A Minplus Derivation of the Fundamental Car-Traffic Law

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Abstract: We give deterministic models and a stochastic model of the traffic on a circular road without overtaking. The average speed is the eigenvalue of a minplus matrix describing the dynamics of the system in the first case and a Lyapounov exponentof a minplus stochastic matrix in the second one. The eigenvalue and the Lyapounov exponent are computed explicitly. From these formulae we derive the fundamental law that links the flow to the density of vehicle on the road. Numerical experiments using the maxplus toolbox of Scilab confirm the theoretical results obtained.

Key-words: maxplus algebra, transportation, statistical mechanics

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Obtention de la loi fondamentale du trafic routier dans le cadre de l'algèbre minplus

Résumé : On donne des modèles déterministes et un modèle stochastique du trafic sur une route circulaire sans dépassement. La vitesse moyenne correspond à la valeur propre de matrices minplus dans le premier cas et à l'exposant de Lyapounov d'une matrice minplus stochastique dans le second. Les valeurs propres ou l'exposant de Lyapounov sont calculés explicitement. De ces formules on peut déduire la loi fondamentale du trafic routier liant le flôt à la densité de véhicules dans ces différents cas. On présente également des simulations numériques utilisant la boîte à outils maxplus de Scilab qui confirment les résultats théoriques obtenus.

Mots-clés : algèbre maxplus, transport, mécanique statistique

1 Introduction

For simple traffic models we derive the relation existing between the average flow and the density of vehicles called sometimes the fundamental traffic law. This law has been studied empirically and theoretically using exclusion processes (for example see [3, 4, 5, 2, 9, 6]). Here we show that the simplest deterministic model is purely minplus linear and that this fundamental law may be deduced from the explicite computation of the minplus eigenvalue of the matrix describing the dynamics of the system[1].

Then we study the simplest stochastic model and show that the average speed is the Lyapounov exponent of a stochastic minplus matrix. In general, it is very difficult to compute a Lyapounov exponent. In our case, it is possible to characterize completely the stationary regime. From this characterization it is straightforward to obtain the Lyapounov exponent. The fundamental traffic law is easily derived from this result.

The traffic is modelled by N cars in a circular road of unitary length. In the deterministic case the cars want to move at the given velocity v and must respect a security distance with the car ahead. In the stochastic model the cars are allowed to move at velocities w and v (with $w < v$) chosen randomly and independently with probability (μ, λ) . We consider here only the case where the cars are not allowed to overtake other cars and we use discrete time dynamic models.

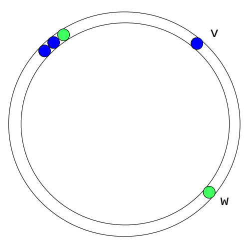

Figure 1: Traffic line without overtaking.

2 Deterministic Modelling

In this model, we consider N cars moving on a one way circular road of length 1. Each car indexed by $n = 1, \cdots, N$ has a desired speed v, a size 0 (without loss of generality), and must respect a security distance σ with the car ahead. A discrete time dynamic model is used where, at each time step t, the driver tries to cover the largest possible distance, denoted x_n^t, taking into account the constraints imposed by the car ahead. In order to determine the dynamics of the system we have to know at what precise instant the security distances have to be verified. We consider two cases :

[1] Here there is no asymptotic study, the computation can be done for a fi x number of vehicles

1. *The move of the driver ahead is not anticipated* then the covered distances by the cars are :

$$x_n^{t+1} = \begin{cases} \min\left(x_n^t + v\ ,\ x_{n+1}^t - \sigma\right), & \text{if } n < N\ , \\ \min\left(x_N^t + v\ ,\ x_1^t + 1 - \sigma\right), & \text{if } n = N\ . \end{cases} \tag{1}$$

2. *The move of the driver ahead is anticipated* then the covered distances are given by :

$$x_n^{t+1} = \begin{cases} \min\left(x_n(t) + v\ ,\ x_{n+1}(t+1) - \sigma\right), & \text{if } n < N\ , \\ \min\left(x_N(t) + v\ ,\ x_1(t+1) + 1 - \sigma\right), & \text{if } n = N\ . \end{cases} \tag{2}$$

For these two models we want to derive the fundamental traffic law giving the relation between the average car density and the average car flow.

3 Minplus Algebra

The fundamental traffic law may be derived easily from the eigenvalue computation of a minplus matrix describing the dynamic of the traffic system. Before showing this result, let us make some recalls about minplus algebra. The minplus algebra is, by definition [1], the set $\mathbb{R} \cup \{+\infty\}$ together with the laws min (denoted by $\oplus$) and $+$ (denoted by $\otimes$). The element $\epsilon = +\infty$ satisfies $\epsilon \oplus x = x$ and $\epsilon \otimes x = \epsilon$ (ϵ acts as zero). The element $e = 0$ satisfies $e \otimes x = x$ (e is the unit). The main discrepancy with the conventional algebra is that $x \oplus x = x$. We denote $\mathbb{R}_{\min} = (\mathbb{R} \cup \{+\infty\}, \oplus, \otimes)$ this structure. $\mathbb{R}_{\min}$ is a special instance of diod (semiring with idempotent addition).

This minplus structure on scalars induces a dioid structure on square matrices with matrix product $A \otimes B$, for two compatible matrices with coefficients in $\mathbb{R}_{\min}$, defined by $(A \otimes X)_{ik} = \min_j \left(A_{ij} + B_{jk}\right)$. Then the unit matrix is denoted E. We may associate to a square matrix A an incidence graph $\mathcal{G}(A)$ where the nodes corresponds to the columns and the rows of the matrix A and the arcs to the nonzero entries (the weight of the arc (j, i) being the non zero entry A_{ji}). Then the weight of a path in $\mathcal{G}(A)$ is the sum of the weights of the arcs composing the path. We will use the three following fundamental results.

1. If the weights of all the circuits $\mathcal{C}$ of $\mathcal{G}(A)$ (where A is a $N \times N$ matrix with entries in $\mathbb{R}_{\min}$) are positive the equation $x = A \otimes x \oplus b$ admits a unique solution $x = A^* \otimes b$ with $A^* = E \oplus A \oplus \cdots \oplus A^{N-1}$.

2. If $\mathcal{G}(A)$ is strongly connected, the matrix A admits a unique eigenvalue ρ satisfying $\exists x \in \mathbb{R}_{\min}^N : A \otimes x = \rho \otimes x$ which has the graph interpretation $\rho = \min_{c \in \mathcal{C}} |c|_w / |c|_l$ where $|c|_w$ denotes the weight of the circuit c and $|c|_l$ denotes the arc number of the corresponding circuit.

3. The eigenvalue of A gives the asymptotic behaviour of the dynamic minplus linear system $X^{t+1} = A \otimes X^t$ in the following sense. There exists $T \leq N$ and K such that for all $k \geq K$ we have $A^{k+T} = \rho^T \otimes A^k$.

4 The Fundamental Traffic Law in the Deterministic Non Anticipative Case

Using minplus notation, the dynamics of the traffic in the non anticipative case may be written in scalar form

$$x_n^{t+1} = \begin{cases} v \otimes x_n^t \oplus (-\sigma) \otimes x_{n+1}^t, & \text{if } n < N, \\ v \otimes x_N^t \oplus (1-\sigma) \otimes x_1^t, & \text{if } n = N \ . \end{cases} \tag{3}$$

In matrix form we have

$$X^{t+1} = A \otimes X^t \ , \tag{4}$$

where

$$A = \begin{bmatrix} v & -\sigma & & \\ & \ddots & \ddots & \\ & & \ddots & -\sigma \\ 1-\sigma & & & v \end{bmatrix} .$$

The precedence graph associated to A is given in Figure 2.

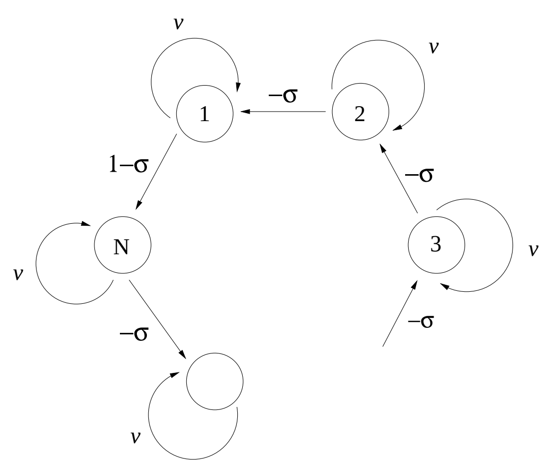

Figure 2: Precedence Graph.

The elementary circuits are the loops (of weight v) and the complete circuit (of weight $1 - N\sigma$), so we have:

$$\rho = \min\left(v,\ \frac{1-N\sigma}{N}\right) \ .$$

Considering that the minimal space needed by a car on the road is σ, the average car density d is $N\sigma$ divided by the length of the road, taken equal to 1, therefore $d = N\sigma$. The average flow is equal to the car density times the average speed that is $f = \rho N\sigma$. Using the eigenvalue formula we obtain the fundamental traffic law :

$$f = \min\{\frac{v}{d}, \sigma(1-d)\} \ .$$

Using this minplus model, we find the results presented in [2].

5 The fundamental Traffic Law in the Deterministic Anticipative Case

Using minplus notations the dynamic of the traffic in the anticipative case may be written

$$X^{t+1} = B \otimes X^{t+1} \oplus C \otimes X^t \ . \tag{5}$$

where

$$B = \begin{bmatrix} \epsilon & -\sigma & & \\ & \epsilon & \ddots & \\ & & \ddots & -\sigma \\ 1-\sigma & & & \epsilon \end{bmatrix}, \quad C = \begin{bmatrix} v & & & \\ & \ddots & & \\ & & \ddots & \\ & & & v \end{bmatrix}.$$

This system is implicit, we have to compute X^{t+1} as a function of X^t. The existence of B^* is verified if and only if $1 - N\sigma \geq 0$, which is true otherwise we could not have N σ-sized cars inside a circular road of length 1. This condition being verified, the explicit form of the system may be computed using the star operation :

$$X^{t+1} = B^* \otimes C \otimes X^t. \tag{6}$$

The mean speed of the cars is the $\mathbb{R}_{\min}$ eigenvalue of $B^* \otimes C$ which is equal to v.

This result could have been guessed without any computation. Indeed in this deterministic case, all the cars may move with speed v (at the starting points the cars respect the security distance and they can move all together at speed v respecting this security distance).

6 Stochastic Modelling

Now we consider that at each time step t, each driver n chooses his wanted speed v_n^t randomly between $\{w, v\}$ with probability $\{\mu, \lambda\}$, $w \leq v$, that is the random variables $\{v_n^t \mid n = 1, \cdots, N; \ t \in \mathbb{N}\}$ are independent identically distributed Bernouilli random variables. We will suppose that –

$w = 0$, without loss of generality, – the security distance is 0 (this means that two cars may be in the same position), – the driver may anticipate the move of the car ahead. Then the dynamic of the system is :

$$x_n^{t+1} = \begin{cases} \min\left(v_n^t + x_n^t\ ,\ x_{n+1}^{t+1}\right)\ , & \text{if } n < N\ , \\ \min\left(v_N^t + x_N^t\ ,\ 1 + x_1^{t+1}\right)\ , & \text{if } n = N\ . \end{cases} \tag{7}$$

This system is still linear in the sense of the minplus algebra but stochastic. Within this algebra the formula (7) becomes

$$x_n^{t+1} = \begin{cases} v_n^t \otimes x_n^t \oplus x_{n+1}^{t+1}\ , & \text{if } n < N\ , \\ v_N^t \otimes x_N^t \oplus 1 \otimes x_1^{t+1}\ , & \text{if } n = N\ . \end{cases} \tag{8}$$

Defining

$$X^t = \begin{bmatrix} x_1^t \\ \vdots \\ x_N^t \end{bmatrix}, \qquad A = \begin{bmatrix} \epsilon & e & & \\ & \ddots & \ddots & \\ & & \ddots & e \\ 1 & & & \epsilon \end{bmatrix}, \qquad B^t = \begin{bmatrix} v_1^t & & & \\ & v_2^t & & \\ & & \ddots & \\ & & & v_N^t \end{bmatrix}$$

where the coefficients not stated are ϵ , we can rewrite the equations in a vectorial way

$$X^{t+1} = A \otimes X^{t+1} \oplus B^t \otimes X^t\ . \tag{9}$$

In our case A^* is easy to compute

$$A^* = \begin{bmatrix} e & e & \cdots & e \\ 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & e \\ 1 & \cdots & 1 & e \end{bmatrix}.$$

Then

$$X^{t+1} = C^t \otimes X^t \tag{10}$$

with $C^t = A^* \otimes B^t$

Using the fact that the matrices C^t are all irreducible we know by Cor. 7.31 of [1] that :

$$\lim_t x_n^t / t = \bar{v}, \forall n\ .$$

Then $\bar{v}$ is called the Lyapounov exponent of the stochastic minplus matrix C (with $(C^t)_{t\in\mathbb{N}}$ independent samples of C).

Computing explicitly the Lyapounov exponent is a difficult task. In [8] explicit formulas involving computation of expectations are given. Here we are able to characterize the stationary regime of X^t and to compute explicitly the expectation appearing in $\bar{v}$.

7 Jam Regime

In order to represent the state of the system we use diagrams where :

1. each segment outside the outer circle represents the amount of cars in that position;
2. the black [resp. grey]segments between the circles are proportional to the proportion of cars with wanted speed 0 [resp. v];
3. the cars numbered 1, $N/2$ and N are represented by a light-grey, grey and dark dot respectively.

In the Figure 3 we show the evolution of the system for 100 cars with speeds 0 and $v = 1/3$, at times $t = 0, 10, 100, 1000$

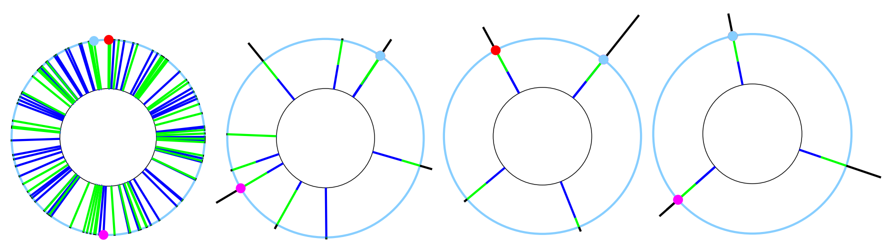

Figure 3: Example of evolution of the system (v=1/3).

In the Figure 3 we show the evolution of the system for 50 cars with speeds 0 and $v = 0.3$, at times $t = 0, 10, 100, 500$.

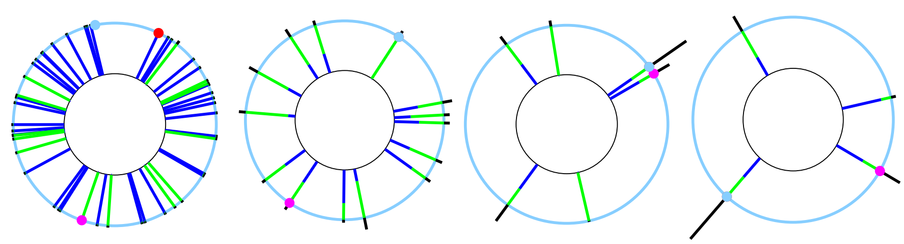

Figure 4: Example of evolution of the system (v=0.3).

Definition 1 *1. A* jam state *is a state where the cars are concentrated in k possibly empty clusters. The number $k = \left\lceil \frac{1}{v} \right\rceil$ is the upper round of $1/v$) $\{\pi_1, \cdots \pi_k\}$ with $\pi_{i+1} - \pi_i = v$ for $i = 1, \ldots, k-1$. In such a jam state the distance between two clusteres is v except for atmost one pair. When $1/v \in \mathbb{N}$ we say that the jam state is* regular*) and the distance between the clusters is always v.*

2. *When for all $t \geq T$ the system stays in jam states we say that after T the system is in a* jam regime.

Proposition 1 *A jam state is characterized by $d_v(x) = 0$ with :*

$$d_v(x) = \min_h \left(\sum_{j \neq h} \left\{ \frac{x_{j+1} - x_j}{v} \right\} \right); \ \textit{with} \quad \{x\} = x - [x] \ , \tag{11}$$

where $[x]$ denotes the integer part of x and therefore $\{x\}$ denotes the decimal part of x. For non jam states we have $d_v(x) > 0$. Moreover

$$d_v(X^T) = 0 \Rightarrow d_v(X^t) = 0, \ \forall t \geq T \ ,$$

that is after to be entered in a jam state we stay in a jam regime.

Proof: It is easy to see that $d_v(x) = 0$ for a jam state x. The question is then to show the converse. Let us suppose that $d_v(x) = 0$ by definition of d_v there is an h_0 such that

$$\sum_{j \neq h_0} \left\{ \frac{x_{j+1} - x_j}{v} \right\} = 0 \, ,$$

then for every $j \neq h_0$ we have that

$$\left\{ \frac{x_{j+1} - x_j}{v} \right\} = 0 \, .$$

So we are in a jam state.

After having reached a jam state the system stays in a jam regime because the wanted moving size of the cars are v or 0. In a jam state only two clusters (at the most) h and $h+1$ may be at a distance different of v. After a moving of only one car there is only two possibilities : – the cluster stays in the same position – the cluster $h+1$ change in such a way that the distance of cluster h and $h+1$ becomes v. Then the new state is still a jam state with one cluster in a new position $\pi_{h+1} - \pi_h = v$ and $\pi_{h+2} - \pi_{h+1} \leq v$. $\square$

The function $d_v(x)$ can be seen as a sort of distance to a jam regime.

Some numerical experiments (Figure 5) suggest the following result which is not proved.

Conjecture : The sequence $\left(d_v(X^t)\right)_t$ is non increasing.

Theorem 1 *A jam regime is reached with probability one.*

IIn order to prove that a jam regime is reachable, we construct a finite sequence of independent event with positive probability after which the system reaches a jam state. Then this finite sequence will appear with probability one in an infinite sequence of events.

The dynamic of the system is given by the matrix $C(\omega) = A^* B(v(\omega))$ where B is the diagonal matrix of car wanted-speeds chosen randomly and independently between 0 and v. Let us consider

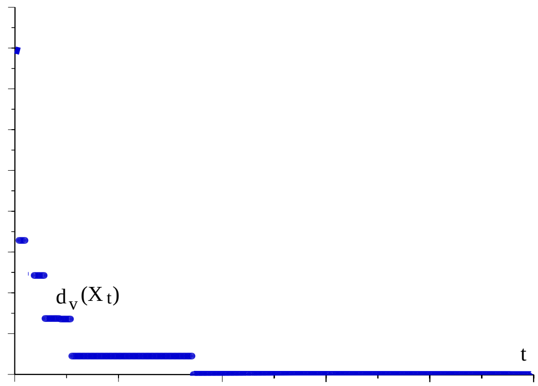

Figure 5: Evolution of $d_v(X^t)$ with t.

the matrix C_j associated to the speed $v = (0 \cdots 0, v, 0 \cdots 0)$ with v in position j. All the matrices C_j, $j = 1 \cdots N$ have a strictly positive probability of occurrence.

Consider the finite sequence of independent events associated to the following matrix product

$$C_1^k C_2^k ......C_{N-2}^k C_{N-1}^k \ ,$$

it is easy to understand that after these events all the cars are together in only one cluster which is a jam state. Then the proposition gives the result. □

This proof suggests that the time needed to reach a jam regime is very long, however there are other jam states reachable with a higher probability. The particular jam state in the proof has only the property to be easily characterized.

8 The Stationary Car Distribution

Let us determine the stationary distribution of the population of cars in the k clusters denoted $\mathcal{N} = (N_1 \cdots N_k)$.

Theorem 2 *The stationary distribution of $\mathcal{N}$ is uniform on the simplex :*

$$S = \left\{ (n_1 \cdots n_k) \ \mid \ \sum_{i=1,k} n_i = N, \ n_i \in \mathbb{N} \right\} \ .$$

Proof: Let us consider the Markov chain where the states belongs to the solution set of the previous diophantic equation. Then we have C_N^{N+k-1} nodes, where k is the number of clusters. Let us show that for each output arc in a node with transition probability p there is an incoming arc with the same transition probability (which show that the transition matrix is bistochastic). This property is clearly a local balance property.

To prove this local balance property let us consider the state $(n_1 \cdots n_k)$ then all the possible transitions following it are of the form

$$(n_1 - d_1 + d_k, \cdots, n_k - d_k + d_{k-1}) \quad \text{with } 0 \leq d_j \leq n_j$$

this means that there are d_j cars that leave the cluster j to the cluster $j+1$. The probability of that event is

$$\lambda^{\sum d_j} \mu^{\sum \phi(d_j, n_j)} \text{ where } \phi(d_j, n_j) = \begin{cases} 0 & \text{if } d_j = n_j \\ 1 & \text{otherwise} \end{cases} .$$

If we consider now the state $(n_1 - d_1 + d_2 \cdots n_k - d_k + d_1)$ then we can reach the state $(n_1 \cdots n_k)$ making leave d_2 cars from the cluster 1, d_3 cars from the cluster 3 and so on until the last one in which we make leave d_1 cars. Now the probability of this event is

$$\lambda^{\sum d_j} \mu^{\sum \phi(d_{j+1}, n_j - d_j + d_{j+1})}$$

but $\phi(d_{j+1}, n_j - d_j + d_{j+1}) = \phi(d_j, n_j)$and so we have the same probability.

To complete the proof we have to show that this construction which associates to each output arc an input one is a bijective mapping. For that, since the mapping is injective, let us show that the number of outgoing arcs from particular state $\mathcal{N} = (n_1 \cdots n_k)$ is equal to the number of incoming arcs to this state. The number of outgoing arcs from $\mathcal{N}$ is the number of elements of the set

$$\{(d_1 \cdots d_k) \mid 0 \leq d_i \leq n_i, \; i = 1 \cdots k\} .$$

The number of incoming arcs to $\mathcal{N}$ is the number of elements of the set

$$\left\{(d_1 \cdots d_k) \mid 0 \leq n'_i - d_i + d_{i-1} \leq n_i, \; n'_i - d_i \geq 0, \; i = 1 \cdots k\right\} .$$

These two numbers are equal because

$$0 \leq n'_i - d_i + d_{i-1} \leq n_i, \quad n'_i - d_i \geq 0, \quad 0 \leq n'_i \;\Leftrightarrow\; 0 \leq d_{i-1} \leq n_i .$$

□

9 Mean Speed Computation

Knowing the stationary measure we are able to compute explicitely the mean speed.

Theorem 3 *If* $k \stackrel{\text{def}}{=} 1/v \in \mathbb{N}$ *then the mean speed satisfies*

$$\bar{v}_\lambda(N, k) = \frac{\lambda v}{N\mu}(k - S_k(N))$$

with

$$(N + k)S_k(N + 1) = k - 1 + (N + 1)\lambda S_k(N), \;\; S_k(0) = k, \;\; \forall k, N \in \mathbb{N} .$$

Moreover for large N *we have the asymptotic*

$$\bar{v}_\lambda(N, k) = \frac{\lambda}{N\mu} + o(1/N) .$$

Example 1 *1.* $\bar{v}_\lambda(3,3) = v(6\lambda + 3\lambda^2 + \lambda^3)/10$.

2. $\bar{v}_\lambda(4,4) = v(\lambda^4 + 4\lambda^3 + 10\lambda^2 + 20\lambda)/35$.

In Figure 6 we show a plot of the mean speed as a function of N and k when $\lambda = 0.5$.

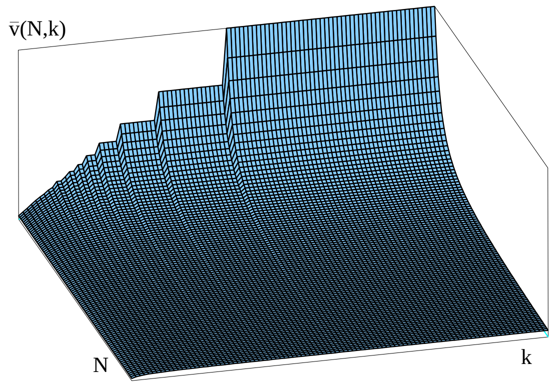

Figure 6: $\bar{v}_{0.5}(N,k)$.

Proof: Knowing the distribution of $\mathcal{N}$ let us compute the mean speed in the following way : the first car of a queue leaves with probability λ increasing the mean speed of v/N, then the second car leaves this queue with probability λ^2 increasing the mean speed of v/N and so on. Then the mean speed $\overline{v} = \mathbb{E}(V)$ with

$$V = \sum_{s=1}^{k} \left(\sum_{j=1}^{N_s} \lambda^j \frac{v}{N} \right) ,$$

that is

$$V = \lambda \frac{v}{N} \sum_{s=1}^{k} \frac{1-\lambda^{N_s}}{1-\lambda} = \frac{\lambda}{\mu} \frac{v}{N} \left(k - \sum_{s=1}^{k} \lambda^{N_s} \right) .$$

If $k = 1$ we easily obtain that

$$V = \frac{\lambda - \lambda^{N+1}}{1-\lambda} \frac{v}{N}.$$

Let us assume that $k \geq 2$ and let us denote

$$S_k(N) = \mathbb{E}\left(\sum_{s=1}^{k} \lambda^{N_s} \right) = \frac{1}{\mathrm{C}_N^{N+k-1}} \sum_{\sum_s N_s = N} \lambda^{N_s} = \frac{1}{\mathrm{C}_N^{N+k-1}} \sum_{s=1}^{k} \sum_{h=0}^{N} \sum_{\sum_{j \neq s} N_j = N-h} \lambda^h .$$

Then counting we obtain :

$$S_k(N) = \frac{Z(N)}{\mathsf{C}_N^{N+k-1}} \text{ with } Z(N) = \sum_{h=0}^{N} \mathsf{C}_{N-h}^{N+k-h-2} \lambda^h \ .$$

If we call $z = 1/\lambda$ and D_z the derivative with respect to z we obtain that

$$Z(N) = \frac{\lambda^N}{(k-2)!} \sum_{h=0}^{N} D_z^{k-2} \, z^{N+k-h-2} \ . \tag{12}$$

Therefore

$$Z(N+1) = \frac{\lambda^{N+1}}{(k-2)!} \sum_{h=0}^{N+1} D_z^{k-2} \, z^{N+1+k-h-2} \, ,$$

$$Z(N+1) = \frac{\lambda^{N+1}}{(k-2)!} D_z^{k-2} \left(z^{N+1+k} + \sum_{h=0}^{N} z^{N+k-h-2} \right) ,$$

$$Z(N+1) = \frac{\lambda^{N+1}}{(k-2)!} \left(D_z^{k-2} z^{N-1+k} + \sum_{h=0}^{N} D_z^{k-2} \, z^{N+k-h-2} \right) ,$$

but from (12) we have that

$$Z(N+1) = \frac{\lambda^{N+1}}{(k-2)!} \left(D_z^{k-2} z^{N-1+k} + \mathsf{C}_N^{N+k-1} (k-2)! \lambda^{-N} S_k(N) \right) \ .$$

Computing the derivative and simplifying we obtain

$$S_k(N+1) = \frac{k-1}{N+k} + \frac{N+1}{N+k} \lambda S_k(N).$$

which proves the first part of th theorem.

To find the asymptotic we remark that $S_k(N)$ goes to 0 when N goes to ∞. □

In Figure 7 we show a simulation of X_t/t converging towards the computed Lyapounov exponent $\bar{v}$.

10 Extensions and Numerical Results

The previous analysis of the stochastic model may be done also in the non anticipative case. It can be extended to the case where the cars have a non negligible size σ. The models are still stochastic minplus linear, for example in the latter case, we have :

$$x_n^{t+1} = \begin{cases} v_n^t \, x_n^t \oplus (-\sigma) x_{n+1}^{t+1} \, , & \text{if } n < N \\ v_N^t x_N^t \oplus (1-\sigma) x_1^{t+1} \, , & \text{if } n = N \end{cases}$$

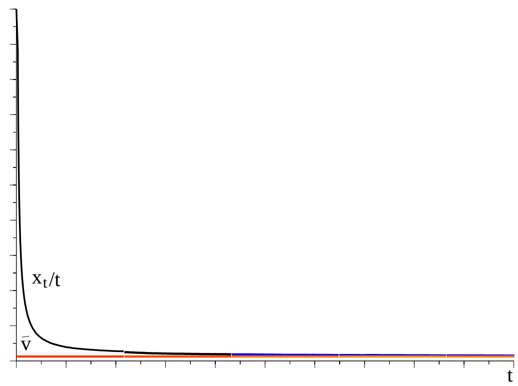

Figure 7: Convergence of X_t/t towards $\bar{v}$.

The formula giving the mean speed can be extended to the case where $1/v \notin \mathbb{N}$ and when overtaking is allowed.

Then, using the formulæ obtained, or a simulator using the maxplus Scilab toolbox [11] we can plot the fundamental traffic law in different cases.

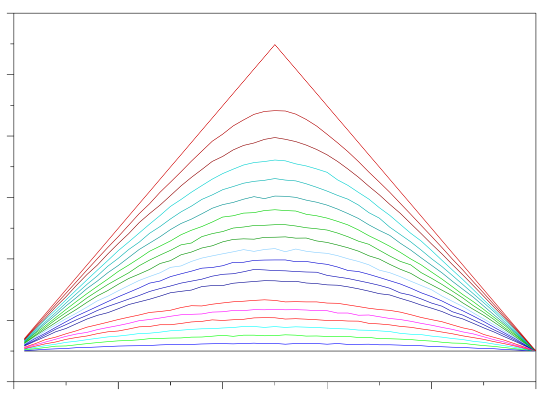

Figure 8: Flow as a function of the density in the stochastic non anticipative case for a continuation of λ when $v = \sigma$.

In conclusion we see that the stochastic model introduce mainly a smoothing of the deterministic case and that the deterministic case is trivial to obtain in the framework of minplus algebra.

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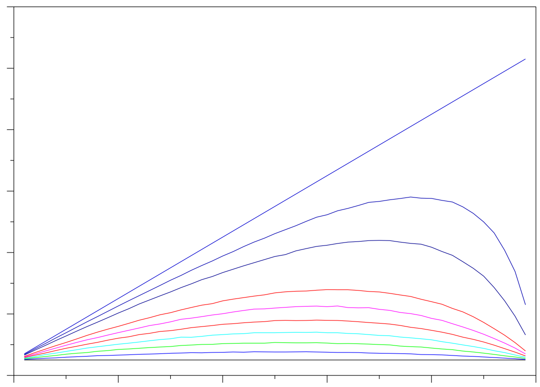

Figure 9: Flow as a function of the density in the stochastic anticipative case for a continuation of λ when $v = \sigma$.

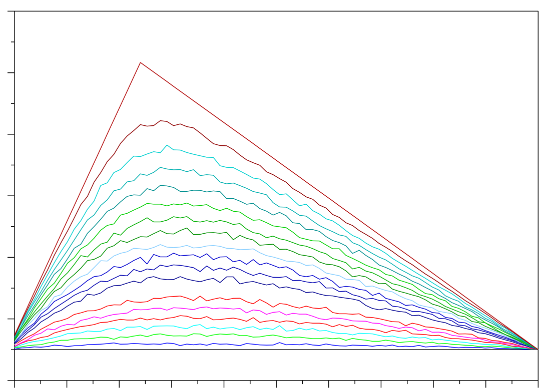

Figure 10: Flow as a function of the density in the stochastic non anticipative case for a continuation of λ when $v = 3\sigma$.

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