Practical and Polytopic Observers for Nonlinear Uncertain Systems

Alain Rapaport, Jean-Luc Gouzé

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Practical and polytopic observers for nonlinear uncertain systems

Alain Rapaport — Jean-Luc Gouzé

N° 4079

Novembre 2000

THÈME 4

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--4079--FR+ENG

Practical and polytopic observers for nonlinear uncertain systems

Alain Rapaport *, Jean-Luc Gouzé

Thème 4 — Simulation et optimisation
de systèmes complexes
Projet Comore

Rapport de recherche n° 4079 — Novembre 2000 — 29 pages

Abstract: For a class of dynamical systems, with uncertain nonlinear terms considered as "unknown inputs", we give sufficient conditions for observability. We show also that there does not exist any exact observer independent of the unknown inputs. Under the additional assumption that the uncertainty is bounded, we build *practical observers* whose error converges exponentially towards an arbitrary small neighborhood of the origin. For the general case, when the system might be not observable with unknown inputs, we build *polytopic observers* providing time-varying bounds (depending on the uncertainty bounds) on the state variables. We illustrate these results on a biological model of a structured population.

Key-words: robust estimation, nonlinear observer, unknown inputs, practical observer, polytopic observer, biological models

* INRA Biométrie, 2, place Viala, BP 93, 34060 Montpellier, France

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles, B.P. 93, 06902 Sophia Antipolis Cedex (France)
Téléphone : 04 92 38 77 77 - International : +33 4 92 38 77 77 — Fax : 04 92 38 77 65 - International : +33 4 92 38 77 65

Observateurs pratiques et polytopiques pour des systèmes non-linéaires incertains

Résumé : Nous donnons des conditions suffisantes d'observabilité pour une classe de systèmes dynamiques avec des termes mal connus. Nous montrons qu'il n'existe pas d'observateur exact classique; sous l'hypothèse que l'incertitude est bornée, nous construisons un observateur pratique, dont l'erreur converge vers un voisinage arbitrairement petit de l'origine. Dans un cas plus général, quand le système peut ne pas être observable, nous construisons des observateurs polytopiques donnant des bornes dynamiques (dépendant des incertitudes) sur les variables. Nous appliquons ces résultats à un exemple de modèle biologique d'une population structurée en stades ou en âge.

Mots-clés : estimation robuste, observateur non-linéaire, entrées inconnues, observateur pratique, observateur polytopique, modèles biologiques

Practical and polytopic observers for nonlinear uncertain systems

A. Rapaport
INRA Biométrie
2, place Viala
34060 Montpellier, France
e-mail: rapaport@ensam.inra.fr

J.L. Gouzé
INRIA COMORE
BP 93
06902 Sophia-Antipolis, France
e-mail: gouze@sophia.inria.fr

November 30, 2000

1 Introduction

In this paper, we are interested in the problem of observation of uncertain (partially known) systems. Let us consider a nonlinear system on a domain Ω, subset of $\mathbb{R}^n$ $(n \geq 2)$, with a scalar measurement y of the following form :

$$(\mathcal{S}) : \left\{ \begin{array}{lcl} \dot{x} & = & f(t,x), \quad x(0) = x_0 \\ y & = & Cx \end{array} \right. \tag{1}$$

where $C \in \mathcal{M}^{n\times 1}(\mathbb{R})$. We shall assume that the map $(t,x) \mapsto f(t,x)$ is measurable w.r.t to t and Lipschitz w.r.t. x. When the map f is perfectly known and the system is observable, one might be able to design an observer of the following form :

$$\dot{\hat{x}} = f(t,\hat{x}) + K(t,\hat{x})(C\hat{x} - y)$$

with an asymptotic or even exponential error dynamics (under some additional properties, see the "high-gain" observers in [4, 9]). The rate of convergence towards zero of the error can then be adjusted, under good hypotheses, with the gain vector K.

We now consider the case when the function f is, in some sense, "uncertain". Our aim will be to estimate the state variables, taking into account the uncertainty on the model. It is often the case in modeling that some part of the models are not well known : for instance, in mathematical models of biological systems [21], the analytical form of the involved functions

*INRA Biométrie, 2, place Viala, BP 93, 34060 Montpellier, France

may be difficult to obtain; or some parameters can fluctuate with time in an unknown way. Consider for instance the dynamics of a biological population structured in three age classes (larvae, young adults and adults), whose stocks (or densities) are : $(x_1, x_2, x_3) \in \mathbb{R}_+^3$:

$$\left\{ \begin{array}{lcl} \dot{x}_1 & = & -\alpha_1 x_1 - m_1 x_1 + r(t, x_2, x_3) \\ \dot{x}_2 & = & \alpha_1 x_1 - \alpha_2 x_2 - m_2 x_2 \\ \dot{x}_3 & = & \alpha_2 x_2 - m_3 x_3 \end{array} \right.$$

where the positive coefficients α_i and m_i represent respectively the growth and mortality rates. The reproduction law $r(t, x_2, x_3)$ is usually not well known but can often be assumed to fluctuating over the time between two known functions $\underline{r}$, $\overline{r}$:

$$\underline{r}(t, x_2, x_3) \leq r(t, x_2, x_3) \leq \overline{r}(t, x_2, x_3),$$

The only available measurement is the stock of the adults population : $y(t) = x_3(t)$. What can we say about the estimation of x_1 and x_2 ? This example is studied in the last section.

More generally, we will suppose that the nonlinear map f is not well known, but is functionally bounded, *i.e.* there exist known maps $\underline{f}$, $\overline{f}$ such that :

$$\underline{f}(t, x) \leq f(t, x) \leq \overline{f}(t, x), \quad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n \tag{2}$$

What are the classical approaches for observation when dealing with such an uncertainty ?

A possible approach to design robust observers with respect to the uncertainty on f is to choose an arbitrary map $\widehat{f}$ (bounded below by $\underline{f}$ and above by $\overline{f}$) and to consider the vector

$$w(t) = \widehat{f}(t, x(t)) - f(t, x(t)) \tag{3}$$

as a disturbance. Then, we can study conditions under which there exist observers that do not depend on w (what is usually called in the literature *observers with unknown inputs*). For the linear case, necessary and sufficient conditions for the existence of such observers (with the same dimension as the state of the system) are known (see for instance Darouach and al. [6]). But, for the nonlinear case, even when the system $(\mathcal{S})$ is differentially observable with unknown f, we show that there does not exist any *exact* observer (see section 2). Designing discontinuous or *sliding mode* observers is another available technique but the convergence of the error requires restrictive matching conditions of the uncertainty (see for instance [7]).

The most classical approach may be the linear optimal filtering or Kalman filter [1], but under quite restrictive assumptions : the nominal system has to be linear, and the disturbance $w(t)$ must be a white gaussian noise. In its deterministic (or robust) version, the disturbances have to fulfill an integral quadratic constraint [22].

Another approach could be to design an observer $\widehat{x}$ attenuating as much as possible the effect of the disturbance w on the error $e = \widehat{x} - x$. For this purpose, the H^∞ filtering theory proposes to study the following optimization problem (see Başar, Bernhard [2]) : given γ a positive number and R a definite symmetric matrix, find an observer $\widehat{x}$ such that

$$\sup_{w()} \int_0^\infty ||\widehat{x}(t) - x(t)||_R^2 - \gamma^2 ||w(t)||^2 \, dt \ \leq 0.$$

Then, the solution (when it exists) leads to an error dynamics of the following form :

$$\dot{e} = F_\gamma e + G_\gamma w \quad \text{with } F_\gamma \text{ Hurwitz.}$$

When w belongs to L^2, the error exponentially converges towards 0. Unfortunately, in our case w does not necessarily belong to L^2 (which would imply that $\lim_{t\to\infty} w(t) = 0$, *i.e.* that the disturbance vanishes with time).

Nevertheless, it is easy to show that there exist sufficient conditions on the set $\mathcal{F}$ of unknown f for the system $(\mathcal{S})$ to be differentially observable with unknown f. When these conditions are fulfilled, the problem is then to build an observer, that cannot be of the usual form, as we have seen above. We propose to consider a weaker kind of observers than the usual one, that we shall call *practical observers*, in the sense that we no longer require the error to converge asymptotically towards 0 but to an arbitrary small neighborhood of the origin. Then, we show that this is feasible when the uncertainty on f is bounded (*i.e.* $\overline{f} - \underline{f}$ uniformly bounded).

For the general case (when the system might be not observable), our approach is based on a *dynamical interval analysis*, in the spirit of the interval computation of solutions of o.d.e. [5, 24] but with control theoretical tools : given guaranteed intervals on uncertainties on the components of f, we compute guaranteed intervals for the variables in some coordinates; we build therefore what we call interval observers (see [11]). In the original coordinates, we obtain then a guaranteed time varying polytope for the unmeasured variables. For the differentially observable case, the asymptotic size of this polytope can be made arbitrary small.

In the spirit, our approach is close to set-valued estimation techniques. For noisy discrete time systems, there exist algorithms to compute recursively an exact or guaranteed set of possible states (cf. [25, 15]); some of them are based on interval analysis (see [20]). For continuous time linear systems, ellipsoidal techniques provide an effective way to compute sets (see [3, 17]). For nonlinear systems, the set of possible states conditioned to past observations can be characterized by a level set of the solution of a non-stationary p.d.e. [18] or "information state" [13], but this approach does not seem to be very practicable due to its infinite dimension.

Our aim is, at the expense of stronger hypotheses, to obtain mathematical conditions for some kind of observability, to build explicitly observers as finite dimensional systems, and to characterize their asymptotic behavior. The present work is a generalization of a simpler case [23], where the uncertain functions depend only on the measurement.

What could be the advantages of such an approach, compared with classical ones, where one seeks bounds on the norm of the error? Firstly, we are able to characterize in a more subtle way the influence of the uncertainty on the dynamical behavior (cf. the decomposition of f in section 5). Moreover, when the bounds on the uncertainties on the inputs are provided independently on each coordinate (*i.e.* by intervals), we would like to use this information. So we naturally seek guaranteed bounds on each coordinate of the unmeasured variables (*i.e.* in terms of intervals or polytopes).

The paper is organized as follows: after some definitions (section 2), we propose conditions for the observability with unknown inputs (section 3), compute practical observers if the system is observable (section 4), and, if not, compute "polytopic observers" and study their asymptotic behavior (section 5). Finally, we illustrate these results on an example inspired by biology (section 6).

2 Definitions

We assume, without any loss of generality (see the remarks below), that the nonlinear system (1) can be written in the following way :

$$(\mathcal{S}) : \left\{ \begin{array}{lcl} \dot{x} & = & Ax + \psi(t,x), \quad x(0)=x_0 \\ y & = & Cx \end{array} \right. \qquad x \in \mathbb{R}^n \tag{4}$$

where $A \in \mathcal{M}^{n\times n}(\mathbb{R})$ such that :
Assumption A1 : The pair (A,C) is observable.

Remarks :
1. If the system (1) is not defined on all $\mathbb{R}^n$ or if ψ is not Lipschitz w.r.t. x on all $\mathbb{R}^n$ but only on the domain Ω, then one can consider a Lipschitz extension of ψ on all $\mathbb{R}^n$ (see for instance the MacShane formula [19]).

2. When the assumption A1 is not fulfilled, it is always possible to choose a matrix $\widetilde{A}$ such that the pair $(\widetilde{A},C)$ is observable and define $\widetilde{\psi}(t,x)=\psi(t,x)+(A-\widetilde{A})x$. Then, the system (1) can be rewritten as follows :

$$(\mathcal{S}) : \left\{ \begin{array}{lcl} \dot{x} & = & \widetilde{A}x + \widetilde{\psi}(t,x), \\ y & = & Cx \end{array} \right.$$

We shall denote $\mathcal{O}$ the observability matrix of the pair (A,C), S the anti-shift operator and $\kappa(P)$ the condition number of a nonsingular matrix P :

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}, \qquad \kappa(P) = ||P^{-1}||\,||P||\,.$$

Although solutions of the system $(\mathcal{S})$ are well defined for ψ which are only measurable w.r.t. t and Lipschitz w.r.t. x, we shall consider, when studying differential observability with inputs, a dense subset of smooth "inputs" ψ (see Gauthier and Kupka [10]), for which the map $t \to y(t)$ can be differentiated $n-1$ times :

Assumption A2 : The set Ψ of unknown ψ is such that $\Psi \cap \mathcal{C}^{n-2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ is dense in Ψ [1].

We define now a notion of observability (cf. [10]) that will be sufficient for our needs.

Definition 1 : For $\psi \in \Psi \cap \mathcal{C}^{n-2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$, consider the map :

$$x \longmapsto \Gamma_t(\psi, x) = \begin{pmatrix} y(t) \\ \dot{y}(t) \\ \vdots \\ \vdots \\ y^{(n-1)}(t) \end{pmatrix},$$

1. $(\mathcal{S})$ is said to be *differentially observable (of rank n)* (for known $\psi \in \Psi$) if and only if the map $x \mapsto \Gamma_t(\psi, x)$ is injective for any time t and any $\psi \in \Psi \cap \mathcal{C}^{n-2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$.

2. $(\mathcal{S})$ is said to be *differentially observable (of rank n) for unknown* $\psi \in \Psi$ if and only if

$$\forall \psi^a, \psi^b \in \Psi \cap \mathcal{C}^{n-2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n), \{\, \Gamma_t(\psi^a, x^a) = \Gamma_t(\psi^b, x^b) \,\} \implies \{\, x^a = x^b \,\}$$

for any time t.

3 Observability with unknown inputs

It is easy to write down sufficient conditions for the differential observability for unknown ψ.

Proposition 1 : *If*

1. *$(\mathcal{S})$ is differentially observable (for known $\psi \in \Psi$),*
2. *For any ψ^a, ψ^b in Ψ and $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$,*
   $CA^i(\psi^a(t,x) - \psi^b(t,x)) = 0$ for all $i \in 0, \cdots, n-1$,

then the system $(\mathcal{S})$ is differentially observable for unknown $\psi \in \Psi$.

Proof For $\psi \in \Psi \cap \mathcal{C}^{n-2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$, we have :

$$\Gamma_t(\psi, x) = \mathcal{O}x \; + \sum_{k=0}^{n-2} S^{k+1} \mathcal{O} \, \frac{d^k \psi}{dt^k}(t, x).$$

[1] $\mathcal{C}^{n-2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ is the set of maps $\mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}^n$ continuously differentiable $n-2$ times.

So, for $\psi^a, \psi^b \in \Psi \cap \mathcal{C}^{n-2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$, we have :

$$\begin{aligned}\Gamma_t(\psi^a, x^a) = \Gamma_t(\psi^b, x^b) \quad &\Rightarrow \quad \mathcal{O}(x^a - x^b) + \sum_{k=0}^{n-2} S^{k+1}\mathcal{O}\left(\frac{d^k\psi^a}{dt^k}(t,x^a) - \frac{d^k\psi^b}{dt^k}(t,x^b)\right) = 0 \\ &\Rightarrow \quad \mathcal{O}(x^a - x^b) + \sum_{k=0}^{n-2} S^{k+1}\mathcal{O}\left(\frac{d^k\psi^a}{dt^k}(t,x^a) - \frac{d^k\psi^a}{dt^k}(t,x^b)\right) \\ &\qquad + \sum_{k=0}^{n-2} S^{k+1}\mathcal{O}\left(\frac{d^k\psi^a}{dt^k}(t,x^b) - \frac{d^k\psi^b}{dt^k}(t,x^b)\right) = 0\end{aligned}$$

From condition 2, we have :

$$\sum_{k=0}^{n-2} S^{k+1}\mathcal{O}\left(\frac{d^k\psi^a}{dt^k}(t,x^b) - \frac{d^k\psi^b}{dt^k}(t,x^b)\right) = 0.$$

Then,

$$\begin{aligned}\Gamma_t(\psi^a, x^a) = \Gamma_t(\psi^b, x^b) \quad &\Rightarrow \quad \mathcal{O}(x^a - x^b) + \sum_{k=0}^{n-2} S^{k+1}\mathcal{O}\left(\frac{d^k\psi^a}{dt^k}(t,x^a) - \frac{d^k\psi^a}{dt^k}(t,x^b)\right) = 0 \\ &\Rightarrow \quad \Gamma_t(\psi^a, x^a) = \Gamma_t(\psi^a, x^b).\end{aligned}$$

According to condition 1, $(\mathcal{S})$ is differentially observable for ψ^a, so $x^a = x^b$, thus the differential observability of $(\mathcal{S})$ for unknown ψ. ∎

Remark : It is clear that as soon as there exist two maps $\psi^a, \psi^b \in \Psi \cap \mathcal{C}^{n-2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$ such that $\psi^a(t,x) - \psi^b(t,x) \notin \ker(S\mathcal{O})$, then $(\mathcal{S})$ is not differentially observable.

As $\mathcal{O}$ is full rank, $\ker(S\mathcal{O})$ is of dimension 1, we consider now particular systems for which the set Ψ has the following structure :

Hypothesis H1 : There exist a known map $\varphi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^n$ and a fixed vector d_1 in $\ker(S\mathcal{O}) \setminus \{0\}$ such that :

$$\forall \psi \in \Psi, \quad \psi(t,x) = \varphi(t, Cx) + \psi_1(t,x)d_1, \quad \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$$

where ψ_1 are unknown scalar functions.

Remark : This hypothesis means that we consider systems in output feedback form : $\dot{x} = Ax + \varphi(t,y)$ (for which the observability of the pair (A,C) is a sufficient condition to build an observer with linear error dynamics [12]) with an additional uncertain term that belongs to $\ker(S\mathcal{O})$.

Corollary 1 : *Under hypothesis H1, the system* $(\mathcal{S})$ *is differentially observable for unknown* $\psi \in \Psi$.

Proof For any known $\psi \in \Psi \cap \mathcal{C}^{n-2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$, we have

$$\Gamma_t(\psi, x) = \mathcal{O}x \ + \sum_{k=0}^{n-2} S^{k+1}\mathcal{O}\frac{d^k\varphi}{dt^k}(t, y)$$

which is independent of ψ. $\mathcal{O}$ being invertible, we have the following equality :

$$x = \mathcal{O}^{-1}\left(\Gamma_t(\psi, x) - \sum_{k=0}^{n-2} S^{k+1}\mathcal{O}\frac{d^k\varphi}{dt^k}(t, y)\right)$$

whose right member depends only on the measurement $y(t)$ and its derivatives, thus the map $x \to \Gamma_t(\psi, x)$ is injective. Condition 1 of Proposition 1 is then fulfilled. Condition 2 is also fulfilled by hypothesis H1. So, by Proposition 1, $(\mathcal{S})$ is differentially observable for unknown $\psi \in \Psi$. ∎

Under the hypothesis H1, we now naturally look for the existence of *exact* observers *i.e.* dynamical systems in finite dimension of the following form :

$$(\widehat{\mathcal{S}}) : \begin{cases} \dot{z} = \widehat{f}(t, z, y), \quad z(0) = z_0 \\ \widehat{x} = g(z, y) \end{cases} \qquad z \in \mathbb{R}^p \quad (p \geq n) \tag{5}$$

with $\widehat{f}$ measurable w.r.t. t and $\widehat{f}, g$ Lipschitz w.r.t. (z, y), such that for any pair $(x_0, z_0) \in \mathbb{R}^n \times \mathbb{R}^p$, the solution of the coupled system $(\mathcal{S}, \widehat{\mathcal{S}})$ verifies $\lim_{t\to\infty} ||\widehat{x}(t) - x(t)|| = 0$ for any unknown $\psi \in \Psi$.

Proposition 2 : *Under hypothesis H1, there does not exist any exact observer for unknown* ψ.

Proof An exact observer has to leave the manifold $x = g(z, Cx)$ invariant for any ψ, which implies, at points (z, y) where g is differentiable (by Rademacher [8] theorem, the Lipschitz map g is almost everywhere differentiable) :

$$Ax + \varphi(t, y) + \psi_1 d_1 = g_z \widehat{f}(t, z, Cx) + g_y C(Ax + \varphi(t, y) + \psi_1 d_1)$$
or
$$(I - g_y C)(Ax + \varphi(t, y)) - g_z \widehat{f}(t, z, Cx) = \psi_1 (g_y C - I) d_1$$

for any unknown scalar ψ_1. So, necessarily $d_1 \in \ker(g_y C - I)$ *i.e.* $d_1 = g_y C d_1$. But, by hypothesis H1, $Cd_1 = 0$ which implies $d_1 = 0$, thus a contradiction. ∎

Remarks :
1. On the contrary to differentially observable systems with known inputs, for which it is possible to build an exact observer under some additional regularity properties (see [9]), this is not possible in the presence of unknown inputs.
2. For the scalar output case, it is not a surprise that there does not exist any observer with unknown inputs, as the conditions found by Darouach and al. [6] are never fulfilled.

4 Practical observers

We assume that the hypothesis of Proposition 2 is fulfilled, so the system is differentially observable with unknown inputs. Since an exact observer does not exist (cf. above), we build practical observers, converging towards a small neighborhood of the actual state variables.

We consider observers $\widehat{x}$ for the system $(\mathcal{S})$ as outputs of a family (indexed by K) of finite dimensional dynamical systems $(\widehat{\mathcal{S}})$ of the form (5) parameterized by a vector of gains K.

Definitions 2 :
1. $\widehat{x}$ *is a* **weak practical observer** *if for any* $\epsilon > 0$*, there exists a* $(\widehat{\mathcal{S}})$ *such that*

$$\forall (x_0, \widehat{x}_0),\ \exists T > 0,\ ||\widehat{x}(t) - x(t)|| \leq \epsilon, \quad \forall t > T$$

The second definition imposes that the rate of convergence towards the ball of radius ϵ can be adjusted arbitrarily (with the gain K).
2. $\widehat{x}$ *is a* **strong practical observer** *if for any* $\epsilon > 0$*, there exists a* $(\widehat{\mathcal{S}})$ *such that*

$$\exists \lambda(\epsilon) > 0,\ \forall (x_0, \widehat{x}_0),\ ||\widehat{x}(t) - x(t)|| \leq \epsilon + e^{-\lambda(\epsilon)t}(||\widehat{x}(0) - x(0)|| - \epsilon)$$

with $\lambda(\epsilon) \to +\infty$ *when* $\epsilon \to 0$.

We shall now consider observers $(\widehat{\mathcal{S}})$ of the following form :

$$\dot{\widehat{x}} = A\widehat{x} + \varphi(t, y) + \widehat{\psi}_1(t, \widehat{x})d_1 + K(C\widehat{x} - y) \tag{6}$$

where the vector K and the function $\widehat{\psi}_1$ have to be chosen.

When (A, C) is observable, it is well-known (see [14]) that A is similar to a companion matrix : take

$$H = \mathcal{O}^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathrm{P} = \begin{pmatrix} H & AH & \ldots A^{n-1}H \end{pmatrix},$$

then $CH = CAH = \ldots = CA^{n-2}H = 0$ and $CA^{n-1}H = 1$. So

$$\mathcal{O}P = \begin{pmatrix} O & & & 1 \\ & & 1 & \\ & \ldots & & \\ 1 & & & * \end{pmatrix}$$

P is non singular and we have :

$$\tilde{A} = P^{-1}AP = \begin{pmatrix} 0 & \ldots & 0 & -a_n \\ 1 & & O & \vdots \\ & \ddots & & \vdots \\ O & & 1 & -a_1 \end{pmatrix} \qquad \tilde{C} = CP = \begin{pmatrix} 0 & \ldots & 0 & 1 \end{pmatrix}$$

where $X^n + a_1 X^{n-1} + \ldots + a_n = 0$ is the characteristic polynomial of A.

We shall use the following notation for diagonal and Vandermonde matrices :

$$\Delta_{\{\lambda_j\}} = \begin{pmatrix} \lambda_1 & & & O \\ & \lambda_2 & & \\ & & \ddots & \\ O & & & \lambda_n \end{pmatrix} \qquad V_{\{\lambda_j\}} = \begin{pmatrix} 1 & \lambda_1 & \ldots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \ldots & \lambda_2^{n-1} \\ \vdots & & & \\ 1 & \lambda_n & \ldots & \lambda_n^{n-1} \end{pmatrix}$$

We consider now the following hypothesis :

Hypothesis H2 : Under hypothesis H1, there exists a positive number L such that ψ_1 is Lipschitz w.r.t. x with constant L, for any $\psi \in \Psi$. Moreover, there exists functions $\underline{\psi}_1$, $\overline{\psi}_1$ measurable w.r.t. t and Lipschitz w.r.t. x such that :

$$\forall \psi \in \Psi, \quad \underline{\psi}_1(t,x) \leq \psi_1(t,x) \leq \overline{\psi}_1(t,x), \quad \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$$

and a positive number M such that $\overline{\psi}_1 - \underline{\psi}_1$ is uniformly bounded by M.

Remark : If the functions $\psi \in \Psi$ are not Lipschitz with constant L globally on $\mathbb{R}^n$, but there exists a compact set $\mathcal{K}$ of $\mathbb{R}^n$ which is positively invariant by $(\mathcal{S})$ for any $\psi \in \Psi$, then one can consider instead of Ψ the set $\widetilde{\Psi}$ of unknown functions $\widetilde{\psi}$ defined as follows :

$$\widetilde{\psi}(t,x) = \left| \begin{array}{ll} \psi(t,x) & \text{if } ||x|| \leq r \\ \psi\left(t, r\dfrac{x}{||x||}\right) & \text{if } ||x|| > r \end{array} \right. \qquad \psi \in \Psi$$

where r is a positive number such that $\mathcal{K} \subset \mathbb{B}(0,r)$. Then, hypothesis H2 is fulfilled.

Proposition 3 : *Under hypotheses H1 and H2, for any function $\widehat{\psi}_1$ measurable w.r.t. t and Lipschitz w.r.t. x such that $\underline{\psi}_1(t,x) \leq \widehat{\psi}_1(t,x) \leq \overline{\psi}_1(t,x)$, $\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, K can be chosen such that the system (6) is a strong practical observer.*

Proof The matrix $P^{-1}(A+KC)P$ is in companion form for any K. If K is chosen such that $A+KC$ has n distinct real negative eigenvalues $\lambda_1, \ldots \lambda_n$, then $P^{-1}(A+KC)P$ is diagonalizable by the Vandermonde matrix $V_{\{\lambda_j\}}$:

$$V_{\{\lambda_j\}} P^{-1}(A+KC)PV_{\{\lambda_j\}}^{-1} = \Delta_{\{\lambda_j\}}.$$

Make the change of variable :

$$z = V_{\{\lambda_j\}} P^{-1} x, \quad \widehat{z} = V_{\{\lambda_j\}} P^{-1} \widehat{x},$$

the dynamics of the error $e = z - \widehat{z}$ is then :

$$\dot{e} \;=\; \Delta_{\{\lambda_j\}} e \;+\; (\psi_1(t,x) - \widehat{\psi}_1(t,\widehat{x})) V_{\{\lambda_j\}} P^{-1} d_1.$$

The hypothesis H1 provides :

$$\mathcal{O}d_1 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \delta \end{pmatrix} \;\Leftrightarrow\; d_1 = \delta H \;\Leftrightarrow\; P^{-1} d_1 = \delta \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \;\Rightarrow\; V_{\{\lambda_j\}} P^{-1} d_1 = \delta \begin{pmatrix} 1 \\ \vdots \\ \vdots \\ 1 \end{pmatrix}$$

Write $\overline{\lambda} = \min_j |\lambda_j|$, we have :

$$\begin{array}{rcl} \dfrac{d}{dt}||e|| & \leq & -\overline{\lambda}||e|| + |\psi_1(t,x) - \widehat{\psi}_1(t,\widehat{x})| \, ||V_{\{\lambda_j\}} P^{-1} d_1|| \\ & = & -\overline{\lambda}||e|| + |\psi_1(t,x) - \widehat{\psi}_1(t,\widehat{x})| \sqrt{n}|\delta| \\ & = & -\overline{\lambda}||e|| + (|\psi_1(t,x) - \psi_1(t,\widehat{x})| + |\psi_1(t,\widehat{x}) - \widehat{\psi}_1(t,\widehat{x})|)\sqrt{n}|\delta| \\ & \leq & (-\overline{\lambda} + L||PV_{\{\lambda_j\}}^{-1}||\sqrt{n}|\delta|)||e|| + M\sqrt{n}|\delta| \quad \text{(by hypothesis H2)} \end{array}$$

The coefficients of $V_{\{\lambda_j\}}^{-1}$ being rational functions in λ_j of degree less or equal to zero, we can choose eigenvalues of the form $\lambda_j(\theta) = \theta\lambda'_j$ (where the set $\{\lambda'_j\}$ is fixed and θ is a positive number) to obtain :

$$\lim_{\theta \to \infty} -\overline{\lambda}(\theta) + L||PV_{\{\lambda_j(\theta)\}}^{-1}||\sqrt{n}|\delta| = -\infty$$

So, for θ large enough, $e(t)$ converges exponentially towards the ball :

$$\mathbb{B}\left(0, \frac{M\sqrt{n}|\delta|}{\left|-\overline{\lambda}(\theta) + L||PV_{\{\lambda_j(\theta)\}}^{-1}||\sqrt{n}|\delta|\right|}\right).$$

Define

$$\rho(\theta) = ||PV^{-1}_{\{\lambda_j(\theta)\}}|| \frac{M\sqrt{n}|\delta|}{\left|-\overline{\lambda}(\theta) + L||PV^{-1}_{\{\lambda_j(\theta)\}}||\sqrt{n}|\delta|\right|}, \text{ then } \rho(\theta) \to 0 \text{ when } \theta \to \infty$$

The error in the original coordinates $(x - \widehat{x})$ converges then exponentially towards the ball $\mathbb{B}(0, \rho(\theta))$, which can be made arbitrary small by taking large values of θ. In the same way, the rate of convergence can be made arbitrarily fast. ∎

5 Guaranteed polytopic observers

We relax now the hypothesis of differential observability and focus on obtaining guaranteed estimates of $x(t)$, conditioned to known bounds on the uncertainty.

Hypothesis H2b : There exists a positive number L such that any $\psi \in \Psi$ is Lipschitz w.r.t. x with constant L. Moreover, there exist known maps $\underline{\psi}$, $\overline{\psi}$ in Ψ and a number $M < +\infty$ such that :

$$\left\{ \begin{array}{l} \underline{\psi}(t,x) \leq \psi(t,x) \leq \overline{\psi}(t,x), \\ ||\overline{\psi}(t,x) - \underline{\psi}(t,x)|| \leq M, \end{array} \right. \qquad \forall \psi \in \Psi, \ \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$$

Note : The operator $\leq$ between two vectors of $\mathbb{R}^n$ should be understood as a collection of n inequalities between their coordinates.

Remark : If the functions ψ in Ψ are not globally Lipschitz but there exists a compact set invariant by the system $(\mathcal{S})$ for any $\psi \in \Phi$, then one might consider an alternative set $\widetilde{\Psi}$ of unknown $\widetilde{\psi}$, as suggested in the remark after Hypothesis H2.

In order to simplify the writing, we shall use in the following the interval notation, introduced in [20] :

$$\psi(t,x) \in [\psi](t,x) = [\underline{\psi}(t,x), \overline{\psi}(t,x)].$$

Then, considering the set of intervals of $\mathbb{R}^n$: $\mathcal{I}^n = \{[\underline{\xi}, \overline{\xi}],\ (\underline{\xi}, \overline{\xi}) \in \mathbb{R}^{2n} \,|\, \underline{\xi} \leq \overline{\xi}\}$, we can also use the notation of interval computation on $\mathcal{I}^n$: when F is a continuous map from $\mathbb{R}^n$ to $\mathbb{R}^m$, the notation $[q] = F([p])$ stands for $[q] = [\underline{q}, \overline{q}]$ where :

$$\underline{q}_i = \min_{\pi \in [p]} F_i(\pi), \quad \overline{q}_i = \max_{\pi \in [p]} F_i(\pi) \quad (i = 1 \cdots m).$$

When p is function of time and $[p(t)] = [\underline{p}(t), \overline{p}(t)]$, then $[\dot{p}]$ denotes by convention $[\dot{\underline{p}}, \dot{\overline{p}}]$.

Lemma 1 : If a function $\phi : \mathbb{R}^n \mapsto \mathbb{R}$ is Lipschitz with constant L, then the functions $\underline{\phi}$, $\overline{\phi} : \mathbb{R}^{2n} \mapsto \mathbb{R}$ such that :

$$\phi([\underline{\xi},\overline{\xi}]) = \left[\, \underline{\phi}(\underline{\xi},\overline{\xi}) \,,\, \overline{\phi}(\underline{\xi},\overline{\xi}) \,\right], \quad \forall [\underline{\xi},\overline{\xi}] \in \mathcal{I}^n$$

are also Lipschitz with constant L.

Proof Fix $\underline{\xi}$ and consider $\overline{\xi}_1$, $\overline{\xi}_2$.
If $\overline{\phi}(\underline{\xi},\overline{\xi}_1) = \overline{\phi}(\underline{\xi},\overline{\xi_2})$, there is nothing to prove.
Assume that $\overline{\phi}(\underline{\xi},\overline{\xi}_2) > \overline{\phi}(\underline{\xi},\overline{\xi}_1)$ (otherwise, we exchange the role of $\overline{\xi}_1$ and $\overline{\xi}_2$). Consider $\widehat{\xi}_2$ such that $\overline{\phi}(\underline{\xi},\overline{\xi}_2) = \phi(\widehat{\xi}_2)$. Necessarily, $\widehat{\xi}_2 \in [\underline{\xi},\overline{\xi}_2] \setminus [\underline{\xi},\overline{\xi}_1]$. Take now $\xi_1 \in [\underline{\xi},\overline{\xi}_1]$ such that $||\xi_1 - \widehat{\xi}_2|| \leq ||\overline{\xi}_1 - \overline{\xi}_2||$ (for instance $\xi_1 = \mathrm{Proj}_{[\underline{\xi},\overline{\xi}_1]}(\widehat{\xi}_2)$). Then, we have $0 \leq \overline{\phi}(\underline{\xi},\overline{\xi}_2) - \overline{\phi}(\underline{\xi},\overline{\xi}_1) \leq \phi(\widehat{\xi}_2) - \phi(\xi_1) \leq L||\xi_1 - \widehat{\xi}_2|| \leq L||\overline{\xi}_1 - \overline{\xi}_2||$. So, $\overline{\phi}$ is Lipschitz with constant L, w.r.t. $\overline{\xi}$. In the same way, $\overline{\phi}$ is again Lipschitz with constant L, w.r.t. $\underline{\xi}$.
Exchanging ϕ by $-\phi$, we obtain that $\underline{\phi}$ is also Lipschitz with constant L. ∎

We consider again the matrix P defined in the previous section.

Definition 3 : For any $\psi \in \Psi$, we define the map :

$$\gamma : \ (t,x) \mapsto \gamma(t,x) = P^{-1}\psi(t,x).$$

Under Hypothesis H2b, the maps γ are Lipschitz with constant $||P^{-1}||L$ and bounded by two maps $\underline{\gamma}, \overline{\gamma}$, measurable w.r.t. t and Lipschitz w.r.t. x :

$$[\underline{\gamma}(t,x), \overline{\gamma}(t,x)] = [\gamma](t,x) = P^{-1}\left([\psi](t,x)\right) \tag{7}$$

such that $\overline{\gamma} - \underline{\gamma}$ is uniformly bounded by $||P^{-1}||M$.

Remark : The coordinates of γ are precisely the decomposition of ψ on the columns of the matrix P :

$$\psi(t,x) = \sum_{i=1}^{n} \gamma_i(t,x) A^{i-1} H$$

that could be written:

$$\psi(t,x) = \sum_{i=1}^{n} \gamma_i(t,x) d_i$$

where the vectors d_i belong to $\ker(S^i\mathcal{O}) \setminus \{0\}$. We recognize in this decomposition the vector d_1 of the preceding section. If the hypothesis H1 is fulfilled, then $\gamma_i = 0$, $i = 2 \ldots n$.

Definition 4 : For $(y,[z]) \in \mathbb{R} \times \mathcal{I}^n$, we define the polytope in $\mathbb{R}^n$:

$$\mathsf{Q}(y,[z]) = \left\{ PV_{\{\lambda_j\}}^{-1} z \mid z \in [z] \right\} \cap \{ x \mid Cx = y \}$$

and the set $\mathcal{X} \subset \mathbb{R} \times \mathcal{I}^n$ of compatible $(y,[z])$ *i.e.* such that $\mathsf{Q}(y,[z]) \neq \emptyset$. We define also the maps γ^-, γ^+ :

$$\left[\gamma^-(t,y,[z]), \gamma^+(t,y,[z])\right] = [\gamma](t, \mathsf{Q}(y,[z]))$$

Proposition 4 : *Under hypothesis H2b, given a vector K such that $A+KC$ has n distinct eigenvalues λ_j in $\mathbb{R} \setminus \{0\}$ and a compact set X_0 of $\mathbb{R}^n$, then there exists $[z_0] \in \mathcal{I}^n$ such that $X_0 \subset PV_{\{\lambda_j\}}^{-1}[z_0]$ and, for any unknown initial condition $x(0) \in X_0$ and unknown $\psi \in \Psi$, the trajectory $x(.)$ of the system $(\mathcal{S})$ belongs to the time-varying polytope :*

$$x(t) \in \mathsf{Q}(y(t),[z](t)), \qquad \forall t \geq 0 \tag{8}$$

where $[z](.)$ is solution of the interval observer :

$$(\mathsf{O}): \left\{ \begin{array}{lcl} [\dot{z}] & = & \Delta_{\{\lambda_j\}}[z] - V_{\{\lambda_j\}}P^{-1}Ky + V_{\{\lambda_j\}}\left[\gamma^-(t,y,[z]), \gamma^+(t,y,[z])\right] \\ [z](0) & = & [z_0] \end{array} \right. \tag{9}$$

Proof Take $[x_0] \in \mathcal{I}^n$ such that $X_0 \subset [x_0]$ and consider :

$$[z_0] = V_{\{\lambda_j\}}P^{-1}[x_0]. \tag{10}$$

Denote :

$$\begin{aligned} & z = V_{\{\lambda_j\}}P^{-1}x, \\ & [\underline{z}, \overline{z}] = [z], \\ & \left[V_{\{\lambda_j\}}\widetilde{\gamma}^-(t,y,[z]), V_{\{\lambda_j\}}\widetilde{\gamma}^+(t,y,[z])\right] = V_{\{\lambda_j\}}\left[\gamma^-(t,y,[z]), \gamma^+(t,y,[z])\right]. \end{aligned} \tag{11}$$

By hypothesis H2b, γ is Lipschitz w.r.t. x with constant $||P^{-1}||L$, and by Lemma 1, γ^+ and γ^- are also Lipschitz w.r.t. $(\underline{z}, \overline{z})$ with constant $||P^{-1}||L$. So, the second members of the equations (9) are Lipschitz w.r.t. $(\underline{z}, \overline{z})$ and the solutions of (O) are then well defined as long as $(y, [\underline{z}, \overline{z}]) \in \mathcal{X}$.

We have :

$$\left\{ \begin{array}{lcl} \dfrac{d}{dt}(\overline{z} - z) & = & \Delta_{\{\lambda_j\}}(\overline{z} - z) + V_{\{\lambda_j\}}\left(\widetilde{\gamma}^+(t,y,[z]) - \gamma(t,x)\right) \\ \dfrac{d}{dt}(z - \underline{z}) & = & \Delta_{\{\lambda_j\}}(z - \underline{z}) + V_{\{\lambda_j\}}\left(\gamma(t,x) - \widetilde{\gamma}^-(t,y,[z])\right) \end{array} \right. \quad \text{where } x = PV_{\{\lambda_j\}}^{-1}z$$

But,

$$x \in \mathsf{Q}(Cx, \underline{z}, \overline{z}) \Rightarrow \left\{ \begin{array}{lcl} V_{\{\lambda_j\}}\left(\widetilde{\gamma}^+(t,y,[z]) - \gamma(t,x)\right) & \geq & 0 \\ V_{\{\lambda_j\}}\left(\gamma(t,x) - \widetilde{\gamma}^-(t,y,[z])\right) & \geq & 0 \end{array} \right.$$

So, we have $\dot{\overline{z}} - \dot{\underline{z}} \geq \Delta_{\{\lambda_j\}}(\overline{z} - \underline{z})$ *i.e.* the solution of (O) leaves $\mathcal{I}^n$ invariant. Furthermore, we can write :

$$z \in [z] \quad \Longrightarrow \quad x \in \mathsf{Q}(Cx, [z]) \quad \Longrightarrow \quad \left\{ \begin{array}{lcl} \frac{d}{dt}(\overline{z} - z) & \geq & \Delta_{\{\lambda_j\}}(\overline{z} - z) \\ \frac{d}{dt}(z - \underline{z}) & \geq & \Delta_{\{\lambda_j\}}(z - \underline{z}) \end{array} \right.$$

So, for any $x(0) \in X_0$ we have $z(0) \in [z_0] \;\Rightarrow\; z(t) \in [z](t),\ \forall t > 0$ and we conclude that $x(t) \in \mathsf{Q}(Cx(t), [z](t))$ and that the solution of (O) is well defined : $(y(t), \underline{z}(t), \overline{z}(t)) \in \mathcal{X}$ for any time t. ∎

Remarks :
1. This result means that, if we are able to obtain initial bounds for the state variables to estimate, then we are able to build an interval observer, providing a guaranteed frame for the state variables of the uncertain system. Of course, these initial estimates can be very loose, and consequently not difficult to obtain.

2. The interval bounds of $[\gamma]$ in (7) and $[z_0]$ in (10) when $X_0 \subset [\underline{x}_0, \overline{x}_0]$ can be computed in the following way :

$$\begin{array}{lcl} \overline{\gamma}(t,x) & = & \frac{1}{2}\left[P^{-1}(\overline{\psi}(t,x) + \underline{\psi}(t,x)) + |P^{-1}|(\overline{\psi}(t,x) - \underline{\psi}(t,x))\right] \\ \underline{\gamma}(t,x) & = & \frac{1}{2}\left[P^{-1}(\overline{\psi}(t,x) + \underline{\psi}(t,x)) - |P^{-1}|(\overline{\psi}(t,x) - \underline{\psi}(t,x))\right] \\ \overline{z}_0 & = & \frac{1}{2}\left[V_{\{\lambda_j\}}P^{-1}(\overline{x}_0 + \underline{x}_0) + |V_{\{\lambda_j\}}P^{-1}|(\overline{x}_0 - \underline{x}_0)\right] \\ \underline{z}_0 & = & \frac{1}{2}\left[V_{\{\lambda_j\}}P^{-1}(\overline{x}_0 + \underline{x}_0) - |V_{\{\lambda_j\}}P^{-1}|(\overline{x}_0 - \underline{x}_0)\right] \end{array}$$

where $|P^{-1}|$ stands for the matrix whose coefficients are the absolute values of the coefficients of the matrix P^{-1}.

When the eigenvalues λ_j are negative, the maps $\tilde{\gamma}^{\pm}(t, y, [z])$ defined in (11) are :

$$(\tilde{\gamma}_i^+(t,y,[z]), \tilde{\gamma}_i^-(t,y,[z])) = \left| \begin{array}{ll} \left(\max\limits_{x \in \mathsf{Q}(y,[z])} \overline{\gamma}_i(t,x),\ \min\limits_{x \in \mathsf{Q}(y,[z])} \underline{\gamma}_i(t,x)\right) & \text{if } i \equiv 1 \mod 2 \\ \left(\min\limits_{x \in \mathsf{Q}(y,[z])} \underline{\gamma}_i(t,x),\ \max\limits_{x \in \mathsf{Q}(y,[z])} \overline{\gamma}_i(t,x)\right) & \text{if } i \equiv 0 \mod 2 \end{array} \right.$$

3. We could have added a known function $\varphi(t, y)$ to the function $\psi(t, x)$, as in hypothesis H1.

Of course, the proposed bounds can become unbounded with time. So, we give now sufficient conditions that guarantee an asymptotic upper bound on the size of the set $\mathsf{Q}(y(t), \underline{z}(t), \overline{z}(t))$. We shall use the decomposition of ψ mentioned above, that shows explicitly and gradually where the asymptotic bounds come from :

$$\psi(t,x) = \sum_{i=1}^{n} \gamma_i(t,x) d_i$$

where the vectors d_i belong to $\ker(S^i\mathcal{O}) \setminus \{0\}$.

The first step is already known : when only γ_1 is non-null, we are in the case of hypothesis H1 :

Corollary 2 *Under hypotheses H1 and H2, K can be chosen such that the size of* $\mathsf{Q}(y(t),[z](t))$ *converges arbitrary fast towards an arbitrary small value.*

Proof Write $l = \overline{z} - \underline{z}$, we have :

$$\dot{l}_j = \lambda_j l_j + \sum_{i=1}^{n} \lambda_j^{i-1}(\widetilde{\gamma}_i^+(t,y,[z]) - \widetilde{\gamma}_i^-(t,y,[z])), \quad j = 1\ldots n. \tag{12}$$

As $\ker(S\mathcal{O}) = \operatorname{span}\{H\}$, we have $\gamma_j \equiv 0$ and $\gamma_j^+ \equiv \gamma_j^- \equiv 0$ for $j \neq 1$. So, with λ_j negative and $\overline{\lambda} = \min_j |\lambda_j|$,

$$\begin{aligned}
\frac{d}{dt}||l|| &\leq -\overline{\lambda}||l|| + (\widetilde{\gamma}_1^+(t,y,[z]) - \widetilde{\gamma}_1^-(t,y,[z]))\sqrt{n} \\
&\leq -\overline{\lambda}||l|| + ||P^{-1}||(M + L||PV_{\{\lambda_j\}}^{-1}||\,||l||)\sqrt{n} \\
&\leq (-\overline{\lambda} + \kappa(P)L||V_{\{\lambda_j\}}^{-1}||\sqrt{n})||l|| + ||P^{-1}||M\sqrt{n}
\end{aligned}$$

As in proposition 3, we can choose eigenvalues of the form $\theta\lambda_j$ and obtain the exponential convergence of the set $\mathsf{Q}(y(t),[z](t))$ towards a ball of radius arbitrary small (with an arbitrary fast convergence), by taking large values of θ. ∎

Now we consider the next step, when γ_2 is non null.

Hypothesis H1b : $\forall \psi \in \Psi,\ \psi(t,x) \in \ker(S^2\mathcal{O}),\ \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$.

Proposition 5 : *Under hypotheses H1b and H2b, when $L < (\kappa(P)\sqrt{n})^{-1}$, for any $\epsilon > 0$, K can be chosen such that the size of $\mathsf{Q}(y(t),[z](t))$ converges arbitrarily fast towards $[0, M/((\kappa(P)\sqrt{n})^{-1} - L) + \epsilon]$.*

Proof As $\ker(S^2\mathcal{O}) = \operatorname{span}\{H, AH\}$, we have $\gamma_j \equiv 0$ and $\gamma_j^+ \equiv \gamma_j^- \equiv 0$ for $j > 2$ in (12). So

$$\begin{aligned}
\frac{d}{dt}||l|| &\leq -\overline{\lambda}||l|| + \left(\widetilde{\gamma}_1^+(t,y,[z]) - \widetilde{\gamma}_1^-(t,y,[z]) + \overline{\lambda}(\widetilde{\gamma}_2^-(t,y,[z]) - \widetilde{\gamma}_2^+(t,y,[z]))\right)\sqrt{n} \\
&\leq -\overline{\lambda}||l|| + ||P^{-1}||(M + L||PV_{\{\lambda_j\}}^{-1}||\,||l||)(1+\overline{\lambda})\sqrt{n} \\
&\leq (-\overline{\lambda} + \kappa(P)L||V_{\{\lambda_j\}}^{-1}||(1+\overline{\lambda})\sqrt{n})||l|| + ||P^{-1}||M(1+\overline{\lambda})\sqrt{n}
\end{aligned}$$

From [4], if the n eigenvalues of $A + KC$ are chosen as $\lambda_j = \lambda_j(\sigma) = -\sigma^j$, then

$$\lim_{\sigma\to\infty} ||V_{\{\lambda_j(\sigma)\}}^{-1}|| = 1.$$

So, when $\kappa(P)L\sqrt{n} < 1$, there exists $\sigma = \overline{\lambda}$ large enough such that $\mu_\sigma = -\sigma + \kappa(P)L||V^{-1}_{\{\lambda_j(\sigma)\}}||(1+\sigma)\sqrt{n} < 0$, then $l(t)$ converges towards the ball :

$$\mathbb{B}\left(0, \frac{||P^{-1}||M(1+\sigma)\sqrt{n}}{|\mu_\sigma|}\right),$$

thus the convergence of $\mathsf{Q}(y(t),[z](t))$ towards a ball of radius :

$$\rho_\sigma = \frac{\kappa(P)||V^{-1}_{\{\lambda_j(\sigma)\}}||M(1+\sigma)\sqrt{n}}{|\mu_\sigma|}.$$

We have also :

$$\lim_{\sigma\to+\infty} \mu_\sigma = -\infty \quad \text{and} \quad \lim_{\sigma\to+\infty} \rho_\sigma = \frac{M}{1/(\kappa(P)\sqrt{n}) - L}.$$

∎

Remark : The hypotheses of Proposition 5 imply that $(\mathcal{S})$ is differentially observable for any *known* $\psi \in \Psi$: in the coordinates $\xi = P^{-1}x$, $(\mathcal{S})$ can be rewritten as follows :

$$\left\{\begin{array}{lcl} \dot{\xi} & = & \widetilde{A}\xi + \gamma(t, P\xi) \\ y & = & \widetilde{C}\xi \end{array}\right.$$

Thus

$$\Gamma_t(\psi, x^a) = \Gamma_t(\psi, x^b) \quad \Rightarrow \quad (\xi^a - \xi^b) + \left(\begin{array}{c} \gamma_2(t, P\xi^a) - \gamma_2(t, P\xi^b) \\ 0 \\ \vdots \\ 0 \end{array}\right) = 0$$

that implies that $\xi^a_j = \xi^b_j$ for $j = 2 \ldots n$. There remains one equation in ξ_1. If we consider the scalar function $\xi_1 \to \xi_1 + \gamma_2(t, P\xi)$, where the other coordinates of ξ are fixed, it is easy to see that this function is increasing (because γ_2 is Lipschitz with constant $||P^{-1}||L$ with $L < (\kappa(P)\sqrt{n})^{-1}$) and therefore $\xi^a_1 = \xi^b_1$. The two observers defined in Proposition 4 are then strong observers (when $M = 0$). Note that we do not require Γ_t to be a global diffeomorphism neither the system $(\mathcal{S})$ to leave a compact invariant, as it is required for the exact convergence of the "high-gain" observers (see [10]).

The following steps (γ_3 non null, ...) give unbounded dynamics when the eigenvalues become large. Further work needs to be done for the determination of the optimal eigenvalues, with respect to a compromise between speed and accuracy of the error.

6 An example : a biological model

Consider the dynamics of a biological population structured in three classes [21] whose stocks are : $(x_1, x_2, x_3) \in \Omega = \mathbb{R}^3_+$:

$$\left\{ \begin{array}{lcl} \dot{x}_1 & = & -\alpha_1 x_1 - m_1 x_1 + r(t, x_2, x_3) \\ \dot{x}_2 & = & \alpha_1 x_1 - \alpha_2 x_2 - m_2 x_2 \\ \dot{x}_3 & = & \alpha_2 x_2 - m_3 x_3 \end{array} \right.$$

where the positive coefficients α_i and m_i represent respectively the growth and mortality rates. The classes could be age classes (i.e. larvae, young adults and old adults) or stages in the life cycle. The same model exists within the metabolic domain [21].

We assume that the births in class x_1 are produced only by the classes x_2 and x_3 (the young and old adults) with a reproduction law of the classical Beverton-Holt type [21] :

$$r(t, x_2, x_3) = a(t)\, \frac{x_2 + x_3}{b + x_2 + x_3} \qquad a(t) > 0,\ b > 0 \tag{13}$$

Note that the system leaves Ω invariant. We suppose that online measurements on the stocks are available only for the adults population :

$$y(t) = x_3(t), \quad \forall t \geq 0$$

One can always choose loose bounds on initial values of unmeasured variables :

$$x(0) \in [\underline{x}(0), \overline{x}(0)].$$

For simplicity of notation, we shall write $\beta_1 = \alpha_1 + m_1$, $\beta_2 = \alpha_2 + m_2$ and $\beta_3 = m_3$. Write then :

$$A = \begin{pmatrix} -\beta_1 & 0 & 0 \\ \alpha_1 & -\beta_2 & 0 \\ 0 & \alpha_2 & -\beta_3 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

For simplicity of calculi, we shall study only the case where

$$\beta_1 = \beta_2 = \beta_3 = \beta \quad \text{and} \quad \alpha_1 = \alpha_2 = \alpha \quad \text{with} \quad \beta \geq \alpha > 0.$$

Then, the matrix P is :

$$P = \begin{pmatrix} 1/\alpha^2 & -\beta/\alpha^2 & \beta^2/\alpha^2 \\ 0 & 1/\alpha & -2\beta/\alpha \\ 0 & 0 & 1 \end{pmatrix}$$

6.1 Uncertainty on the reproduction law

In the reproduction law (13), the parameter b is assumed to be known but $a(t)$ is unknown and fluctuating over the time between two known functions :

$$a(t) \in [\underline{a}(t), \overline{a}(t)].$$

Then, the dynamics of the system can be written in the following way :

$$(\mathcal{S}): \left\{ \begin{array}{rcl} \dot{x} & = & Ax + \psi_1(t,x)d_1 \\ y & = & Cx \end{array} \right.$$

with

$$d_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

One can easily check that the pair (A, C) is observable and $d_1 \in \ker(S\mathcal{O})$, so hypothesis H1 is fulfilled : the system is observable for unknown ψ_1. Furthermore,

$$\psi_1(t,x) \in \left[\underline{\psi}_1(t,x_2,x_3), \overline{\psi}_1(t,x_2,x_3)\right]$$

where

$$(\underline{\psi}_1(t,x_2,x_3), \overline{\psi}_1(t,x_2,x_3)) = \left(\underline{a}(t)\ \frac{x_2+x_3}{b+x_2+x_3}, \overline{a}(t)\ \frac{x_2+x_3}{b+x_2+x_3}\right)$$

with

$$\overline{\psi}_1(t,x_2,x_3) - \underline{\psi}_1(t,x_2,x_3) \leq \overline{a}(t) - \underline{a}(t) \leq M,$$

where M is a given constant, so hypothesis H2 is fulfilled. Consider then the matrix

$$A + KC = -\beta I + \alpha I \begin{pmatrix} 0 & 0 & \alpha^{-1}k_1 \\ 1 & 0 & \alpha^{-1}k_2 \\ 0 & 1 & \alpha^{-1}k_3 \end{pmatrix}$$

We can fix the eigenvalues of $A + KC$ to be $\lambda_1, \lambda_2, \lambda_3$ with :

$$k_i = \alpha(-1)^{i+1}\sigma_i\left(\mu_1,\mu_2,\mu_3\right), \qquad i = 1\ldots 3$$

where $\mu_i = (\lambda_i + \beta)/\alpha$ and σ_i are the symmetric functions of the roots :

$$\left\{ \begin{array}{rcl} \sigma_1(\mu_1,\mu_2,\mu_3) & = & \mu_1\mu_2\mu_3 \\ \sigma_2(\mu_1,\mu_2,\mu_3) & = & \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 \\ \sigma_3(\mu_1,\mu_2,\mu_3) & = & \mu_1 + \mu_2 + \mu_3 \end{array} \right.$$

For any set $\{\lambda_j\}_{j=1\ldots3}$ of distinct negative real numbers, $A + KC$ is diagonalizable by the Vandermonde matrix $V_{\{\mu_j\}}$ and we can write the interval observer in the coordinates $z = V_{\{\mu_j\}}x$:

$$\left\{ \begin{array}{rcl} \dot{\overline{z}}_i & = & \lambda_i\overline{z}_i - (k_1 + k_2\mu_i + k_3\mu_i^2)y + \psi_1^+(t,y,[\underline{z},\overline{z}]) \\ \dot{\underline{z}}_i & = & \lambda_i\underline{z}_i - (k_1 + k_2\mu_i + k_3\mu_i^2)y + \psi_1^-(t,y,[\underline{z},\overline{z}]) \end{array} \right. \qquad i = 1\ldots 3$$

with :

$$\begin{cases} \psi_1^+(t, y, [\underline{z}, \overline{z}]) &= \overline{\psi}_1 \left(t, y, \sum_{i=1}^{3} \max\{w_{2i}\overline{z}_i, w_{2i}\underline{z}_i\} \right) \\ \psi_1^-(t, y, [\underline{z}, \overline{z}]) &= \underline{\psi}_1 \left(t, y, \sum_{i=1}^{3} \min\{w_{2i}\overline{z}_i, w_{2i}\underline{z}_i\} \right) \end{cases}$$

where w_{ij} are the coefficients of the matrix $V_{\{\mu_j\}}^{-1}$:

$$\begin{cases} w_{21} &= -\dfrac{\mu_2 + \mu_3}{\mu_1^2 + \mu_2\mu_3 - \mu_1\mu_2 - \mu_1\mu_3} \\ w_{22} &= \dfrac{\mu_1 + \mu_3}{-\mu_2^2 + \mu_2\mu_3 + \mu_1\mu_2 - \mu_1\mu_3} \\ w_{23} &= -\dfrac{\mu_1 + \mu_2}{\mu_3^2 - \mu_2\mu_3 + \mu_1\mu_2 - \mu_1\mu_3} \end{cases}$$

because $\underline{\psi}_1$, $\overline{\psi}_1$ are non decreasing functions w.r.t. x_2.

With the initial condition :

$$\begin{cases} \overline{z}_i(0) &= \overline{x}_1(0) + \max\{\mu_i\overline{x}_2(0),\ \mu_i\underline{x}_2(0)\} + \mu_i^2 y(0) \\ \underline{z}_i(0) &= \underline{x}_1(0) + \min\{\mu_i\overline{x}_2(0),\ \mu_i\underline{x}_2(0)\} + \mu_i^2 y(0) \end{cases} \qquad i = 1 \dots 3,$$

the Proposition 4 guarantees that $x(t)$ belongs to the polytope :

$$\mathsf{Q}(y(t), [\underline{z}(t), \overline{z}(t)]) = \left\{ x \,|\, V_{\{\mu_j\}} x \in [\underline{z}(t), \overline{z}(t)] \text{ and } x_3 = y(t) \right\}, \quad \forall t \geq 0.$$

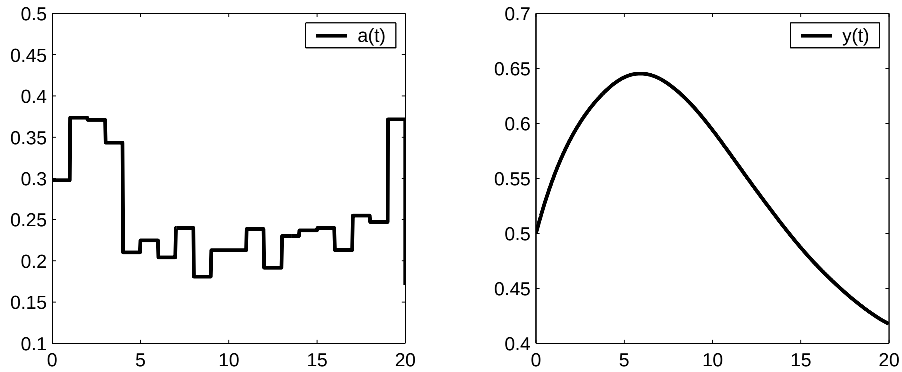
and Corollary 2 ensures that the asymptotic size of this polytope can be reduced as desired by taking large negative values λ_j.

Remark : In this example, the conditions of [6] are not satisfied.

Numerical experimentations :

$$\left. \begin{array}{lllllll} \alpha_1 &=& 0.3, & m_1 &=& 0 \\ \alpha_2 &=& 0.3, & m_2 &=& 0 \\ & & & m_3 &=& 0.3 \end{array} \right\} \Rightarrow \alpha = 0.3,\ \beta = 0.3$$

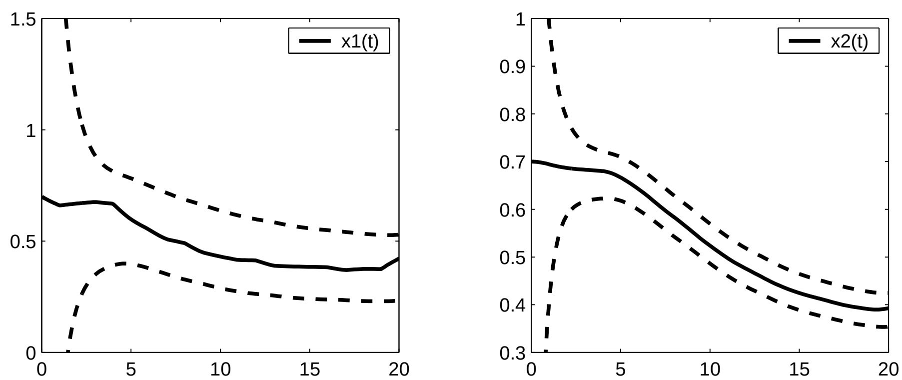
$$[\underline{a}, \overline{a}] = [0.1, 0.4], \quad b = 1$$



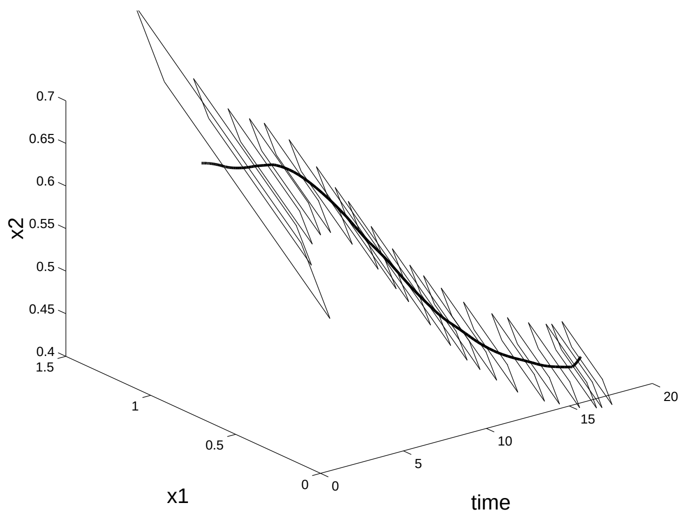
Unknown input $a(t)$ and observation $y(t)$

The eigenvalues λ_i have been chosen such that $\mu_i = -\theta^i$, which ensure reasonable values for the norm of the matrix $V^{-1}_{\{\mu_j\}}$.

Simulation 1 : $Sp(A + KC) = \{-1.1, -2.4, -6\}$

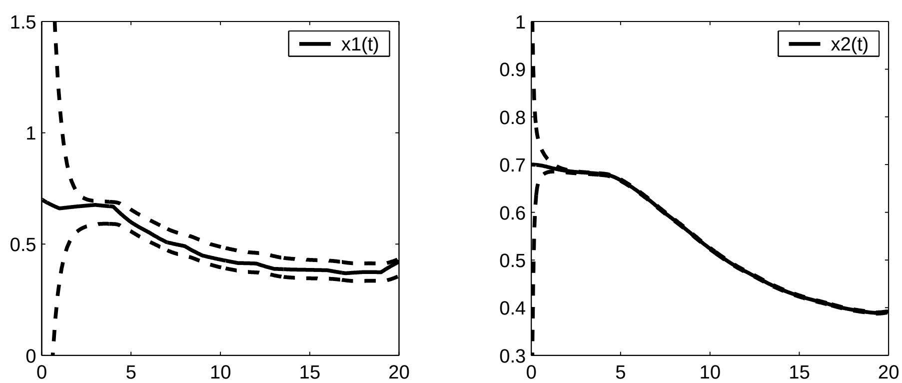


Unmeasured variables $x_1(t)$ and $x_2(t)$ with the guaranteed intervals

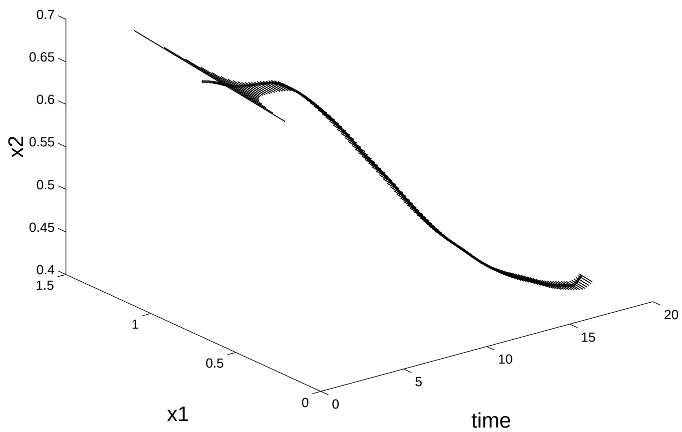


State trajectory in the (x_1, x_2) plane, boxed by the time varying polytope

Simulation 2 : $Sp(A + KC) = \{-2, -10, -55\}$



Unmeasured variables $x_1(t)$ and $x_2(t)$ with the guaranteed intervals



State trajectory in the (x_1, x_2) plane, boxed by the time varying polytope

We see on these simulations that the convergence of the observer is a practical one : the asymptotic error is not equal to 0 but can be made arbitrarily small, taking large negative eigenvalues λ_i. We see also that the effect of the unknown input on the width of the guaranteed polytope is larger in the direction of the vector $(1\,0\,0)'$ (the first column of P), which corresponds to the image of the vector $(1\,1\,1)'$ by the matrix $PV^{-1}_{\{\mu_j\}}$.

6.2 Uncertainty on an harvesting effort

In addition to the uncertainty on the reproduction law, we consider that an uncertain mortality or harvesting effort $e(t) \geq 0$ exists on the class x_2 :

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & -\alpha_1 x_1 - m_1 x_1 + r(t, x_2, x_3) \\ \dot{x}_2 & = & \alpha_1 x_1 - \alpha_2 x_2 - m_2 x_2 \ - \ e(t) x_2 \\ \dot{x}_3 & = & \alpha_2 x_2 - m_3 x_3 \end{array} \right.$$

where $e(t)$ is bounded between two known functions :

$$e(t) \in [\underline{e}(t), \overline{e}(t)] \subset [0, E^{\max}], \quad \forall t \geq 0.$$

Moreover, it is easy to check that the variables are bounded (and non negative). With the same notations as before, the dynamics of the system can be written :

$$(\mathcal{S}) : \ \left\{ \begin{array}{rcl} \dot{x} & = & Ax + \psi_1(t,x) d_1 + \psi_2(t,x) d_2 \\ y & = & Cx \end{array} \right.$$

with

$$d_2 = \left(\begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right).$$

One can easily check that $d_2 \in \ker(S^2\mathcal{O})$, so the system is not observable for the unknown input ψ_2. But hypotheses H1b and H2b are fulfilled :

$$\psi_2(t,x) \in [\underline{\psi}_2(t,x_2), \overline{\psi}_2(t,x_2)] = [\underline{e}(t)x_2, \overline{e}(t)x_2]$$

and

$$\overline{\psi}_2(t,x_2) - \underline{\psi}_2(t,x_2) \leq H^{\max}$$

where $H^{\max}$ is some constant depending on the bounds on the variables, for instance $H^{\max} = E^{\max}x_2^{\max}$ (where $x_2^{\max}$ is a global upper bound on x_2). Then, a guaranteed interval observer can be written in the coordinates z :

$$\left\{ \begin{array}{lcl} \dot{\overline{z}}_i & = & \lambda_i \overline{z}_i - (k_1 + k_2\mu_i + k_3\mu_i^2)y + \overline{\psi}_1(t,y,\overline{x}_2) - \mu_i\overline{\psi}_2(t,\underline{x}_2) \\ \dot{\underline{z}}_i & = & \lambda_i \underline{z}_i - (k_1 + k_2\mu_i + k_3\mu_i^2)y + \underline{\psi}_1(t,y,\underline{x}_2) - \mu_i\underline{\psi}_2(t,\overline{x}_2) \end{array} \right. \qquad i = 1\dots3$$

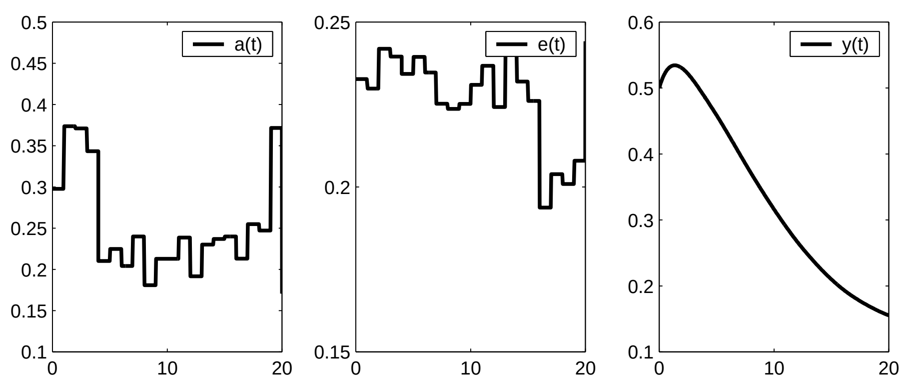
where

$$(\overline{x}_2, \underline{x}_2) = \left(\sum_{i=1}^{3} \max\{w_{2i}\overline{z}_i, w_{2i}\underline{z}_i\}, \sum_{i=1}^{3} \min\{w_{2i}\overline{z}_i, w_{2i}\underline{z}_i\} \right)$$

if the λ_i are large enough (then the μ_i are non positive) and Proposition 5 guarantees that the polytope $\mathsf{Q}(y(t), \underline{z}(t), \overline{z}(t))$ converges exponentially towards a bounded set when $E^{\max} < 1/(\sqrt{3}\kappa(P))$.

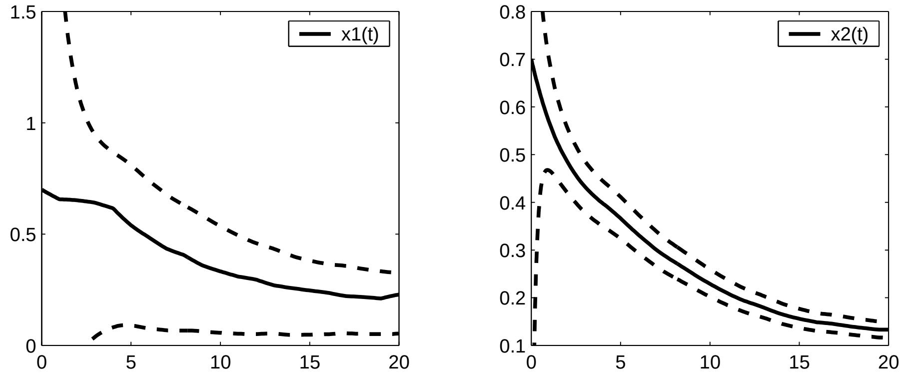
Numerical experimentations : We used the same parameters as before with :

$$E^{\max} = 0.25 \quad \text{and} \quad H^{\max} = 1.$$

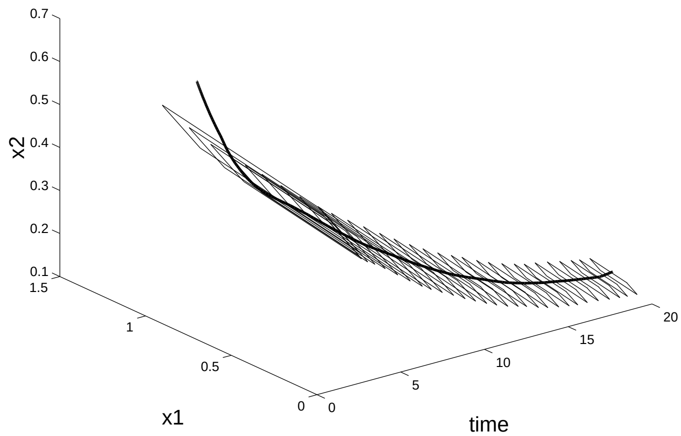


Unknown inputs $a(t)$, $e(t)$ and observation $y(t)$

Simulation 1 : $Sp(A+KC) = \{-1.5, -5.1, -19\}$

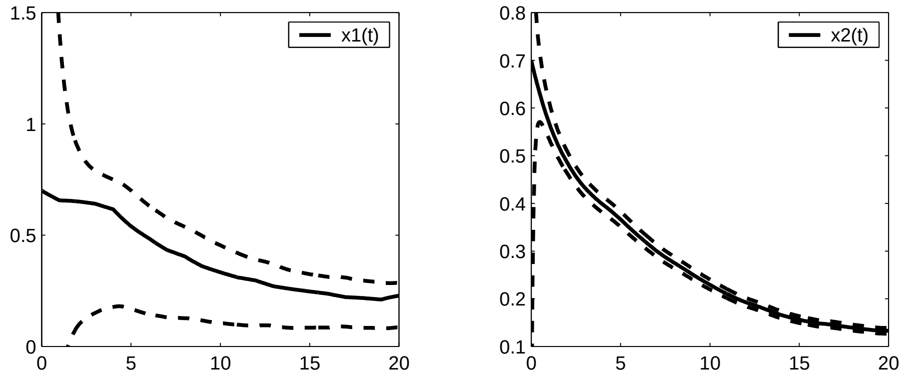


Unmeasured variables $x_1(t)$ and $x_2(t)$ with the guaranteed intervals

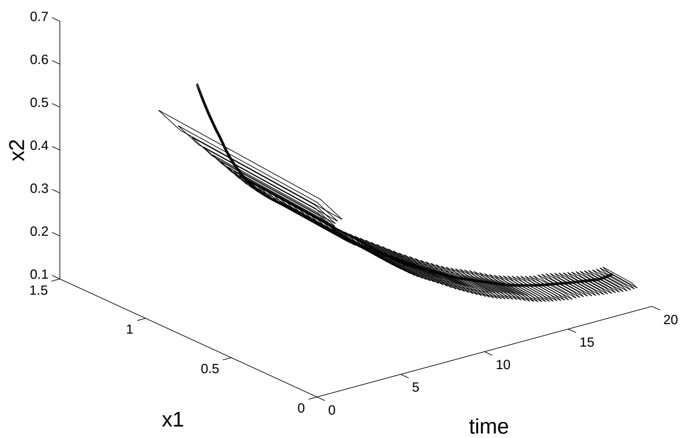


State trajectory in the (x_1, x_2) plane, boxed by the time varying polytope

Simulation 2 : $Sp(A+KC) = \{-3, -25, -220\}$



Unmeasured variables $x_1(t)$ and $x_2(t)$ with the guaranteed intervals



State trajectory in the (x_1, x_2) plane, boxed by the time varying polytope

We see on these simulations that the system is not observable for the unknowns inputs : even when taking large negative eigenvalues λ_i, there is always a residual error on the variable x_1, but the guaranteed polytope stays bounded.

7 Conclusion

We have studied the observability for unknown inputs for a class of nonlinear systems, for which we proposed practical observers. For the general case, we have proposed guaranteed polytopic observers. In this work, we have considered that the measurements $y(t)$ were available free from any error. It is also possible to consider deterministic and bounded uncertainties on the measurements : $y(t) \in [\underline{y}(t), \overline{y}(t)]$ using similar interval techniques. This will be the purpose of a forthcoming work.

Acknowledgments : The authors would like to thank the *G.d.R. Automatique* (French Research Program in Control Science) for its support.

References

[1] B.D.O. Anderson and J.B. Moore. *Optimal Filtering.* Prentice Hall, Englewood Cliffs, N.J., 1979.

[2] T. Başar and P. Bernhard. *H^∞-Optimal control and Related Minimax Design Problems: A Dynamic Game Approach.* 2nd ed., Birkhäuser, Boston, 1991.

[3] D.P. Bertsekas and I.B. Rhodes. *Recursive state estimation for a set-membership description of uncertainty.* IEEE Trans. on Automatic Control, vol. 16, pp. 117–128, 1971.

[4] C. Ciccarella, M. Dalla and A. Germani. *A Luenberger-like observer for nonlinear systems.* Int. J. Control, vol. 57, no. 3, pp. 537–556, 1993.

[5] G.F. Corliss. *Survey of Interval Algorithms for Ordinary Differential Equations.* Appl. Math. Comput. 31, pp. 112–120, 1989.

[6] M. Darouach, M. Zasadzinski and S.J. Liu. *Full-Order Observers for Linear Systems with Unknown Inputs.* IEEE Trans. Aut. Control. vol. 39, no. 3, pp. 606–609, 1994.

[7] C. Edwards and S.K. Spurgeon. *On the development of discontinuous observers.* Int. J. Control, vol. 59, no. 5, pp. 1211–1229, 1994.

[8] L.C. Evans and R.F. Gariepy. *Measure Theory and Fine Properties of Functions.* CRC Press, Boca Raton (Florida), 1992.

[9] J.P. Gauthier and I. Kupka. *Observability and observers for nonlinear systems.* SIAM J. Control and Optimization, vol. 32, no. 4, pp. 975–994, 1994.

[10] J.P. Gauthier and I. Kupka. *Deterministic Observation Theory and Applications.* Rapport de recherche, Université de Bourgogne, no. 187, 1999.

[11] J.L. Gouzé, A. Rapaport and M.Z. Hadj-Sadok. *Interval observers for uncertain biological systems.* Ecological Modelling, 133 (1), pp. 45–56, 2000.

[12] A. Isidori. *Nonlinear control systems.* 3rd ed., Springer Verlag, London, 1995.

[13] M.R. James and J.S. Baras. *Partially observed differential games, infinite-dimensional Hamilton-Jacobi-Isaacs equations, and nonlinear H_∞ control.* SIAM J. Control Optimization vol. 34, no. 4, pp. 1342–1364, 1996.

[14] T. Kailath. *Linear Systems.* Prentice Hall, 1980.

[15] M. Kieffer, L. Jaulin and E. Walter. *Guaranteed recursive nonlinear state estimation using interval analysis.* 37th IEEE Conference on Decision and Control, Tampa (Florida), 16–18 Dec. 1998.

[16] A.J. Krener and W. Respondek. *Nonlinear observers with linearizable error dynamics.* SIAM J. Control and Optimization, vol. 23, no. 2, pp. 197–216, 1985.

[17] A. Kurzhanski and I. Valyi. *Ellipsoidal Calculus for Estimation and Control.* Birkhäuser, 1997.

[18] W. McEneaney. *Uniqueness for viscosity solutions of nonstationary Hamilton-Jacobi-Bellman equations under some a priori conditions (with applications).* SIAM J. Control Optimization vol. 33, no. 5, pp. 1560–1576, 1995.

[19] E. McShane. *Extension of range of functions.* Bull. Amer. Math. Soc. 40, pp. 837–842, 1934.

[20] R.E. Moore. *Interval Analysis.* Prentice Hall, 1966.

[21] J.D. Murray. *Mathematical Biology.* Springer-Verlag, 1990.

[22] I.R. Petersen and A.V. Savkin. *Robust Kalman Filtering for Signals and Systems with Large Uncertainties.* Birkhäuser Boston, 1999.

[23] A. Rapaport and J.L. Gouzé. *Practical observers for uncertain affine output injection systems.* European Control Conference, Karlsruhe (Germany), 31 Aug.–3 Sept. 1999.

[24] R. Rihm. *Interval methods for initial value problems in ODEs.* Topics in Validated Computations (J. Herzberger, ed.), Elsevier Science B.V., pp. 173–208, 1994.

[25] F.C. Schweppe. *Recursive state estimation : Unknown but bounded errors and systems inputs.* IEEE Trans. Aut. Control. vol. 13, no. 1, pp. 22–28, 1968.

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles - B.P. 93 - 06902 Sophia Antipolis Cedex (France)

Unité de recherche INRIA Lorraine : Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - B.P. 101 - 54602 Villers lès Nancy Cedex (France)
Unité de recherche INRIA Rennes : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex (France)
Unité de recherche INRIA Rhône-Alpes : 655, avenue de l'Europe - 38330 Montbonnot St Martin (France)
Unité de recherche INRIA Rocquencourt : Domaine de Voluceau - Rocquencourt - B.P. 105 - 78153 Le Chesnay Cedex (France)

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, B.P. 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399