Dynamic Routing in the Mean-Field Approximation

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Abstract: A queuing system is considered, with stations 1, 2, where station i contains a collection S_i of infinite-buffer FCFS single servers. The number of servers in each S_i is N, and we assume that N is large (formally, $N \to \infty$). The system is fed by a Poisson flow of rate $2\lambda N$, with i.i.d. exponential service times of mean one. Upon arrival, a customer chooses two servers according to the following rule. He performs two independent trials, each time choosing station 1 with probability p_1 and 2 with probability $p_2 = 1 - p_1$ and then choosing a server at random from the corresponding collection S_i. He then selects the server with a shortest queue from the two, breaking ties at random if necessary. We assume that $p_1 \geq p_2 \geq 0$; the main result is that (a) the inequalities $\lambda < 1$, $2\lambda p_1^2 < 1$, are necessary and sufficient for the existence (and uniqueness) of a (stable) stationary regime, and (b) under these inequalities, a stationary queue-size distribution has a super-exponentially decaying tail.

These results are in a striking contrast with a 'linear' model where a customer simply chooses a station with probabilities p_1 and p_2 and then selects a server at random from the corresponding S_i. Here, the necessary and sufficient condition is $2\lambda p_1 < 1$, and the queue-size distribution is geometric.

Key-words: queuing system, single FCFS server, dynamic routing, Markov process, generator, convergence, differential equations, initial-boundary value problem, fixed points

(Résumé : tsvp)

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Routage dynamique sous approximation du champ moyen

Résumé : Nous considèrons un système de files d'attente regroupées en deux stations 1 et 2, où la station i contient un ensemble S_i de files d'attente non bornées servies dans l'ordre d'arrivée des clients. Le nombre de serveurs dans chaque classe S_i est N, et nous supposons que N est large. (formellement $N \to \infty$). Le système est soumis à une arrivée de clients suivant une loi de Poisson d'un taux $2\lambda N$ et les services sont distribuées de manière i.i.d. suivant une loi exponentielle de moyenne un. À son arrivée, le client choisit deux serveurs. Chaque serveur est choisi de la manière suivante. D'abord le client choisit sa station: la station 1 avec probabilité p_1, ou la station 2 avec probabilité $p_2 = 1 - p_1$. Ensuite le client choisit son serveur de manière uniforme dans sa station. Ayant choisi ses deux serveurs, le client choisit d'aller dans la file d'attente la plus courte des deux serveurs. En cas d'égalité dans les longueurs des files d'attente, le serveurs choisit alors à pile ou face. Nous supposons que $p_1 \geq p_2 \geq 0$; notre résultat principal est que (a) les inégalités $\lambda < 1$ et 2^2_1 sont nécessaires et suffisantes pour garantir l'existence (et l'unicité) d'un régime stationnaire stable; et (b) sous ces conditions la distribution stationnaire des longueur des files d'attente a une décroissance super-exponentielle.

Ces résultats sont en contraste très fort avec le modèle "linéaire" où le client choisit aléatoirement un seul serveur sans chercher à faire de comparaison avec la file d'attente d'un autre serveur choisi aussi aléatoirement. Dans ce cas la condition nécessaire et suffisante pour l'existence d'un régime stationaire est $\lambda < 1$ et $2\lambda p_1 < 1$, et la distribution stationaire des longueurs des files d'attente a une décroissance simplement géométrique.

Mots-clé : système de file d'attente, service dans l'ordre d'arrivée, routage dynamique, processus de Markov, générateurs, convergence, équations différentielles, problèmes aux limites, points fixes.

DYNAMIC ROUTING IN THE MEAN-FIELD APPROXIMATION[0]

Ph. Jacquet[1], **Yu.M. Suhov**[2)] **and N.D. Vvedenskaya**[3)]

1 Introduction

Consider the following queuing system with dynamic routeing. The system includes two stations (or nodes), and station S_i, $i = 1, 2$ contains a collection of parallel single FCFS servers, each server with its own infinite buffer. The number of servers in each S_i is the same and equals N; the main assumption throughout the paper is that N is large (formally, we will consider $N \to \infty$).

The system is fed by a Poisson flow of rate $2\lambda N$, with i.i.d. exponential service-times of mean one (the last assumption is not important and taken only for simplicity of notation). Upon arrival, each customer chooses a server to join, taking into account an information about the current state of the system. Namely, an independent random trial is performed two times; each time the customer initially chooses station 1 with probability p_1 and 2 with probability $p_2 = 1 - p_1$ and then a particular server from corresponding collection S_i with probability $1/N$ (all choices with replacement). As a result, the customer has a sample of two servers (possibly with repetitions). He then selects the server with the shortest queue-size and joins the corresponding queue; if there is more than one such server, the selection is performed at random. In what follows we assume that $p_1 \geq p_2$.

The system described bears some distinctive features of dynamic routing, and one can expect that it performs better than a 'linear' model where the customer makes just one trial and then joins the chosen server. The linear model is exactly solvable: here, $\forall\, N \geq 1$, (a) the

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necessary and sufficient condition for the existence and uniqueness of a (proper) stationary regime reads $2\lambda p_1 < 1$, and (b) the stationary distribution of the queue-size at a given server from collection S_i is geometric, with parameter $2\lambda p_1$ for $i = 1$ and $2\lambda p_2$ for $i = 2$.

Our main results about the dynamic routing system focus on similar properties. We check that
(a) $\forall N \geq 1$ the necessary and sufficient condition for the existence and uniqueness of a (proper) stationary regime is given by a pair of inequalities

$$\lambda < 1, \ \ 2\lambda p_1^2 < 1, \tag{1}$$

(b) as $N \to \infty$, the stationary probability that a given server from collection S_i had at least l customers in the buffer does not exceed $C(2\lambda p_1^2)^{2^l}$ for $i = 1$ and $C(p_2/p_1)^l(2\lambda p_1^2)^{2^l}$ for $i = 2$, where $C > 0$ is a constant (which can be assessed numerically). We also define and study a limiting model emerging as $N \to \infty$.

These results support rather popular heuristics that it is the 'dedicated' traffic that determines how a dynamic routing network performs, in terms of its capacity, the size of buffers and the end-to-end delay distribution.

We are not able to derive an exact limiting form of the stationary queue-size distribution in the limiting dynamic routing model, unless $p_1 = p_2 = 1/2$ (in which case the above probability equals λ^{2^l-1}, ≥ 0). For $p_1 > 1/2$ we cannot give an explicit expression for the values $a_1(1)$, $a_2(1)$, that show in what proportion the total arrival flow is divided in the stationary regime between stations 1 and 2. But we present a functional equation that has the root equal to $a_1(1)$. This equation can be easily solved numerically.

Even in the symmetric case, there are interesting open questions, e.g., what is the form of the process of arrivals at a given server? (Some information about the last problem is contained in [4]).

The paper is organized as follows. In Section 2 we give formal definitions and state our results. Sections 3 and 4 contain proofs. Section 3 concerns with various properties of the limiting dynamic routing model. In Section 4 we establish the existence of the unique invariant distribution of the process for $\forall\ N \geq 1$ and check the convergence as $N \to \infty$. Consequently, Section 3 focuses mainly on techniques from the theory of differential equations whereas Section 4 on techniques of probabilistic and functional-analytic origin. In Section 5 the equation for $a_1(1)$ is presented.

The limit $N \to \infty$ exploited in this paper is closely related to the so-called mean-field limit in statistical physics. Its main features are distinctive averaging and decoupling phenomena where a given queue becomes independent of the rest of the system and is governed by a 'master' equation (which, as a rule, is a single non-linear differential equation or a system of such equations).

We dedicate this and subsequent papers to the memory of Roland Dobrushin (1929–1995). R. Dobrushn pioneered rigorous studying of the queuing networks, in particular the mean-field approximation; the interested reader can find more details in the review [3]. See also [1] and [7].

2 Preliminaries and results

The above description of the dynamic routing system, with given N, λ and p, leads to a denumerable continuous-time Markov process representing the evolution of a state of the system. Due to the symmetry within each group S_i, it is convenient to associate such a state with a 'vector' $\mathbf{g} = (g_1, g_2)$, where each g_i, $i = 1, 2$, is in turn a sequence: $g_i = (g_i(l))$, $l \in \mathbf{Z}_+ := \{0, 1, 2, \ldots\}$, whose entries $g_i(l)$ are monotonically non-increasing ($g_i(l+1) \leq g_i(l)$) numbers of the form r/N, $r = 0, 1, \ldots, N$, with $g_i(0) = 1$ and $g_i(l) = 0$ for l large enough. 'Physically', $g_i(l)$ represents the portion of servers from S_i with at least l customers in buffer (including the one currently under service). The set of such $\mathbf{g}$'s (the phase space of the Markov process) is denoted by $\mathcal{U}^{(N)}$.

The process under consideration is determined by its generator which is a linear operator $\mathbf{A}^{(N)}$ acting on functions $f \colon \mathcal{U}^{(N)} \to \mathbf{C}^1$ and given by

$$\begin{array}{rcl} \mathbf{A}_N f(\mathbf{g}) & = & N\sum_{l\geq 1}\sum_{i=1,2}\left[g_i(l)-g_i(l+1)\right]\left(f\left(\mathbf{g}-\frac{\mathbf{e}_{i,(l)}}{N}\right)-f(\mathbf{g})\right)+ \\ & & +2\lambda N\sum_{l\geq 1}\Big(p_1^2\left[\left(g_1(l-1)\right)^2-\left(g_1(l)\right)^2\right]+ \\ & & +p_1p_2\left[g_1(l-1)-g_1(l)\right]\left[g_2(l-1)+g_2(l)\right]\Big)\left(f\left(\mathbf{g}+\frac{\mathbf{e}_{1,(l)}}{N}\right)-f(\mathbf{g})\right)+ \\ & & +2\lambda N\sum_{l\geq 1}\Big(p_2^2\left[\left(g_2(l-1)\right)^2-\left(g_2(l)\right)^2\right]+ \\ & & +p_1p_2\left[g_2(l-1)-g_2(l)\right]\left[g_1(l-1)+g_1(l)\right]\Big)\left(f\left(\mathbf{g}+\frac{\mathbf{e}_{2,(l)}}{N}\right)-f(\mathbf{g})\right),\ \mathbf{g}\in\mathcal{U}^{(N)}. \end{array} \tag{2}$$

Here, $\mathbf{e}_{1,(l)}$ stands for the vector $\left(e_{(l)}, 0\right)$ and $\mathbf{e}_{2,(l)}$ for $\left(0, e_{(l)}\right)$ where $e_{(l)}$ is the sequence whose only non-zero entry is at position l and equals one. The addition and division of the vectors and sequences under the sign of function f in (2) is component wise; although $\mathbf{e}_i(l) \notin \mathcal{U}^{(N)}$, the resulting vectors belong to $\mathcal{U}^{(N)}$, and the right-hand side of (2) is correctly defined.

The Markov process with generator (2) is denoted by $U^{(N)}(t)$ and its semi-group of operators by

$$\{\mathbf{T}_t^{(N)},\ t \in [0, \infty)\} :\ \mathbf{T}_t^{(N)} f(\mathbf{g}) = \mathbf{E}_{\mathbf{g}} f\left(U^{(N)}(t)\right), g \in \mathcal{U}^{(N)}.$$

(Here, $\mathbf{E}_{\mathbf{g}}$ stands for the expectation with respect to the distribution of the Markov process with the initial state $\mathbf{g}$.) We study the limiting behavior of $U^{(N)}(t)$ as $N \to \infty$. A formal limit in (2) leads to the following equation

$$\begin{array}{rcl} \mathbf{A} f(\mathbf{g}) & = & \sum_{l\geq 1}\sum_{i=1,2}\left[g_i(l+1)-g_i(l)\right]\frac{\partial}{\partial g_i(l)}f(\mathbf{g})+ \\ & & +2\lambda\sum_{l\geq 0}\Big(p_1^2\left[\left(g_1(l-1)\right)^2-\left(g_1(l)\right)^2\right]+ \\ & & +p_1p_2\left[g_1(l-1)-g_1(l)\right]\left[g_2(l-1)+g_2(l)\right]\Big)\frac{\partial}{\partial g_1(l)}f(\mathbf{g})+ \\ & & +2\lambda\sum_{l\geq 0}\Big(p_2^2\left[\left(g_2(l-1)\right)^2-\left(g_2(l)\right)^2\right]+ \\ & & +p_1p_2\left[g_2(l-1)-g_2(l)\right]\left[g_1(l-1)+g_1(l)\right]\Big)\frac{\partial}{\partial g_2(l)}f(\mathbf{g}). \end{array} \tag{3}$$

This suggests that Markov process $U^{(N)}(t)$ converges, as $N \to \infty$, to a deterministic process. The limiting process lives in a larger space $\bar{\mathcal{U}} \supset \mathcal{U}^{(N)}$. Space $\bar{\mathcal{U}}$ consists of vectors $\mathbf{g} = (g_1, g_2)$ as above, where now g_i is an arbitrary non-decreasing sequence of numbers $g_i(l)$ from $[0,1]$, with $g_i(0) = 1$. We endow $\bar{\mathcal{U}}$ with the distance $\rho\big(\mathbf{g}, \mathbf{g}'\big)$, $\mathbf{g} = (g_1, g_2)$, $\mathbf{g}' = ({g'}_1, {g'}_2)$, given by

$$\rho\big(\mathbf{g}, \mathbf{g}'\big) = \sup_{i,l} \frac{1}{l+1} |g_i(l) - g_i'(l)|, \tag{4}$$

this turns $\bar{\mathcal{U}}$ into a complete compact metric space. We denote by $\mathrm{C}(\bar{\mathcal{U}})$ the space of bounded continuous functions on $\bar{\mathcal{U}}$. We will be also interested in the subset $\mathcal{U} \subset \bar{\mathcal{U}}$ consisting of those $\mathbf{g} \in \bar{\mathcal{U}}$ for which

$$\sum_l g_i(l) < \infty, \quad i = 1, 2, \tag{5}$$

and we set

$$V(\mathbf{g}; l) = \sum_{\tilde{l} \geq l} \sum_{i=1,2} g_i(\tilde{l}), \ l \geq 1, \ \ i = 1, 2. \tag{6}$$

Returning to (3), one can expect that if the (non-random) initial state $U^{(N)}(0) = \mathbf{g}^{(N)} \in \mathcal{U}^{(N)}$ converges to some $\mathbf{g}$ then $U^{(N)}(t)$ converges to $\mathbf{u}\big(t, \mathbf{g}\big)$ where $\mathbf{u}\big(t, \mathbf{g}\big) = \big(u_1(t, \mathbf{g}),$ $u_2(t, \mathbf{g})\big)$ and $u_i(t, \mathbf{g}) = (u_i(t, \mathbf{g}; l))$, $l = 0, 1, \ldots$ give a solution to the following Cauchy problem (to ease the notation we write here $u_j(t; l)$ instead of $u_j(t, \mathbf{g}; l)$)

$$\begin{aligned} \dot{u}_1(t;l) \;&=\; \big[u_1(t;l+1) - u_1(t;l)\big] + 2\lambda\Big(p_1^2\big[\big(u_1(t;l-1)\big)^2 - \big(u_1(t;l)\big)^2\big] + \\ &\quad + p_1p_2\big[u_1(t;l-1) - u_1(t;l)\big]\big[u_2(t;l-1) + u_2(t;l)\big]\Big), \ t > 0, \ l \geq 1, \\ \dot{u}_2(t;l) \;&=\; \big[u_2(t;l+1) - u_2(t;l)\big] + 2\lambda\Big(p_2^2\big[\big(u_2(t;l-1)\big)^2 - \big(u_2(t;l)\big)^2\big] + \\ &\quad + p_1p_2\big[u_2(t;l-1) - u_2(t;l)\big]\big[u_1(t;l-1) + u_1(t;l)\big]\Big), \ t > 0, \ l \geq 1, \end{aligned} \tag{7}$$

with the boundary condition

$$u_i(t; 0) = 1, \ t \geq 0, \ i = 1, 2, \tag{8}$$

and the initial condition

$$u_i(0; l) = g_i(l), \ l \geq 1, \ i = 1, 2. \tag{9}$$

The convergence to and properties of the limiting problem (7)-(9) are main topics of this paper. A particular attention is paid to fixed points of (7)-(9), i.e., vectors $\mathbf{a} = \big(a_1, \, a_2\big)$, with components $a_i = \big(a_i(l)\big)$ satisfying

$$a_1(0) = a_2(0) = 1$$

and

$$
\begin{aligned}
a_1(l)-a_1(l+1) &= 2\lambda\Big(p_1^2\big[\big(a_1(l-1)\big)^2-\big(a_1(l)\big)^2\big]+\\
&\quad +p_1p_2\big[a_1(l-1)-a_1(l)\big]\big[a_2(l-1)+a_2(l)\big]\Big),\ l\geq 1,\\
a_2(l)-a_2(l+1) &= 2\lambda\Big(p_2^2\big[\big(a_2(l-1)\big)^2-\big(a_2(l)\big)^2\big]+\\
&\quad +p_1p_2\big[a_2(l-1)-a_2(l)\big]\big[a_1(l-1)+a_1(l)\big]\Big),\ l\geq 1.
\end{aligned}
\tag{10}
$$

An obvious solution to (10) is given by $\mathbf{a}^0=\big(a_1^0,\,a_2^0\big)$, where $a_i^0=\big(a_i^0(l)\big)$ and $a_i^0(l)=1$, $l\geq 0$, $i=1,2$; it corresponds to an infinite 'saturation' of the whole system. In a similar fashion, one can consider 'partially' saturated fixed points.

Theorem 1 *For $\forall\ \mathbf{g}\in\bar{\mathcal{U}}$ there exists in $\bar{\mathcal{U}}$ a unique solution $\mathbf{u}(t,\mathbf{g})$, $t\in[0,\infty)$, to problem (7)-(9). If $\mathbf{g}\in\mathcal{U}$ then $\mathbf{u}(t,\mathbf{g})\in\mathcal{U}\ \forall\ t\in[0,\infty)$.*

Theorem 2 *For $\forall\ f\in C(\bar{\mathcal{U}})$, uniformly in t within a bounded interval in $\mathbf{R}_+:=[0,\infty)$,*

$$
\lim_{N\to\infty}\ \sup_{\mathbf{g}\in\mathcal{U}^{(N)}}\ \big|\mathbf{T}_t^{(N)}f(\mathbf{g})-f\big(\mathbf{u}(\mathbf{g},t)\big)\big|=0. \tag{11}
$$

Theorem 3 *Assume that inequalities (1) hold. Then*
a) in $\mathcal{U}$ there exists a unique fixed point $\mathbf{a}$ satisfying (10),
b) $\exists\ C>0$ such that entries $a_i(l)$ of $\mathbf{a}$ obey

$$
a_i(l)\leq C\big(2\lambda p_1\big)^{2^l},\ l\geq 1,\ i=1,2, \tag{12}
$$

c) $\forall\ \mathbf{g}\in\mathcal{U}$, the solution $\mathbf{u}(t,\mathbf{g})$ to (7)-(9) converges, as $t\to\infty$, to $\mathbf{a}$ in metric (10).
d)If, on the other hand, some of inequalities (1) fails, then there is no fixed point $\mathbf{a}\in\mathcal{U}$.

Theorem 4 *Assume that inequalities (1) hold. Then process $U^{(N)}(t)$ has a unique equilibrium probability, $\pi^{(N)}$, and the distribution of $U^{(N)}(t)$ converges as $t\to\infty$ to $\pi^{(N)}=(\pi_i^{(N)})$ regardless of the initial distribution. If $\mathbf{E}^{(N)}$ denotes the expectation with respect to $\pi^{(N)}$ then*

$$
\begin{array}{ll}
(a) & \mathbf{E}^{(N)}V_1<\infty,\ \ V_1=\sum_{i=1,2}\sum_{l\geq 1}u_i(t,\mathbf{g},l),\\
(b) & \mathbf{E}^{(N)}\pi_i(l)\leq C(2\lambda p_1^2)^{2^l},\\
(c) & \lim_{N\to\infty}\mathbf{E}^{(N)}u_i(t,\mathbf{g},l)=\pi_i(l),\ l\geq 1,\ 1\leq i\leq 2.
\end{array}
\tag{13}
$$

In Section 3 we discuss the proof of Theorems 1 and 3 and in Section 4 of Theorems 2 and 4.

3 Properties of the limiting system

In addition to (7)-(9), it is convenient to consider, for a given $M \in \mathbf{Z}_+$, a truncated problem:

$$\begin{aligned}\dot{u}_1(t;l) &= \big[u_1(t;l+1)-u_1(t;l)\big]+2\lambda\Big(p_1^2\big[\big(u_1(t;l-1)\big)^2-\big(u_1(t;l)\big)^2\big]+\\ &\quad +p_1p_2\big[u_1(t;l-1)-u_1(t;l)\big]\big[u_2(t;l-1)+u_2(t;l)\big]\Big),\ t>0,\ 1\le l\le M,\\ \dot{u}_2(t;l) &= \big[u_2(t;l+1)-u_2(t;l)\big]+2\lambda\Big(p_2^2\big[\big(u_2(t;l-1)\big)^2-\big(u_2(t;l)\big)^2\big]+\\ &\quad +p_1p_2\big[u_2(t;l-1)-u_2(t;l)\big]\big[u_1(t;l-1)+u_1(t;l)\big]\Big),\ t>0,\ 1\le l\le M,\end{aligned} \tag{14}$$

$$u_i(t;0)=1,\ u_i(t;l)=0,\ t\ge 0,\ l\ge M+1,\ i=1,2, \tag{15}$$

$$u_i(0;l)=g_i(l),\ 1\le l\le M,\ i=1,2. \tag{16}$$

A fixed point $\mathbf{a}^{[M]}$ for (14)-(16) should obey

$$\begin{aligned}a_1^{[M]}(l)-a_1^{[M]}(l+1) &= 2\lambda\Big(p_1^2\big[\big(a_1^{[M]}(l-1)\big)^2-\big(a_1^{[M]}(l)\big)^2\big]+\\ &\quad +p_1p_2\big[a_1^{[M]}(l-1)-a_1^{[M]}(l)\big]\big[a_2^{[M]}(l-1)+a_2^{[M]}(l)\big]\Big),\ 1\le l\le M,\\ a_2^{[M]}(l)-a_2^{[M]}(l+1) &= 2\lambda\Big(p_2^2\big[\big(a_2^{[M]}(l-1)\big)^2-\big(a_2^{[M]}(l)\big)^2\big]+\\ &\quad +p_1p_2\big[a_2^{[M]}(l-1)-a_2^{[M]}(l)\big]\big[a_1^{[M]}(l-1)+a_1^{[M]}(l)\big]\Big),\ 1\le l\le M,\end{aligned} \tag{17}$$

with

$$a_i^{[M]}(0)=1,\ a_i^{[M]}(l)=0,\ l\ge M,\ i=1,2. \tag{18}$$

In what follows the inequality $\mathbf{g}^{(1)}\ge\mathbf{g}^{(2)}$ between $\mathbf{g}^{(1)}$, $\mathbf{g}^{(2)}\in\bar{\mathcal{U}}$ is understood entry-wise: $g_i^{(1)}(l)\ge g_i^{(2)}(l)$, $i=1,2$. Given $M\in\mathbf{Z}_+$, we also denote by $\mathbf{g}^{[M]}$ the truncated vector, with $g_i^{[M]}(l)=g_i(l)\mathbf{1}(l\le M)$.

Lemma 1 *For any $M\in\mathbf{Z}_+$ and $\mathbf{g}\in\bar{\mathcal{U}}$, there exists a unique solution $\mathbf{u}^{[M]}(t,\mathbf{g})$, $t\ge 0$, to (14)-(16). Furthermore, $\forall t$, vector $\mathbf{u}^{[M]}(t,\mathbf{g})\in\mathcal{U}$. If $\mathbf{g}^{(1)}$, $\mathbf{g}^{(2)}\in\bar{\mathcal{U}}$ obey $\mathbf{g}^{(1)}\ge\mathbf{g}^{(2)}$, the corresponding solutions $\mathbf{u}^{[M]}(t,\mathbf{g}^{(1)})$ and $\mathbf{u}^{[M]}(t,\mathbf{g}_2)$ obey $\mathbf{u}^{[M]}(t,\mathbf{g}_1)\ge\mathbf{u}^{[M]}(t,\mathbf{g}^{(2)})$ $\forall t\ge 0$. If $g_1(l)\ge g_2(l)$, then $\forall\ t$ $u_1^{[M]}(t;l)\ge u_2^{[M]}(t;l)$ $l=1,2,\ldots.$*

Proof of Lemma 1. System (14) contains finitely many first-order differential equations with smooth coefficients; the existence and uniqueness of the (global) solution to (14)-(16), as well as the continuity with respect to the initial data follow from standard general theorems; see, e.g., textbooks [2] or [5]. However, we have to check that for $\mathbf{g} \in \bar{\mathcal{U}}$, the solution $u^{[M]}(t;l)$ are monotone non-increasing in l $\forall$ t and the rest of the lemma. The argument that follows essentially repeats that from [6] (see the proof of Lemmas 1 and 2 there), although we now deal with two components u_i, $i = 1, 2$, instead of one.

Owing to the continuity in $\mathbf{g}$, we can assume until the end of the proof that $\mathbf{g}$ is such that $g_i(l+1) < g_i(l) < 1$, $1 \le l \le M$, $i = 1, 2$, and so are $\mathbf{g}^{(1)}$ and $\mathbf{g}^{(2)}$, and furthermore, $g_i^{(1)}(l) > g_i^{(2)}(l)$, $1 \le l \le M$, $i = 1, 2$. We will then check that with time the inequalities remain strict.

Assume that $t^0 \in (0, \infty)$ is the first time when equality $u_i^{[M]}(t, \mathbf{g}; l+1) = u_i^{[M]}(t, \mathbf{g}; l)$ occurs for some $i = 1, 2$ and $0 \le l \le M$ and pick up i^0, say $i_0 = 1$, and l^0 with this property (for simplicity, symbols ${}^{[M]}$ and $\mathbf{g}$ are temporarily omitted from the notation). As $u_1(t^0; 0) = 1 > 0 = u_1(t^0; M+1)$, we can always choose l^0 so that either $l^0 \ge 1$ and $u_1(t^0; l^0 - 1) > u_1(t^0; l^0) = u_1(t^0; l^0 + 1)$ or $l^0 \le M - 1$ and $u_1(t^0; l^0) = u_1(t^0; l^0 + 1) > u_1(t^0; l^0 + 2)$. In the first case, from the equation for $\dot{u}_1(t; l^0)$, we find that

$$\dot{u}_1(t^0; l^0) = 2\lambda\Big(p_1^2\big[\big(u_1(t^0; l^0 - 1)\big)^2 - \big(u_1(t^0; l^0)\big)^2\big]$$

$$+ p_1 p_2\left[u_1(t^0; l^0 - 1) - u_1(t^0; l^0)\right]\left[u_2(t^0; l^0 - 1) + \big(u_2(t^0; l^0)\big]\right) > 0$$

and from the equation for $\dot{u}_1(t; l^0 + 1)$ that

$$\dot{u}_1(t^0; l^0 + 1) = \left[u_1(t^0; l^0 + 2) - u_1(t^0; l^0 + 1)\right] \le 0,$$

which makes this case impossible. A similar argument is applied in the second case. This yields the proof that $\mathbf{u}^{[M]}(t; \mathbf{g}) \in \mathcal{U}$ $\forall t$.

Now assume that $t^0 \in (0, \infty)$ is the first time when equality $u_i^{[M]}(t, \mathbf{g}^1; l) = u_i^{[M]}(t, \mathbf{g}^2; l)$ occurs for some $i = 1, 2$ and $1 \le l \le M$ and pick up i^0, again say $i^0 = 1$, and l^0 with this property (symbol ${}^{[M]}$ is again omitted). It is convenient to take the largest l^0 for which the above equality holds; in this case, comparing the equations for $\dot{u}_1(t, \mathbf{g}^1; l^0)$ and $\dot{u}_1(t, \mathbf{g}^2; l^0)$, we find, as before, that

$$\dot{u}_1(t^0, \mathbf{g}^1; l^0) > \dot{u}_1(t^0, \mathbf{g}^2; l^0),$$

which contradicts the choice of t^0.

Similar arguments are used to prove that from $g_1(l) \ge g_2(l)$ follows $u_1(t; l) \ge u_2(t; l)$, $l = 1, 2, ...$

Lemma 2 *For any $\mathbf{g} \in \bar{\mathcal{U}}$ and $t \in [0, \infty)$, there exists the limit, in metric (4),*

$$\mathbf{u}(t; \mathbf{g}) = \lim_{M \to \infty} \mathbf{u}^{[M]}(t, \mathbf{g}), \tag{19}$$

and vector $\mathbf{u}(t;\mathbf{g}) \in \bar{\mathcal{U}}$. *Furthermore,* $\mathbf{u}(t;\mathbf{g})$, $t \in [0,\infty)$, *gives a unique in* $\bar{\mathcal{U}}$ *solution to (7)-(9). If* $\mathbf{g}^{(1)}$, $\mathbf{g}^{(2)} \in \bar{\mathcal{U}}$ *obey* $\mathbf{g}^{(1)} \geq \mathbf{g}^{(2)}$, *the corresponding solutions* $\mathbf{u}(t,\mathbf{g}^{(1)})$ *and* $\mathbf{u}(t,\mathbf{g}^{(2)})$ *obey* $\mathbf{u}(t,\mathbf{g}^{(1)}) \geq \mathbf{u}(t,\mathbf{g}^{(2)})$ $\forall t$. *If* $\mathbf{g} \in \mathcal{U}$, $\mathbf{u}(t;\mathbf{g}) \in \mathcal{U}$ $\forall t$. *Finally, if* $g_1(l) \geq g_2(l)$, *then* $\forall$ t $u_1(t;l) \geq u_2(t;l)$, $l = 1,2,\ldots$.

Proof of Lemma 2. First, observe that if $\mathbf{u}^{[M_1]}(t,\mathbf{g}^{(1)})$ and $\mathbf{u}^{[M_2]}(t,\mathbf{g}^{(2)})$ are two solutions, with $\mathbf{g}^{(1)} \geq \mathbf{g}^{(2)}$ and $u_i^{[M_1]}(t,\mathbf{g}^{(1,M_1)};l_0) \geq u_i^{[M_2]}(t,\mathbf{g}^{(2,M_2)};l_0)$ for some $l_0 \leq \min\,[M_1,M_2]$ and $\forall$ $t \geq 0$, then $u_i^{[M_1]}(t,\mathbf{g}^{(1,M_1)};l) \geq u_i^{[M_2]}(t,\mathbf{g}^{(2,M_2)};l)$ for $\forall$ $t \geq 0$ and $1 \leq l \leq l_0 - 1$. This follows when one compares the right-hand sides of the corresponding systems (14) subsequently for $l = l_0 - 1$, $l_0 - 2$, etc.

Next, note that $u_i^{[M+1]}(t,\mathbf{g};M) \geq u_i^{[M]}(t,\mathbf{g};M) = 0$ and use the above fact to get that $u_i^{[M+1]}(t,\mathbf{g};l) \geq u_i^{[M]}(t,\mathbf{g};l)$, $t \geq 0$, $1 \leq l \leq M$. Continuing, obtain the monotonicity in M and hence the convergence $\lim_{M\to\infty} u_i^{[M]}(t,\mathbf{g};l) = u_i(t;\mathbf{g},l)$. The limit $u_i(t;\mathbf{g},l)$ satisfy (7)-(9) and form a vector $\mathbf{u}(t,\mathbf{g}) \in \bar{\mathcal{U}}$. It is clear that $u_1(t;l) \geq u_2(t;l)$ if $g_1(l) \geq g_2(l)$. The uniqueness of the solution in $\bar{\mathcal{U}}$ may be established by standard methods.

Finally, if $\mathbf{g} \in \mathcal{U}$, we write the equations for

$$V(t,\mathbf{g};l) = \sum_{i=1,2} \sum_{\widetilde{l}\geq l} u_i(t,\mathbf{g};\widetilde{l}) \ :$$

$$\dot{V}(t,\mathbf{g};l) = 2\lambda\Big(\sum_{i=1,2} p_i u_i(t,\mathbf{g};l-1)\Big)^2 - \sum_{i=1,2} u_i(t,\mathbf{g};l),\ l \geq 1. \tag{20}$$

As $\mathbf{u}(t,\mathbf{g}) \in \bar{\mathcal{U}}$, the right-hand side of (20) is uniformly bounded in time. Thus, $V(t,\mathbf{g};l)$ grows at most linearly with t and therefore remains finite. This completes the proof of Lemma.

Lemma 3 *For any* $M \in \mathbf{Z}_+$ *there exists in* $\mathcal{U}$ *a unique solution* $\mathbf{a}^{[M]}$ *to (17),(18), and* $\forall$ $\mathbf{g} \in \mathcal{U}$, *the solution* $\mathbf{u}^{[M]}(t,\mathbf{g})$ *converges, as* $t \to \infty$, *to* $\mathbf{a}^{[M]}$ *in metric (4). Furthermore, under condition (1), there exists a constant* $C > 0$, *independent of* M, *such that*

$$a_i^{[M]}(l) \leq C(2\lambda p_1^2)^{2^l},\ l \geq 1,\ i = 1,2. \tag{21}$$

Finally, $a_i^{[M]}(l)$ *is non-decreasing in* M $\forall$ i *and* l.

Proof of Lemma 3. Set $g_i^0(l) = \mathbf{1}(l=0)$, $l \geq 0$, $i = 1,2$. Owing to the monotonicity of $\mathbf{u}^{[M]}(t;\mathbf{g})$ with respect to the initial data $\mathbf{g}$, the entries $u_i^{[M]}(t,\mathbf{g}^0;l)$ of solution $\mathbf{u}^{[M]}(t,\mathbf{g}^0)$ are nondecreasing in t, and as $\mathbf{u}^{[M]}(t,\mathbf{g}^0) \in \mathcal{U}$, they are bounded from above by one. Therefore, there exist the limits $a_i^{[M]}(l) = \lim_{t\to\infty} u_i^{[M]}(t,\mathbf{g}^0;l)$, $1 \leq l \leq M$, $i = 1,2$, and limiting values $a_i^{[M]}(l)$ obey (17),(18). We will show that $a_i^{[M]}(l) = \lim_{t\to\infty} u_i^{[M]}(t,\mathbf{g};l)$ for all $\mathbf{g} \in \mathcal{U}$; this will imply the uniqueness of the fixed point $\mathbf{a}^{[M]}$. For simplicity, the index $^{[M]}$ will be temporarily omitted.

First, assume that $\mathbf{g} \geq \mathbf{a}$, then $\mathbf{u}(t, \mathbf{g}) \geq \mathbf{a}\, \forall\, t$. We will check that $\int_0^\infty \sum_{i=1,2}(u_i(t, \mathbf{g}; l)) - a_i(l))\mathrm{d}t < \infty$ by using the induction in l, $0 \leq l \leq M$. For $l = 0$ we have this bound automatically owing to the boundary condition. Also, for $l = 1$,

$$\begin{aligned} \dot{V}(t, \mathbf{g}; 1) &= 2\lambda\Big[- \Big(\textstyle\sum_{i=1,2} p_i u_i(t; \mathbf{g}; M) \Big)^2 + \Big(\textstyle\sum_{i=1,2} p_i a_i(M) \Big)^2 \Big] \\ &\quad - \textstyle\sum_{i=1,2} \big[u_i(t; \mathbf{g}; 1) - a_i(1) \big] \leq 0, \end{aligned}$$

i.e., $V(t, \mathbf{g}); 1)$ remains bounded in t.

Now assume the induction hypothesis: $\int_0^\infty \sum_{i=1,2}(u_i(t, \mathbf{g}; l-1)) - a_i(l-1))dt < \infty$. Then

$$\begin{aligned} \dot{V}(t, \mathbf{g}; l) &= 2\lambda\Big[\Big(\textstyle\sum_{i=1,2} p_i u_i(t; \mathbf{g}; l-1) \Big)^2 - \Big(\textstyle\sum_{i=1,2} p_i a_i(l-1) \Big)^2 - \\ &\quad - \Big(\Big(\textstyle\sum_{i=1,2} p_i u_i(t; \mathbf{g}; M) \Big)^2 - \Big(\textstyle\sum_{i=1,2} p_i a_i(M) \Big)^2 \Big) \Big] - \textstyle\sum_{i=1,2} \big[u_i(t; \mathbf{g}; l) - a_i(l) \big] \\ &\leq 2\lambda\Big[\Big(\textstyle\sum_{i=1,2} p_i u_i(t; \mathbf{g}; l-1) \Big)^2 - \Big(\textstyle\sum_{i=1,2} p_i a_i(l-1) \Big)^2 \Big] - \\ &\quad - \textstyle\sum_{i=1,2} \big[u_i(t; \mathbf{g}; l) - a_i(l) \big] \Big) \end{aligned} \tag{22}$$

and

$$\begin{aligned} V(t, \mathbf{g}; l) - V(0, \mathbf{g}; l)) &\leq 2\lambda \textstyle\int_0^t \Big(\Big[\Big(\textstyle\sum_{i=1,2} p_i u_i(s; \mathbf{g}; l-1) \Big)^2 - \\ &\quad - \Big(\textstyle\sum_{i=1,2} p_i a_i(l-1) \Big)^2 \Big] - \textstyle\sum_{i=1,2} \big[u_i(s; \mathbf{g}; l) - a_i(l) \big] \Big] \Big) ds. \end{aligned} \tag{23}$$

Owing to the induction hypothesis, the integral

$$\int_0^\infty \Big[\Big(\sum_{i=1,2} p_i u_i(s; \mathbf{g}; l-1) \Big)^2 - \Big(\sum_{i=1,2} p_i a_i(l-1) \Big)^2 \Big] ds < \infty.$$

As the left-hand side of (23) is bounded uniformly in t, it follows that the integral

$$\int_0^\infty \sum_{i=1,2} \big[u_i(s; \mathbf{g}; l) - a_i(l) \big] ds < \infty.$$

A similar integral converges when $\mathbf{g} \leq \mathbf{a}$. Thus, if either $\mathbf{g} \leq \mathbf{a}$ or $\mathbf{g} \geq \mathbf{a}$, the solution $\mathbf{u}(t, \mathbf{g})$ approaches $\mathbf{a}$ as $t \to \infty$. In a general case we pass to $\mathbf{g}^+ = \max\ \big[\mathbf{g}, \mathbf{a}\big]$ and $\mathbf{g}^- = \min\ \big[\mathbf{g}, \mathbf{a}\big]$ and use the monotonicity of $\mathbf{u}(t, \mathbf{g})$ in $\mathbf{g}$. This completes the proof of convergence $\mathbf{u}(t, \mathbf{g}) \to \mathbf{a}$.

Now assume that inequalities (1) hold and set $y(l) = \sum_{i=1,2} a_i(l)$. We then have $y(0) = 2$, $y(1) = 2\lambda$ and

$$y(l+1) = 2\lambda\Big[\Big(\sum_{i=1,2} p_i a_i(l) \Big)^2 - \Big(\sum_{i=1,2} p_i a_i(M) \Big)^2 \Big] \leq 2\lambda \Big(\sum_{i=1,2} p_i a_i(l) \Big)^2,\ l \geq 1. \tag{24}$$

From (24) we conclude that

$$\begin{aligned} y(l+1) &\leq 2\lambda\Big(p_1 a_1(l) + p_2 a_2(l)\Big)^2 \\ &= 2\lambda\Big(\big[p_1 a_1(l) + p_2(1-a_1(l))\big] + p_2(y(l)-1)\Big)^2 \\ &\leq 2\lambda\Big(\big[p_1 a_1(l) + p_1(1-a_1(l))\big] + p_2(y(l)-1)\Big)^2 \\ &= 2\lambda\Big(p_1 + p_2(y(l)-1)\Big)^2. \end{aligned}$$

Thus, $y(l) \leq z(l)$, $l \geq 0$, where $z(0) = 2$, and

$$z(l+1) = 2\lambda\left(p_1 + p_2(z(l)-1)\right)^2, \ l \geq 0. \tag{25}$$

One can expect that, as $l \to \infty$, $z(l) \to w$ where w is the root of

$$w = \phi(w), \ \text{with } \phi(x) = 2\lambda\left(p_1 + p_2(x-1)\right)^2, \ x \in [0,\infty).$$

Note that the quadratic function ϕ takes the following values: (i) $\phi(0) = 2\lambda\Big(p_1 - p_2\Big)^2 \geq 0$, (ii) $\phi(2) = 2\lambda < 2$ and (iii) $\phi(1) = 2\lambda p_1^2 < 1$ (two last inequalities are exactly (1)). Therefore, iterating the map $x \mapsto \phi(x)$ with the initial point $x = 2$, we approach a fixed point w which is necessarily < 1. Hence, $z(l)$ is < 1 for $l \geq L$, and so is $y(l)$. Observe that L does not depend on M.

Owing to (24), $y(l+1) \leq 2\lambda p_1^2\big(y(l)\big)^2$. Thus, iterating, we get that $y(L+r) \leq \big(2\lambda p_1^2\big)^{2^r-1}$, $r \geq 0$, which leads to (21).

The final assertion of Lemma 3, that $a_i^{[M]}(l)$ increases in M follows from the similar property of $u_i^{[M]}(t,\mathbf{g};l)$.

Lemma 4 *Under condition (1), there exists in $\mathcal{U}$ a unique solution* $\mathbf{a}$ *to (10). It obeys (21) (with the same constant C), and $\forall$ $\mathbf{g} \in \mathcal{U}$, the solution $\mathbf{u}(t,\mathbf{g})$ of (7)-(9) converges, as $t \to \infty$, to $\mathbf{a} \in \mathcal{U}$.*

Proof of Lemma 4. Take the initial condition $g_i^0(l) = \mathbf{1}(l=0)$, $l \in \mathbf{Z}_+$, $i = 1,2$. The solution $\mathbf{u}(t,\mathbf{g}^0) = \lim_{M\to\infty} \mathbf{u}^{[M]}(t;\mathbf{g}^0)$ increases with t. $\forall$ M and t, $u_i^{[M]}(t;\mathbf{g}^0;l) \leq a_i^{[M]}(l)$ $\leq C(2\lambda p_1^2)^{2^l-1}$, $l \geq 1$, $i = 1,2$ (see (21)).

Therefore, $\exists$ the limit $\lim_{t\to\infty} u_i(t,\mathbf{g}^0;l) = a_i(l)$, $l \geq 1$, $i = 1,2$, satisfying (10) and (21). This gives a fixed point of the infinite system in $\mathcal{U}$. Its uniqueness will follow if we check that $\lim_{t\to\infty} \mathbf{u}(t,\mathbf{g}) = \mathbf{a}$ $\forall \mathbf{g}$. This can be done by repeating the argument from the proof of Lemma 3.

Lemma 5 *If one of inequalities (1) fails, system (10) does not have solutions in $\mathcal{U}$.*

Proof of Lemma 5 Assume, for definiteness, that $2\lambda p_1^2 > 1$. Then, if $\mathbf{a} \in \mathcal{U}$ is a solution to (10), we have:

$$\begin{aligned} a_1(1) &= \sum_{l\geq 1}\big(a_1(l) - a_1(l+1)\big) \\ &= 2\lambda \sum_{l\geq 1}\Big[p_1^2\Big(\big(a_1(l-1)\big)^2 - \big(a_1(l)\big)^2\Big) + \\ &\quad + p_1p_2\Big(a_1(l-1) - a_1(l)\Big)\Big(a_2(l-1) + a_2(l)\Big)\Big] > 2\lambda\big(p_1 a_1(0)\big)^2 > 1 \end{aligned}$$

which is impossible as $a_i(1)$ must be ≤ 1.

4 Convergence to the limiting system

The proof of Theorems 1 and 2 repeats without change that of Theorems 1 a) and 2 from [6] and Theorem 2.4 from [7], and we omit it from the paper.

Theorem 3 is proved in Lemmas 3-5 of the last section.

The proof of Theorem 4 follows the ideas of [6], [7], but as here there are several complications partially due to the absence of the explicit formulas for solutions to (10), we will discuss it in more detail.

Markov process $U^{(N)}$ has countably many states forming a single class, hence it cannot have more than one equilibrium distribution. To check the existence of and the convergence to the equilibrium distribution, it suffices to verify that, under condition (1), $\forall$ initial state $\mathbf{g} \in \mathcal{U}^{(N)}$ the process lives in a compact subset of $\mathcal{U}^{(N)}$. As in the proof of Theorem 5 a) from [6], it is enough to establish that

$$\sup_{t\geq 0} V^{(N)}(t, \mathbf{g}; 1) < \infty. \tag{26}$$

where

$$V^{(N)}(t, \mathbf{g}; \mathbf{l}) = \Big(\mathbf{T}_t^{(N)} V(t; \mathbf{g}; l)\Big) \tag{27}$$

(when $t = 0$, we will omit both the argument t and upper index (N) in agreement with the previously introduced notation).

Bound (26) will follow from Lemma 7 (see below). Extending the definition of $y(l)$ (see Section 3), we set $y(\mathbf{g}; l) = g_1(l) + g_2(l)$, $\mathbf{g} \in \bar{\mathcal{U}}$, $l \leq 0$. We also set $y^{(N)}(t, \mathbf{g}; l) = V^{(N)}(t, \mathbf{g}, l)$ $- V^{(N)}(t, \mathbf{g}, l+1)$, $\mathbf{g} \in \bar{\mathcal{U}}$, $l \geq 1$, $t \geq 0$.

Let $w(t, \mathbf{g}; l)$, $t \geq 0$, $\mathbf{g} \in \mathcal{U}$, denotes the solution to the following *linear* Cauchy problem:

$$\dot{w}(t; l) = \bar{\lambda}\big(w(t; l-1) - w(t; l)\big) + w(t; l+1) - w(t; l), \;\; t > 0, \; l \geq 1,) \tag{28}$$

with the coefficient

$$\bar{\lambda} = \lambda \max\,\big[1,\, 2p_1^2\big], \tag{29}$$

and the initial-boundary values

$$w(0; l) = y(\mathbf{g}; l), \; l \geq 1, \;\; w(t; 0) - w(t; 1) = 2, \; t \geq 0. \tag{30}$$

The sequence $\{w(t,\mathbf{g};l),\ l\geq 0\}$ is denoted by $\mathbf{w}(t,\mathbf{g})$.

It will be convenient to introduce the space $\bar{\mathcal{U}}^*$ formed by sequences $x=\{x(l),\ l\geq 0\}$ such that $x(0)-x(1)=2$, $x(l)\geq 0$ and $x(l-1)\geq x(l)$, $l\geq 1$. In addition, $\mathcal{U}^*$ will denote the subspace of $\bar{\mathcal{U}}^*$ formed by sequences $x=\{x(l)\}$ with $\sum_l x(l)<\infty$.

We will be interested in properties of solution $\mathbf{w}(t,\mathbf{g})$ for large $t>0$.

To this end, we consider a fixed point $\mathbf{b}=\{b(l),\ l\geq 0\}$ of system (28), whose entries satisfy

$$b(l)=\bar{\lambda}b(l-1),\ l\geq 1,\ \ b(0)-b(1)=2. \tag{31}$$

If $\bar{\lambda}<1$ then $\mathbf{b}\in\mathcal{U}^*$.

Lemma 6 *Assume that condition (1) holds. Then $\bar{\lambda}<1$, and for any $\mathbf{g}\in\mathcal{U}$ there exists in $\bar{\mathcal{U}}^*$ a unique solution $\mathbf{w}(t,\mathbf{g})$ to (28), (30), which in fact belongs to $\mathcal{U}^*$ $\forall\ t>0$. This solution approaches $\mathbf{b}$ as $t\to\infty$, and therefore has*

$$\sup_{t\geq 0}\sum_{\widetilde{l}\geq 1} w(t,\mathbf{g};\widetilde{l})<\infty. \tag{32}$$

The proof of Lemma 6 is standard (cf [6], Section 3, Proposition 1, Lemmas 9–14), and is omitted.

From now on, condition (1) will be assumed throughout the rest of this section.

Lemma 7 $\forall\ N,l\geq 1$ *the following bound holds true:*

$$V(t,\mathbf{g},l)\leq w_l(t,\mathbf{g};l). \tag{33}$$

Proof of Lemma 7 (cf. [6], Lemma 16). We begin with the following observation:

$$\begin{array}{rcl}\mathbf{A}^{(N)}V(\mathbf{g};\mathbf{l}) & = & 2\lambda\big(p_1g_1(l-1)+p_2g_2(l-1)\big)^2-y(\mathbf{g};l)\\ & \leq & \bar{\lambda}y(\mathbf{g};l-1)-y(\mathbf{g};l),\end{array} \tag{34}$$

that follows from an elementary inequality $(pr+qs)^2\leq\max\,[1/2,p^2](r+s)$ held $\forall\ r,s,p,q\in[0,1]$ with $p=1-q\geq q$.

Therefore

$$\begin{array}{rcl}\dot{V}(t,\mathbf{g};l) & \leq & \bar{\lambda}y(t,\mathbf{g};l-1)-y(t,\mathbf{g};l)\\ & = & \bar{\lambda}(V^{(N)}(t,\mathbf{g};l-1)-V^{(N)}(t,\mathbf{g};l))-V^{(N)}(t,\mathbf{g};l)+V^{(N)}(t,\mathbf{g};l+1).\end{array} \tag{35}$$

Comparing the right-hand sides of (28) and (35) leads to the assertion of Lemma 7.

Continuing with the proof of Theorem 4, note that, as in the proof of Theorem 5 b,c) of [6], it suffices to check that any limiting point π of the sequence of equilibrium distributions π_N is a probability measure concentrated on set $\mathcal{U}$. But, according to (34), $\forall N \geq 1$,

$$\begin{aligned} \mathbf{E}^{(N)} y(\mathbf{g}; l) &= 2\lambda \mathbf{E}^{(N)} \big(p_1 g_1(l-1) + p_2 g_2(l-1)\big)^2 \\ &\leq \bar{\lambda} \mathbf{E}^{(N)} y(\mathbf{g}, l-1),\ l \geq 1, \end{aligned}$$

i.e.,

$$\mathbf{E}^{(N)} y(\mathbf{g}; l) \leq (\bar{\lambda})^l,\ l \geq 1.$$

Thus, the expected values in measure π are bounded in the same way which implies that π is concentrated on $\mathcal{U}$.

5 Analytical and numerical computation of $\mathbf{a_1(1)}$

Equations (10) show that any given $a_1(1), a_2(1)$ (with $a_1(0) = a_2(0) = 1$) determine the values of $a_1(i), a_2(i)$ for all $i,\ i > 1$, and that $a_j(i),\ \ j = 1, 2,\ i > 1$ can be represented as polynomials on $a_1(1), a_2(1)$.

For the solution to (10) for $\forall\ M > 1$

$$\begin{aligned} a_1(1) &= 2\lambda p_1 + 2\lambda p_1 p_2 \textstyle\sum_{i=1}^{M} [a_1(i-1)a_2(i) - a_1(i)a_2(i_1 - 1)] + \\ &\quad + a_1(M+1) - 2\lambda[p_1^2 a_1^2(M) + p_1 p_2 a_1(M) a_2(M)] \\ a_2(1) &= 2\lambda p_2 - 2\lambda p_1 p_2 \textstyle\sum_{i=1}^{M} [a_1(i-1)a_2(i) - a_1(i)a_2(i_1 - 1)] + \\ &\quad + a_2(M+1) - 2\lambda[p_2^2 a_2^2(M) + p_1 p_2 a_1(M) a_2(M)]. \end{aligned} \tag{36}$$

The sum of equations (36) gives

$$a_1(1) + a_2(1) = 2\lambda + (a_1(M+1) + a_2(M+1)) - 2\lambda\Big(p_1 a_1(M) + p_2 a_2(M)\Big)^2. \tag{37}$$

For the solution $\mathbf{a} = \mathbf{a}^* = (a_i^*(l))$ to (10) (that is the limit for solutions to (7)-(9) as $t \to \infty$) we have $a_1(1) + a_2(1) = 2\lambda$, and therefore

$$a_1^*(l+1) + a_2^*(l+1) = 2\lambda\Big(p_1 a_1^*(l) + p_2 a_2^*(l)\Big)^2,\ \forall\ \ l \geq 1. \tag{38}$$

In this case all $a_j^*(l),\ j = 1, 2,\ l > 1$ can be represented as polynomials on $a^* = a_1^*(1)$ only.

If the solutions $a_i(l)$ to (10) (that may be not in $\mathcal{U}$) has a limit $\lim_{l\to\infty} a_j(l) = a_j(\infty),\ j = 1, 2$, and the sum $\sum_{i=1}^{M}[...]$ in (36) converges as $M \to \infty$, then $a_1(1)$ can be represented as

$$\begin{aligned} a_1(1) &= 2\lambda p_1 + 2\lambda p_1 p_2 \textstyle\sum_{i=1}^{\infty} [a_1(i-1)a_2(i) - a_1(i)a_2(i_1)] + \\ &\quad + \{a_1(\infty) - 2\lambda[p_1^2 a_1^2(\infty) + p_1 p_2 a_1(\infty) a_2(\infty)]\}. \end{aligned} \tag{39}$$

Set

$$\begin{aligned} F^{[M]}(a_1(1)) &= 2\lambda\{p_1 + p_1p_2\textstyle\sum_{i=1}^{M}[a_1(i-1)a_2(i) - a_1(i)a_2(i-1)]\},\\ a_2(1) &= 2\lambda - a_1(1). \end{aligned}$$

Let $a^{[M]}$ be the root of the equation $a^{[M]} - F^{[M]}(a^{[M]}) = 0$. Here

$$\begin{aligned} a^{[0]} &= 2p_1\lambda,\\ a^{[1]} &= 2\lambda p_1(1+2p_2\lambda)/(1+4p_1p_2\lambda), \end{aligned}$$

$a^{[M]}, M > 1$, are the roots of the polynomials and can be calculated numerically.

The numerical computations show that $a^{[M]}$ tend to $a^* = a_1^*(1)$ rather fast. Figures 1 and 2 displays the numerical evaluation of $a^{[M]}$ for λ respectively equal to 0.9 and 1.6. Notice that the computations are less stable when λ tends to 1 as illustrated in F figure 2.. Figure 3 summarizes the overall analytical model by displaying a 3D plot of quantity $a_1(1)$.

We will show that the polynomials $F^{[M]}(a)$ converge as $M \to \infty$ to a function $F(a)$, and $F(a)$ is an analytical function in some neighborhood of the point a^*.

It is useful to rewrite (10) in the form $(i = 1, 2)$

$$a_i(l) - a_i(l+1) = 2\lambda p_i[a_i(l-1) - a_i(l)]\Big(p_1[a_1(l-1) + a_1(l)] + p_2[a_2(l-1) + a_2(l)]\Big). \quad (40)$$

Denote

$$B_l = B_l(a_1(1), a_2(1)) = 2\lambda p_1[p_1(a_1(l-1) + a(l)) + p_2(a_2(l-1) + a_2(l))]. \quad (41)$$

Note that $B_l(a_1^*(1), a_2^*(1)) \to 0$ as $l \to \infty$.

Lemma 8 *If for some solution* $\mathbf{a}$ *to (10) with* $0 < a_1(1) < 1,\ 0 < a_2(1) < 1$ *there exists a* $\lim_{l\to\infty} a_j(l) = a_j(\infty),\ j = 1, 2,$ *such that*

$$|B_\infty(a_1(1), a_2(1))| = 4\lambda p_1(p_1|a_1(\infty)| + p_2|a_2(\infty)|) < b < 1, \quad (42)$$

then there exists in $\mathbf{C}^2$ *a neighborhood* $\mathcal{O}$ *of the point* $(a_1(1), a_2(1))$ *such that for* $(a_1'(1), a_2'(1)) \in \mathcal{O}$ *all* $|a_i'(l)|,\ l > 1$, *determined by (10) are uniformly bounded, there exists* $\lim_{l\to\infty} a_j'(l) = a_j'(\infty),\ j = 1, 2$, *and the sum* $\sum_1^\infty[...]$ *in (39) converges.*

Proof of Lemma 8. For any $b',\ b < b' < 1$ and for any $\delta > 0$ there exists such L that for the solution $\mathbf{a}$ the inequalities $|a_j(l) - a_j(\infty)| < \delta(1-b')/8,\ j = 1, 2$, hold for $l \geq L - 1$, and that for any $a_j'',\ j = 1, 2$, with $|a_j(\infty) - a_j''| < \delta$

$$4\lambda p_1(p_1|a_1''| + p_2|a_2''|) < b'. \quad (43)$$

Owing to the uniform continuity of $a_j'(l),\ 1 < l \leq L$, on $a_j'(1)$, there exist such neighborhood $\mathcal{O}$ of the point $(a_1(1), a_2(1))$ that $|a_i'(l) - a_i(l)| < \delta(1-b')/8,\ \forall\, l \leq L$, for $(a_1'(1), a_2'(1) \in \mathcal{O}$.

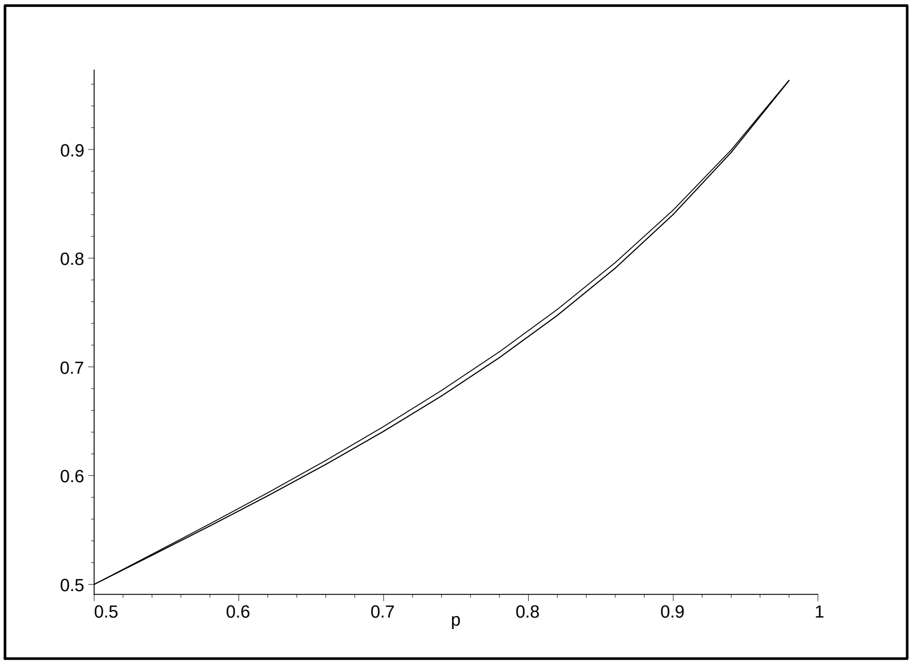

Figure 1: Quantity $a^{[M]}$ versus p_1 for $2\lambda = 0.9$ for $M = 2, 3, 4$, obtained via the analytical model.

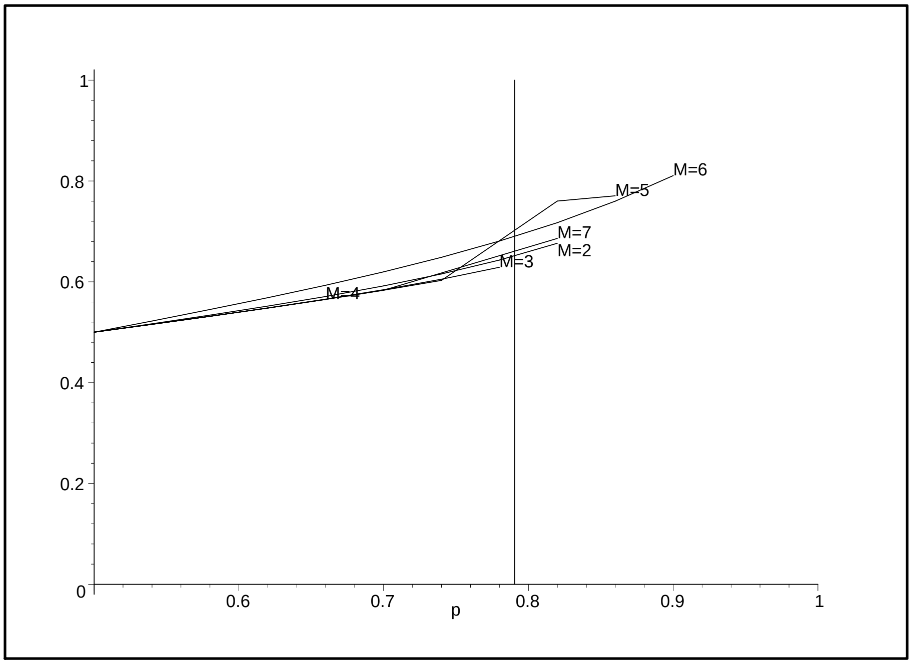

Figure 2: Quantity $a^{[M]}$ versus p_1 for $2\lambda = 1.6$ for $M = 3, 4, 5, 6, 7$, obtained via the analytical model. The border of the validity domain of the model $p_1 = (2\lambda)^{-1/2}$ is given by the vertical line. For small M there exists an other root near the needed one, therefore the instability of curves

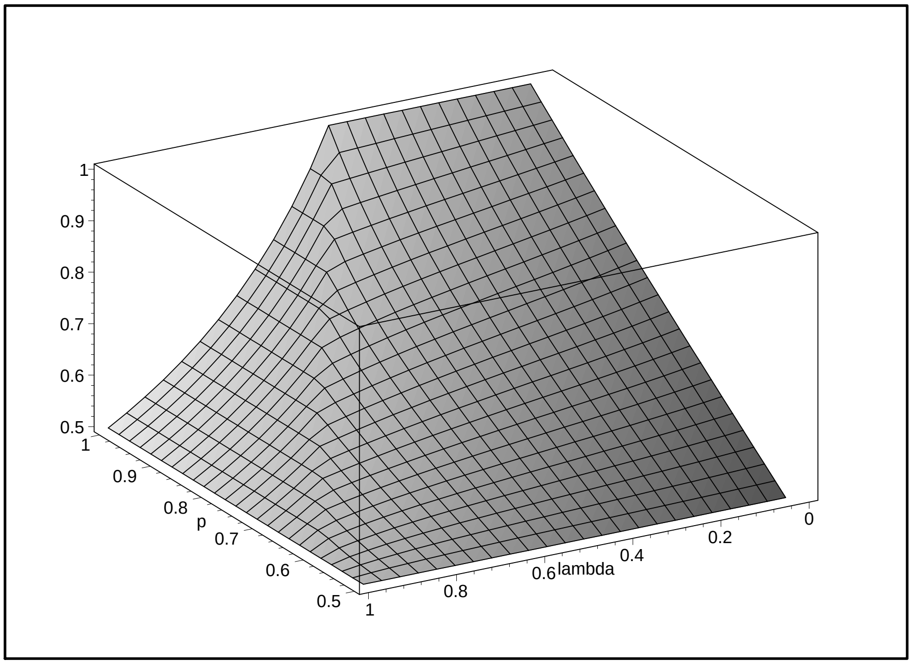

Figure 3: Quantity $a_1(1)/2\lambda$ versus p_1 and λ, obtained via the analytical model. Note that $a_1(1)/2\lambda = 0.5$ as $p_1 = 0.5$; $a_1(1)/2\lambda \to p_1$ as $\lambda \to 0$.

For a geometrically decreasing sequence $\{\alpha_l\}$, $\alpha_{l+1} = b'\alpha_l$, $l \geq L$, $\forall K > L$ $\sum_L^K \alpha_l < \alpha_L/(1-b')$. This brings an a priori estimates $\sum_{l=L}^K |a'_j(l-1) - a'_j(l)| < \delta/2$ and $|a'_j(l) - a_j(\infty)| < \delta$. Therefore $|B_l(a'_1, a'_2| < 1$, $l > L$, an there exists $\lim_{l\to\infty} a'_j(l) = a'_j(\infty)$.

Consider now $\sum_{l=L}^{\infty} \alpha(l-1/2)$, $\alpha(l-1/2) = (a'_1(l-1)a'_2(l) - a'_1(l)a'_2(l-1))$. The values $|\alpha(l-1/2)| \leq |(a'_1(l-1) - a'_1(l))a'_2(l)| + |(a'_2(l-1) - a'_2(l))a'_1(l)|$ decrease geometrically, therefore the sum in (39) converges.

Corollary of Lemma 8 *Let $(a_1(1), a_2(1)) = (a_1^*(1), a_2^*(1))$ and let b', δ, $\mathcal{O}$ satisfy the conditions of the proof of Lemma 8. Then for $a' = (a'_1, a'_2) \in \mathcal{O}$, with $a'_2(1) = 2\lambda - a'_1(1)$ the values of $|F^{[M]}(a'(1)|$ are uniformly bounded for all M.*

Lemma 9 *In the neighborhood $\mathcal{O}' \subset \mathcal{O}$ of the point $(a_1^*(1), a_2^*(1))$, $a_2^*(1) = 2\lambda - a_1^*(1)$, there exists a analytical function $F(a') = lim_{M\to\infty} F^{[M]}(a')$, $(a'_1(1), a'_2(1)) \in \mathcal{O}$ and in some neighborhood $\mathcal{O}'' \subset \mathcal{O}'$ of the point $(a_1^*(1), a_2^*(1))$ the equation $a' = F(a')$ has a unique root $(a_1^*(1), a_2^*(1))$.*

Proof of Lemma 9. The set of polynomials $F^{[M]}$ forms a set of bounded analytical functions in $\mathcal{O}$. Therefore this set is compact in sub-neighborhood $\mathcal{O}'$ and there exists a point of condensation, namely an analytical function $F(a')$. The differences $|F^{[M]}(a') - F^{[M+1]}(a')|$, $(a') \in \mathcal{O}'$, decrease geometrically therefore $F(a') = lim_{M\to\infty} F^{[M]}(a')$. The root a^* is bounded away from the other roots (if any) in $\mathcal{O}'$, thus is unique in some sub-neighborhood $\mathcal{O}''$.

The numerical computations show that the neighborhood $\mathcal{O}''$ is in fact not too small, and that permits to perform the computations of a^* easily.

We can mention here that the boundary-value problems (7)-(9) and (10) for ODE can be solved for boundary conditions different from the ones we needed for our original problem.

For example, consider the boundary value problem (7)-(9) with initial values $g_j(l)$, $j = 1, 2$, $l \geq 1$, $g_j(l) \geq g_j(l+1)$, $\lim_{l\to\infty} g_j(l) = g_j(\infty)$, and boundary conditions

$$u_1(0) = u_2(0) = 1, \;\; u_1(\infty) = g_1(\infty), \;\; u_2(\infty) = g_2(\infty), \tag{44}$$

where

$$0 < 4p_1(p_1 g_1(\infty) + p_2 g_2(\infty)) < 1, \tag{45}$$

$$0 < 2\lambda + g_1(\infty) + g_2(\infty) - 2\lambda(p_1 g_1(\infty) + p_2 g_2(\infty))^2 < 2, \tag{46}$$

$$0 < 2\lambda(p_1)^2 + g_1(\infty) + g_2(\infty) - 2\lambda(p_1 g_1(\infty) + p_2 g_2(\infty))^2 < 1. \tag{47}$$

$$0 < 2\lambda(p_1 - p_2)^2 + g_2(\infty) - 2\lambda(p_1 g_1(\infty) + p_2 g_2(\infty))^2 \tag{48}$$

Proposition Under the conditions (44)-(48) there exists for $0 < t < \infty$ a solution $(u_1(t;l), u_2(t;l))$ to problem (7)-(9), $u_j(t;l) \geq u_j(l+1), j = 1, 2$, $u_j(t;l) \to g_j(\infty)$ as $l \to \infty$, and $u_j(t;l) \to a_j(l)$ as $t \to \infty$, where $a_j(l)$, $j = 1, 2$ is the solution to (10) with the boundary conditions (44). .

Proof of this Proposition repeats the proofs of Lemmas of Section 2.

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