Bayesian Modelling of PWR Vessels Flaw Distributions

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Abstract: We present a full Bayesian method for estimating the density and size distribution of subclad-flaws in French Pressurized Water Reactor (PWR) vessels. This model takes into account in service inspection (ISI) data, a flaw size-dependent probability of detection function (different function types are possible) with a threshold of detection, and a flaw sizing error distribution (different distribution types are possible). It is identified through a Markov Chain Monte Carlo (MCMC) algorithm. The article includes discussion for choosing the prior distribution parameters and an illustrative application is presented highlighting its ability to provide good parameter estimates even when a small number of flaws is observed.

Key-words: Flaw size, Flaw density, Weibull distribution, log-normal distribution, missing data, Bayesian inference, Markov Chain Monte Carlo.

(Résumé : tsvp)

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Estimation bayésienne des distributions de défauts des cuves REP

Résumé : Nous présentons une modélisation bayésienne complète de la densité et de la distribution de probabilité de la taille des défauts des cuves de réacteurs à eau pressurisée (REP). Ce modèle prend en compte les résultats d'inspection en service des cuves, une fonction de probabilité de détection d'un défaut en fonction de sa taille (différents types de fonction sont possibles), un seuil de détection d'un défaut et une distribution de probabilté pour les erreurs de mesure (différents types de distribution sont possibles). Ce modèle est estimé par une méthode de type chaînes de Markov de Monte-Carlo (MCMC). Le choix des paramètres des lois a priori est discuté dans l'article et un exemple illustrant les bonnes performances de cette approche même pour des échantillons de faible taille est présenté.

Mots-clé : taille de défaut, densité de défaut, lois de Weibull et lognormale, données manquantes, inférence bayésienne, chaînes de Markov de Monte-Carlo.

1 Introduction

The estimation of flaw size and flaws number distributions to be used in structural integrity analysis of nuclear Pressurized Water Reactor (PWR) vessels has received increasing attention (see, for instance Foulds *et al.* 1992, Ammirato and Dimitrijevic 1993). The flaws characteristics play effectively an important role in the probabilistic fracture mechanics model of the vessel and they consequently are very influent upon the estimated risk of failure. It is so of high importance to have a realistic representation of these relatively small flaws (a few mm) which are generally not very well known. As far as we know, the most achieved mathematical approach has been proposed by Foulds *et al.* (1992) which gathered the key ingredients of a probabilistic model for PWR vessel flaws. They used a maximum likelihood methodology:

1. They used the versatile Weibull distribution as flaw size distribution function.
2. They considered a threshold detection limit below which flaw sizes are considered unreliable.
3. They took into account the probability of detection of a flaw during a vessel inspection, from which they deduce an estimation of the true number of flaws using the observed number of flaws.
4. They introduced an additive flaw sizing error random variable.

However their approach suffers some limitations. They only considered a constant probability of detection of a flaw regardless its size, and they limit attention on uniform sizing error. And, especially, it is doubtful that maximum likelihood inference provides reliable estimates in a domain where there can be very few observations. Actually, Foulds *et al.* (1992), assumed that all indications represent flaws and moreover, to get enough data to make their methodology workable, they considered indications (conservatively interpreted as flaws) provided by the vendors which were beyond the ASME Code threshold of reportable indications.

In this article, we present a methodology answering all the above mentioned limitations. This methodology is a fully Bayesian analysis of the problem taking account in a proper way all the elements considered in Foulds *et al.* (1992). (For other partially Bayesian methodologies see Ammirato and Dimitrijevic, 1993 and Ammirato *et al.*, 1992.) This Bayesian methodology makes use of Markov Chain Monte Carlo (MCMC) algorithms (a comprehensive reference for MCMC algorithms is Gilks, Richardson and Spiegelhalter 1996). It allows to discard limitations on the parametric flaw distribution function, on the flaw probability of detection function, and on the flaw sizing error distribution function.

The article is organized as follows. The probabilistic model is presented in Section 2. The fully Bayesian estimation of the model is presented in Section 3. The choices of the prior distributions are discussed in Section 4. Section 5 is devoted to a numerical illustration. A discussion section summarizing the main points concludes the article.

2 The probabilistic model

The model that we now describe is aimed to allow the analysis of vessel specific in-service inspection (ISI) data. We assume a parametric density $f(.|\theta)$ for the flaw size random variable H. As Foulds *et al.*(1992), we favor the Weibull distribution with vector parameter $\theta = (\beta, \gamma)$, $\beta > 0$ being a shape parameter and $\gamma > 0$ a scale parameter $\gamma > 0$, with density

$$f(h|\theta) = \gamma\beta(\gamma h)^{\beta-1} \exp\left(-(\gamma h)^{\beta}\right), h \geq 0,$$

h denoting a flaw size. But, we also consider an exponential distribution, where θ is reduced to a scale parameter γ,

$$f(h|\gamma) = \gamma \exp\left(-\gamma h\right), h \geq 0,$$

and a log-normal distribution, with $\theta = (\mu, \sigma^2)$ where μ is a position parameter and σ^2 is a dispersion parameter, with density

$$f(h|\theta) = \frac{1}{\sqrt{2\pi}\sigma h} \exp\left(-1/2\frac{(\log(h) - \mu)^2}{\sigma^2}\right), h \geq 0.$$

All the flaws on a vessel are not observed and the actual number of flaws n is to be estimated. For each flaw, we consider a binary random variable D indicating if the flaw has been measured. By convention, $D = 1$ if the flaw has been detected and $D = 0$ otherwise. The detection of a flaw depends on its size h and is defined with the probability of detection function $\mathrm{POD}(h)$ which, for each h, is the conditional probability

$$\mathrm{POD}(h) = P(D = 1|H = h).$$

The function POD is important and our modelling allows any step or continuous POD function. For instance, we consider the following POD functions:

- an exponential POD function (Ammirato and Dimitrijevic, 1993)

  $$\mathrm{POD}(h) = \begin{cases} A[1 - \exp(-b(h - s))] & \text{if } h > s \\ 0 & \text{otherwise} \end{cases}$$

  where A is a saturation level, that is, the maximum of the POD function, b is a non negative real number measuring the quality of the detection, and s is a threshold of detection;

- a logistic POD function (Temple, 1982)

  $$\mathrm{POD}(h) = 1 - \frac{1 + \exp(-qh*)}{1 + \exp(q(h - h*)}$$

  where q is a non negative real number measuring the quality of the detection, and $h*$ is a threshold of "good" detection (for large q, $h*$ is approximatively the median of the flaw size distribution);

- a normal POD function (Harris, 1977) defined as follows

$$\mathrm{POD}(h) = 1 - \Phi\left[q(\log h - \log h_m)\right] \tag{1}$$

  where q is again a non negative real number measuring the quality of the detection, h_m is approximatively the median of the flaw size distribution and Φ is the cumulative distribution function of the standard normal distribution with mean 0 and variance 1.

The probability that a flaw is detected regardless its size depends on the chosen flaw size density $f(.|\theta)$. It is given by the formula

$$p_f(\theta) = P(D = 1) = \int_0^{+\infty} \mathrm{POD}(h) f(h|\theta) dh. \tag{2}$$

The number of measured flaws on a vessel is the realization of a random variable N_0 with a binomial distribution $B(n, p_f(\theta))$

$$P(N_0 = n_0) = \begin{pmatrix} n \\ n_0 \end{pmatrix} (p_f(\theta))^{n_0} (1 - p_f(\theta))^{n-n_0}$$

whose parameters n (the actual number of flaws) and $p_f(\theta)$ are unknown.

Moreover, the statistical flaw distribution model is made more complicated because measured flaw sizes include a random error E with density $f_E(e)$ and the analysis of ISI data requires knowing the lower flaw size s below which the data are considered unreliable in size estimate. This threshold s is a parameter of the model which can be taken into account easily. It just leads to modify the POD function by truncating it below s. The random error E can be represented with a uniform distribution or a Gaussian distribution. In our methodology, the choice of the random size error distribution f_e does not include additional difficulty. For simplicity, we essentially pay attention, in the following, to a uniform size error distribution on the interval $[-c/2, c/2], c > 0$ being given by the user,

$$f_E(e) = \begin{cases} 1/c & \text{if } -c/2 \leq e \leq c/2 \\ 0 & \text{otherwise.} \end{cases}$$

In this framework, the problem is to estimate the flaw size distribution parameter θ and the actual number of flaws from the detected flaws and their sizes. Since there is in general a very small number of flaws and that often a priori knowledge on both the size and the number of flaws is available, Bayesian inference is desirable to solve this estimation problem. We present it now.

3 Bayesian Inference

The probabilistic model described in Section 2 involves many parameters. Let $\xi = (\theta, n)$ be the resulting vector parameter and let $\mathbf{h}$ denote the n_0 measured flaw sizes $\mathbf{h} = (h_1, \ldots, h_{n_0})$.

From the choice of a prior distribution $\pi(\xi)$ for ξ which is considered as a random vector, Bayesian inference consists in deriving parameter estimates from features of the posterior distribution $\pi(\xi|\mathbf{h})$ which, using Bayes theorem, is defined as

$$\pi(\xi|\mathbf{h}) = \frac{\pi(\xi)P(\mathbf{h}|\xi)}{\int \pi(\xi)P(\mathbf{h}|\xi)d\xi},$$

where $P(\mathbf{h}|\xi)$ denotes the likelihood of the parameter ξ for the data $\mathbf{h}$. Then, Bayesian estimates can be expressed in terms of posterior expectations of functions of ξ. The posterior expectation of a function $g(\xi)$ is

$$E(g(\xi)|\mathbf{h}) = \frac{\int g(\xi)\pi(\xi)P(\mathbf{h}|\xi)d\xi}{\int \pi(\xi)P(\mathbf{h}|\xi)d\xi}.$$

In this high-dimensional setting calculating such posterior expectations is a difficult task. In fact, our model is enough complicated (see the directed acyclic graph (DAG) summarizing the complete Bayesian representation of the model in Fig. 1) to prevent the calculation of the integral $\int \pi(\xi)P(\mathbf{h}|\xi)d\xi$ for any choice of prior distribution $\pi(\xi)$. Thus we know that $\pi(\xi|\mathbf{h}) \propto \pi(\xi)P(\mathbf{h}|\xi)$, but we cannot easily evaluate the normalizing constant $\int \pi(\xi)P(\mathbf{h}|\xi)d\xi$.

Markov Chain Monte Carlo (MCMC) methods are aimed to evaluate posterior expectations by Monte Carlo integration using Markov chains. They provide a unifying framework within which many complex problems can be analyzed using generic software. For evaluating the flaw distributions of PWR vessels, we essentially make use of the Gibbs sampling (see for instance Gilks *et al.*, 1996 or Casella and George, 1992) associated with the Hastings-Metropolis algorithm (see for instance Chib an Greenberg, 1995) when the flaw size distribution is a Weibull distribution. The Gibbs sampling consists in sampling from the full conditional distributions involved in the model to get a Markov chain whose stationary distribution is precisely the desired unconditional posterior distribution.

Before presenting detailed formula for the Bayesian analysis of flaw size distribution via Gibbs sampling, we want to sketch the main structure of the calculation. Equation (2) shows that the actual number of flaws n depends on θ since the mean of the random variable N_0 is $np_f(\theta)$. Thus, Bayesian inference on n and θ will consist in the two iterated steps:

- Step **a**. Computation of the conditional distribution of n knowing θ.
- Step **b**. Computation of the conditional distribution of θ knowing n.

We know detail how these two steps are performed. Note that each time it is possible, we make use of conjugate prior distributions (see Robert, 1994).

3.1 Bayesian modelling of the actual number of flaws

We assume that the observed number of flaws n_0 on a vessel is the realization of a random variable N_0 with a binomial distribution $\mathcal{B}(n, p_f(\theta))$. We assume that n is a random variable

with a Poisson distribution $\mathcal{P}(\lambda)$. Since it is difficult to guess a reasonable value for the parameter λ, we introduce a second level of prior distribution producing a hierarchical Bayes model (see for instance Robert, 1994). We consider that λ is a random variable following a Gamma distribution $\mathcal{G}(a,b)$ with the parameters a and b defined such that the mean of this distribution is a/b and its variance is a/b^2. The posterior distribution $\pi(n,\lambda|\theta,n_0)$ is approximated through Gibbs sampling which consists in repeating the following steps from an initial vector (n^0,p^0,λ^0):

- Sample λ from $\pi(\lambda|n,\theta,n_0) = \mathcal{G}(a+n,b+1)$.
- Sample n from $\pi(n|\lambda,\theta,n_0) = \mathcal{P}((1-p_f(\theta))\lambda) + n_0$, namely a Poisson distribution translated in n_0.

In this scheme, the quantity $p_f(\theta)$ given by (2) is approximated through Monte Carlo integration. It leads to

$$p_f(\theta) = \frac{1}{T}\sum_{i=1}^{T} \mathrm{POD}(y_i)$$

where the y_i's are random independent realizations sampled from $f(.|\theta)$ with the convention $\mathrm{POD}(y_i) = 0$ if $y_i < s$ for taking account of the detection threshold s. Typically, we take $T = 100$.

3.2 Bayesian modelling of the flaw size distribution

We consider a parameterized flaw size distribution $f(h|\theta)$. The choice of the particular form of this distribution is important. In the following, we will detail computation for the exponential distribution, the Weibull distribution and the log-normal distribution. Before that, there is the need to derive the likelihood $L(\mathbf{h}|\theta)$ since Bayesian inference is based on the posterior distribution $\pi(\theta)L(\mathbf{h}|\theta)$.

The likelihood of the observed data It is worth noting that we are faced with a missing data problem. Some flaws have not been detected (with the probability $1-\mathrm{POD}(h)$, h being the size of the flaw), and each measured flaw size h_i is littered with an additive random error measure e_i with density f_e: $h_i = h_i^* + e_i$, where h_i^* is the exact flaw size with distribution $f(h_i^*|\theta)$. Thus, the density $f_o(h)$ of the measured flaw size random variable H_o is obtained as the product of convolution of f and f_e, $f_o(h) = \int f(y|\theta)f_e(h-y)dy$. As a consequence, the conditional density of a measured flaw size is obtained as follows

$$\begin{aligned} P(H_o \in [h,h+dh]|D=1) &= \frac{P\left[(H_o \in [h,h+dh]) \cap (D=1)\right]}{P(D=1)} \\ &= \frac{P\left[(D=1)|(H_o \in [h,h+dh])\right] P\left(H_o \in [h,h+dh]\right)}{P(D=1)}. \end{aligned}$$

Thus the conditional density of a measured flaw size is

$$\begin{aligned}
\lim_{dh\to 0} \frac{P(H_o \in [h, h+dh] | D=1)}{dh} &= \lim_{dh\to 0} \frac{P\left[(D=1)|(H_o \in [h,h+dh])\right] P\left(H_o \in [h,h+dh]\right)}{P(D=1)dh} \\
&= \lim_{dh\to 0} \frac{P\left[(D=1)|(H_o \in [h,h+dh])\right]}{P(D=1)} \times \\
&\quad \lim_{dh\to 0} \frac{P\left(H_o \in [h,h+dh]\right)}{dh} \\
&= \frac{\mathrm{POD}(h)}{p_f(\theta)} \int f(y|\theta) f_e(h-y) dy.
\end{aligned}$$

Finally the likelihood, for a sample of sizes $\mathbf{h} = h_1, \ldots, h_{n_0}$ of detected flaws, is

$$L(\mathbf{h}|\theta) = \frac{1}{(p_f(\theta))^{n_0}} \prod_{i=1}^{n_0} \mathrm{POD}(h_i) \int f(y|\theta) f_e(h_i - y) dy. \tag{3}$$

For instance, in the case where the error distribution is the uniform distribution on $[-c/2, c/2]$, the likelihood becomes

$$L(\mathbf{h}|\theta) = \frac{1}{(p_f(\theta))^{n_0}} \prod_{i=1}^{n_0} \frac{POD(h_i)}{c} \int_{h_i - c/2}^{h_i + c/2} f(y|\theta) dy.$$

In the extreme case where there is no measured flaw, the likelihood reduces to $1 - p_f(\theta)$.

Remark: The likelihood function (3) is generally difficult to maximize. (Imagine for instance a Weibull distribution for the flaw size distribution with a Gaussian error distribution and a logistic POD function.) For Bayesian inference, we will make use of a Data Augmentation version of the Gibbs sampling (see Tanner and Wong, 1987) which consists in simulating both the parameters and the missing data in an iterative procedure. And, we will not encounter difficulties when approximating the posterior distribution $\pi(\theta|\mathbf{h})$ since at each iteration, the missing data (the unmeasured flaws and the measurement errors) will be simulated according to their current conditional distributions. Thus, contrary to likelihood inference there is no need to restrict attention for numerical reasons to a particular POD function or error distribution.

Prior distributions for the possible flaw size distributions We develop Bayesian inference for the most currently used flaw size distributions, namely the exponential distribution, the Weibull distribution and the log-normal distribution.

For the exponential distribution with scale parameter $\theta = \gamma$, we consider a conjugate gamma distribution $\mathcal{G}(A, B)$.

For the Weibull distribution with shape parameter β and scale parameter γ, there is no conjugate prior distribution. Following Bacha *et al.* (1998), we consider that the prior

distribution on $\theta = (\beta, \gamma)$ is the product of a prior distribution for β and a prior distribution for γ. For the shape parameter, we consider a beta distribution with parameters r and t defined on the support $[\beta_l, \beta_r]$ where β_l (resp. β_r) is the minimum (resp. maximum) of possible values for the shape parameter β. The parameters r and t are such that the mean and variance of the distribution are respectively $\beta_l + r(\beta_r - \beta_l)/(r+t)$ and $rt(\beta_r - \beta_l)^2/[(r+t+1)(r+t)^2]$. We denote $Be(r, t, [\beta_l, \beta_r])$ this distribution. For the scale parameter prior distribution, we consider a Gamma distribution $\mathcal{G}(A, B)$.

For the log-normal distribution with position parameter m and dispersion parameter σ^2, we consider the following conjugate prior distributions. The prior distribution for $1/\sigma^2$ is supposed to be a Gamma distribution $\mathcal{G}(\nu/2, S/2)$ and the prior distribution of m knowing σ^2 is supposed to be a normal distribution with mean M and variance σ^2/R, R being a parameter to be tuned by the user.

The complete hierarchical model The directed acyclic graph (DAG) displayed in Fig. 1 describes the complete hierarchical Bayesian model assuming a Weibull flaw size distribution and a uniform error distribution. In this figure, square boxes represent fixed or observed quantities and circles the unknown. More precisely, the involved quantities are

- the observed data which are n_0 the number of detected flaws and the sample of flaw sizes $\mathbf{h} = (h_1, \ldots, h_{n_0})$;
- the missing data which are the actual number n of flaws, the undetected $(n - n_0)$ flaw sizes $\mathbf{z} = (z_1, \ldots, z_{n-n_0})$ and the error vector, associated to the measured flaws, $\mathbf{e} = (e_1, \ldots, e_{n_0})$;
- fixed constants inherent to some data description, namely the parameters defining the POD function (grouped under the name "POD" in the DAG), the notification threshold s and the parameters defining the error distribution (reduced to c for a uniform error);
- the parameters to be estimated in the probabilistic model, they are n and θ, with $\theta = (\beta, \gamma)$ in the Weibull case;
- the parameters used for defining the prior distributions, to be fixed by the user, in the Bayesian representation of the model; we will call them "hyperparameters" in the following. They are λ, a, b and the hyperparameters used to define the prior distribution of θ. (In the Weibull case, those hyperparameters are A, B, r, t, β_l and β_r.)

Each quantity in the model appears as a node in the graph, and directed links correspond to direct, probabilistic or deterministic, dependences. In the present graph, the most important link is represented by the arrow from n, the actual number of flaws, to $\mathbf{z}$ the sizes of the unmeasured flaws. This link is of crucial importance since it allows to convey inference on the number of flaws to the flaw sizes. We are now in position to detail the Gibbs sampling steps for the model depicted in Fig. 1.

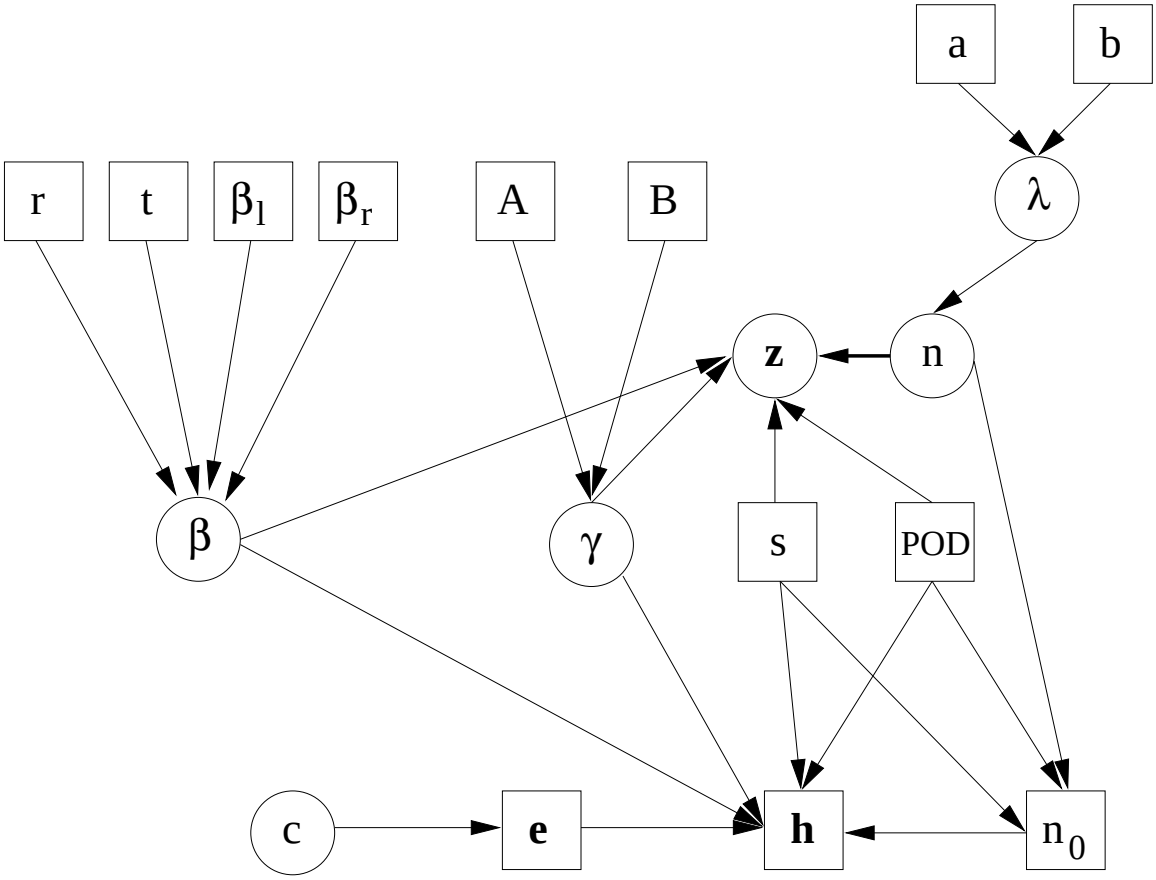

Figure 1: DAG of the flaw size model specific to the Weibull distribution.

3.3 The Gibbs sampling implementation

We describe the $(j+1)$ iteration of the Gibbs sampling (j denotes the iteration index). Step **a** concerning the simulation according to the full conditional posterior distribution of n is decomposed as follows:

- Sample λ^{j+1} from $\pi(\lambda|n^j, \theta^j, \mathbf{z}^j, \mathbf{e}^j, \mathbf{h}, n_0) = \mathcal{G}(a + n^j, b + 1)$.
- Sample n^{j+1} from $\pi(n|\lambda^{j+1}, \theta^j, \mathbf{z}^j, \mathbf{e}^j, \mathbf{h}, n_0) = \mathcal{P}\left((1-p^j)\lambda^{j+1}\right) + n_0$. It is achieved in the following way: $\tilde{n}$ is sampled from $\mathcal{P}\left((1-p^j)\lambda^{j+1}\right)$ and $n^{j+1} = \tilde{n} + n_0$. Here, according to equation (2), p^j is approximated through Monte Carlo integration

$$p^j = \frac{1}{100}\sum_{i=1}^{100} \mathrm{POD}(y_i^j)$$

  where $y_1^j, \ldots, y_{100}^j$ are independent realizations from the current flaw size distribution $f(.|\theta^j)$.

The implementation of Step **b**, concerning the simulation of the full conditional posterior distribution of the flaw size parameter θ, is as follows

- Simulate $n^{j+1} - n_0$ "unmeasured" flaw sizes from the current flaw size distribution $f(h|\theta^j)$. Those simulations are performed as follows. Let $\tilde{z}$ be a random drawing from $f(h|\theta^j)$. If $\tilde{z} < s$ accept this value, otherwise accept it with probability $1 - \mathrm{POD}(\tilde{z})$. This process is continued until $n^{j+1} - n_0$ simulated flaw sizes are accepted. Let $\mathbf{z}^{j+1}$ be the resulting simulated flaw size vector.
- Simulate $\mathbf{e}^{j+1} = (e_1^{j+1}, \ldots, e_{n_0}^{j+1})$, a vector of n_0 independent errors from the distribution $f_e(e)$.
- The sampling of θ^{j+1} depends on the flaw size distribution at hand. It is detailed for the three considered distributions hereunder.

  Exponential distribution: γ^{j+1} is simulated from a $\mathcal{G}(A + n^{j+1}, B + \sum_{i=1}^{n_0}(h_i + e_i^{j+1}) + \sum_{i=1}^{n^{j+1}-n_0} z_i^{j+1})$.

  Weibull distribution: Since there is no conjugate prior distribution for the parameter $\theta = (\beta, \eta)$ of a Weibull distribution, the simulation of θ^{j+1} from its full conditional distribution is more complicated. We make use of an Hasting-Metropolis algorithm to achieve this simulation. More precisely, we first sample a candidate vector $\tilde{\theta} = (\tilde{\beta}, \tilde{\gamma})$ from the proposal distribution $\pi(\beta)\pi(\gamma)$ and we accept it with probability

$$\min(1, \frac{\pi(\tilde{\beta})\pi(\tilde{\gamma})LC(\mathbf{h}, \mathbf{e}^{j+1}, \mathbf{z}^{j+1}|\tilde{\theta})}{\pi(\beta^j)\pi(\gamma^j)LC(\mathbf{h}, \mathbf{e}^{j+1}, \mathbf{z}^{j+1}|\theta^j)})$$

where,

$$LC(\mathbf{h}, \mathbf{e}^{j+1}, \mathbf{z}^{j+1}|\theta) = \prod_{i=1}^{n_0} \gamma\beta(\gamma(h_i + e_i^{j+1}))^{\beta-1} \exp[-(\gamma(h_i + e_i^{j+1}))^{\beta}] \prod_{i=1}^{n^{j+1}-n_0} \gamma\beta(\gamma(z_i^{j+1})^{\beta-1} \exp[-(\gamma(z_i^{j+1}))^{\beta}]$$

is the likelihood of θ for the completed flaw size sample. If the candidate vector $\tilde{\theta}$ is accepted, the next state becomes $\theta^{j+1} = \tilde{\theta}$, if the candidate is rejected, the state remains the same, i.e. $\theta^{j+1} = \theta^j$.

Log-normal distribution: $1/(\sigma^2)^{j+1}$ is simulated from a $\mathcal{G}(\nu^{j+1}/2, S^{j+1}/2)$ and m^{j+1} is simulated from a normal distribution with mean M^{j+1} and variance $(\sigma^2)^{j+1}/R^{j+1}$, with

$$\nu^{j+1} = \nu + 1 + n^{j+1}$$

$$R^{j+1} = R + n^{j+1}$$

$$M^{j+1} = \frac{RM + n^{j+1}\bar{h}}{R^{j+1}}$$

$$S^{j+1} = S + V + \left(\frac{1}{R} + \frac{1}{n^{j+1}}\right)^{-1} (M - \bar{h})^2$$

$$\bar{h} = \frac{\sum_{i=1}^{n_0} \log(h_i + e_i^{j+1}) + \sum_{i=1}^{n^{j+1}-n_0} \log(z_i^{j+1})}{n^{j+1}}$$

and

$$V = \frac{\sum_{i=1}^{n_0} [\log(h_i + e_i^{j+1}) - \bar{h}]^2 + \sum_{i=1}^{n^{j+1}-n_0} [\log(z_i^{j+1}) - \bar{h}]^2}{n^{j+1}}.$$

Finally the sequence of iterates $(n^j, p^j, \theta^j), j = j_0, \ldots, J$ can be regarded as a sample of the posterior distribution $\pi(n, p, \theta|\mathbf{h}, n_0)$. The integer $j_0 > 1$ defines the length of the burn-in. It must be chosen large enough to ensure that the chain has forgotten its starting position. The theoretical specification of this burn-in length is a difficult problem (see for instance Gilks *et al.*, 1996). Visual inspections of plots is the most commonly used method for determining burn-in. We used such visual inspection in our experiments, and it seems that $j_0 = 100$ is a reasonable length value for our problem. But, in some circumstances, this value can appear to be dramatically too small and visual inspection is always necessary to choose a feasible length of the burn-in j_0 ensuring that the chain has reached stationarity. The stopping time J is an important practical parameter. It must be chosen large enough to obtain adequate precision in the estimates derived from the MCMC approximation of the posterior distribution $\pi(n, \theta|\mathbf{h}, n_0)$. For our particular problem, this point will be discussed further in Section 5 devoted to the presentation of a numerical experiment.

4 Choosing the prior parameters

In this section, we give simple guidelines for choosing the hyperparameters of the prior distributions.

4.1 Choosing the flaw number hyperparameters

For the actual number n of flaws, we considered a hierarchical Bayes analysis: the distribution of n is a Poisson distribution $\mathcal{P}(\lambda)$, and the distribution of the random parameter λ is a gamma distribution $\mathcal{G}(a,b)$ with mean value a/b and variance a/b^2. Thus the hyperparameters a and b can be chosen from a guessed value of the mean number $\bar{n}$ of flaws regardless their sizes. As we will see in the following, $\bar{n}$ appears to be a sensitive parameter of the Bayesian analysis. Note that the Jeffreys non informative prior for λ (see Robert 1994) is proportional to $\lambda^{-1/2}$ and, thus, the non informative strategy leads to take $a = 1/2$ and $b = 0$.

4.2 Choosing the flaw size hyperparameters

For choosing the hyperparameters of the prior distribution concerning the flaw size we make use of prior information on the so-called "reference flaws". This information is defined with equations of the following type

$$P(\text{there exits on a vessel a flaw of size} > h_0) = 10^{-d}. \tag{4}$$

Denoting N and H the random variables associated respectively with the actual number of flaws n and the size of a flaw, we have

$$\begin{aligned} P(\text{there exits on a vessel a flaw of size} > h_0) &= \sum_{n=1}^{+\infty} P(N=n)(1-P(H \leq h_0)^n) \\ &= \sum_{i=1}^{+\infty} P(N=n)(1-(1-P(H > h_0))^n) \\ &\approx \sum_{n=1}^{+\infty} P(N=n) n P(H > h_0), \end{aligned}$$

for $P(H > h_0)$ small enough compared to one. Finally we have

$$P(\text{there exits on a vessel a flaw of size} > h_0) = P(H > h_0)\bar{n}. \tag{5}$$

This equation highlights the importance of the quantity $\bar{n}$ for choosing the flaw size hyperparameter prior values in the Bayesian analysis.

We now examine how the equation (5) can be exploited to derive the hyperparameters of the prior distribution defined for θ. The idea is to define intervals, derived from extreme

values for θ, producing a reasonable range of values for this parameter from several equations of the form (4). This is achieved as described hereafter for the three considered flaw size distributions.

Exponential distribution: We need two equations (4) to derive the bounds γ_l and γ_r of an interval of possible values for the scale parameter γ. From (5), they take the form

$$\bar{n}\exp(-h\gamma_r) = 10^{-d}$$

and

$$\bar{n}\exp(-h'\gamma_l) = 10^{-d'}$$

with $h < h'$ and $d < d'$. It leads to

$$\gamma_r = \frac{1}{h}[\log\bar{n} + d\log 10]$$

and

$$\gamma_l = \frac{1}{h'}[\log\bar{n} + d'\log 10].$$

Approximating the prior distribution $\mathcal{G}(A, B)$ with a normal distribution and assuming that $[\gamma_l, \gamma_r]$ is a centered ± 2 standard deviation interval, we get

$$A = 4\frac{(\gamma_l + \gamma_r)^2}{(\gamma_l - \gamma_r)^2}$$

and

$$B = 8\frac{(\gamma_l + \gamma_r)}{(\gamma_l - \gamma_r)^2}.$$

Weibull distribution: Since in this situation, the parameter θ is bivariate, we need four equations (4) to derive the extreme values of θ. They are of the form

$$\bar{n}\exp[-(h_{11}\gamma_1)^{\beta_1}] = 10^{-d_{11}},$$

$$\bar{n}\exp[-(h_{12}\gamma_1)^{\beta_1}] = 10^{-d_{12}},$$

$$\bar{n}\exp[-(h_{21}\gamma_2)^{\beta_2}] = 10^{-d_{21}},$$

and

$$\bar{n}\exp[-(h_{22}\gamma_2)^{\beta_2}] = 10^{-d_{22}}, \tag{6}$$

with $h_{11} < h_{12}$, $h_{21} < h_{22}$, $d_{11} < d_{12}$, and $d_{21} < d_{22}$. This four-equation system leads to

$$\beta_1 = \frac{D_{11} - D_{12}}{\log(h_{11}) - \log(h_{12})},$$

and

$$\beta_2 = \frac{D_{21} - D_{22}}{\log(h_{21}) - \log(h_{22})},$$

$$\gamma_1 = \frac{1}{h_{11}} \exp\left(\frac{D_{11}}{\beta_1}\right)$$

$$\gamma_2 = \frac{1}{h_{21}} \exp\left(\frac{D_{21}}{\beta_2}\right)$$

where for $i = 1, 2$ and $j = 1, 2$

$$D_{ij} = \log\left[\log(\bar{n}) + d_{ij}\log(10)\right].$$

Then, we put $\gamma_l = \min(\gamma_1, \gamma_2)$, $\gamma_r = \max(\gamma_1, \gamma_2)$, $\beta_l = \min(\beta_1, \beta_2)$ and $\beta_r = \max(\beta_1, \beta_2)$. The coherence of the equations of the form (4) provided by the experts ensures that the above written solutions are well defined.

Log-normal distribution: As in the Weibull case, we need four equations of type (4), to get the values m_l, m_r, σ_l^2 and σ_r^2. In this situation, we have

$$P(H > h) = 1 - \Phi\left(\frac{\log(h) - m}{\sigma}\right),$$

where Φ is the cumulative distribution function of a normal distribution with mean zero and variance one. The four equations (5) take the form

$$\bar{n}\left[1 - \Phi\left(\frac{\log(h_{11}) - m_l}{\sigma_l}\right)\right] = 10^{-d_{11}},$$

$$\bar{n}\left[1 - \Phi\left(\frac{\log(h_{12}) - m_l}{\sigma_l}\right)\right] = 10^{-d_{12}},$$

$$\bar{n}\left[1 - \Phi\left(\frac{\log(h_{21}) - m_r}{\sigma_r}\right)\right] = 10^{-d_{21}},$$

and

$$\bar{n}\left[1 - \Phi\left(\frac{\log(h_{22}) - m_r}{\sigma_r}\right)\right] = 10^{-d_{22}},$$

with $h_{11} < h_{12}$, $h_{21} < h_{22}$, $d_{11} < d_{12}$, and $d_{21} < d_{22}$. Those equations lead to

$$m_l = \frac{\log h_{12}\Phi^{-1}(1 - 10^{-d_{11}}/\bar{n}) - \log h_{11}\Phi^{-1}(1 - 10^{-d_{12}}/\bar{n})}{\Phi^{-1}(1 - 10^{-d_{11}}/\bar{n}) - \Phi^{-1}(1 - 10^{-d_{12}}/\bar{n})}$$

$$m_r = \frac{\log h_{22}\Phi^{-1}(1 - 10^{-d_{21}}/\bar{n}) - \log h_{21}\Phi^{-1}(1 - 10^{-d_{22}}/\bar{n})}{\Phi^{-1}(1 - 10^{-d_{21}}/\bar{n}) - \Phi^{-1}(1 - 10^{-d_{22}}/\bar{n})}$$

$$\sigma_r = \frac{\log(h_{11}) - \log(h_{12})}{\Phi^{-1}(1 - 10^{-d_{11}}/\bar{n}) - \Phi^{-1}(1 - 10^{-d_{12}}/\bar{n})}$$

and

$$\sigma_l = \frac{\log(h_{21}) - \log(h_{22})}{\Phi^{-1}(1 - 10^{-d_{21}}/\bar{n}) - \Phi^{-1}(1 - 10^{-d_{22}}/\bar{n})}.$$

5 Numerical experiments

To illustrate the behavior of our methodology, we draw a sample of size $n = 10$ from a Weibull distribution with shape parameter $\beta = 2$ and scale parameter $\gamma = 0.5$. Its theoretical mean value is 1.77. The observations are littered with a uniform random error of amplitude $c = 1$. The resulting data set is $\{1.0, 1.9, 1.1, 4.4, 3.2, 1.8, 1.7, 1.5, 2.5, 2.4\}$. Note that the maximum likelihood estimates from those noisy data are $\hat{\beta} = 2.4$ and $\hat{\gamma} = 0.41$. Then the data are filtered with a normal POD function defined by equation (1) whose parameters q and h_m have been computed by solving the equations $\mathrm{POD}(2) = 0, \mathrm{POD}(4) = 0.5, \mathrm{POD}(6) = 0.9, \mathrm{POD}(9) = 1$, with the mean square error method. Using a detection threshold $s = 4$ mm, we get a unique ($n_0 = 1$) flaw size $h_1 = 4.4$. Using a detection threshold $s = 2$ mm we get three ($n_0 = 3$) flaw sizes $h_1 = 4.4$, $h_2 = 3.2$ and $h_3 = 2.4$.

For the inference, we use the same random error distribution e and the same normal POD function as we considered when simulating the data. This choice is made for simplicity. Actually, extensive numerical experiments that are not reported here showed that those parameters are not very sensitive (see Celeux *et al.* 1997). On the contrary, the assumption on the expected number $\bar{n}$ of actual flaws is sensitive. In this experiment, we consider three different values $\bar{n} = 50, 10$, and 4.

The prior distribution of the Poisson parameter λ is a Gamma distribution $\mathcal{G}(a = \bar{n}, b = 1)$. The prior distributions for the shape parameter β and the scale parameter γ of the Weibull distribution have been defined according to the calculations described in Section 4.2. The bound β_l, β_r, γ_l and γ_r have been deduced from the four-equation system (6) with $h_{11} = 2$, $h_{12} = 3$, $d_{11} = 1$, $d_{12} = 7$, and $h_{21} = 3$, $h_{22} = 4$, $d_{21} = 2$, $d_{22} = 3$. Moreover, we set $r = t = 1$ for the parameters of the beta distribution $Be(r, t, [\beta_l, \beta_r])$, which becomes the uniform distribution on $[\beta_l, \beta_r]$.

The Gibbs sampler described in Section 3.3 is then used with $J = 10000$ iterations a burn-in period of $j_0 = 100$ iterations. To control that the stopping time J is large enough to ensure that posterior distribution $\pi(n, \theta|\mathbf{h}, n_0)$ is estimated in a satisfactory way, we use the following heuristic rule:

- At each iteration j, we compute the mean value $\bar{h}^j$ of the simulated and measured flaw sizes and we compute

$$\bar{h} = \frac{1}{J - j_0} \sum_{j=j_0+1}^{j_0+J} \bar{h}^j.$$

- We compare this empirical mean value $\bar{h}$ (denoted EMV in Tables 1 and 2) of the flaw sizes with the theoretical mean value (denoted TMV in Tables 1 and 2) of the Weibull distribution with parameters

$$\bar{\beta} = \frac{1}{J - j_0} \sum_{j=j_0+1}^{j_0+J} \beta^j,$$

and

$$\bar{\gamma} = \frac{1}{J - j_0} \sum_{j=j_0+1}^{j_0+J} \gamma^j.$$

When J is large enough to ensure a reasonable approximation of the posterior distribution $\pi(n, \theta | \mathbf{h}, n_0)$, the values EMV and TMV are expected to be close. Thus, if EMV and TMV are notably different, it can be guessed that J is too small to ensure the Gibbs sampler convergence for the chosen prior distributions.

Typically, such a situation occurs when the Gamma prior distributions $\mathcal{G}(A, B)$ and $\mathcal{G}(a, b))$ of γ and λ are too tight around their mode. Thus, when EMV and EMT are different we multiply A and B by M ($M < 1$) and a and b by m ($m < 1$). Consequently, the means of the prior Gamma distributions are unchanged, but their variance are multiplied by $1/M$ and $1/m$ respectively. (This is the reason why we take $M, m < 1$.) The values M and m are chosen by trial and error to get close EMV and TMV values. This modification of the prior distributions results in a more mixing Markov chain and produces an acceleration of the Gibbs sampler convergence, and consequently allows to take a larger J value.

Table 1, respectively Table 2, displays the posterior means of the actual number of flaws $est(n)$, the shape parameter $\bar{\beta}$ and the scale parameter $\bar{\gamma}$ of the Weibull distribution for three different prior values of $\bar{n} = 50, 10$, and 4 and a 4 mm detection threshold (one measured flaw), respectively a 2 mm detection threshold (three measured flaws). Those tables display also the corresponding EMV and TMV values and the variance dilatation coefficients M and m.

$\bar{n}$	$est(n)$	$\bar{\beta}$	$\bar{\gamma}$	EMV	TMV	M	m
50	45	2.4	0.46	2.03	1.93	0.1	0.1
10	10	2.0	0.43	2.24	2.06	0.1	0.1
4	4	2.5	0.36	2.99	2.46	0.1	0.1

Table 1: For a detection threshold $s = 4$ mm and different prior assumptions $\bar{n}$ on the actual number of flaws, posterior means of the actual number of flaws $est(n)$, the shape parameter $\bar{\beta}$ and the scale parameter $\bar{\gamma}$ of the Weibull distribution. EMV : empirical mean value of the flaw size; TMV : theoretical mean value of the flaw size; M and m are the dilatation coefficients chosen for the prior Gamma distributions.

$\bar{n}$	$est(n)$	$\bar{\beta}$	$\bar{\gamma}$	EMV	TMV	M	m
50	44	2.1	0.49	1.87	1.81	0.1	0.1
10	11	2.4	0.40	2.42	2.22	0.1	0.1
4	6	2.5	0.38	2.77	2.34	0.1	0.1

Table 2: For a detection threshold $s = 2$ mm and different prior assumptions $\bar{n}$ on the actual number of flaws, posterior means of the actual number of flaws $est(n)$, the shape parameter $\bar{\beta}$ and the scale parameter $\bar{\gamma}$ of the Weibull distribution. EMV : empirical mean value of the flaw size; TMV : theoretical mean value of the flaw size; M and m are the dilatation coefficients chosen for the prior Gamma distributions.

Typically, the variance dilatation factors $M = 0.1$ and $m = 0.1$ produce good results in these numerical experiments. The estimates posterior values of n, β and γ are quite reasonable according to the small sample size in both situations. Note for instance that with a detection threshold $s = 2$ mm (three measured flaws) and the prior assumption $\bar{n} = 10$, we get Bayesian estimates of β and γ very close to the maximum likelihood estimates obtained from the complete flaw data (ten actual flaws). Not surprisingly, the estimates are slightly better when $s = 2$ mm. But the influence of the prior assumption concerning the actual number of flaws seems to be a more important factor. For instance, as expected, the mean flaw sizes EMV and TMV are decreasing functions of $\bar{n}$.

6 Discussion

In this paper, we proposed a fully Bayesian analysis of flaw size distribution and flaws density of PWR vessels. This Bayesian modelling allows various assumptions concerning the flaw size distributions, the flaw sizing error distribution and the probability of detection of a flaw. The resulting model is estimated with a Gibbs sampler avoiding technical difficulties. Moreover, numerical experiments exhibit good performances even for poor sample sizes. The price to be paid is twofold:

- The Gibbs sampler is high time consuming.
- The tuning of some of the prior hyperparameters must be performed by trial and error to get reliable estimates. But, we have conceived an efficient rule of thumb to achieve this tuning operation.

In our experiments, we revealed a sensitive prior parameter to be fixed: the number $\bar{n}$ of actual flaws guessed by the experts. In our opinion, it is possible to go further by taking into account the "indications" beyond the detection threshold s. Indeed, the indications are either small flaws or artifacts. It is natural to assume that the artifacts are the realizations of a Poisson process with mean value μs. Thus, the extension to a model handling indications

does not seem to produce major additional difficulties. And designing a Gibbs sampler to identify such a model will be possible without tears.

In this framework, the sensitive parameter of the model will be the mean measure μ of the Poisson process rather than the mean number $\bar{n}$ of actual flaws. Choosing the prior distribution of μ will be facilitated by the fact that we know that μs is smaller than the number of observed indications. And, clearly, a model taking account of this new information (the number of indications smaller than s) will be a more accurate model than the present one. Our future work will consist in designing such a model handling the indications in a fully Bayesian perspective.

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