A Convergent Algorithm for the Multimodal Traffic Equilibrium Problem

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Abstract: The multimodal traffic equilibrium problem is described in general by non-monotone, asymmetric, variational inequality problem. We show that in spite of the non-monotonicity of the cost operator, the problem may admit a different weaker property, which describes the hierarchical nature of the travel cost interactions. This property permits a natural decomposition approach, not otherwise available, which admits provably convergent algorithms. We present one such algorithm, easily implementable using a solver for the single-class traffic equilibrium problem, and a convergence proof. This represents the first provably convergent method for solving the non-monotone, asymmetric, traffic equilibrium problem.

Key-words: asymmetric multiclass traffic assignment, nested monotonicity, descent method

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Un algorithme convergeant pour le problème d'affectation multiclasse du trafic

Résumé : Le problème d'affectation multiclasse du trafic s'exprime en général comme une inéquation variationelle non-monotone, avec un opérateur asymétrique. Nous montrons que malgré la non-monotonie de l'opérateur, une autre propriété, moins restrictive, peut être satisfaite. Cette propriété décrit la nature hiérarchique des interactions entre classes de trafic. Elle permit également une décomposition naturelle de l'opérateur par classe, et avec cette décomposition des algorithmes convergeants. Nous présentons un tel algorithme, qui est par ailleurs facile à implémenter à partir d'un solveur pour le problème d'affectation uni-classe, et sa preuve de convergence.

Mots-clés : affectation asymmetrique du trafic, monotonie enboîtée, méthode de descente

1 Introduction

The multimodal (or multiclass) traffic equilibrium problem is a natural and important generalization of the classic traffic equilibrium problem. While the latter can be expressed as a convex optimization program under the relatively mild assumption of strictly increasing travel cost functions, the former admits no such formulation, in general. Instead, the multimodal problem is expressed mathematically as a variational inequality problem, as introduced by Smith (1979). Indeed, in the case of a multimodal equilibrium problem, the cost operator for each class depends on the flow of vehicles of more than one class, and, in particular, these dependencies are generally not symmetric. As a result, the jacobian matrix of the overall cost operator is itself asymmetric. When this occurs, the jacobian matrix is not diagonalisable; this implies that the variational inequality defining the Wardrop equilibrium conditions is not the optimality condition of a particular optimization problem. That is, the overall cost operator is not the gradient of another function defined over the entire feasible region, and there is no analogous "Beckmann transformation" into an equivalent convex optimization problem.

While the multimodal traffic equilibrium problem, by definition, then, does not permit the application of the favorite algorithms for the separable, single-class traffic assignment problem, efficient algorithmic approaches, can in principle, be applied. Methods for solving monotone variational inequality problems are the focus of a large body of research in the nonlinear programming community. The interested reader is referred to Patriksson (1999) for a summary of methods for monotone variational inequality problems.

Another popular resolution strategy for the asymmetric traffic assignment problem is the so-called *diagonalization method*, based on the idea of the Jacobi or the Gauss-Seidel decomposition approaches used for solving systems of nonlinear equations.

The idea behind the method is to fix the off-diagonal, or cross terms, of the multivariate cost functions and to then solve a sequence of diagonal (and thus separable) problems. In the case of the traffic assignment problem, the diagonalised, single-class assignments can be solved, following a simple transformation, by various methods for convex programming. One of the most widely-used in this case is the popular Frank-Wolfe, or convex combinations, method. This approach leads to solving a sequence of problems of the form

$$\min_{x_j \in \mathcal{X}} \sum_{a \in \mathcal{A}} \int_0^{x_{aj}} t(x_1^k, x_2^k, \ldots, s_j, \ldots, x_m^k) d(s_j) \tag{1}$$

where $\mathcal{A}$ is the set of arcs in the network, the travel cost function is $t : \mathcal{X} \mapsto \mathcal{X}$, and the model includes m user classes, or modes. The iteration number is referred to as k. Thus, since at each inner iteration of the decomposition algorithm, the problem to be solved is a single-class traffic equilibrium problem, the favorite algorithms can be used in a sequential (or even parallel) manner.

However, this decomposition method, as is the case for the algorithms for variational inequalities alluded to above, is provably convergent only under relatively stringent conditions on the travel cost operator. These conditions take the form of lower semi-continuity and some form of monotonicity, depending on the particular algorithm used. The most straightforward algorithms require strict monotonicity, and variants exist which allow the equilibrium problem to be defined in terms of monotone operators. In the case of the diagonalization method above, Ahn and Hogan (1982), Florian and Spiess (1982), and Pang and Chan (1982) (see also Dupuis and Darveau 1986) give local convergence results based upon two assumptions on the travel cost functions:

- The cost functions are (Frechet) differentiable in the neighborhood of the equilibrium solution x^*.
- $\|I - D[\nabla t(x^*)]^{-1/2} B D[\nabla t(x^*)]^{-1/2}\|^2 < 1$,

where D refers to the diagonal part of the matrix, $t(x^*)$ is the travel time function evaluated at an equilibrium solution x^*, $B = \nabla t(x^*) - D[\nabla t(x^*)]$ and $\| \ \|$ represents the matrix norm. These two conditions ensure that the diagonalisation algorithm converges to a locally unique solution x^* (Florian and Spiess 1982). Global convergence requires a strict monotonicity assumption over the entire feasible region.

The second condition implies that $\nabla t(x^*)$ is positive definite (Ahn 1979), though the converse is not necessarily true. A sufficient condition for it to hold, however, is that $\nabla t(x^*)$ is both row and column diagonally dominant, that is, letting $M = \nabla t(x^*)$:

$$|M_{ii}| > \max\{\sum_{i \neq j} |M_{ij}|, \sum_{i \neq j} |M_{ji}|\} \quad \forall\, i, j = 1, \ldots, |A|. \tag{2}$$

In other words, the interactions due to members of one's own class are much greater than the cross interactions.

It is clear that if a model includes cost functions that incorporate more than one class of traffic, mode, or flow on more than one arc, then these interactions will not be all negligeable. (Or, in that case, a separable approximation of the multimodal model would have been sufficient in the first place). Numerical experiments have confirmed the non-monotonicity of the travel cost operator, in practice. Indeed, it has been observed numerous times (see Toint and Wynter, 1996, for one such example) that the cost operator in a multimodal traffic equilibrium problem is almost never monotone. In this context, the majority of algorithms used to solve multimodal problems are no longer provably convergent, and indeed, it can be shown that such algorithms may frequently converge to points which are not equilibria at all.

In this paper, we examine the nature of the cost operator in a multimodal traffic equilibrium problem. We show, in section 2, the behavior that can be exhibited by these algorithms when the monotonicity requirement is not met, that is, that without the necessary monotonicity property, algorithms may converge to non-equilibrium points. Next, we show that a different property, strictly weaker than monotonicity, does hold in a many cases, and provides a very elegant interpretation of the interactions across variables in the multivariate travel cost functions. A characterization of this property is provided in section 3. Of significant importance is that this property has been shown to admit provably convergent algorithms. This class of algorithms is described in section 4. Section 5 concludes with a discussion of the consequences of this result for both the traffic equilibrium problem, and the more general optimal pricing and network design problems.

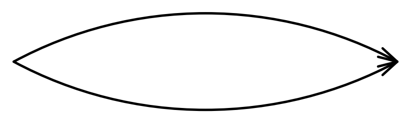

FIGURE 1: A two-arc network

2 Difficulties with current algorithms requiring monotonicity

Let us begin by recalling the basic definition of traffic equilibrium. Given a network defined over a graph $G = (N, A)$, N being a set of nodes, A a set of arcs, and a set of origin-destination pairs, W, each OD pair is said to have associated with it a demand d_w that must be satisfied. As in most other network flow problems, the arcs have associated with them cost functions, $t_a(x)$, that increase with the flow on that (and possibly another) arc, x_a. An equilibrium is reached on the network when the flow distrubutes itself in such a way so that all demands are satisfied, and no single user can reduce his travel cost by changing path.

The multiclass traffic assignment problem has the added complexity that demands exist between each OD pair for more than one class of flow; as such, the flows are vector valued. More precisely, each OD pair demand d_w and each arc flow variable x_a is identified by two indices, that is, d_{iw} and x_{ia}, where i is the class index. Similarly, the travel costs on the network links are functions of more than one variable, that is, $t_a(x) = t_a(x_1, x_2, \ldots, x_K)$ for some number K of classes. This problem is referred to as *asymmetric* when the effect of one class on another is not the same as vice-versa; that is, $\partial t_i^1(x)/\partial x_j^1 \neq \partial t_j^1(x)/\partial x_i^1$ for all classes $i \neq j$ on the graph G.

The traffic assignment problem is of practical interest for predicting the likely outcome of a change in network infrastructure or services. The multimodal traffic assignment problem has received particular attention, as numerous planning scenarios require making the distinction between different flow types on the network, such as buses versus cars, heavy versus light vehicles, guided versus unguided vehicles, as well as urban network models in which the flows on intersecting links interact through their travel cost functions.

Next, we examine analytically a simple example that illustrates the presence of multiple equilibria, stable and unstable. Consider the two-arc network of Figure 1 with linear cost functions for two classes and origin-destination demand as given below:

$$\begin{aligned} t_{1i}(x) &= 1.5x_{1i} + 5x_{2i} + 30 \\ t_{2i}(x) &= 1.3x_{1i} + 2.6x_{2i} + 28 \\ d &= [16, 4] \end{aligned} \tag{3}$$

where $t_{ki}(x)$ is the travel time and x_{ki} the flow (or the number of vehicles) of class k on arc i, $i = 1, 2$, and d_k is the demand for class k. Then, the feasible domain $\mathcal{X}$ consists of those x_{ji} for which $\sum_{i\in[1,2]} x_{1i} = 16$ and $\sum_{i\in[1,2]} x_{2i} = 4$ and $x_{ji} \geq 0$ for all $i, j \in [1, 2]$. It is clear that the partial derivative, or Jacobian, matrix $\nabla t(x)$ of these cost functions is not positive definite, and thus the functions are not monotonic. That is, $\langle t(x) - t(y), (x - y)\rangle \not\geq 0 \;\; \forall x, y \in \mathcal{X}$. Consequently, we cannot

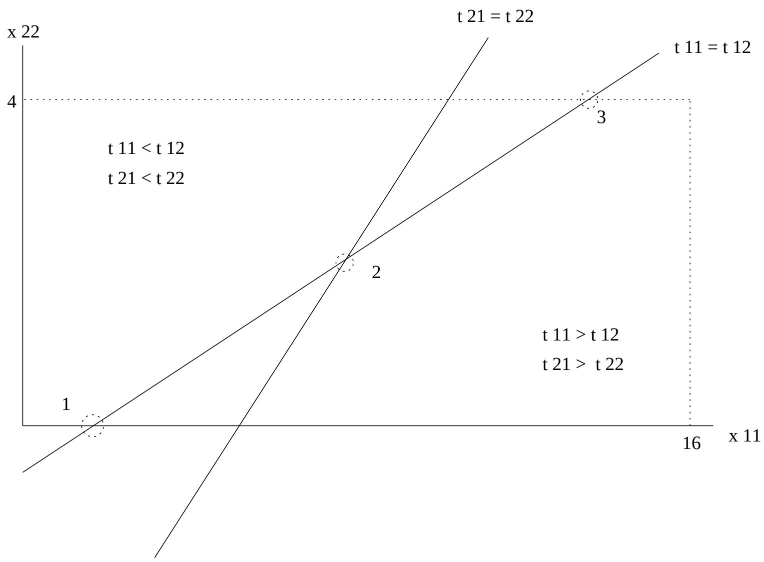

FIGURE 2: Three equilibrium solutions

be certain that the solution to the multimodal traffic equilibrium problem defined by (3) is unique. In fact, it turns out that there are three equilibrium solutions, shown in Figure 2.

The three solutions correspond to the following flows, where the first row gives the amount of the first class on arcs 1 and 2, and the second row the amount of the second class:

1.	1.33	14.67	2.	8	8	3.	14.67	1.33
	4	0		2	2		0	4

Netter (1972) illustrated a similar example, also based on a two-arc network with linear cost functions. The article is important in that it was the first to have discussed the presence of multiple equilibria. Furthermore, Netter indicated that the middle solution, which corresponds to the two classes sharing fully the two arcs, was "unstable".

The costs for each of the three solutions are:

1.	52	52	2.	52	52	3.	52	52
	40	47		44	44		47	40

where again, the first row gives the travel times for a vehicle of class 1 on arcs 1 and 2, and the second gives the travel times for class 2.

Solution 2 is given by an equal split of the two classes on the two arcs. Although the equilibrium condition is satisfied at this point, it is an unstable solution in that altering even slightly the flow of class 2 on the network will result in vehicles of that class seeking a different, lower cost, flow pattern.

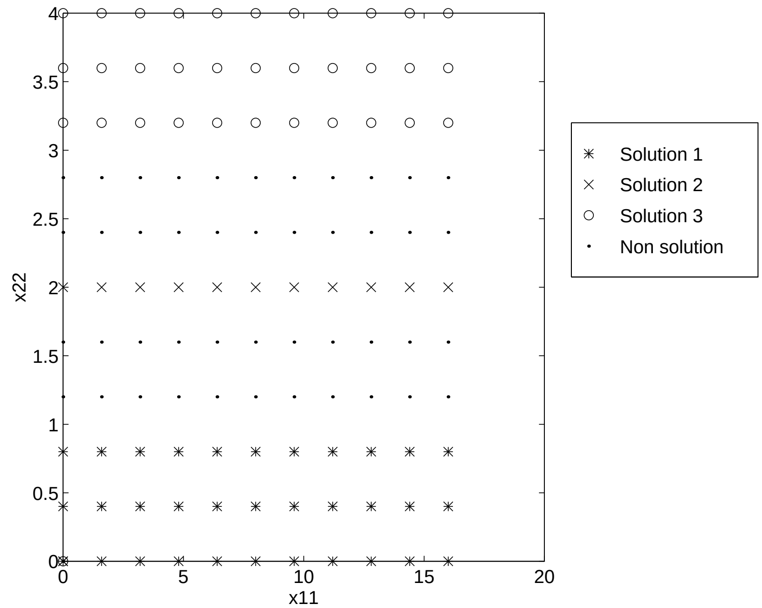

FIGURE 3: The convergence to different solutions according to the algorithm starting point

Consequently, vehicles of both classes will re-equilibrate into a new, stable equilibrium, and solution 2 is a flow that would probably never be observed, according to the cost function used.

In Figure 3, we illustrate the solutions found by the diagonalisation method according to 121 different starting points in the feasible region. (For example, starting point ($x_{11} = 0$ and $x_{22} = 0$) corresponds to starting the algorithm at $x_{11} = 0$, $x_{12} = d_1 = 16$, $x_{21} = d_2 = 4$ and $x_{22} = 0$, so that all of the class 1 flow is initially on arc 2 and all of the class 2 flow on arc 1.) The convergence criterion of the algorithm is defined as the aggregate flow differences between successive iterations, and is set to 10^{-5} (or a maximum of 100 diagonalisation iterations).

It is clear that the solution chosen differs according to the starting point of the algorithm. Aside from the origin, which is a special case, starting the algorithm on a point along the first three rows causes it to converge to Solution 1, and similarly for the last three rows and Solution 3. When the algorithm is started along the line $x_{22} = 2$, and at the origin, it converges to the unstable solution. Furthermore, when the algorithm is started around the line $x_{22} = 2$, it stops at a non-equilibrium flow solution.

The preceding example shows clearly that algorithms for monotone variational inequalities can fail to converge, or may even converge to a non-equilibrium value, when applied to the usual, non-monotone traffic equilibrium problems. Furthermore, Figure 3 gives an algorithmic interpretation to the “unstability” of the Solution 2: the vector fields (here, given by the Frank-Wolfe direction vector) point towards Solution 2 only along the line $x_{22} = 2$.

3 A weaker form of monotonicity

While monotonicity can be shown not to hold for most multimodal traffic equilibrium problems by construction, we will show in this section that the notion of nested monotonicity, introduced in Cohen and Chaplais (1988), offers an elegant explanation of the process of interactions in the multimodal travel cost functions. Furthermore, this class of functions can be shown to admit provably convergent algorithms.

First we introduce a number of definitions, notation, and assumptions.

ASSUMPTION 3.1 *The following properties of the problem (4) will be assumed to hold.*

1. *The subset $\mathcal{X} \subset \Re^n$ is closed, bounded, and convex.*
2. *Suppose further that the operator $t : \Re^n_+ \mapsto \Re^n_+$ is continuously differentiable on $\mathcal{X}$*

While the boundedness of $\mathcal{X}$ is not needed, and a coercivity condition would suffice, the (fixed demand) traffic equilibrium is by construction defined over a polyhedral, convex, compact set. Note that, by using one of the many transformations available (see, e.g. [18]), the elastic demand model also fits into this framework.

The variational inequality that defines the multimodal traffic equilibrium problem is given by:

DEFINITION 3.2 [Variational inequality problem] *Find $x^* \in \mathcal{X}$ such that*

$$\langle t(x^*), x - x^* \rangle \geq 0, \quad \forall x \in \mathcal{X}. \tag{4}$$

In the traffic equilibrium problem, the feasible set $\mathcal{X}$ is a cartesian product of component subspaces, that is:

$$x = (x_1, \ldots, x_n) \in \mathcal{X} \tag{5}$$
$$x_i \in \mathcal{X}_i \tag{6}$$
$$\mathcal{X} = \prod_{i=1..n} \mathcal{X}_i \tag{7}$$

Let $n_i := |\mathcal{X}_i|$ be the dimension of the ith component subspace. Furthermore, we have that

$$t(x) = (t_1(x), \ldots, t_n(x)). \tag{8}$$

Using the above notation, the variational inequality (4) is equivalent to the following system of coupled variational inequalities:

DEFINITION 3.3 [Coupled system of variational inequalities] *Find $x^* \in \mathcal{X}$ such that for $i = 1, \ldots, n$,*

$$\langle t_i(x^*), x_i - x_i^* \rangle \geq 0, \quad \forall x_i \in \mathcal{X}_i. \tag{9}$$

DEFINITION 3.4 [Strong monotonicity] *An operator t is* strongly monotone *on $\mathcal{X}$ if there exists a constant $m_t > 0$ such that:*

$$\langle t(x) - t(y), x - y \rangle \geq m_t \|x - y\|^2, \quad \forall x, y \in \mathcal{X}. \tag{10}$$

DEFINITION 3.5 *An operator t is* Lipschitz *on $\mathcal{X}$ if there exists a constant $M_t > 0$ such that:*

$$\| t(x) - t(y) \| \leq M_t \| x - y \|, \quad \forall x, y \in \mathcal{X}. \tag{11}$$

In order to define nested monotonicity, it is necessary to further decompose the operator t and feasible set $\mathcal{X}$ into the following. Let

$$x_{<i} := (x_1, \ldots, x_{i-1}) \tag{12}$$
$$x_{\geq i} := (x_i, \ldots, x_n) \tag{13}$$
$$\mathcal{X}_{<i} := \prod_{i \in [1, i-1]} \mathcal{X}_i \tag{14}$$
$$\mathcal{X}_{\geq i} := \prod_{i \in [i, n]} \mathcal{X}_i. \tag{15}$$

Analogous definitions hold for $x_{>i}$ and $\mathcal{X}_{>i}$.

Then, the definition of a nested monotone operator relative to the decomposition above is as follows (Cohen and Chaplais, 1988):

DEFINITION 3.6 [Strong nested monotonicity] *An operator $t : \mathcal{X} \mapsto \mathcal{X}$ is* strongly nested monotone *on $\mathcal{X}$ relative to the decomposition (6) if and only if the following three conditions hold:*

1. *The operator $t_1(x_1, x_{>1})$ is strongly monotone in x_1 on $\mathcal{X}_1$, uniformly in $x_{>1}$ on $\mathcal{X}_{>1}$. Then, under the assumptions 1.1–3, the VIP (9) in t_1 has a unique solution, parameterized by $x_{>1}$. Denote this parametric solution $x_1^*(x_{>1})$.*
2. *The operators $t_i(x^*_{<i}(x_{\geq i}), x_i, x_{>i})$ are strongly monotone in x_i over $\mathcal{X}_i$, uniformly in $x_{>i}$ over $\mathcal{X}_{>i}$. Each such VIP then also has, under assumptions 1.1–3, a unique parametric solution denoted by $x_i^*(x_{>i})$.*
3. *The operator $t_n(x^*_{<n}(x_n), x_n)$ is strongly monotone in x_n on $\mathcal{X}_n$. Then, under the assumptions 1.1–3, the VIP (9) in t_n has a unique solution.*

It is easy to see that any strongly monotone function is strongly nested monotone, but that the converse is not true. A proof is nonetheless provided in [2]. Examples in [2] and in the following section illustrate that a strongly nested monotone function need not even be monotone.

In the next section, we show that nested monotonicity will hold in many cases of the multimodal traffic equilibrium problem, and further provides an elegant interpretation of the cross-class interactions.

In full generality, the decomposition used in defining the coupled system of variational inequalities reduces each subset $\mathcal{X}_i \quad i \in [1, n]$ to a one-dimensional subspace. This may not, however, be the most computationally advantageous approach, even if that decomposition admits the strong nested monotonicity property; it corresponds, in some sense, to a $n-$level hierarchical problem, which is clearly difficult to manipulate. In many cases, a small number of classes is found to underly a hierarchical problem, and this is particularly so for traffic networks. Indeed, one can often identify two distinct classes, and a natural hierarchy between their cost functions.

For this reason, as well as for clarity of exposition, we therefore simplify the multimodal equilibrium problem to one having two interacting classes. Then, the decomposition is immediate and the definition of nested monotonicity reduces to the following:

DEFINITION 3.7 [Nested monotonicity over two subspaces] *The operator t is nested monotone if and only if:*

1. *$t_1(x_1, x_2)$ is strongly monotone in x_1 on $\mathcal{X}_1$, uniformly in x_2 on $\mathcal{X}_2$. Let $x_1^*(x_2)$ solve VIP($t_1(x_1, x_2)$,$\mathcal{X}_1$).*
2. *$t_2(x_1^*(x_2), x_2)$ is strongly monotone in x_2 over $\mathcal{X}_2$.*

This decomposition corresponds to a multimodal traffic network with two traffic classes, x_1 and x_2. Note however, that t_1 and t_2 are still vector functions, where $\|\mathcal{X}_1\| = n_1$ and $\|\mathcal{X}_2\| = n_2$. In general, both n_1 and n_2 will be equal to the number of arcs of the network, though this is not needed for the results or methods proposed in this paper.

4 Nested monotonicity of multimodal traffic cost operators

Consider now a network with two traffic classes, e.g., buses and cars, or heavy and light vehicles. Recall the two arc network of figure 1. In the following examples, suppose that the total origin-destination demand from node 1 to node 2 of class "c" is equal to 10, and 20 for class "b". All numerical results in this section were obtained using Maple, version 6.

Let the multimodal cost operators be given below:

$$t_1^c := 2(x_1^c/6)^3 + 2 + 1.5(x_1^b) \tag{16}$$
$$t_2^c := (x_2^c/8)^3 + 5 + 1.3(x_2^b) \tag{17}$$
$$t_1^b := 2(x_1^c/6)^2 + 2 + 2.3(x_1^b)^{1.2} \tag{18}$$
$$t_2^b := (x_2^c/8)^2 + 5 + 2.2(x_2^b)^{1.2} \tag{19}$$

where t_i^c gives the travel time of the "car" class, c on arc i, and t_i^b gives the travel time of the "bus" class, b on arc i. Similarly, x_i^c and x_i^b give the flows of the car and bus classes on arc i.

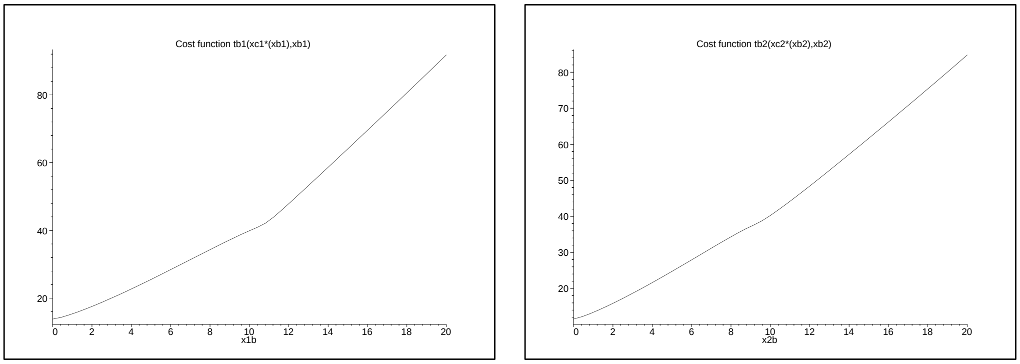

FIGURE 4: The nested operators $t_1^b(x_1^{c^*}(x_1^b), x_1^b)$ and $t_2^b(x_2^{c^*}(x_2^b), x_2^b)$ of equations (16)–(19) are monotonically increasing functions

We observe that these functions do not define a globally monotonic cost operator $t := (t^c, t^b)$. Indeed, the jacobian matrix ∇t is given by:

$$\nabla \mathbf{t} = \begin{vmatrix} 1/36\,(x_1^c)^2 & 0 & 1.5 & 0 \\ 0 & 3/512(x_2^c)^2 & 0 & 1.3 \\ 1/9\,x_1^c & 0 & 2.76\,(x_1^b)^{0.2} & 0 \\ 0 & 1/32\,x_2^c & 0 & 2.64\,(x_2^b)^{0.2} \end{vmatrix}$$

The determinant of ∇t is then

$$\det(\nabla t) = .00118 x_1^c x_2^c x_1^b x_2^b - .00311 x_1^c x_2^c x_1^b - .00257 x_1^c x_2^c x_2^b + .00677 x_1^c x_2^c$$

which, evaluated at $x_1^c = 9.9$, $x_1^b = 19.9$ takes the value -0.04764. Similarly, at $x_1^c = 9.9$, $x_1^b = 0.1$, the determinant is equal to $-.01168$. Hence, the cost operator t is not monotone.

However, one may suspect a natural hierarchy underlying the cost structure. The strong nested monotonicity property is clearly satisfied by the first class since each cost function is increasing in the arguments of its own class variable. To test whether the overall cost operator does indeed have the strong nested monotonicity property, we must check whether the second class cost function is increasing when the optimal parametric solution of the first class is substituted for the first class variable.

We therefore solve the parametric traffic equilibrium problem in the variables x_i^c to obtain the solution $x_1^{c^*}(x_1^b)$, $x_2^{c^*}(x_2^b)$. Then, we can evaluate and plot the cost functions $t_1^b(x_1^{c^*}(x_1^b), x_1^b)$ and $t_2^b(x_2^{c^*}(x_2^b), x_2^b)$. The plots of the two functions are provided in Figures 4.

Observe that both functions are strictly increasing in their respective arguments. That is, the overall cost operator does indeed satisfy the property of nested monotonicity, even though it is globally non-monotone.

A second set of cost functions is given below:

$$t_1^c \quad := \quad 2(x_1^c/6)^3 + 2 + 1.5(x_1^b) \tag{20}$$

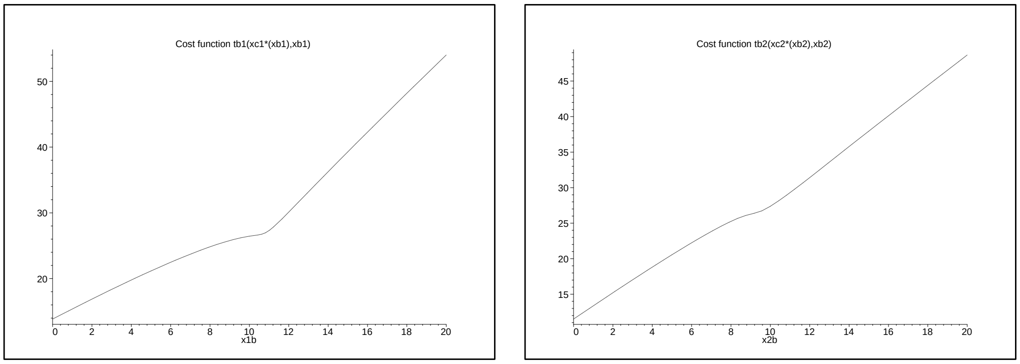

FIGURE 5: The nested operators $t_1^b(x_1^{c^*}(x_1^b), x_1^b)$ of equations (20)–(23) and $t_2^b(x_2^{c^*}(x_2^b), x_2^b)$ are monotonically increasing functions

$$t_2^c := (x_2^c/8)^3 + 5 + 1.3(x_2^b) \tag{21}$$
$$t_1^b := 2(x_1^c/6)^2 + 2 + 2.3(x_1^b) \tag{22}$$
$$t_2^b := (x_2^c/8)^2 + 5 + 2.2(x_2^b) \tag{23}$$

where the variables are defined as above. Note that the difference occurs in the value of the exponent of the terms t_i^b, where now the dependence is linear in flow of class b. As before, we confirm the non-monotonicity of the overll operator by taking the determinant of the jacobian of t and evaluating it at several extreme points of the demand polyhedron. We obtain for $x_1^c = 9.9$, $x_1^b = 19.9$ that $\det(\nabla t) = -.01814$ and for $x_1^c = 0.1$, $x_1^b = 19.9$, that $\det(\nabla t) = -.01380$; that is, the jacobian is not everywhere positive definite.

On the other hand, we can show, as above, that the operator t defined by equations 20–23 is indeed nested monotone. Again, solving for equilibrium in terms of the parametric variable $x^c(x^b)$, and substituting the optimal values into the cost functions $t^b(x^{c^*}(x^b),\, x^b)$, we obtain the functions illustrated in Figure 5.

The reader may have a doubt as to the monotonicity of the functions of Figure 5, namely at the inflection points. A zoom around the inflection point is therefore provided in Figure 6. Indeed, the nested functions are monotonically increasing functions.

Not all multimodal traffic cost functions satisfy this property, however. Consider the following set of functions.

$$t_1^c := 2(x_1^c/6)^3 + 2 + 1.5(x_1^b) \tag{24}$$
$$t_2^c := (x_2^c/8)^3 + 5 + 1.3(x_2^b) \tag{25}$$
$$t_1^b := 2(x_1^c/6)^2 + 2 + 2.3(x_1^b)^2 \tag{26}$$
$$t_2^b := (x_2^c/8)^2 + 5 + 2.2(x_2^b)^2 \tag{27}$$

The determinant of the jacobian for equations 24–(27) is equal to $-.161054$ when evaluated at $x_1^c = 9.9$, $x_1^b = 19.9$ and to $-.004224$ when evaluated at $x_1^c = 0.1$, $x_1^b = 19.9$. The overall operator is clearly then non-monotone.

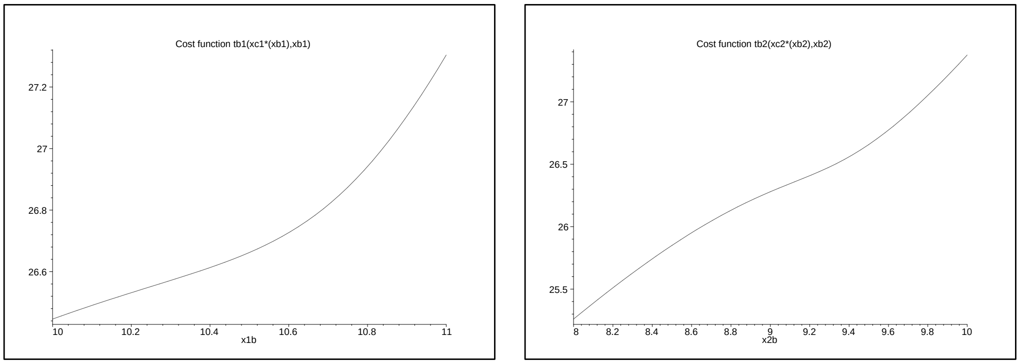

FIGURE 6: A zoom around the inflection points of the nested operators of equations (20)–(23)

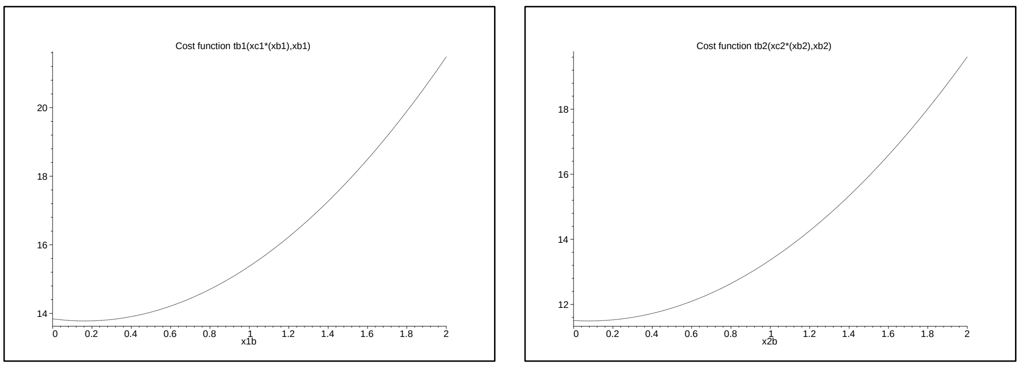

FIGURE 7: The nested operators $t_1^b(x_1^{c^*}(x_1^b), x_1^b)$ and $t_2^b(x_2^{c^*}(x_2^b), x_2^b)$ of equations (24)–(27) are not monotonically increasing functions

The plots of nested functions obtained from the system above are provided in Figure 7, where a zoom has been used to illustrate the portion of the two curves that is not monotonically increasing (between 0 and 2).

REMARK 4.1 (TESTING FOR NESTED MONOTONICITY) *In order to test for the property of nested monotonicity, it is necessary, in principle, to solve for the parametric equilibrium solution of the first class, in terms of the second, and then examine the cost function of the second class, as a function of this parametric solution. Clearly, over a large-scale network, such a study is highly impractical, if not impossible. If, however, the cost functions on each arc are structurally similar, up to additive and/or multiplicative constants, then a test for the nested monotonicity property over a small subnetwork, such as possibly one arc of the network, as in these examples, offers a good approximate test.*

5 An algorithm for the multimodal equilibrium problem

In this section, we describe a convergent algorithm for the multimodal traffic equilibrium problem. This is important in that the multimodal equilibrium problem has, to date, been considered to be a fundamentally non-monotone variational inequality problem, and therefore to satisfy no known conditions for algorithmic convergence. We have shown, in the previous section, that, while the problem is indeed non-monotone, in general, it can at the same time satisfy a different, weaker, property. We provide algorithmic results for multimodal equilibrium problems satisfying this weaker property of strong nested monotonicity.

Note that class indices c and b will at times throughout this section be expressed as subscripts (rather than superscripts) to permit the use of iteration counter superscripts.

The following assumptions will be required in this section.

ASSUMPTION 5.1 *Assume that the following positive constants $M_* > 0$ exist for all $x_c,\ \hat{x}_c \in \mathcal{X}_c$ and for all $x_b,\ \hat{x}_b \in \mathcal{X}_b$:*

$$\begin{aligned} \|t_c(x_c, x_b) - t_c(\hat{x}_c, x_b)\| &\leq M_{t_{cc}} \|x_c - \hat{x}_c\|, & (28)\\ \|t_c(x_c, x_b) - t_c(x_c, \hat{x}_b)\| &\leq M_{t_{cb}} \|x_b - \hat{x}_b\|, & (29)\\ \|t_b(x_c, x_b) - t_b(\hat{x}_c, x_b)\| &\leq M_{t_{bc}} \|x_c - \hat{x}_c\|, & (30)\\ \|t_b(x_c, x_b) - t_b(x_c, \hat{x}_b)\| &\leq M_{t_{bb}} \|x_b - \hat{x}_b\|, & (31) \end{aligned}$$

In the light of these assumptions, we have the following property of the optimal value function $x_c^*(x_b)$.

LEMMA 5.2 *Under assumptions (28)–(31), the strongly nested monotone operator t has the following Lipschitz properties:*

$$\begin{aligned} \|x_c^*(x_b) - x_c^*(\hat{x}_b)\| &\leq M_{x_{cb}} \|x_b - \hat{x}_b\|, & (32)\\ \|t_b(x_c^*(x_b), x_b) - t_b(x_c^*(\hat{x}_b), \hat{x}_b)\| &\leq M_{t_b} \|x_b - \hat{x}_b\|, & (33) \end{aligned}$$

Proof: The strong monotonicity of $t_c(\cdot, x_b)$ and $t_b(x_c^*(x_b), x_b)$ follow from the definition of a strongly nested monotone operator. Then the result follows from Lemma 5.1 of [2].

We will make use of the following lemma.

LEMMA 5.3 *Assume that $t_c \in C^1$ and*

$$\partial t_i^c(x^c, x^b)/\partial x_j^c = \partial t_j^c(x^c, x^b)/\partial x_i^c \qquad (34)$$

for all $i,j \in [0,n_c]$ *and for all* $x_c \in \mathcal{X}_c$. *Then, it is possible to solve VIP*$(t_c,\mathcal{X}_c)$ *by solving the following convex optimization problem.*

$$x_c^*(x_b) = \arg\min \int_{\mathbf{0}}^{\mathbf{x_c}} t_b(x, x_b^k)dx \tag{35}$$

subject to

$$x_c \in \mathcal{X}_c. \tag{36}$$

Proof: For x^b fixed, by (34), the mapping t is symmetric and therefore diagonalisable. Then, the mapping is the derivative of a potential operator, which is given in (35). By the strong monotonicity of $t_1(\cdot, x_b)$, the minimum of (35) is attained at a unique point.

Remark 5.4 *The above lemma states that the cost operator* t_c *is asymmetric only in the dependency upon class variables, and not with respect to flow variables on other arcs. That is, (asymmetric) interactions exist across vehicles on the same arc only for class c. This is with some loss of generality, for urban networks, but is nonetheless applicable to a large number of multimodal traffic equilibrium models.*

Remark 5.5 *The restriction imposed by Lemma 5.3 will be removed in algorithm 2, the* Decomposed descent method, *at the expense of a somewhat more involved algorithm and convergence proof.*

Consider next the pair of gap functions for the system of variational inequality problems, defined by the nested decomposition over two component subspaces. Let

$$g_c(x_c, x_b) := \max_{y\in\mathcal{X}_c} \phi_c(x_c) + \langle t_c(x_c, x_b) - \nabla\phi_c(x_c), x_c - y\rangle - \phi_c(y), \tag{37}$$

where $\phi_c : \Re^{n_c} \mapsto \Re$ is strongly convex and continuously differentiable. Let $y_c(x_c, x_b)$ be the unique solution to (37).

We define analogously the gap function for the VIP$(t_b, \mathcal{X}_b)$ as:

$$g_b(x_c^*(x_b), x_b) := \max_{y\in\mathcal{X}_b} \phi_b(x_b) + \langle t_b(x_c^*(x_b), x_b) - \nabla\phi_b(x_b), x_b - y\rangle - \phi_b(y), \tag{38}$$

where $\phi_b : \Re^{n_b} \mapsto \Re$ is strongly convex and continuously differentiable. Let $y_b(x_c^*(x_b), x_b)$ be the unique solution to (38).

One interesting special case of the gap functions (37) and (38) is when $\phi_c(x_c) := \frac{1}{2} \parallel x_c \parallel_{G_c}^2$, where $G_c \in \Re^{n_c\times n_c}$ is an a priori defined symmetric positive definite matrix, and similarly, $\phi_b(x_b) := \frac{1}{2} \parallel x_b \parallel_{G_b}^2$, with $G_b \in \Re^{n_b\times n_b}$.

Using this particular choice of auxiliary function ϕ leads to the following gap function for class i, as in [7]:

$$g_i(x_c, x_b) := \min_{y\in\mathcal{X}_i} \langle t_i(x_c, x_b), y - x_i\rangle + \frac{1}{2}\langle y - x_i, G_i(y - x_i)\rangle, \tag{39}$$

which can then be expressed as:

$$g_i(x_c, x_b) := \min_{y \in \mathcal{X}_i} \|y - (x_i - G_i^{-1} t_i(x_c, x_b))\|_{G_i}^2. \tag{40}$$

The differentiable, nonconvex optimization problems associated with (37) and (38), or (39) or (40) are given by:

$$\min_{x_c \in \mathcal{X}_c} g_c(x_c, x_b), \tag{41}$$

and

$$\min_{x_b \in \mathcal{X}_b} g_b(x_c^*(x_b), x_b). \tag{42}$$

Essential properties of the gap functions are given below. Using Lemma 5.2, the proofs may be obtained as direct extensions of those found, for example, in [12], Theorems 2.2 and 4.4 and have therefore been omitted.

THEOREM 5.6 *Under the assumptions of (28)–(31) and Lemma 5.2, the following properties of the optimization problems (37) and (38) hold:*

1. *The gap function $g_c(x_c, x_b) \geq 0$, for all $x_c \in \mathcal{X}_c$ and $g_c(x_c, x_b) = 0$ if and only if $x_c^*(x_b) \in \mathcal{X}_c$ solves the parametric VIP($t_c(x_c, x_b), \mathcal{X}_c$).*
2. *Similarly, the gap function $g_b(x_c^*(x_b), x_b) \geq 0$ for all $x_b \in \mathcal{X}_b$, and $g_b(x_c^*(x_b), x_b) = 0$ if and only if $x_b \in \mathcal{X}_b$ solves the VIP($t_b(x_c^*(x_b), x_b), \mathcal{X}_b$).*
3. *Furthermore, if $\nabla\phi(x_c)$ is Lipschitz on $\mathcal{X}_c$ with constant $M_{\nabla\phi_c} < m_{\phi_c} + m_{t_{cc}}$, then if $x_c^*(x_b)$ is a stationary point of problem (41), then $x_c^*(x_b)$ is a global optimal solution of (41), and therefore solves the parametric variational inequality VIP($t_c(x_c, x_b), \mathcal{X}_c$).*
4. *Similarly, if $\nabla\phi(x_b)$ is Lipschitz on $\mathcal{X}_b$ with constant $M_{\nabla\phi_b} < m_{\phi_b} + m_{t_b}$, then if x_b^* is a stationary point of problem (42), then x_b^* is a global optimal solution of (42), and therefore solves the variational inequality VIP($t_b(x_c^*(x_b), x_b), \mathcal{X}_b$).*

THEOREM 5.7 (DIRECTIONAL DIFFERENTIABILITY OF g_b) *Under Assumptions (28)–(31) and Lemma 5.2, suppose further that $t_b \in C^1$ and the auxiliary function $\phi_b \in C^2$ on $\mathcal{X}_b$. Then, the implicit gap function g_b is locally Lipschitz continuous with Lipschitz constant M_{g_b} and directionally differentiable.*

Proof: The assumptions and the Lemma 5.2 guarantee the Lipschitz continuity of the implicit mappings $x_b \mapsto x_c^*(x_b)$ and $x_b \mapsto y_b(x_b)$. Then, the result follows from, e.g., [11].

We define a descent direction for the problem (42), based upon the solution to (38). Let

$$d_b := y_b(x_c^*(x_b), x_b) - x_b \tag{43}$$

be the descent direction for the gap function g_b.

Consider now the special case of auxiliary function, $\phi_b(x) := \frac{1}{2} \parallel x_x \parallel^2_{G_b}$, which leads to the gap function of (39). Before showing that d_b is a descent direction, a technical definition is needed.

DEFINITION 5.8 *[20] The* graphical derivative *of a mapping $T : \Re^n \mapsto \Re^n$ at a point $x \in dom(T)$ for some $v \in T(x)$ is the mapping $DT : \Re^n \mapsto \Re^n$ defined by*

$$z \in DT(x|v)(d) \longleftrightarrow (d, z) \in \mathcal{T}_{gph\,T}(x, v), \tag{44}$$

where $\mathcal{T}_{gph\,T}$ is the tangent cone to the graph of the mapping T. When T is single valued at x, that is $T(x) = \{v\}$, then the notation is simplified to $DT(x)(d)$.

REMARK 5.9 *Note that, under the hypothesis of strong nested monotonicity, the mappings t_c and t_b are indeed single valued. In the latter case, in particular, $y_b(x^*_c(x_b), x_b)$ is uniquely determined by the strong monotonicity of the implicit function $t_b(x^*_c(x_b), x_b)$.*

The text [20] provides properties and illustrations of graphical derivatives, which generalize subgradients and directional derivatives to the case of vector functions and multivalued mappings. We are concerned with one particular property of graphical derivatives, presented below.

THEOREM 5.10 *[20] Consider a maximal monotone mapping $T : \Re^n \mapsto \Re^n$, any point $x \in dom(T)$, and $v \in T(x)$. Then, under the continuity of T at x, when T is single valued at x, the mapping $DT(x|v)(d) = DT(x)(d)$ is also maximal monotone and everywhere continuous. Also, the following properties are equivalent:*

1. *T is semi-differentiable at x for v,*
2. *$DT(x|v)(d)$ is single-valued.*

THEOREM 5.11 *Suppose in addition to the stated assumptions that $t_b \in C^1$ on $\mathcal{X}_b$. Then, the vector $d_b := y_b(x^*_c(x_b), x_b) - x_b$ satisfies the following descent condition*

$$g'_b(x^*_c(x_b), x_b\,; d_b) < -\mu_{t_b} \|d_b\|^2 \tag{45}$$

for every $x_b \in \mathcal{X}_b$, where

$$\begin{aligned} g'_b(x^*_c(x_b), x_b\,; d_b) \quad = \quad & \sup_{z \in \{y_b(x^*_c(x_b), x_b)\}} \{\langle t_b(x^*_c(x_b), x_b) \\ & -Dt(x^*_c(x_b), x_b)(z - x_b) + G_b(z - x_b), y - x\rangle. \end{aligned} \tag{46}$$

Proof: Since the $z \in Y(x_b)$ is attained uniquely, then, letting $y_b(x_b) := y_b(x_c^*(x_b), x_b)$ one can express (46) as

$$\begin{aligned} g_b'(x_c^*(x_b), x_b\,; d_b) &= \langle t_b(x_c^*(x_b), x_b) + G_b(y_b(x_b) - x_b), y_b(x_b) - x_b\rangle \\ &\quad - Dt_b(x_c^*(x_b), x_b)(y_b(x_b) - x_b). \end{aligned} \tag{47}$$

By the optimality of $y_b(x_b)$ to the subproblem, we have that

$$\langle x - G_b^{-1} t_b(x_c^*(x_b), x_b) - y_b(x_b),\, G_b(z - y_b(x_b))\rangle \leq 0, \tag{48}$$

for every $z \in \mathcal{X}$. Which can be re-written as

$$\langle t_b(x_c^*(x_b), x_b) + G_b(y_b(x_b) - x_b), y_b(x_b) - x_b\rangle \leq 0. \tag{49}$$

Combining (47) and (49), we obtain

$$g_b'(x_c^*(x_b), x_b\,; d_b) \leq -Dt_b(x_c^*(x_b), x_b)^T (y_b(x_b) - x_b), \tag{50}$$

$$\leq -\mu_{t_b} \|d_b\|. \tag{51}$$

where the final inequality hold due to the monotonicity property of Dt_b when Dt_b is semi-differentiable, which in turn holds when the mapping t_b is maximal monotone, continuous, and single-valued, according to Theorem 5.10. Since the $z \in Y(x_b)$ is attained uniquely, and $t_b(x_c^*(x_b), x_b)$ is strongly monotone, it is maximal monotone and continuous.

A Mixed descent algorithm for solving the coupled pair of optimization programs is provided next. For simplicity, we will continue to assume that the auxiliary function chosen is $\phi_b(x) := \frac{1}{2} \parallel x_b \parallel_{G_b}^2$.

Mixed descent method

1. Initialization: Obtain an initial feasible pair of flow vectors (x_c^0, x_b^0). Set the iteration counter $t = 0$.
2. Solve the following parametric, strongly convex optimization problem: find $x_c^*(x_b^t) \in \mathcal{X}_c$ from (35)–(36).
3. Given the solution $x_c^*(x_b^t)$, perform the following update: $x_b^{t+1} := x_b^t + s_b^t d_b^t$, where d_b^t is the search direction, given by

$$d_b := y_b(x_b) - x_b \tag{52}$$

$$= \mathrm{Proj}_{\mathcal{X}_b, G_b}(x_b - G_b^{-1} t_b(x_c^*(x_b), x_b)) - x_b. \tag{53}$$

4. If a stop test is not met, then set $t = t + 1$ and return to step 2.

This algorithm is referred to as a *mixed* descent algorithm since it combines the resolution of a convex optimization problem for obtaining the optimal response $x_c^*(x_b)$ with a descent procedure to update x_b.

REMARK 5.12 *The Mixed descent method is a straightforward, convergent algorithm for the multimodal traffic equilibrium problem, and has the further advantage that it requires little additional programming if one already has a solver for the single-class traffic assignment problem.*

THEOREM 5.13 *Under the stated assumptions, starting from any point,* (x_c^0, x_b^0)*, the mixed descent algorithm generates a well-defined and bounded sequence* $\{(x_c^t, x_b^t)\}$ *for which every cluster point of the sequence* $\{x_b^t\}$ *is a solution to* $VIP(t_b(x_c^*(x_b), x_b), \mathcal{X}_b)$*. Furthermore, the entire sequence* $\{(x_c^t, x_b^t)\}$ *converges to a solution of the* $VIP(t_c(x_c, x_b) + t_b(x_c, x_b), \mathcal{X}_c \times \mathcal{X}_b) = VIP(t, \mathcal{X})$*.*

Proof: We have by construction that the feasible set $\mathcal{X}_b$ is compact, and so the iterates $\{x_b^k\}$ remain in a compact subset of $\Re_+^{n_2}$. The closedness of the algorithmic map follows from the use of an exact line search step, along with the closedness of the implicit mappings $x_b \mapsto y_b(x_b)$ and $x_b \mapsto x_c^*(x_b)$, where the latter follows from the continuity of the mappings t_b and t_c. The fact that $g_b(x_c^*(x_b^{t+1}), x_b^{t+1}) < g_b(x_c^*(x_b^t), x_b^t)$ follows from Theorem 5.11 and the use of an exact line search.

6 Numerical Results

The Mixed descent method was implemented in the Scilab programming environment, and tested on the cost functions provided in equations 16-19. A projection method was used to solve the single-class subproblem of Step 2, as well as the direction finding subproblem of Step 3.

The algorithm converged rapidly and in 16 iterations satisfied the convergence criterion of 10^{-8}, defined as the norm of the difference between successive flow values.

The actual differences between successive iterates, $x_b^{t+1} - x_b^t$ is illustrated in Figure 8, where the upper curve is the difference of successsive flow values (for class b on arc 1, and the lower curve gives the difference between successive values on arc 2. Note that the difference between successive iterates rapidly approaches zero for both arcs (one from above, and one from below).

In Figure 9, the norm of this difference over both arcs is presented.

7 Conclusions

This paper has provided a first convergent method for solving the non-monotone, asymmetric, traffic equilibrium problem, by making use of a different, weaker, property. This property has an elegant interpretation in terms of hierarchical interactions between the traffic classes. If the cost functions on all arcs of the network are of a similar form, then a check of whether or not the property holds on the network is relatively easy to perform. We have shown that not all multimodal travel cost mappings will satisfy the property, but when it is satisfied, simple, and provably convergent algorithms are available.

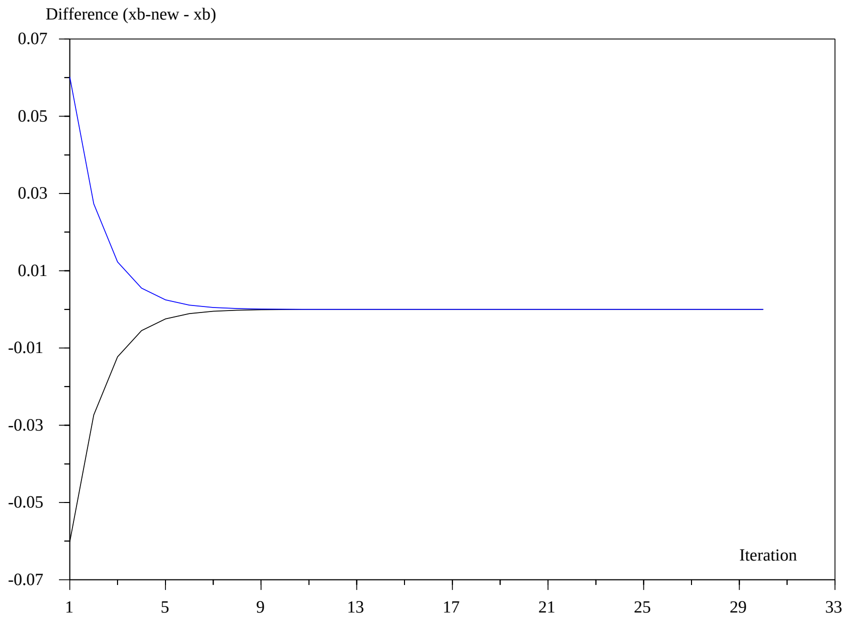

FIGURE 8: Convergence of the Mixed descent method on the example of equations (16)–(19): difference between successive flow values on each of the two arcs

We have provided one convergent algorithm for this class of traffic equilibrium problems which is easily implementable and permits the use of any existing code for solving the separable traffic equilibrium problem as a subroutine.

A worthwhile topic for future study is to examine the consequences of these results within a bilevel framework, that is, when the multimodal traffic equilibrium problem is at the lower-level of a hierarchical optimization problem, such as is the case in optimal pricing, or network design, problems.

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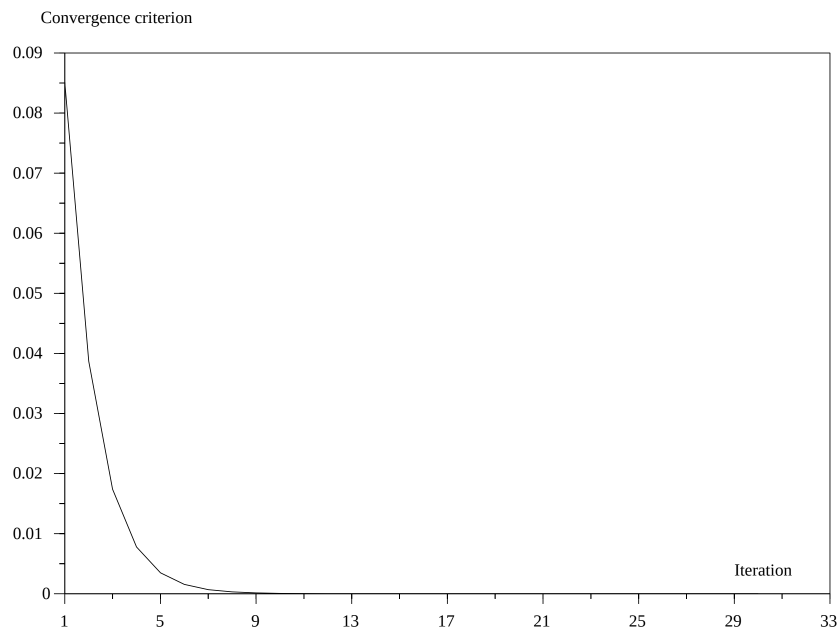

FIGURE 9: Convergence of the Mixed descent method on the example of equations (16)–(19): norm of the flow differences at successive iterations

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