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Abstract: In this note is proposed an analogue for linear delay systems of the characterization of asymptotic stability of the rational systems by the solvability of associated Lyapunov equation. It is shown that strong delay-independent stability of delay system is equivalent to the feasibility of certain linear matrix inequality (LMI), related to quadratic Lyapunov-Krasovskii functionals.

Key-words: linear delay systems, delay-independent stability, quadratic Lyapunov-Krasovskii functionals, linear matrix inequalities.

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Caractérisation par des inégalités linéaires matricielles de la stabilité fortement indépendante du retard des systèmes linéaires à retard, *via* des fonctionnelles de Lyapunov-Krasovskii quadratiques

Résumé : Dans cette note, on propose pour les systèmes linéaires à retard un résultat analogue à la caractérisation de la stabilité asymptotique des systèmes rationnels par la solvabilité d'une équation de Lyapunov associée. On montre que la propriété de stabilité forte indépendante du retard (*strong delay-independent stability* en anglais) est équivalente à la solvabilité d'une inégalité linéaire matricielle, obtenue à partir de fonctionnelles quadratiques de Lyapunov-Krasovskii.

Mots-clés : Systèmes linéaires à retard, stabilité indépendante du retard, fonctionnelles de Lyapunov-Krasovskii quadratiques, inégalités linéaires matricielles.

1 Introduction

It is a well-known property that the asymptotic stability of the rational system

$$\dot{x} = Ax \ , \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, may be equivalently characterized by the spectral property:

$$\forall s \in \mathbb{C}, \mathrm{Re}\, s \geq 0 \Rightarrow \det(sI_n - A) \neq 0 \ ,$$

or by the solvability of the Lyapunov inequation

$$\exists P \in \mathbb{R}^{n\times n}, P = P^T > 0, A^T P + PA < 0 \ .$$

The link between the two properties is made by the use of the quadratic function $V(x) \stackrel{\text{def}}{=} x^T Px$, which is a Lyapunov function for system (1) when P is solution of the previous inequality.

For delay systems, such a result does not exist. However, spectral characterization is known: the asymptotic stability of the system

$$\dot{x} = Ax(t) + Bx(t-h) \ , \tag{2}$$

$A, B \in \mathbb{R}^{n\times n}$, is equivalent to

$$\forall s \in \mathbb{C}, \mathrm{Re}\, s \geq 0 \Rightarrow \det(sI_n - A - e^{-sh}B) \neq 0 \ .$$

When the delay is imperfectly known, there is need for stability results *robust* wrt this uncertainty. The notion of *delay-independent* (asymptotic) stability has been introduced: system (2) is said delay-independently stable [11, 12, 13] if

$$\forall h \geq 0, \forall s \in \mathbb{C}, \mathrm{Re}\, s \geq 0 \Rightarrow \det(sI_n - A - e^{-sh}B) \neq 0 \ .$$

This property has been proved to be equivalent to [6, 8]

$$\forall (s,z) \in \mathbb{C}^2, \mathrm{Re}\, s \geq 0, s \neq 0, |z| \leq 1 \text{ or } s = 0, z = 1 \Rightarrow \det(sI_n - A - zB) \neq 0 \ .$$

A slightly stronger property has also been introduced [16]: system (2) is called *strongly* delay-independently stable if

$$\forall (s,z) \in \mathbb{C}^2, \mathrm{Re}\, s \geq 0, |z| \leq 1 \Rightarrow \det(sI_n - A - zB) \neq 0 \ .$$

On the other hand, generalizations of the Lyapunov method to delay differential equations have been found, notably by Krasovskii [15]. As an example (see [6, Corollary 3.1]), for a general system

$$\dot{x} = f(x_t), \quad f : \mathcal{C}([-h,0];\mathbb{R}^n) \to \mathbb{R}^n \ , \tag{3}$$

under usual regularity assumptions, the existence of a so-called Lyapunov-Krasovskii functional $V : \mathcal{C}([-h,0]) \to \mathbb{R}$ and of $\alpha_1, \alpha_2 : \mathbb{R}^+ \to \mathbb{R}^+$, α_1 unbounded, α_2 definite, such that

$$\alpha_1(|x(t)|) \leq V(x_t), \ \frac{dV(x_t)}{dt} \leq -\alpha_2(|x(t)|) \tag{4}$$

along the trajectories, ensures asymptotic stability of the origin.

In particular, simple quadratic Lyapunov-Krasovskii functionals of the type

$$V(x_t) = x^T(t)Px(t) + \int_{t-h}^{t} x^T(s)Qx(s)\ ds \ , \tag{5}$$

for positive definite matrices $P, Q \in \mathbb{R}^{n\times n}$, have been used early [15, 19, 7]. As a matter of fact, the derivative of V given by (5) along the trajectories of (2) is

$$\frac{dV(x_t)}{dt} = \begin{pmatrix} x(t) \\ x(t-h) \end{pmatrix}^T \begin{pmatrix} A^TP+PA+Q & PB \\ B^TP & -Q \end{pmatrix} \begin{pmatrix} x(t) \\ x(t-h) \end{pmatrix} .$$

The result by Krasovskii cited above implies that solvability of the linear matrix inequality (LMI)

$$P, Q \in \mathbb{R}^{n\times n}, P = P^T > 0, Q = Q^T > 0, R = \begin{pmatrix} A^TP+PA+Q & PB \\ B^TP & -Q \end{pmatrix} < 0 \tag{6}$$

is sufficient for asymptotic stability of system (2). Remark that the value of the delay h does not appear explicitly: solvability of (6) is hence sufficient for *delay-independent* stability of (2).

There hence exist for delay systems analogue of the results known for the delay-free case, namely spectral criterion and LMI criterion, the second one being obtained by use of quadratic Lyapunov-Krasovskii functionals. However, it is known that condition (6) constitutes a *conservative* criterion of delay-independent stability: it is sufficient, but not necessary. This fact has been already observed [9]. It has been established formally in [2, 3], in terms closely related to the approach exposed in [4]. It has also been obtained by Agathoklis *et al.* [1] without using Lyapunov approach.

Theorem 1. • *Solvability of LMI* (6) *is equivalent to*

$$\operatorname{Re}\sigma(A) < 0 \text{ and } \min_{\substack{M \\ \text{invertible}}} \max_{s\in j\mathbb{R}} \|M(sI_n - A)^{-1}BM^{-1}\| < 1 . \tag{7}$$

- *Strong delay-independent stability of system* (2) *is equivalent to*

$$\operatorname{Re}\sigma(A) < 0 \text{ and } \max_{s\in j\mathbb{R}} \min_{\substack{M \\ \text{invertible}}} \|M(sI_n - A)^{-1}BM^{-1}\| < 1 . \tag{8}$$

- *Delay-independent stability of system* (2) *is equivalent to*

$$\operatorname{Re}\sigma(A) < 0, \max_{s\in j\mathbb{R}\setminus\{0\}} \min_{\substack{M \\ \text{invertible}}} \|M(sI_n - A)^{-1}BM^{-1}\| < 1 \text{ and } \det(A+B) \neq 0 . \tag{9}$$

■

For sake of completeness, a rapid proof of the first two points is given here. See also the proofs in [4] (formulas (8), (9)) and [1] and [2, 3] (formula (7)).

Proof. By Kalman-Yakubovich-Popov lemma, solvability of (6) is equivalent to

$$\operatorname{Re}\sigma(A) < 0, \text{ and } \exists Q = Q^T > 0, \forall s \in j\mathbb{R},\ B^T(s^*I - A^T)^{-1}Q(sI-A)^{-1}B - Q < 0 ,$$

which is itself equivalent to (7).

When A is Hurwitz, one has: $\forall s \in \mathbb{C}$ with $\operatorname{Re} s \geq 0$,

$$\begin{aligned} \min\{|z| \ :\ z \in \mathbb{C},\ \det(sI - A - zB)\} &= \max\{|z| \ :\ z \in \mathbb{C},\ \det(zI - (sI-A)^{-1}B)\} \\ &= \rho((sI-A)^{-1}B) = \min_{\substack{M \\ \text{invertible}}} \|M(sI-A)^{-1}BM^{-1}\| , \end{aligned}$$

using [21, p. 282]. Using maximum modulus principle, one proves the equivalence of the strong delay-independent stability with (8). The delay-independent stability is treated similarly. □

Theorem 1 clearly enlightens that solvability of (6) is indeed sufficient for strong delay-independent stability as well. An interesting feature is the fact [4] that, for any $s \in \mathbb{C}$,

$$\min_{\substack{M \\ \text{invertible}}} \|M(sI_n - A)^{-1}BM^{-1}\| = \rho((sI_n - A)^{-1}B) = \mu_\Delta((sI_n - A)^{-1}B) \ , \tag{10}$$

where μ_Δ represents the structured singular value associated to the uncertainty structure $\Delta = \{\delta I_N \ : \ \delta \in \mathbb{C}\}$. This robust control interpretation shows that strong delay-independent stability is equivalent to the stability of all the uncertain systems obtained when replacing the delay by any proper real rational stable perturbation of $\mathcal{H}_\infty$-norm not greater than 1.

In view of these results, it may be proved that the delay-independent stability property is not robust wrt perturbations of the parameters A, B. More precisely [3], using the right-hand side of these equivalences, one shows that the subsets of the matrices (A, B) fulfilling (7) or (8) are *open* for the product topology induced by the spectral norm. Moreover, the set of the matrices (A, B) representing strongly delay-independently stable systems is the *interior* of the set of the matrices associated to delay-independently stable systems.

Remark that, in order to improve the accuracy of the Lyapunov-Krasovskii approach, one may consider larger classes of candidate functionals than (5). This search is not hopeless, as Krasovskii has proved what is now called a *converse Lyapunov theorem*, ensuring under reasonable regularity assumptions the existence of a functional V fulfilling (4), provided that system (3) is asymptotically stable [15, §30]. Notably, Infante *et al.* [10] and Whenzhang [20] have shown results which imply that asymptotic stability of system (2) is equivalent to the existence of a quadratic Lyapunov-Krasovskii functional (depending upon the delay h), in certain classes generalizing (5). The generalization consists essentially in allowing nonconstant integral kernels. A similar idea is exploited by Gu [5], with piecewise linear discretizations, giving rise to sufficient stability conditions. For review on the available methods and results on stability of linear delay systems, the reader is refered to the recent surveys [14, 17].

So far, the question of the existence of an exact correspondence between a spectral definition of delay-independent stability and a condition obtained via simple quadratic Lyapunov-Krasovskii functionals, hence remains unanswered. This is our concern in the present paper.

We provide an analogue of the equivalence between spectral characterization of the asymptotic stability for rational systems and solvability of the Lyapunov function. A seemingly original method is used for improving Lyapunov-Krasovskii approach, which may be appropriate in other stability problems for delay systems. The principle is as follows. Instead of considering the state variable $\{x(t+s) \ : \ -h \leq s \leq 0\}$ as usual, we allow the use of more ancient part of the trajectories, and consider rather $\{x(t+s) \ : \ -kh \leq s \leq 0\}$, for some $k \in \mathbb{N}$. Of course, any such function, even sufficiently smooth, cannot be part of a trajectory of system (2): the supplementary available information is reintroduced when estimating the derivative of the candidate Lyapunov-Krasovskii functional.

In Section 2, the main result (Theorem 2) is enunciated and commented. Two direct applications are provided, to strong delay-independent stability of delay systems with commensurate delays (Corollary 3), and to strong delay-independent stabilizability (Corollary 5). A detailed proof of Theorem 2 is given in Section 3.

2 A LMI characterization of strong delay-independent stability

Denoting $\otimes$ the Kronecker product, one defines by LMI(k), $k \in \mathbb{N} \setminus \{0\}$, the following linear matrix inequality.

$$\mathcal{P}_k, \mathcal{Q}_k \in \mathbb{R}^{kn\times kn}, \mathcal{P}_k = \mathcal{P}_k^T > 0, \mathcal{Q}_k = \mathcal{Q}_k^T > 0, \mathcal{R}_k < 0, \text{ where } \mathcal{R}_k \in \mathbb{R}^{(k+1)n\times(k+1)n} \text{ is given by}$$

$$\mathcal{R}_k \stackrel{\text{def}}{=} \begin{pmatrix} I_{kn} & 0_{kn\times n} \\ 0_{kn\times n} & I_{kn} \end{pmatrix}^T \begin{pmatrix} \mathcal{P}_k(I_k\otimes A) + (I_k\otimes A)^T\mathcal{P}_k + \mathcal{Q}_k & \mathcal{P}_k(I_k\otimes B) \\ (I_k\otimes B)^T\mathcal{P}_k & -\mathcal{Q}_k \end{pmatrix} \begin{pmatrix} I_{kn} & 0_{kn\times n} \\ 0_{kn\times n} & I_{kn} \end{pmatrix} . \tag{11}$$

Our main result is the following.

Theorem 2. *The two following properties are equivalent.*

1. *System* (2) *is strongly delay-independently stable, i.e. for any $(s,z) \in \mathbb{C}^2$,*

$$\operatorname{Re} s \geq 0, |z| \leq 1 \Rightarrow \det(sI_n - A - zB) \neq 0 \ .$$

2. *There exists $k \in \mathbb{N} \setminus \{0\}$ such that LMI(k) is feasible .*

Moreover,

- *if LMI(k) is feasible, then, for any $k' \in \mathbb{N}$, $k' \geq k$, LMI(k') is feasible;*
- *for any $h \geq 0$, for any $k \in \mathbb{N}\setminus\{0\}$, for any $t \geq (k-1)h$, denoting, for any trajectory x of system* (2)

$$V_k(x)(t) \stackrel{\text{def}}{=} \begin{pmatrix} x(t) \\ \vdots \\ x(t-(k-1)h) \end{pmatrix}^T \mathcal{P}_k \begin{pmatrix} x(t) \\ \vdots \\ x(t-(k-1)h) \end{pmatrix} + \int_{t-h}^t \begin{pmatrix} x(s) \\ \vdots \\ x(s-(k-1)h) \end{pmatrix}^T \mathcal{Q}_k \begin{pmatrix} x(s) \\ \vdots \\ x(s-(k-1)h) \end{pmatrix} ds \ , \tag{12}$$

one has

$$\frac{dV_k(x)(t)}{dt} = \begin{pmatrix} x(t) \\ \vdots \\ x(t-kh) \end{pmatrix}^T \mathcal{R}_k \begin{pmatrix} x(t) \\ \vdots \\ x(t-kh) \end{pmatrix} , \tag{13}$$

where $\mathcal{R}_k$ is defined as in (11).

■

Theorem 2 establishes a natural extension of the celebrated result valid for delay-free systems: the strong delay-independent stability is equivalent to the feasibility of a certain LMI, whose unknowns are the parameters of a quadratic Lyapunov-Krasovskii functional. The Lyapunov-Krasovskii functionals solving the problem, instead of involving only the values of $\{x(t+s) \ : \ -h \leq s \leq 0\}$, are based on a larger, but *finite*, memory, of length at least kh.

Theorem 2 furnishes a sequence of LMI criteria, of arbitrary precision. The precision may be estimated as follows. One may check from the proof that condition 1. is fulfilled if and only if

$$\operatorname{Re}\sigma(A) < 0 \text{ and } \lim_{k\to+\infty} \|[(sI_n - A)^{-1}B]^k\|_\infty = 0 \ .$$

On the other hand, it will be shown that a sufficient condition for solvability of LMI(k), $k \in \mathbb{N} \setminus \{0\}$, is just

$$\mathrm{Re}\, \sigma(A) < 0 \text{ and } \|[(sI_n - A)^{-1}B]^k\|_\infty < 1 \ .$$

Remark that, as in the rational case, asymptotic stability of linear delay system is equivalent to exponential stability. However, the decay rate is not uniform wrt the delay h. At last, notice that a major benefit of being able to use Lyapunov-Krasovskii framework, is the possibility to deduce results of stability robustness wrt various, possibly nonlinear, perturbations.

We propose in the sequel two direct corollaries of Theorem 2.

2.1 Strong delay-independent stability of systems with commensurate delays

The first application concerns the linear delay systems having commensurate delays. Consider the system

$$\dot{x} = \sum_{l=0}^{l_0} A_l x(t - lh) \ , \tag{14}$$

where $l_0 \in \mathbb{N} \setminus \{0\}$ and $A_l \in \mathbb{R}^{n\times n}, l = \overline{0, l_0}$. To handle system (14), one transforms it into a new, augmented, system having a unique delay. More precisely, defining

$$X(t) \stackrel{\text{def}}{=} \begin{pmatrix} x(t) \\ x(t-h) \\ \vdots \\ x(t-(l_0-1)h) \end{pmatrix} , \tag{15}$$

one may "rewrite" system (14) as :

$$\dot{X} = \mathcal{A}X + \mathcal{B}X(t - l_0 h) \ , \tag{16}$$

where $\mathcal{A}, \mathcal{B} \in \mathbb{R}^{l_0 n\times l_0 n}$ are defined by

$$\mathcal{A} \stackrel{\text{def}}{=} \sum_{l=0}^{l_0} E_{l_0}^l \otimes A_l, \ \mathcal{B} \stackrel{\text{def}}{=} \sum_{l=0}^{l_0} E_{l_0}^{(l_0-l)T} \otimes A_l, \tag{17a}$$

$$E_{l_0} \in \mathbb{R}^{l_0\times l_0}, \ (E_{l_0})_{ij} \stackrel{\text{def}}{=} 1 \text{ if } i+1=j, \ 0 \text{ otherwise}, \ i,j = \overline{1, l_0} \ . \tag{17b}$$

It is clear that the trajectories of system (14) may be obtained as projections of some trajectories of system (16). Asymptotic stability of (16) hence implies asymptotic stability of (14). It turns out that strong delay-independent stability of (16) is indeed *equivalent* to strong delay-independent stability of (14), see below the proof of Corollary 3.

Corollary 3. *The two following properties are equivalent.*

1. *System* (14) *is strongly delay-independently stable, i.e. for any* $(s,z) \in \mathbb{C}^2$*,*

$$\mathrm{Re}\, s \geq 0, |z| \leq 1 \Rightarrow \det(sI_n - \sum_0^{l_0} z^l A_l) \neq 0 \ .$$

2. *There exists* $k \in \mathbb{N} \setminus \{0\}$ *for which the following LMI is feasible.*

$\mathcal{P}_k, \mathcal{Q}_k \in \mathbb{R}^{kl_0 n\times kl_0 n}, \mathcal{P}_k = \mathcal{P}_k^T > 0, \mathcal{Q}_k = \mathcal{Q}_k^T > 0, \mathcal{R}_k < 0,$ *where* $\mathcal{R}_k \in \mathbb{R}^{(k+1)l_0 n\times (k+1)l_0 n}$ *is given by*

$$\mathcal{R}_k \stackrel{\text{def}}{=} \begin{pmatrix} I_{kl_0 n} & 0_{kl_0 n\times l_0 n} \\ 0_{kl_0 n\times l_0 n} & I_{kl_0 n} \end{pmatrix}^T \begin{pmatrix} \mathcal{P}_k(I_k\otimes\mathcal{A}) + (I_k\otimes\mathcal{A})^T\mathcal{P}_k + \mathcal{Q}_k & \mathcal{P}_k(I_k\otimes\mathcal{B}) \\ (I_k\otimes\mathcal{B})^T\mathcal{P}_k & -\mathcal{Q}_k \end{pmatrix} \begin{pmatrix} I_{kl_0 n} & 0_{kl_0 n\times l_0 n} \\ 0_{kl_0 n\times l_0 n} & I_{kl_0 n} \end{pmatrix} ,$$

where $\mathcal{A},\mathcal{B}$ *are defined in* (17).

Moreover, for any $h\geq 0$, *for any* $k\in\mathbb{N}\setminus\{0\}$, *for any* $t\geq (k-1)h$, *denoting, for any trajectory* x *of system* (14)

$$\mathcal{V}_k(x)(t) \stackrel{\text{def}}{=} \begin{pmatrix} x(t)\\ \vdots \\ x(t-(kl_0-1)h)\end{pmatrix}^T \mathcal{P}_k \begin{pmatrix} x(t)\\ \vdots \\ x(t-(kl_0-1)h)\end{pmatrix} + \int_{t-h}^{t} \begin{pmatrix} x(s)\\ \vdots \\ x(s-(kl_0-1)h)\end{pmatrix}^T \mathcal{Q}_k \begin{pmatrix} x(s)\\ \vdots \\ x(s-(kl_0-1)h)\end{pmatrix} ds ,$$

one has

$$\frac{d\mathcal{V}_k(x)(t)}{dt} = \begin{pmatrix} x(t)\\ \vdots \\ x(t-((k+1)l_0-1)h)\end{pmatrix}^T \mathcal{R}_k \begin{pmatrix} x(t)\\ \vdots \\ x(t-((k+1)l_0-1)h)\end{pmatrix} .$$

■

Similar to Theorem 2, the property stating that the LMI is indeed feasible for $k'\geq k$, has been omitted for sake of space.

Proof. After applying Theorem 2, it only remains to compare $\det(sI_n-\sum_0^{l_0} z^l A_l)$ and $\det(sI_n-\mathcal{A}-z^{l_0}\mathcal{B})$. One has :

$$sI_n-\mathcal{A}-z^{l_0}\mathcal{B} = sI_n - \sum_{l=0}^{l_0}(E_{l_0}^l + z^{l_0}E_{l_0}^{(l_0-l)T})\otimes A_l .$$

Now, for any $z\in\mathbb{C}$, denote

$$r=r(l_0)\stackrel{\text{def}}{=} e^{-\frac{2i\pi}{l_0}},\ M=M(z,l_0)\stackrel{\text{def}}{=}(M_{ij}),\ M_{ij}\stackrel{\text{def}}{=}(r^{j-1}z)^{i-1},\ i,j=\overline{1,l_0} .$$

The next lemma is proved easily :

Lemma 4. *The following formula is true, for any* $l=\overline{0,l_0}$ *:*

$$E_{l_0}^l + z^{l_0}E_{l_0}^{(l_0-l)T} = M\ \text{diag}\{z^l;(rz)^l;\dots;(r^{l_0-1}z)^l\}M^{-1} .$$

■

One deduces from Lemma 4, that

$$\begin{aligned}\det(sI_n-\mathcal{A}-z^{l_0}\mathcal{B}) &= \det(sI_n-\sum_{l=0}^{l_0}\text{diag}\{z^lA_l;(rz)^lA_l;\dots;(r^{l_0-1}z)^lA_l\})\\ &= \prod_{l'=0}^{l_0-1}\det(sI_n-\sum_{l=0}^{l_0}(r^{l'}z)^lA_l) .\end{aligned}$$

As $|r|=1$, the preceding formula yields

$$\forall(s,z)\in\mathbb{C}^2 \text{ with } \operatorname{Re} s\geq 0, |z|\leq 1,\ \det(sI_n-\sum_0^{l_0} z^lA_l)\neq 0$$
$$\Leftrightarrow \forall(s,z)\in\mathbb{C}^2 \text{ with } \operatorname{Re} s\geq 0, |z|\leq 1,\ \det(sI_n-\mathcal{A}-z^{l_0}\mathcal{B})\neq 0 ,$$

which permits to conclude the proof of Corollary 3. □

2.2 Strong delay-independent stabilizability

The other application of Theorem 2 concerns stabilizability by static output feedback. Its proof is straightforward.

Corollary 5. *Consider the open-loop system*

$$\dot{x} = A_0x(t) + A_1x(t-h) + Bu(t),\ y(t) = C_0x(t) + C_1x(t-h)\ , \tag{18}$$

for $A_0, A_1 \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times p}, C_0, C_1 \in \mathbb{R}^{q\times n}$, $n, p, q \in \mathbb{N}\setminus\{0\}$. *There exists a static output feedback* $u(t) = Ky(t)$ *ensuring strong delay-independent stability of the corresponding closed-loop system if and only if there exist* $K \in \mathbb{R}^{p\times q}$ *and* $k \in \mathbb{N}\setminus\{0\}$, *for which the following LMI is feasible.*

$$\mathcal{P}_k, \mathcal{Q}_k \in \mathbb{R}^{kn\times kn}, \mathcal{P}_k, \mathcal{Q}_k > 0 \text{ and } \mathcal{R}_k < 0, \text{ where } \mathcal{R}_k \in \mathbb{R}^{(k+1)n\times(k+1)n} \text{ is given by}$$

$$\mathcal{R}_k \stackrel{\text{def}}{=} \begin{pmatrix} I_{kn} & 0_{kn\times n}\\ 0_{kn\times n} & I_{kn}\end{pmatrix}^T \begin{pmatrix} \mathcal{P}_k(I_k\otimes A_K) + (I_k\otimes A_K)^T\mathcal{P}_k + \mathcal{Q}_k & \mathcal{P}_k(I_k\otimes B_K)\\ (I_k\otimes B_K)^T\mathcal{P}_k & -\mathcal{Q}_k\end{pmatrix} \begin{pmatrix} I_{kn} & 0_{kn\times n}\\ 0_{kn\times n} & I_{kn}\end{pmatrix}\ ,$$

where $A_K \stackrel{\text{def}}{=} A_0 + BKC_0, B_K \stackrel{\text{def}}{=} A_1 + BKC_1$. ■

For sake of space, the result on existence of Lyapunov-Krasovskii functional is not repeated. Necessary and sufficient condition for strong delay-independent stabilizability by use of dynamic output feedback of fixed order may be deduced as well.

3 Proof of Theorem 2

3.1 Notations

Define, for any $k \in \mathbb{N}\setminus\{0\}$ and $s, z \in \mathbb{C}$, the following matrices

$$J_{0,k} \stackrel{\text{def}}{=} \begin{pmatrix} I_{kn} & 0_{kn\times n}\end{pmatrix},\quad J_{1,k} \stackrel{\text{def}}{=} \begin{pmatrix} 0_{kn\times n} & I_{kn}\end{pmatrix},\quad J_k \stackrel{\text{def}}{=} \begin{pmatrix} J_{0,k}\\ J_{1,k}\end{pmatrix}\ ,$$

$$(E_k)_{ij} \stackrel{\text{def}}{=} 1 \text{ if } i+1=j,\ 0 \text{ otherwise},\ i,j=\overline{1,k},\quad (e_k)_i \stackrel{\text{def}}{=} 1 \text{ if } i=k,\ 0 \text{ otherwise},\ i=\overline{1,k}\ ,$$

$$(v_k(z))_i \stackrel{\text{def}}{=} z^{i-1},\ i=\overline{1,k},\quad \mathcal{S}=\mathcal{S}(s) \stackrel{\text{def}}{=} (sI_{kn} - (I_k\otimes A) - (E_k\otimes B))^{-1}\ .$$

Also, for any trajectory of system (2), denote, for $t \geq (k-1)h$,

$$\mathcal{X}_k(t) \stackrel{\text{def}}{=} \begin{pmatrix} x(t)\\ \vdots\\ x(t-(k-1)h)\end{pmatrix}\ .$$

The size of the previous matrices is respectively

$$J_{0,k}, J_{1,k} : kn\times(k+1)n,\qquad J_k : 2kn\times(k+1)n,\qquad E_k : k\times k\ ,$$
$$e_k : k\times 1,\qquad v_k(z) : k\times 1,\qquad \mathcal{S}(s) : kn\times kn\ ,$$
$$\mathcal{X}_k(t) : kn\times 1\ .$$

3.2 Evolution of the candidate Lyapunov-Krasovskii functionals along the trajectories

Let $k \in \mathbb{N}\setminus\{0\}$. One deduces from (2) that

$$\dot{\mathcal{X}}_k(t) = (I_k\otimes A)\mathcal{X}_k(t) + (I_k\otimes B)\mathcal{X}_k(t-h)\ .$$

On the other hand, V_k defined in (12) writes

$$V_k(x)(t) = \mathcal{X}_k(t)^T \mathcal{P}_k \mathcal{X}_k(t) + \int_{t-h}^{t} \mathcal{X}_k(s)^T \mathcal{Q}_k \mathcal{X}_k(s)\, ds ,$$

so

$$\begin{aligned}\frac{dV_k(x)(t)}{dt} &= \dot{\mathcal{X}}_k(t)^T \mathcal{P}_k \mathcal{X}_k(t) + \mathcal{X}_k(t)^T \mathcal{P}_k \dot{\mathcal{X}}_k(t) + \mathcal{X}_k(t)^T \mathcal{Q}_k \mathcal{X}_k(t) - \mathcal{X}_k(t-h)^T \mathcal{Q}_k \mathcal{X}_k(t-h)\\ &= \begin{pmatrix} \mathcal{X}_k(t) \\ \mathcal{X}_k(t-h) \end{pmatrix}^T \begin{pmatrix} \mathcal{P}_k(I_k \otimes A) + (I_k \otimes A)^T \mathcal{P}_k + \mathcal{Q}_k & \mathcal{P}_k(I_k \otimes B) \\ (I_k \otimes B)^T \mathcal{P}_k & -\mathcal{Q}_k \end{pmatrix} \begin{pmatrix} \mathcal{X}_k(t) \\ \mathcal{X}_k(t-h) \end{pmatrix} . \end{aligned} \quad (19)$$

As

$$\mathcal{X}_k(t) = J_{0,k} \mathcal{X}_{k+1}(t), \quad \mathcal{X}_k(t-h) = J_{1,k} \mathcal{X}_{k+1}(t) ,$$

one then gets

$$\begin{pmatrix} \mathcal{X}_k(t) \\ \mathcal{X}_k(t-h) \end{pmatrix} = J_k \mathcal{X}_{k+1}(t) ,$$

and

$$\frac{dV_k(x)(t)}{dt} = \mathcal{X}_{k+1}^T(t) J_k^T \begin{pmatrix} \mathcal{P}_k(I_k \otimes A) + (I_k \otimes A)^T \mathcal{P}_k + \mathcal{Q}_k & \mathcal{P}_k(I_k \otimes B) \\ (I_k \otimes B)^T \mathcal{P}_k & -\mathcal{Q}_k \end{pmatrix} J_k \mathcal{X}_{k+1}(t) .$$

Formula (13) is then proved. Remark that matrix J_k is precisely put to take into account the additional constraints on the trajectory, as explained in the introduction: if the components of $\mathcal{X}_k$ were independent, one would simply get (19), and deduce a LMI condition which is indeed equivalent to (6) (Hint: use Theorem 1).

3.3 Feasibility of LMI(k) implies strong delay-independent stability of system (2)

We now prove that the feasibility of LMI(k) for a certain $k \in \mathbb{N}\backslash\{0\}$ not only implies delay-independent stability, but also strong delay-independent stability. Notice that the stability of the system may thus be proved directly by spectral criterion, without the use of the Krasovskii's result enunciated in Section 1.

Let $k \in \mathbb{N} \setminus \{0\}$ such that LMI(k) is feasible, let $z \in \mathbb{C}$. One shows easily that

$$J_k(v_{k+1}(z) \otimes I_n) = \begin{pmatrix} v_k(z) \\ z v_k(z) \end{pmatrix} \otimes I_n \in \mathbb{R}^{2kn \times n} .$$

One then deduces from the feasibility of LMI(k) that, for any $z \in \mathbb{C}$,

$$\begin{aligned} 0 &> (v_{k+1}(z) \otimes I_n)^* J_k^T \begin{pmatrix} \mathcal{P}_k(I_k \otimes A) + (I_k \otimes A)^T \mathcal{P}_k + \mathcal{Q}_k & \mathcal{P}_k(I_k \otimes B) \\ (I_k \otimes B)^T \mathcal{P}_k & -\mathcal{Q}_k \end{pmatrix} J_k(v_{k+1}(z) \otimes I_n) \\ &= (v_k(z) \otimes I_n)^* \mathcal{P}_k (v_k(z) \otimes I_n)(I_k \otimes (A + zB)) + (I_k \otimes (A + zB))^* (v_k(z) \otimes I_n)^* \mathcal{P}_k (v_k(z) \otimes I_n) \\ &\quad + (1 - |z|^2)(v_k(z) \otimes I_n)^* \mathcal{Q}_k (v_k(z) \otimes I_n) . \end{aligned}$$

In particular, if $|z| \leq 1$, this yields

$$0 > (v_k(z) \otimes I_n)^* \mathcal{P}_k (v_k(z) \otimes I_n)(I_k \otimes (A + zB)) + (I_k \otimes (A + zB))^* (v_k(z) \otimes I_n)^* \mathcal{P}_k (v_k(z) \otimes I_n) .$$

The matrix $(v_k(z) \otimes I_n)^* \mathcal{P}_k (v_k(z) \otimes I_n)$ is positive definite. For any nonzero complex eigenvector u of $A + zB$, associated to a complex eigenvalue s, one gets

$$0 > (s + s^*) u^* (v_k(z) \otimes I_n)^* \mathcal{P}_k (v_k(z) \otimes I_n) u = (s + s^*) \|\mathcal{P}_k^{1/2} (v_k(z) \otimes I_n) u\|^2 .$$

This means exactly that, for any $z \in \mathbb{C}$ such that $|z| \leq 1$, the eigenvalues of $A + zB$ have negative real part. This hence proves that the feasibility of LMI(k) is sufficient for strong delay-independent stability of (2). In other words, condition 2. implies condition 1.

3.4 Decreasing conservativity of the criteria

Let us now study the behavior of the sequence of criteria LMI(k). We shall establish here that, for any $k \in \mathbb{N} \setminus \{0\}$,

$$\text{LMI}(k) \text{ is feasible } \Rightarrow \text{LMI}(k+1) \text{ is feasible} .$$

Let $k \in \mathbb{N}\setminus\{0\}$ and $(\mathcal{P}_k, \mathcal{Q}_k)$ be a pair of positive definite matrices from $\mathbb{R}^{kn\times kn}$, solution of LMI(k). We shall check that $(\mathcal{P}_{k+1}, \mathcal{Q}_{k+1})$ is solution of LMI($k+1$), where $\mathcal{P}_{k+1} = \mathcal{P}_{k+1}^T > 0$, $\mathcal{Q}_{k+1} = \mathcal{Q}_{k+1}^T > 0$ are defined by

$$\mathcal{P}_{k+1} \stackrel{\text{def}}{=} \text{diag}\{\mathcal{P}_k; 0_n\} + \text{diag}\{0_n; \mathcal{P}_k\}, \quad \mathcal{Q}_{k+1} \stackrel{\text{def}}{=} \text{diag}\{\mathcal{Q}_k; 0_n\} + \text{diag}\{0_n; \mathcal{Q}_k\} .$$

One has

$$J_{k+1}^T \begin{pmatrix} \text{diag}\{\mathcal{P}_k;0_n\}(I_{k+1}\otimes A) + (I_{k+1}\otimes A)^T \text{diag}\{\mathcal{P}_k;0_n\} + \text{diag}\{\mathcal{Q}_k;0_n\} & \text{diag}\{\mathcal{P}_k;0_n\}(I_{k+1}\otimes B) \\ (I_{k+1}\otimes B)^T \text{diag}\{\mathcal{P}_k;0_n\} & -\text{diag}\{\mathcal{Q}_k;0_n\} \end{pmatrix} J_{k+1}$$
$$= \begin{pmatrix} J_{0,k+1} \\ J_{1,k+1} \end{pmatrix}^T \begin{pmatrix} \text{diag}\{\mathcal{P}_k(I_k\otimes A) + (I_k\otimes A)^T\mathcal{P}_k + \mathcal{Q}_k; 0_n\} & \text{diag}\{\mathcal{P}_k(I_k\otimes B); 0_n\} \\ \text{diag}\{(I_k\otimes B)^T\mathcal{P}_k; 0_n\} & -\text{diag}\{\mathcal{Q}_k;0_n\} \end{pmatrix} \begin{pmatrix} J_{0,k+1} \\ J_{1,k+1} \end{pmatrix} \quad (20)$$

and similarly for the other term. One proves directly that, for any $\alpha, \beta \in \{0,1\}$, for any matrix $M \in \mathbb{R}^{kn\times kn}$,

$$J_{\alpha,k+1}^T \text{diag}\{M;0_n\}J_{\beta,k+1} = \text{diag}\{J_{\alpha,k}^T M J_{\beta,k}; 0_n\}, \quad J_{\alpha,k+1}^T \text{diag}\{0_n;M\}J_{\beta,k+1} = \text{diag}\{0_n; J_{\alpha,k}^T M J_{\beta,k}\} .$$

The expression in (20) is hence equal to

$$\text{diag}\left\{ \begin{pmatrix} J_{0,k} \\ J_{1,k} \end{pmatrix}^T \begin{pmatrix} \mathcal{P}_k(I_k\otimes A) + (I_k\otimes A)^T\mathcal{P}_k + \mathcal{Q}_k & \mathcal{P}_k(I_k\otimes B) \\ (I_k\otimes B)^T\mathcal{P}_k & -\mathcal{Q}_k \end{pmatrix} \begin{pmatrix} J_{0,k} \\ J_{1,k} \end{pmatrix}; 0_n \right\} ,$$

and this permits to deduce that

$$\mathcal{R}_{k+1} = \text{diag}\{\mathcal{R}_k; 0_n\} + \text{diag}\{0_n; \mathcal{R}_k\} < 0 .$$

As a conclusion, the set of those k for which LMI(k) is feasible, is either void, or an unbounded interval of $\mathbb{N} \setminus \{0\}$.

3.5 Asymptotic precision of the criteria

One now studies the feasibility of LMI(k) when $k \to +\infty$, under the assumption that 1. holds. This will yield the implication 1. $\Rightarrow$ 2.

• One first transforms LMI(k) into a condition expressed as in the robustness interpretation of Chen *et al.* [4], see Theorem 1 above.

Matrix J_k may be decomposed *by blocks* as

$$J_k = \begin{pmatrix} I_{kn} & 0_{kn\times n} \\ E_k \otimes I_n & e_k \otimes I_n \end{pmatrix} .$$

Hence, the matrix $\mathcal{R}_k$ in LMI(k), decomposed by blocks as

$$\mathcal{R}_k = \begin{pmatrix} [\mathcal{R}_k]_{11} & [\mathcal{R}_k]_{12} \\ [\mathcal{R}_k]_{12}^T & [\mathcal{R}_k]_{22} \end{pmatrix} ,$$

is such that

$$\begin{aligned}
[\mathcal{R}_k]_{11} &= \mathcal{P}_k((I_k\otimes A)+(E_k\otimes B))+((I_k\otimes A)+(E_k\otimes B))^T\mathcal{P}_k+\mathcal{Q}_k-(E_k\otimes I_n)^T\mathcal{Q}_k(E_k\otimes I_n),\\
[\mathcal{R}_k]_{12} &= \mathcal{P}_k(e_k\otimes B)-(E_k\otimes I_n)^T\mathcal{Q}_k(e_k\otimes I_n),\\
[\mathcal{R}_k]_{22} &= -(e_k\otimes I_n)^T\mathcal{Q}_k(e_k\otimes I_n)\ .
\end{aligned}$$

Using now Kalman-Yakubovich-Popov lemma (see e.g. [18]), solvability of LMI(k) is proved to be *equivalent* to

$$\exists \mathcal{Q}_k=\mathcal{Q}_k^T>0, \forall s\in\mathbb{C}, \operatorname{Re} s\geq 0 \Rightarrow \det(sI_{kn}-(I_k\otimes A)-(E_k\otimes B))\neq 0 \text{ and}$$
$$\begin{pmatrix}\mathcal{S}(s)(e_k\otimes B)\\ I_n\end{pmatrix}^*\begin{pmatrix}\mathcal{Q}_k-(E_k\otimes I_n)^T\mathcal{Q}_k(E_k\otimes I_n) & -(E_k\otimes I_n)^T\mathcal{Q}_k(e_k\otimes I_n)\\ -(e_k\otimes I_n)^T\mathcal{Q}_k(E_k\otimes I_n) & -(e_k\otimes I_n)^T\mathcal{Q}_k(e_k\otimes I_n)\end{pmatrix}\begin{pmatrix}\mathcal{S}(s)(e_k\otimes B)\\ I_n\end{pmatrix}<0\,,$$

that is

$$\exists \mathcal{Q}_k=\mathcal{Q}_k^T>0, \forall s\in\mathbb{C}, \operatorname{Re} s\geq 0 \Rightarrow \det(sI_n-A)\neq 0 \text{ and}$$
$$(e_k\otimes B)^T\mathcal{S}^*\mathcal{Q}_k\mathcal{S}(e_k\otimes B)<[(E_k\otimes I_n)\mathcal{S}(e_k\otimes B)+(e_k\otimes I_n)]^*\mathcal{S}^*\mathcal{Q}_k\mathcal{S}[(E_k\otimes I_n)\mathcal{S}(e_k\otimes B)+(e_k\otimes I_n)]\ .$$

Now, one proves easily that

$$\mathcal{S}(e_k\otimes B)=\begin{pmatrix}[(sI_n-A)^{-1}B]^k\\ \vdots\\ (sI_n-A)^{-1}B\end{pmatrix},\quad (E_k\otimes I_n)\mathcal{S}(e_k\otimes B)+(e_k\otimes I_n)=\begin{pmatrix}[(sI_n-A)^{-1}B]^{k-1}\\ \vdots\\ I_n\end{pmatrix}\ .$$

Taking $\mathcal{Q}_k=I_{kn}$, one finally gets that a *sufficient* condition for realization of 2. is

$$\exists k\in\mathbb{N}\setminus\{0\}, \forall s\in\mathbb{C}, \operatorname{Re} s\geq 0\Rightarrow \det(sI_n-A)\neq 0 \text{ and } \|[(sI_n-A)^{-1}B]^k\|<1\ . \tag{21}$$

• One now transforms condition 1. From relations (8) and (10), this condition is equivalent to

$$\forall s\in\mathbb{C}, \operatorname{Re} s\geq 0\Rightarrow \det(sI_n-A)\neq 0 \text{ and } \rho((sI_n-A)^{-1}B)<1\ .$$

It is well-known that, for any $M\in\mathbb{R}^{n\times n}$,

$$\rho(M)<1\Leftrightarrow \lim_{k\to+\infty}\|M^k\|=0\ .$$

One hence deduces that condition 1. is indeed *equivalent* to

$$\forall s\in\mathbb{C}, \operatorname{Re} s\geq 0\Rightarrow \det(sI_n-A)\neq 0 \text{ and } \exists k\in\mathbb{N}\setminus\{0\}, \sup_{k'\geq k}\|[(sI_n-A)^{-1}B]^{k'}\|<1\ . \tag{22}$$

• In view of (21) and (22), it remains, in order to prove that 1. implies 2., to show that one may choose in (22) the index k *uniformly* wrt $s\in\mathbb{C}$, $\operatorname{Re} s\geq 0$.

For $k\in\mathbb{N}\setminus\{0\}$, let

$$K_k\stackrel{\text{def}}{=}\{s\in\mathbb{C}\ :\ \operatorname{Re} s\geq 0,\ \|[(sI_n-A)^{-1}B]^k\|\geq 1\}\ .$$

By continuity, the sets K_k are closed. As

$$\|[(sI_n-A)^{-1}B]^k\|\leq\left(\frac{\|B\|}{|s|-\|A\|}\right)^k \quad \text{if } |s|>\|A\|\ ,$$

the sets K_k are bounded, and hence compact. Moreover,

$$s \in K_{2k} \Rightarrow 1 \leq \|[(sI_n - A)^{-1}B]^{2k}\| \leq \|[(sI_n - A)^{-1}B]^k\|^2 \Rightarrow s \in K_k \ .$$

Hence $K_{2k} \subset K_k$, for any $k \in \mathbb{N} \setminus \{0\}$. The sequence K_{2^k} is thus a sequence of nested compact sets.

Assume now that (21) does not hold. If $\mathrm{Re}\,\sigma(A) \not< 0$, then (22) does not hold. Otherwise, for any $k \in \mathbb{N} \setminus \{0\}$, the sets K_k are nonempty. In particular,

$$\exists s_0 \in \bigcap_{k \in \mathbb{N}} K_{2^k} \ ,$$

that is

$$\exists s_0 \in \mathbb{C}, \forall k \in \mathbb{N}, \ \|[(s_0 I - A)^{-1}B]^{2^k}\| \geq 1 \ .$$

Hence,

$$\forall k \in \mathbb{N} \setminus \{0\}, \sup_{k' \geq k} \|[(s_0 I - A)^{-1}B]^{k'}\| \geq 1 \ ,$$

and (22) does not hold either. One has hence proved by contradiction that (22) implies (21).

To summarize, it has been shown that condition 1. is equivalent to (22), which implies (21), which is itself sufficient to have condition condition 2. This shows finally that condition 1. implies condition 2., and concludes the proof of Theorem 2.

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