial mesh around the well [11]. This method suffers from its lack of flexibility. The cartesian grid does not correctly represent the geology of the reservoir and the locations where to insert the wells are restricted. The second approach can deal with faults, as well as with vertical and horizontal wells. The idea is to connect a well mesh and the reservoir mesh, or two blocks of the reservoir mesh along a fault, through an unstructured mesh which is made up of pyramids, prisms, hexahedral and tetrahedral cells [9]. This method is more flexible than the previous one as faults can be represented but it fails to achieve the orthogonal property.

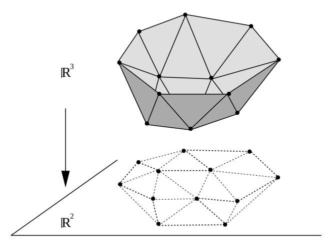
The works in this paper retain the hybrid mesh approach and mainly propose an original method to generate transition meshes. In order to introduce a well in a reservoir, some cells of the reservoir mesh are set inactive. Their union defines a cavity where the well mesh is placed. The transition mesh will have to fill in the domain enclosed between the boundary of the cavity and the boundary of the well mesh. If the faces of these boundaries are regarded as constraints, the problem of generating a transition mesh amounts to generate an orthogonal mesh that strictly conforms to these constraints. In this paper, we consider simplicial constraining complexes in any dimension and mainly show how an orthogonal mesh that strictly conforms to a given simplicial complex $\mathcal{SC}$ can be computed. As orthogonal meshes are known to be strongly related to power diagrams [1], our method heavily relies on the properties of power diagrams and their dual regular triangulations which have been widely studied [8, 2, 6].

Section 2 reviews some known facts about regular triangulations and power diagrams. In Section 3, we define regular constrained triangulations for a constraining simplicial complex and show that finding such a triangulation amounts to solve a linear problem. Then, in Section 4, we show how an orthogonal complex that strictly conforms to a simplicial complex $\mathcal{SC}$ can be computed once a regular constrained triangulation of $\mathcal{SC}$ is known. Finally, section 5 illustrates how this result is used to generate 2D hybrid meshes for oil reservoir simulation.

2 Regular triangulations and power diagrams

2.1 Definitions

A triangulation in $\mathbb{R}^d$ is said to be **regular** if it can be obtained as the orthogonal projection of the lower convex hull of a polytope in $\mathbb{R}^{d+1}$. More precisely, let T be a triangulation in $\mathbb{R}^d$ with vertices at the set of points $\mathcal{P} = \{p_1, \ldots, p_n\}$. The triangulation T is regular iff there exists a set $\mathcal{P}^+ = \{p_1^+, \ldots, p_n^+\}$ of points in $\mathbb{R}^{d+1}$ such that point p_i^+ projects on p_i and the projection of the lower convex hull of $\mathcal{P}^+$ is T.



As a regular triangulation in $\mathbb{R}^d$ is the projection of the lower convex hull of a polytope in $\mathbb{R}^{d+1}$, its domain covers the convex hull of its vertices.

Let $\mathcal{WP} = \{(p_1, \omega_1^2), \ldots, (p_n, \omega_n^2)\}$ be a set of **weighted points** where each p_i is a point in $\mathbb{R}^d$ and each ω_i^2 is a scalar called the weight of point p_i. Alternatively, each weighted point (p_i, ω_i^2) can be regarded as a sphere in $\mathbb{R}^d$ with center p_i and radius ω_i. Notice that the weight ω_i^2 does not have to be positive and that points with negative weights can be handled as well. Points with negative weights are regarded as imaginary spheres whose radii are imaginary complex numbers.

Let us define the **power product** of two weighted points (p_i, ω_i^2) and (p_j, ω_j^2) as :

$$\Pi((p_i, \omega_i^2), (p_j, \omega_j^2)) = (p_i - p_j)^2 - \omega_i^2 - \omega_j^2.$$

Notice that if the weight ω_i^2 is zero, the power product $\Pi((p_i,\omega_i^2),(p_j,\omega_j^2)) = \Pi(p_i,(p_j,\omega_j^2))$ is the power of point p_i with respect to the sphere (p_j,ω_j^2).

The **power diagram** $PD(\mathcal{WP})$, of the set $\mathcal{WP}$, is a space partition. Each cell corresponds to a weighted point (p_i,ω_i^2) of $\mathcal{WP}$ and is the locus of points p in $\mathbb{R}^d$ whose power with respect to (p_i,ω_i^2) is less than its power with respect to any other weighted point (p_j,ω_j^2) in $\mathcal{WP}$. Let $\mathcal{C}_{\mathcal{WP}}(p_i,\omega_i^2)$ be the cell of (p_i,ω_i^2) in the power diagram $PD(\mathcal{WP})$.

$$\mathcal{C}_{\mathcal{WP}}(p_i,\omega_i^2) = \{p \in \mathbb{R}^d \;\mid\; \Pi(p,(p_i,\omega_i^2)) \le \Pi(p,(p_j,\omega_j^2)) \quad \forall (p_j,\omega_j^2) \in \mathcal{WP}\}.$$

The power diagram extends the notion of Voronoi diagram in the sense that a Voronoi diagram is a power diagram of equally weighted points. Notice however, that a weighted point (p_i,ω_i^2) may have no cell (or more exactly an empty cell) in the power diagram of $\mathcal{WP}$.

2.2 Duality

In the following, a set of weighted points $\mathcal{WP}$ is said to be in general position if, for any subset $\mathcal{WP}_k$ of k weighted points in $\mathcal{WP}$, the locus of points in $\mathbb{R}^d$ that have equal power with respect to all the weighted points of $\mathcal{WP}_k$ is empty if $k \ge d+2$ and has dimension $d+1-k$ otherwise. The general position assumption on the set $\mathcal{WP}$ ensures that the complex dual of the power diagram of $\mathcal{WP}$ is a triangulation. We recall in the next lemma a well known fact (see e.g. [5]):

Lemma 1 *If $\mathcal{WP}$ is a set of weighted points of $\mathbb{R}^d \times \mathbb{R}$ in general position, the dual of its power diagram is a regular triangulation. Reciprocally, any regular triangulation in $\mathbb{R}^d$ is the dual of a power diagram.*

Proof Let us first define the lift map Φ, mapping weighted points of $\mathbb{R}^d \times \mathbb{R}$ to points in $\mathbb{R}^{d+1}$, as follows :

$$\begin{array}{rcccl} \Phi & : & \mathbb{R}^d \times \mathbb{R} & \rightarrow & \mathbb{R}^{d+1} \\ & & (p,\omega^2) & \rightarrow & \Phi(p,\omega^2) = (p, p^2 - \omega^2) \end{array}$$

Then, we can combine the lift map Φ with the duality defined by the unit paraboloid of $\mathbb{R}^{d+1}$, obtaining the map Φ^* that maps weighted points in $\mathbb{R}^d \times \mathbb{R}$ to non vertical hyperplanes of $\mathbb{R}^{d+1}$:

$$\begin{array}{rcccl} \Phi^* & : & \mathbb{R}^d \times \mathbb{R} & \rightarrow & \mathbb{R}^{d+1} \\ & & (p,\omega^2) & \rightarrow & \Phi^*(p,\omega^2) = \\ & & & & \{(x, x_{d+1}) \in \mathbb{R}^{d+1} \mid -2p \cdot x + x_{d+1} + (p^2 - \omega^2) = 0\} \end{array}$$

In the following, we call *normalized* the equation of $\Phi^*(p,\omega^2)$ written as above, i.e. where the coefficient of x_{d+1} is equal to unity.

It is then easy to notice that the power product $\Pi((p,\omega^2),(p_i,\omega_i^2))$ of two weighted points can be obtained in plugging the lifted point $\Phi(p,\omega^2)$ into the normalized equation of the hyperplane $\Phi^*(p_i,\omega_i^2)$, or alternatively in plugging the lifted point $\Phi(p_i,\omega_i^2)$ into the normalized equation of $\Phi^*(p,\omega^2)$.

Two weighted points are said to be **orthogonal** if their power product is null. Thus, the hyperplane $\Phi^*(p_i,\omega_i^2)$ is the locus of lifted points corresponding to weighted points orthogonal to (p_i,ω_i^2). The halfspace $\Phi^{*+}(p_i,\omega_i^2)$ is the locus of lifted point $\Phi^*(p,\omega^2)$ associated with the weighted points (p,ω^2) that have a positive power product $\Pi((p,\omega^2),(p_i,\omega_i^2))$ with (p_i,ω_i^2).

Let $\mathcal{WP}$ be a set of weighted points in general position and let $\mathcal{WPS} = \{(p_1,\omega_1^2), \ldots, (p_{d+1},\omega_{d+1}^2)\}$ be a subset of $(d+1)$ weighted points in $\mathcal{WP}$. The general position assumption ensures that the hyperplanes $\{\Phi^*(p_1,\omega_1^2),\ldots,\Phi^*(p_{d+1},\omega_{d+1}^2)\}$ intersect in a unique point in $\mathbb{R}^{d+1}$, which is the image $\Phi(p,\omega^2)$ of the unique weighted point (p,w) orthogonal to $\{(p_1,\omega_1^2),\ldots,(p_{d+1},\omega_{d+1}^2)\}$. This point is called the **weighted power center** of $\{\{(p_1,\omega_1^2), \ldots, (p_{d+1},\omega_{d+1}^2)\}$. The weighted power center (p,w) of $\{(p_1,\omega_1^2),\ldots,(p_{d+1},\omega_{d+1}^2)\}$ projects on a vertex of the power diagram PD of $\mathcal{WP}$ iff its power product $\Pi((p,\omega^2),(p_l,\omega_l^2))$ with respect to any other weighted point (p_l,ω_l^2) in $\mathcal{WP}$ is positive. This is equivalent to say, that for any weighted point $(p_l,\omega_l^2) \in \mathcal{WP} \setminus \mathcal{WPS}$, the lifted point $\Phi(p_l,\omega_l^2)$ belongs to the halfspace $\Phi^{*+}(p,\omega^2)$ above $\Phi^*(p,\omega^2)$. Now, because (p,w) is orthogonal to $\{(p_1,\omega_1^2),\ldots,(p_{d+1},\omega_{d+1}^2)\}$, the lifted points $\{\Phi(p_1,\omega_1^2),\ldots,\Phi(p_{d+1},\omega_{d+1}^2)\}$ belong to $\Phi^*(p,\omega^2)$. Thus $\Phi^*(p,\omega^2)$ is just the hyperplane through $\{\Phi(p_1,\omega_1^2),\ldots,\Phi(p_{d+1},\omega_{d+1}^2)\}$.

Hence, $\mathcal{WPS} = \{(p_1,\omega_1^2),\ldots,(p_{d+1},\omega_{d+1}^2)\}$ defines a vertex in the power diagram of $\mathcal{WP}$ iff the hyperplane through $\{\Phi(p_1,\omega_1^2),\ldots,\Phi(p_{d+1},\omega_{d+1}^2)\}$ supports the convex hull of the set $\Phi(\mathcal{WP})$. Therefore $\{p_1,\ldots,p_{d+1}\}$ forms a simplex in the triangulation dual to the power diagram of $\mathcal{WP}$ iff $\{\Phi(p_1,\omega_1^2),\ldots,\Phi(p_{d+1},\omega_{d+1}^2)\}$ is a facet in the lower convex hull $LCH(\Phi(\mathcal{WP})$. This achieves the proof that the dual of a power diagram is a regular triangulation.

Reciprocally, let T be a regular triangulation in $\mathbb{R}^d$ with vertices at the set of points $\mathcal{P} = \{p_1,\ldots,p_n\}$. By definition, there is a set $\mathcal{P}^+ = \{p_1^+,\ldots,p_n^+\}$ of points in $\mathbb{R}^{d+1}$ such that the projection of the lower convex hull $LCH(\mathcal{P}^+)$ of $\mathcal{P}^+$ is T. Let us note z_i the $(d+1)$th coordinate of p_i^+, then $\omega_i^2 = p_i^2 - z_i$. Each point p_i^+ can be regarded as the lifted image $\Phi(p_i,\omega_i^2)$ of the weighted point (p_i,ω_i^2).

Now, each facet $\{p_1p_2\ldots p_{d+1}\}$ of T is the projection of a facet $\{p_1^+p_2^+\ldots p_{d+1}^+\}$ of $LCH(\mathcal{P}^+)$ whose affine hull supports $LCH(\mathcal{P}^+)$. This proves that each facet in T is dual to a vertex in the power diagram of the set $\mathcal{WP} = \{(p_1,\omega_1^2),\ldots,(p_n,\omega_n^2)\}$. Thus, any regular triangulation is the dual of a power diagram which achieves the proof of lemma 1. □

Lemma 1 allows us to define the regular triangulation of $\mathcal{WP}$ as the dual of the power diagram $PD(\mathcal{WP})$. Let us notice that this triangulation covers the convex hull of the set $\mathcal{P} = \{p_1, \ldots, p_n\}$ but that its set of vertices may be a proper subset of $\mathcal{P}$. A point $p_i \in \mathcal{P}$ that is not a vertex of the regular triangulation of $\mathcal{WP}$ is called a *hidden vertex* and its lifted point $\Phi(p_i, \omega_i^2)$ is above the lower convex hull of $\Phi(\mathcal{WP})$.

3 Regular Constrained Triangulations

Let us consider a given simplicial complex $\mathcal{SC}$. We say that a triangulation T **strictly conforms** to $\mathcal{SC}$ iff any simplex of $\mathcal{SC}$ is a simplex of T.

Let $\mathcal{P}$ be the set of vertices of the complex $\mathcal{SC}$. The above definition implies that any triangulation that strictly conforms to $\mathcal{SC}$ includes $\mathcal{P}$ in its set of vertices. Assume that we are looking for a triangulation T that strictly conforms to $\mathcal{SC}$ and has $\mathcal{P}$ as set of vertices. We may first check if the Delaunay triangulation of $\mathcal{P}$ conforms to $\mathcal{SC}$, in which case we are done. Otherwise we can :

- either build a constrained Delaunay triangulation of $\mathcal{SC}$
- or look for a set of weights $\mathcal{W}$ in one to one correspondance with $\mathcal{P}$, such that the regular triangulation of the resulting weighted point set $\mathcal{WP}$ stricly conforms to $\mathcal{SC}$.

Given a simplicial complex $\mathcal{SC}$ whose set of vertices is $\mathcal{P}$, the regular triangulation of a set $\mathcal{WP}$ of weighted points projecting on $\mathcal{P}$ is called a **regular constrained triangulation** of $\mathcal{SC}$ iff it strictly conforms to $\mathcal{SC}$.

Note that while a constrained Delaunay is not a Delaunay triangulation (it does not fulfill the *emtpy circle property* but only a weaker property), a regular constrained triangulation is a regular triangulation. Also notice that the Delaunay triangulation of $\mathcal{P}$ is just a particular case of a regular triangulation for which all weights are zero (or equal).

In the following, we describe how to find a regular constrained triangulation of a simplicial complex $\mathcal{SC}$ if one exists.

Lemma 2 *Finding a regular constrained triangulation of a simplicial complex $\mathcal{SC}$ reduces to linear programming.*

Proof Finding a regular constrained triangulation of $\mathcal{SC}$ amounts to find the weights to be associated to the vertices of $\mathcal{SC}$ in this triangulation. Let $\mathcal{W} = \{\omega_1^2, \ldots, \omega_n^2\}$ be such a set of weights. The regular triangulation of the set $\mathcal{WP} = \{(p_1, \omega_1^2), \ldots, (p_n, \omega_n^2)\}$ is the projection of the lower convex hull $LCH(\Phi(\mathcal{WP}))$ of the set of lifted points $\Phi(\mathcal{WP})$. Our goal is then to find a set of weights $\mathcal{W}$ such that the projection of $LCH(\Phi(\mathcal{WP}))$ conforms to $\mathcal{SC}$.

A face of $\mathcal{SC}$ that is not included in another face of $\mathcal{SC}$ is said to be a maximal face. Let $\mathcal{E}$ be the set of maximal faces of $\mathcal{SC}$. For a face f of $\mathcal{SC}$, we denote by k_f the dimension of f, by $\mathcal{P}_f$ the set of its vertices and by $\mathcal{WP}_f$ the set of associated lifted points. The goal is now to choose $\mathcal{W}$ such that for each face $f \in \mathcal{E}$, there is a non vertical hyperplane H_f in $\mathbb{R}^{d+1}$ passing through the lifted vertices $\Phi(\mathcal{WP}_f)$ and supporting from below the convex hull of $\Phi(\mathcal{WP})$.

Let H_f be a non vertical hyperplane in $\mathbb{R}^{d+1}$ whose equation is written as follows :

$$H_f = \{(p, z) \in \mathbb{R}^d \times \mathbb{R} \mid h_f \cdot p + z + c_f = 0\}$$

where $(h_f, c_f) \in \mathbb{R}^d \times \mathbb{R}$. The hyperplane H_f passes through the lifted vertices $\Phi(\mathcal{WP}_f)$ and supports $LCH(\Phi(\mathcal{WP}))$, iff

$$\begin{cases} h_f \cdot p_i + w_i^2 + c_f = 0 & \forall p_i \in \mathcal{P}_f \\ h_f \cdot p_i + w_i^2 + c_f > 0 & \forall p_i \in \mathcal{P} \setminus \mathcal{P}_f \end{cases} \tag{1}$$

A face of the convex hull of $\mathcal{P}$ automatically appears in any triangulation of $\mathcal{P}$. Then, if we denote by $\mathcal{E}'$ the faces of $\mathcal{E}$ that are not faces of the convex hull of $\mathcal{P}$, system 1 has to be satisfied for each face f of $\mathcal{E}'$. Each equation of H_f has $d+1$ unknown coefficients which can be reduced to $d - k_f$ unknowns using the $k_f + 1$ equalities of system 1. As the weights w_i^2 of points in $\mathcal{P}$ are also unknowns of the system, we are left with a global linear system with $N_i = \sum_{f \in \mathcal{E}'} (n-(k_f+1))$ inequations ans $N_u = n + \sum_{f \in \mathcal{E}'} (d-k_f)$ unknowns.

Finally, solving this linear system using linear programming will provide, if one exists, a set $\mathcal{W} = \{\omega_1^2, \ldots, \omega_n^2\}$ of weights leading to a regular constrained triangulation of $\mathcal{SC}$. This means that if a solution to the system exists, a regular constrained triangulation of $\mathcal{SC}$ is provided, otherwise we know that no regular constrained triangulation of $\mathcal{SC}$ exists. □

Remark If the simplicial complex $\mathcal{SC}$ is a triangulation T, the problem amounts to decide if T is a regular triangulation or not. In this case, the system 1 reduces to a *power test* writing

$$\frac{\begin{vmatrix} p_0 & p_1 & \cdots & p_{d+1} \\ p_0^2 - w_0^2 & p_1^2 - w_1^2 & \cdots & p_{d+1}^2 - w_{d+1}^2 \\ 1 & 1 & \cdots & 1 \end{vmatrix}}{\begin{vmatrix} p_0 & \cdots & p_d \\ p_0^2 - w_0 & \cdots & p_d^2 - w_d \\ 1 & \cdots & 1 \end{vmatrix}} \geq 0 \tag{2}$$

where $\{p_0, p_1, p_2, \ldots p_d\}$ are the vertices of a d-simplex in T and p_{d+1} is another vertex of T. In this case, the unknowns are just the weights $\{\omega_1^2, \ldots, \omega_n^2\}$. Furthermore, it suffices to solve a system including one power test for each pair of adjacent d-simplexes in T. Indeed, let $\Phi(T)$ be the polyhedral surface obtained by lifting each vertex of T. The considered system guarantees the local convexity of each ridge of $\Phi(T)$ which, because $\Phi(T)$ projects onto the convex hull $CH(\mathcal{P})$ of P, is enough to guarantee the convexity of $\Phi(T)$ (as proved in [4]).

4 Conforming Orthogonal Complexes

As in the previous section, let us consider a given simplicial complex $\mathcal{SC}$ and let $\mathcal{P}$ be the set of its vertices. We say that a cellular complex **strictly conforms** to $\mathcal{SC}$ iff each face of $\mathcal{SC}$ is a face of the cellular complex.

A cellular complex is said to be **orthogonal** if there is an embedding of the dual complex such that each face is orthogonal to its dual face.

The aim of our work is to construct an orthogonal complex that strictly conforms to $\mathcal{SC}$. As the class of orthogonal complexes coincides with the class of power diagrams [1], we will in fact construct a power diagram that strictly conforms to $\mathcal{SC}$. Then, our goal is to define a set of sites whose power diagram strictly conforms to $\mathcal{SC}$.

Theorem 1 *If there exists a regular constrained triangulation of the simplicial complex $\mathcal{SC}$, we can define a set $\mathcal{WS}$ of weighted points, called weighted sites, whose power diagram strictly conforms to $\mathcal{SC}$.*

Proof Let $\mathcal{P}$ be the set of vertices of $\mathcal{SC}$ and let T be a regular constrained triangulation of $\mathcal{SC}$. The triangulation T is the regular triangulation of a set of weighted points $\mathcal{WP}$ projecting on $\mathcal{P}$. We note $PD(\mathcal{WP})$ the power diagram of the set $\mathcal{WP}$ which is the dual of T. Let $\mathcal{WS}$ be the set of weighted sites we are looking for. We denote by $PD(\mathcal{WS})$ the power diagram of the set $\mathcal{WS}$.

Let f be a face of $\mathcal{SC}$. As above, we note k_f the dimension of f, $\mathcal{P}_f$ the set of its vertices and $\mathcal{WP}_f$ the set of associated weighted points. The face f^* of $PD(\mathcal{WP})$ that is dual to f has dimension $d+1-k$. Each point p^* in f^* has equal power with respect to all weighted point in $\mathcal{WP}_f$, and a greater power with respect to any weighted point in $\mathcal{WP} \setminus \mathcal{WP}_f$. For each face f in $\mathcal{SC}$, we choose $d+1-k$ points p_j^* affinely independant in f^* and associate to each of them the weight: $\omega_j^{*2} = \Pi(p_j^*, (p_i, \omega_i^2)),\ \forall p_i \in \mathcal{P}_f$. We thus obtain a subset $\mathcal{WP}_{f^*}$ of $d+1-k$ weighted sites in general position.

Let $\mathcal{WS} = \cup_{f \in \mathcal{SC}} \mathcal{WP}_{f^*}$. We claim that the power diagram $PD(\mathcal{WS})$ conforms to $\mathcal{SC}$. Indeed, for each weighted point $(p_j^*, \omega_j^{*2}) \in \mathcal{WP}_{f^*}$, we have:

$$\begin{cases} \Pi((p_j^*, \omega_j^{*2}), (p_i, \omega_i^2)) = 0 & \forall p_i \in \mathcal{P}_f \\ \Pi((p_j^*, \omega_j^{*2}), (p_i, \omega_i^2)) \geq 0 & \forall p_i \in \mathcal{P} \setminus \mathcal{P}_f \end{cases}$$

The above equations show that each vertex p_i of f belongs to the cell $\mathcal{C}_{\mathcal{WS}}(p_j^*, \omega_j^{*2})$ of the power diagram $PD(\mathcal{WS})$ if the weighted site (p_j^*, ω_j^{*2}) belongs to $\mathcal{WP}_{f^*}$. Thus, by convexity of the faces of a power diagram, we know that the face f is included in the face of $PD(\mathcal{WS})$ that is the intersection of the cells $\mathcal{C}_{\mathcal{WS}}(p_j^*, \omega_j^{*2})$ of the weighted sites in $\mathcal{WP}_{f^*}$. Then, an easy induction on the dimension of f proves that any face f of $\mathcal{SC}$ is a face of $PD(\mathcal{WS})$. □

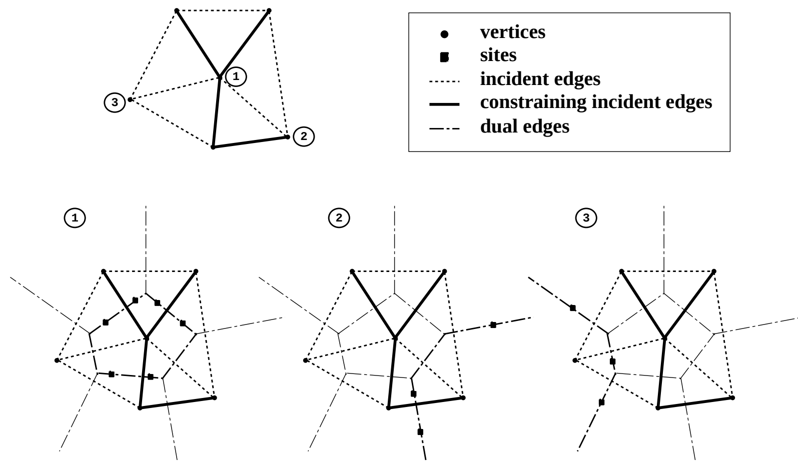


Figure 1: Defining sites

In practice, to define the sites of $\mathcal{WS}$, we iterate on the simplexes of $\mathcal{SC}$ in order of decreasing dimension. First, we define the weighted sites linked to the maximal faces of $\mathcal{SC}$. Then, we consider the simplexes of lower dimensions until we reach the vertices. For each simplex f of $\mathcal{SC}$ that is not maximal, we already have the weighted sites linked to the faces including f and, if necessary, we add the right amount of additional sites. As an example, let us consider, in two dimensions, a simplicial complex $\mathcal{SC}$ made up of vertices and constraining edges. At first, for each constraining edge e_{ij}, we take two sites on its dual edge e_{ij}^*. Subsequently, we consider each vertex p_i of $\mathcal{SC}$ which may be (cf. figure 1) :

1) a vertex incident to at least two constraining edges,
2) a vertex incident to only one constraining edge,
3) an isolated vertex.

In the first case, we have already defined two sites on each edge dual to a constraining edge incident to p_i. Thus, p_i is ensured to belong to four cells of $PD(\mathcal{WS})$ and to be a vertex of $PD(\mathcal{WS})$. In the second case, p_i is already ensured to belong to two cells. We choose a third site in the power diagram cell associated to p_i. In practice, the third site for p_i is choosen on the dual of some non constraining edge of T incident to p_i. This way, p_i lies in three cells of $PD(\mathcal{WS})$ and is a vertex of $PD(\mathcal{WS})$. In the last case, we choose three sites in the power diagram cell associated to p_i. In practice, those sites are choosen on three edges of $PD(\mathcal{WP})$ that are dual to edges of T incident to p_i. Again, p_i lies in three cells and is a vertex of $PD(\mathcal{WS})$.

5 Application

From above, we know how to compute an orthogonal complex that strictly conforms to a set of contraints forming a simplicial complex $\mathcal{SC}$. In the following, we show how this method can be used to generate 2D hybrid meshes for oil reservoir simulation. In the case of 2D meshes, the boundaries of the cavity in the reservoir mesh and of the wells meshes are polygons whose edges form a 1D simplicial complex and the above method is used to generate a conforming orthogonal mesh filling the domain delimited by these boundaries. This calls for two remarks.

First the conforming orthogonal complex that will be used as transition mesh is a power diagram, and the generating sites of such a diagram may lie outside of their respective cells. In the oil reservoir simulation application described here, sites are used as discretization points where physical data are estimated, and sites lying outside their cells are inappropriate. Therefore we have to take into account the additional requirement for the sites of the generated transition mesh to lie in their corresponding cells.

The second remark concerns the dimension of the meshes considered here. Of course, oil companies really need 3D meshes of oil reservoirs to perform 3D simulations. On one hand our method to generate a conforming orthogonal complex works in any dimension. On the other hand it requires a simplicial constraining complex and 3D structured meshes of geological reservoirs are made up of hexahedral cells that have non flat quadrangular faces with vertices that are not cocyclic. Then in 3D meshes, the faces of the cavity won't form a simplicial complex, unless we divide them into triangular subfaces. In that case, we realize that the transition mesh will only conform to the triangular constraints but not to the mesh faces. Therefore the generation of 3D transition meshes for oil reservoir simulation requires more work, and we restrict here to 2D meshes.

5.1 From meshes to the simplicial constraining complex

In the following, we deal with the construction of a transition mesh between a reservoir mesh RM and a well mesh WM. First, we need to create a cavity in RM where to place WM. For this purpose, we superimpose the two given meshes and set inactive a set of appropriate cells of RM. This set includes at least the cells of RM intersected by WM. Then, the boundary of the created cavity and the boundary of the inserted well are two polygons whose vertices and edges form the constraining simplicial complex $\mathcal{SC}$. These successive steps are illustrated by the following figure 2.

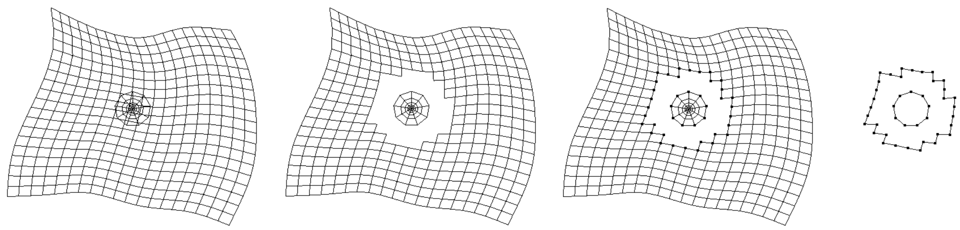


Figure 2: Creation of a cavity

The next step is to find a regular a constrained regular triangulation T of $\mathcal{SC}$, to compute its dual power diagram $PD(\mathcal{WP})$, and to choose the weighted sites of a set $\mathcal{WS}$ whose power diagram $PD(\mathcal{WS})$ conforms to $\mathcal{SC}$.

Let us consider a site s in a power diagram PD and its neighbouring sites $s_{i=1..n}$. The site s lies in its cell C_s iff for each of its neighbour s_i, sites s and s_i lie on different sides of their radical axis.

For each constraining edge $e \in \mathcal{SC}$, we will add to $\mathcal{WS}$ two sites on the edge e^* of $PD(\mathcal{WP})$ dual to e. In the power diagram $PD(\mathcal{WS})$, the two sites choosen on e^* are neighbours and their cells share the edge e. A necessary condition for the sites of $\mathcal{WS}$ to be in their cells is thus that the two sites on a dual edge e^* lie on both side of e. We conclude that each constraining edge e need to intersect its dual e^*. Therefore, for each edge e in $\mathcal{SC}$ with p_i and p_j as vertices, we add constraints to guarantee first that the radical axis of the weighted points (p_i, w_i^2) and (p_j, w_j^2) intersects e and second that the resulting intersecting point is on the common edge of the cells of (p_i, w_i^2) and (p_j, w_j^2) in $PD(\mathcal{WP})$.

$$\left\{ \begin{array}{lll} \omega_i^2 - \omega_j^2 & \leq & (p_i - p_j)^2 \\ \omega_j^2 - \omega_i^2 & \leq & (p_i - p_j)^2 \\ \omega_k^2 - \omega_i^2 & \leq & (p_i - p_k)^2 - (1 + \frac{\omega_i^2 - \omega_j^2}{(p_i - p_j)^2}).\overrightarrow{p_i p_k}.\overrightarrow{p_i p_j} \quad \forall (p_k, \omega_k^2) \in \mathcal{WP} \end{array} \right.$$

Finally, the complete linear system defines, if any, regular constrained triangulations such that each constraining edge intersects its dual. Looking for such a triangulation implies the resolution of a linear optimization problem and we want to avoid this costly step as often as possible. Then, we first compute the Delaunay triangulation $DT(\mathcal{P})$ of the vertices set $\mathcal{P}$ of $\mathcal{SC}$. If $DT(\mathcal{P})$ conforms to $\mathcal{SC}$ and if all edges of $\mathcal{SC}$ are Gabriel edges (which means that they intersect their dual edge) $DT(\mathcal{P})$ is an adequate regular constrained triangulation of $\mathcal{SC}$ and we are done. In most cases, a judicious choice of the cells to be set inactive leads to such a constraining complex. However, in some specific situations, we still have to solve the linear problem which yields a regular constrained triangulation of $\mathcal{SC}$ such that each constraining edge intersects its dual. If no regular constrained triangulation of $\mathcal{SC}$ exists, no conforming power diagram can be computed.

5.2 Construction of the transition mesh

Let T be a regular constrained triangulation of $\mathcal{SC}$ such that each constraining edge intersects its dual edge. Let $PD(\mathcal{WP})$ denotes the dual power diagram of T and let $PD(\mathcal{WS})$ be the constrained power diagram we want to construct.

First, we define a set of site $\mathcal{WS}$ using the method described earlier and presented as an example. For each constraining edge $e \in \mathcal{SC}$, we define two sites on its dual edge in $PD(\mathcal{WP})$ with the constraint that one site has to be defined on each side of e. Then, as $\mathcal{SC}$ is made of two polygons, we only have to deal with the first case illustrated in figure 1. Each vertex of $\mathcal{SC}$ is already associated with four sites and therefore is a vertex of $PD(\mathcal{WS})$.

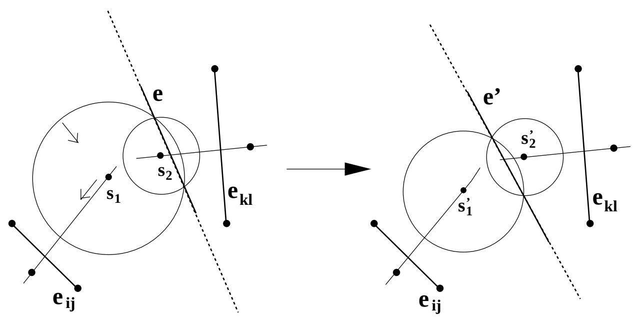


Figure 3: Moving a site

Once sites have been defined, their power diagram $PD(\mathcal{WS})$ strictly conforms to $\mathcal{SC}$ and for each constraining edge e, its two associated sites are on both of its sides. But this last property may not be realized for an edge e' of $PD(\mathcal{WS})$ that is not an edge of $\mathcal{SC}$ and a site may be outside its cell. In that case, two sites are on the same side of e'. Then, we will move one or the other site along its definition edge in order to bring the edge e' back between the two considered sites (cf. figure 3). This processing will be executed until each site is in its cell (which is always possible).

In the end, $PD(\mathcal{WS})$ is a power diagram that strictly conforms to $\mathcal{SC}$, and each site belongs to its cell. Then, the subset of the $PD(\mathcal{WS})$'s cells that covers the space delimited by the boundaries make up the transition mesh between *RM* and *WM* and we finally obtain the global hybrid mesh.

5.3 Results

Given a reservoir mesh and well meshes, we have presented how to compute transition meshes that establish the proper connections. The method has been experimented and tested using the Computational Geometry Algorithms Library CGAL[1].

As a first illustration, figure 4a shows the hybrid mesh that has been computed on the data presented figure 2 : a radial well mesh inserted in a CPG reservoir mesh. Our method deals with a set of vertices and edges extracted from the boundary of the cavity and the boundary of the well. Therefore, we can handle different kinds of wells or/and several wells in a unique cavity. Figure 4b illustrates the example of a horizontal mesh and a vertical mesh inserted in a unique cavity in a reservoir mesh while a vertical mesh is inserted in an other separate cavity.

In a same way, we can deal with faults represented by a sequence of segments and inserted in a cavity of a CPG reservoir mesh. The slight difference lies in the presence of vertices incident to a single constraining edge (ends of the fault). For these vertices, the sites defined for the constraining edges are not sufficient. Each extremity p is associated with two sites and we need to define a third site as described in the general method (see Section 4).

A first step toward 3-dimensional meshes is to generate $2\frac{1}{2}$-dimensional meshes. First, we compute a global hybrid mesh in dimension two on a plane. Then, we project this 2D generated mesh onto geological surfaces. The interpolations between two consecutive surfaces lead to the final $2\frac{1}{2}$-dimensional mesh. An example is presented figure 7.

6 Conclusion

In this paper, we have presented the notion of regular constrained triangulation, a regular triangulation that strictly conforms to a given simplicial complex $\mathcal{SC}$ and has no extra Steiner vertex. Then we have shown that if such a triangulation exists for $\mathcal{SC}$, an orthogonal complex that strictly conforms to $\mathcal{SC}$ can be computed and an appropriate technique has been exhibited. In a practical way, we use this method to generate a new kind of 2D hybrid meshes dedicated to oil reservoir simulation. The method can also yield $2\frac{1}{2}$-dimensional meshes. The generation of real 3D hybrid meshes still requires further works.

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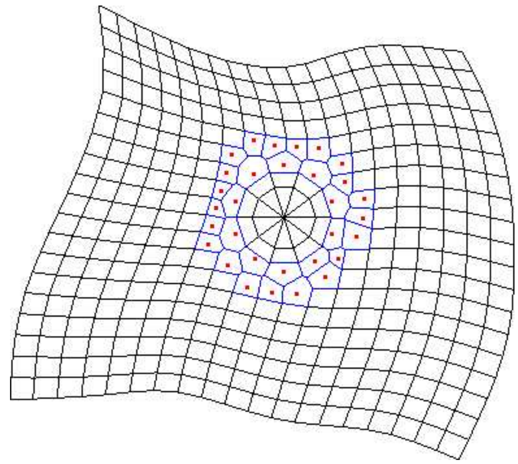
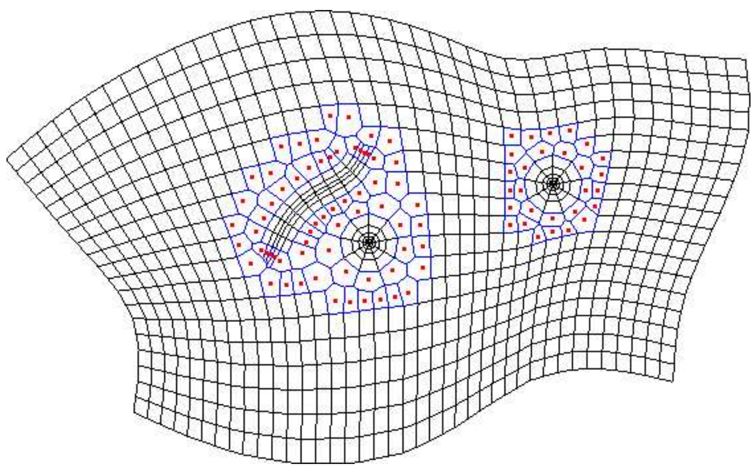


a) a vertical well



b) a horizontal and two vertical wells

Figure 4: *Inserting radial meshes around wells in a reservoir mesh*

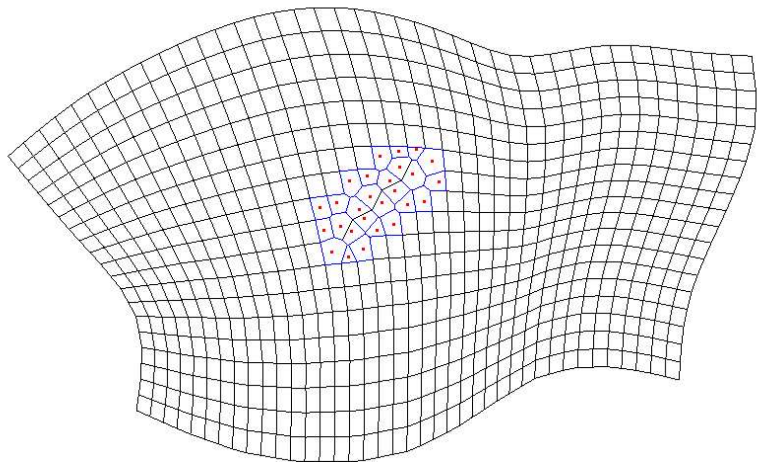


Figure 5: Inserting a fault

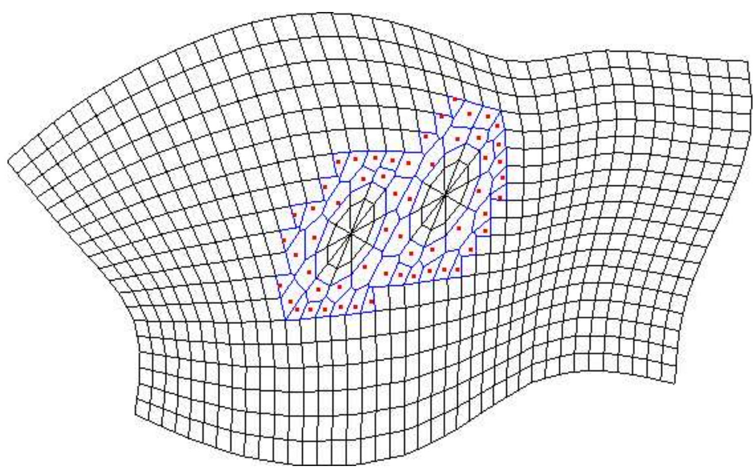


Figure 6: Anisotropic flow at well proximity

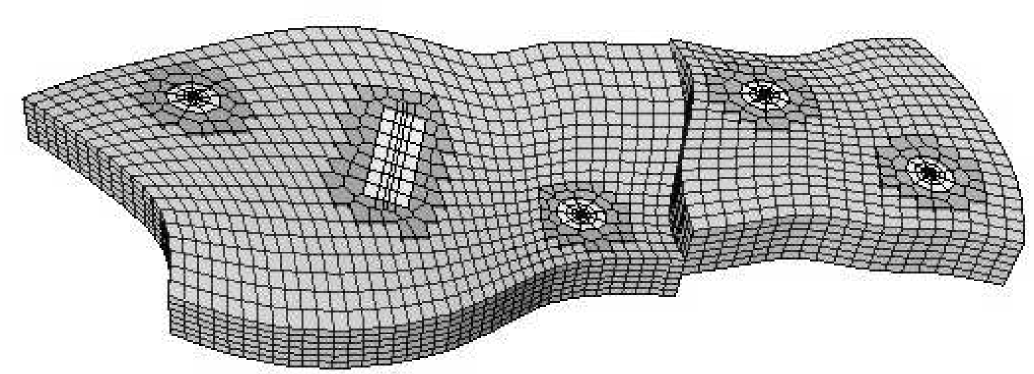


Figure 7: $2\frac{1}{2}$-dimensional grid, a step toward 3-dimensional cases

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