Exponential Stabilization of Nonlinear Driftless Systems with Robustness to Unmodeled Dynamics

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Abstract: Exponential stabilization of nonlinear driftless affine control systems is addressed with the concern of achieving robustness with respect to imperfect knowledge of the system's control vector fields. In order to satisfy this robustness requirement, and inspired by [1] where the same issue was first addressed, we consider a control strategy which consists of applying periodically updated open-loop controls that are continuous with respect to state initial conditions. These controllers are more precisely described as continuous time-periodic feedbacks associated with a specific dynamic extension of the original system. Sufficient conditions which, if they are satisfied by the control law, ensure that the control is a robust exponential stabilizer for the extended system are given. Explicit and simple control expressions which satisfy these conditions in the case of n-dimensional chained systems are proposed. A constructive algorithm for the design of such control laws, which applies to any (sufficiently regular) driftless control system, is described.

Key-words: nonlinear system, asymptotic stabilization, robust control, Chen-Fliess series

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Stabilisation exponentielle de systèmes sans dérive avec robustesse par rapport aux dynamiques non modélisées

Résumé : Pour les systèmes de commande sans dérive, le problème de la stabilisation exponentielle est considéré avec pour principal objectif l'obtention de commandes robustes par rapport aux erreurs de modélisation sur les champs de commande. Motivés par [1], où ce problème de robustesse a déjà été étudié, nous considérons des lois de commande en boucle ouverte, réinitialisées périodiquement, et dépendant continument de l'état initial. Ces lois de commande peuvent aussi être vues comme des retours d'état instationnaires continus définis à partir d'une extension dynamique particulière du système de départ. Nous énonçons des conditions suffisantes portant sur ces retours d'état pour qu'ils stabilisent de façon robuste le système étendu. Nous proposons ensuite des lois de commande explicites et simples qui satisfont ces conditions pour un système chaîné de dimension quelconque. Enfin, nous décrivons un algorithme de synthèse de lois de commande robustes, applicable à tout système régulier localement commandable.

Mots-clés : système non linéaire, stabilisation asymptotique, commande robuste, série de Chen-Fliess

1 Introduction

We consider an analytic driftless system on $\mathbb{R}^n$

$$(S_0) \ : \qquad \dot{x} = \sum_{i=1}^{m} f_i(x) u_i \ , \tag{1}$$

locally controllable around the origin, i.e.

$$\text{Span}\{f(0) : f \in \text{Lie}(f_1, \dots, f_m)\} = \mathbb{R}^n \ , \tag{2}$$

and address the problem of constructing explicit feedback laws which (locally) exponentially stabilize, in some sense specified later, the origin $x = 0$ of the controlled system. A further requirement is that these feedbacks should also be exponential stabilizers for any "perturbed" system in the form

$$(S_\epsilon) \ : \qquad \dot{x} = \sum_{i=1}^{m} (f_i(x) + h_i(\epsilon, x)) u_i \ , \tag{3}$$

with h_i analytic in $\mathbb{R} \times \mathbb{R}^n$ and $h_i(0, x) = 0$, when $|\epsilon|$ is small enough. In other words, given a *nominal* control system (S_0), we would like to find *nominal* feedback controls, derived on the basis of this nominal system, that preserve the property of exponential stability when they are applied to "neighboring" systems (S_ϵ). It is of course assumed that the tangent linear approximation of (S_0) at $(x = 0, u = 0)$ is not stabilizable, since otherwise the problem is simply solved by using an adequate linear feedback $u(x) = Kx$. This assumption implies in particular that the rank of the matrix formed by the column-vectors $f_i(0)$, $1 \leq i \leq m$, is smaller than n.

In this context, the term $\sum_{i=1}^{m} h_i(\epsilon, x) u_i$ represents a class of unmodeled dynamics with respect to which the stabilizing nominal feedback must be robust. As pointed out in [16], such unmodeled dynamics may for example arise in practice, when dealing with nonholonomic wheeled vehicles, because of uncertainties upon the geometrical features of the vehicle. The present study is in fact essentially motivated by this *robustness* requirement. Indeed, explicit "homogeneous" exponential (time-varying) stabilizers $u(x, t)$ for systems (S_0) have been derived in various previous studies (see [13, 15], for example). However, as demonstrated recently in [12], none of these controls solves the robustness problem stated above in the sense that there always exists some $h_i(\epsilon, .)$ for which the origin of the associated controlled system is not stable when $\epsilon \neq 0$. Note that this negative result does not contradict the fact that such controllers are robust against less general unmodeled dynamics, as this is ensured by the existence of a Lyapunov function for the controlled nominal system. It just emphasizes the fact that, contrary to the case of linear control systems stabilized via the use of linear feedbacks, the existence of a Lyapunov function for the controlled system is not sufficient to ensure the type of robustness considered here. This negative result also strongly suggests (although this remains to be rigorously established) that no continuous feedback $u(x, t)$, not necessarily homogeneous, can be a robust exponential stabilizer. However, it does not imply that the problem cannot be handled via an adequate dynamic extension of the original nominal system. Proving the existence, or non-existence, of solutions of this type could thus be a subject for future studies. As a matter of fact, and as explained below, the present study may already be seen as a step in this direction.

Feedbacks which are continuous with respect to the state do not represent the only "reasonable" possibility in order to achieve the desired robustness result. For instance, besides feedbacks which are discontinuous at the origin, as proposed by several authors in the past (see [3, 8], for example) and for which robustness issues have seldom been addressed so far, a possibility consists in considering *hybrid* open-loop/feedback control strategies such as open-loop controls which are periodically updated from the measurement $x(kT)$, $k \in \mathbb{N}$, of the state at discrete time-instants. Such a control has features shared by classical piecewise-constant discrete-time feedbacks, but unlike these (and for the purpose of asymptotic stabilization) the control value between two sampling time-instants cannot, in the present case, be constant because this would contradict the known non-existence (resulting from the violation of Brockett's condition [2]) of asymptotically stabilizing continuous pure-state feedbacks $u(x)$. The idea of using this type of control for the purpose of stabilizing the class of driftless systems considered here is not new. This possibility has sometimes been presented as an extension of solutions obtained when addressing the open-loop steering problem, i.e. the problem of finding an open-loop control which steers the system from an initial state to another desired one (see [14, 17], for example). Hybrid continuous/discrete time exponential stabilizers for chained systems, which do not specifically rely on open-loop steering control, have also been proposed in [19]. However, [1] is to our knowledge the only study where the robustness problem stated above has been formulated in a similar fashion and where it has been shown that this problem can be solved by using a hybrid open-loop/feedback control. More precisely, the above reference i) proposes a methodology for constructing a robust control solution in the specific case (of practical interest) when the nominal system (S_0) is a 2-input n-dimensional *chained system*, ii) describes sufficient conditions for the n-dimensional case which, if they are met by the control expression, ensures robust stabilization, and iii) provides an explicit control expression in the 4-dimensional case (the dimension 4 having been chosen merely to show that the approach remains tractable for dimensions larger than 3). In fact, although this is not specified in the above reference, the proposed control does not "strictly" ensure asymptotic stability, in the usual sense of Lyapunov, of the origin of the perturbed systems (S_ϵ).

In order to be more specific about the latter point, and also clarify the meaning of "periodically updated open-loop control applied to a time-continuous system $\dot{x} = f(x,u)$", it is useful to introduce the following *extended* control system:

$$\begin{cases} \dot{x} = f(x,u) \\ \dot{y} = (\sum_{k\in\mathbb{N}} \delta_{kT})(x - y_{-\alpha}) \qquad 0 < \alpha < T\,, \end{cases} \tag{4}$$

with T denoting the updating time-period of the control part which depends upon y, δ_{kT} the classical dirac impulse at the time-instant kT, and $y_{-\alpha}$ the delay operator such that $y_{-\alpha}(t) = y(t-\alpha)$. Given a continuous feedback control $u(x,y,t)$, an initial condition $(x(t_0), y(t_0))$ to the controlled extended system is defined by i) choosing a point $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^n$, and ii) setting $x(t_0) = x_0$, $y(t_0) = y_0$ if t_0 is not a multiple of T, and $y(t_0) = x_0$ if $t_0 = kT$. The introduction of the extra equation in y just indicates that $y(t)$ is constant and equal to $x(kT)$ on the time-interval $[kT,(k+1)T)$. Therefore, any control the expression of which, on the time-interval $[kT,(k+1)T)$, is a function of $x(kT)$ and t, may just be interpreted as a feedback control $u(y,t)$ for the corresponding extended system. From now on, we will adopt this point of view whenever referring to this type of control. As commonly done elsewhere, we will also say that a feedback control $u(x,y,t)$ is a (uniform) *exponential stabilizer* for the extended system (4) if there exist an open set $U \in \mathbb{R}^n \times \mathbb{R}^n$ containing the point $(0,0)$, a positive real number γ, and a function β of class $\mathcal{K}$ such that:

$$|(x(t),y(t))| \leq \beta(x(t_0),y(t_0))exp(-\gamma(t-t_0)) \quad \forall t \geq t_0 \geq 0;\ \forall (x(t_0),y(t_0)) \in U \tag{5}$$

with $(x(t),y(t))$ denoting any solution of the controlled system. Note that the satisfaction of (5) does not imply that the control is an exponential stabilizer for the original system (while the converse is true). It is thus a slightly weaker property. However, except for asking finite-time convergence to zero, it is nearly the best that can be obtained when using a control which depends continuously upon y, knowing that such a control cannot by construction asymptotically stabilize the original system. For instance $x(t)$ may well cross zero at a time-instant which is not a multiple of T, without stopping there. Note that a similar impossibility holds when the feedback control depends on the integral of the state x: asymptotic stability can only be established for an extended state which contains the integral of x. This has not prevented linear PID controllers from being popular and widely employed in practice.

Let us now come back to the control strategy studied in [1] and interpret it in view of the above definitions. It yields continuous *time-periodic* feedbacks $u(y,t)$ (i.e. such that $u(y,t+T') = u(y,t)$ for some $T' > 0$), the time-periods of which are equal to the updating period T (i.e. $T' = T$). In our opinion, the importance of the contribution in [1] comes from that it convincingly demonstrates the possibility of achieving *robust* (with respect to unmodeled dynamics, as defined earlier) *exponential stabilization* (stability being now taken in the *strict* sense of Lyapunov) of an extended control system

$$(\bar{S}_0) \ : \quad \begin{cases} \dot{x} = \sum_{i=1}^{m} f_i(x) u_i \\ \dot{y} = (\sum_{k \in \mathbb{N}} \delta_{kT})(x - y_{-\alpha}) \qquad 0 < \alpha < T \end{cases} \tag{6}$$

via the use of a continuous time-periodic feedback $u(y, t)$.

In the present paper, the exploration of this possibility is carried further on. The first result provides a sufficient condition under which a time-periodic continuous feedback controller $u(y, t)$ i) exponentially stabilizes the origin of a system (6), and also ii) exponentially stabilizes the origin of any neighboring system

$$(\bar{S}_\epsilon) \ : \quad \begin{cases} \dot{x} = \sum_{i=1}^{m} (f_i(x) + h_i(\epsilon, x)) u_i \\ \dot{y} = (\sum_{k \in \mathbb{N}} \delta_{kT})(x - y_{-\alpha}) \qquad 0 < \alpha < T \end{cases} \tag{7}$$

provided that $|\epsilon|$ is small enough. Then, on the basis of this result, we propose a systematic and complete control design procedure which only involves a finite number of algebraic operations. Following this design procedure thus yields entirely explicit feedback laws. The procedure is itself adapted from existing time-periodic open-loop control design techniques which have been proposed in [22, 11] (see also [10] for an early but complete survey of such techniques) for driftless control systems affine in the control. Although the implementation of the resulting algorithm is somewhat involved in the general case, we show that simple control expressions can be obtained for specific classes of systems, as illustrated in the case where the original system (S_0) is in the chained form. Also, with respect to the solution given in [1] for the 4-dimensional chained system, the single control expression proposed here encompasses all dimensions with no extra work needed.

The paper is organized as follows. The main robustness result is presented in Section 2. This result is used in Section 3 for the design of robust stabilizers. A few final remarks are given in Section 4.

The following notation is used.

The identity function on $\mathbb{R}^n$ is denoted *id*, $|.|$ is the Euclidean norm, and $|.|_P$ is the norm induced by a positive definite matrix P.

The transpose of a row-vector $(x_1, \ldots, x_n)$ is denoted as $(x_1, \ldots, x_n)'$.

For any vector field X and smooth function $f$on $\mathbb{R}^n$, Xf denotes the Lie derivative of f along the vector field X. When $f = (f_1, \ldots, f_n)'$ is a smooth map from $\mathbb{R}^n$ to itself, Xf denotes the map $(Xf_1, \ldots, Xf_n)'$.

A square matrix A is called *discrete-stable* if all its eigenvalues are strictly inside the complex unit circle.

Given a continuous functions g, defined on some neighborhood of the origin in $\mathbb{R}^n$, we denote $o(g)$ (resp. $O(g)$) any function or map such that $\frac{|o(g)(x)|}{|g(x)|} \longrightarrow 0$ as $|x| \longrightarrow 0$ (resp. such that $\frac{|O(g)(x)|}{|g(x)|} \leq K$ in some neighborhood of the origin). When $g = |.|$, we write $o(x)$ (resp. $O(x)$) instead of $o(g)(x)$ (resp. $O(g)(x)$).

2 Sufficient conditions for exponential and robust stabilization

Prior to stating the main result of this section, we review some properties of Chen-Fliess series that will be used in the sequel. The exposition is based on [6, 23], and limited here to driftless systems.

A *m-valued multi-index* I is a vector $I = (i_1, \ldots, i_k)$ with k denoting a strictly positive integer, and $i_1, \ldots, i_k$ integers taken in the set $\{1, \ldots, m\}$. We denote the length of I as $|I|$, i.e. $I = (i_1, \ldots, i_k) \Longrightarrow |I| = k$.

Given piecewise continuous functions $u_1, \ldots, u_m$ defined on some time-interval $[0, T]$, and a m-valued multi-index $I = (i_1, \ldots, i_k)$, we define

$$\int_0^t u_I = \int_0^t \int_0^{t_k} \cdots \int_0^{t_2} u_{i_k}(t_k) u_{i_{k-1}}(t_{k-1}) \cdots u_{i_1}(t_1)\, dt_1 \cdots dt_k \quad (t \in [0, T])\,. \tag{8}$$

Given smooth vector fields $f_1, \ldots, f_m$ on $\mathbb{R}^n$, and a m-valued multi-index $I = (i_1, \ldots, i_k)$, we define the k-th order differential operator $f_I : \mathcal{C}^\infty(\mathbb{R}^n; \mathbb{R}) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n; \mathbb{R})$ by

$$f_I\, g = f_{i_1} f_{i_2} \cdots f_{i_k}\, g\,. \tag{9}$$

The following proposition is a classical result (see e.g.[23] for the proof).

Proposition 1 *[23] Consider the analytic system (S_0) and a compact set $K \subset \mathbb{R}^n$. There exists $\mu > 0$ such that for $M, T \geq 0$ verifying*

$$MT \leq \mu\,, \tag{10}$$

and for any control $u = (u_1, \ldots, u_m)$ piecewise continuous on $[0, T]$ and verifying

$$|u(t)| \leq M\,, \quad \forall t \in [0, T]\,, \tag{11}$$

the solution $x(.)$ of (S_0), with $x_0 \triangleq x(0) \in K$, satisfies

$$x(t) = x_0 + \sum_I (f_I\, id)(x_0) \int_0^t u_I\,, \quad \forall t \in [0, T]\,. \tag{12}$$

Furthermore, the series in the right-hand side of (12) is uniformly absolutely convergent w.r.t. $t \in [0, T]$ and $x_0 \in K$.

Note that the sum in the right-hand side of equality (12) can be developed as

$$\sum_{k=1}^{\infty} \sum_{i_1, \ldots, i_k = 1}^{m} (f_{i_1} \cdots f_{i_k}\, id)(x_0) \int_0^t \int_0^{t_k} \cdots \int_0^{t_2} u_{i_k}(t_k) u_{i_{k-1}}(t_{k-1}) \cdots u_{i_1}(t_1)\, dt_1 \cdots dt_k\,.$$

Let us also remark that the condition (10), which relates the integration time-interval to the control size, is specific to driftless systems. For a system which contains a drift term, it is *a priori* not true that decreasing the size of the control inputs allows to increase the time-interval on which the expansion (12) is valid. The fact that this property holds for driftless systems can be viewed as a consequence of time-scaling invariance properties.

We are now ready to state sufficient conditions under which exponential stabilization robust to unmodeled dynamics is granted.

Theorem 1 *Consider an analytic locally controllable system (S_0), a neighborhood U of the origin in $\mathbb{R}^n$, and a function $u \in \mathcal{C}^0(U \times [0, T]; \mathbb{R}^m)$. Assume that*

1. *there exist $\alpha, K > 0$ such that $|u(x, t)| \leq K|x|^\alpha$ for all $(x, t) \in U \times [0, T]$,*

2. *the solution* $x(.)$ *of*

$$\dot{x} = \sum_{i=1}^{m} f_i(x) u_i(x_0, t)\,, \quad x(0) = x_0 \in U\,, \tag{13}$$

satisfies $x(T) = Ax_0 + o(x_0)$ *with* A *a discrete-stable matrix,*

3. *for any multi-index* I *of length* $|I| \leq 1/\alpha$ *(this assumption is only needed when* $\alpha < 1$*),*

$$\int_0^T u_I(x) = O(x) \tag{14}$$

Then, given a family of perturbed systems (S_ϵ)*, there exists* $\epsilon_0 > 0$ *such that the origin of* $(\bar{S}_\epsilon)$ *controlled by* $u(y,t)$ *is locally exponentially stable for any* $\epsilon \in (-\epsilon_0, \epsilon_0)$ *.*

Proof: It much relies on the following claim.

Claim 1 *Given* $\epsilon_1 > 0$*, there exists* $\delta > 0$ *such that, for* $|x_0| \leq \delta$ *and* $|\epsilon| \leq \epsilon_1$*, the solution of*

$$\dot{x} = \sum_{i=1}^{m} (f_i(x) + h_i(\epsilon, x))\, u_i(x_0, t)\,, \quad x(0) = x_0 \tag{15}$$

is defined on $[0, T]$ *and satisfies*

$$x(T) = Ax_0 + \beta(\epsilon, x_0) + \gamma(\epsilon, x_0) + o(x_0) \tag{16}$$

with

$$\frac{|\beta(\epsilon, x_0)|}{|x_0|} \to 0 \text{ as } \epsilon \to 0 \text{ uniformly in } x_0 \; (|x_0| \leq \delta) \tag{17}$$

$$\frac{|\gamma(\epsilon, x_0)|}{|x_0|} \to 0 \text{ as } x_0 \to 0 \text{ uniformly in } \epsilon \; (|\epsilon| \leq \epsilon_1)\,. \tag{18}$$

(Proof in Appendix)

Existence of the solutions of the system $(\bar{S}_\epsilon)$ controlled by $u(y,t)$, when the initial conditions x_0 and y_0 are close enough to the origin, can easily be established (details are left to the interested reader) once it is proven that these solutions at least exist on a small time-interval and are uniformly bounded with respect to x_0 and y_0. Existence on a small time-interval is in fact granted by application of Proposition 1 (as illustrated in the proof of Claim 1), while uniform boundedness of the solutions simply results from the property of (local) stability which is proven below.

In order to prove that the origin of the system $(\bar{S}_\epsilon)$ controlled by $u(y,t)$ is locally (uniformly) exponentially stable, one must establish that i) the solutions of this system converge exponentially to zero (uniformly with respect to the initial conditions (t_0, x_0, y_0)), and ii) the origin of the controlled system is stable.

Since A is a discrete-stable matrix, there exists a positive definite matrix P such that $|A|_P < 1$ ($\Rightarrow \exists \tau < 1 : |Ax|_P \leq \tau |x|_P$). For the sake of simplifying the notation, and keeping in mind that subsequent normed vectors will refer to the norm induced by the matrix P, we will just drop the index P in the remaining of the proof.

i) exponential convergence

Let $t_0 \in [k_0 T, (k_0+1)T)$ ($k_0 \in \mathbb{N}$), and $(x_\epsilon(., t_0, x_0, y_0), y_\epsilon(., t_0, x_0, y_0))$ denote the solution of the controlled system $(\bar{S}_\epsilon)$ with initial conditions (t_0, x_0, y_0). Then for any $k \in \mathbb{N}$ such that $t_0 \leq kT$, and any $t \in [kT, (k+1)T)$, this solution satisfies

$$\begin{cases} \dot{x} = \sum_{i=1}^{m} (f_i(x) + h_i(\epsilon, x)) u_i(x(kT), t) \\ \dot{y} = 0 \,, \quad y(t) = x(kT) \end{cases} \tag{19}$$

In order to show the exponential convergence to zero of the trajectories of (19), it is clearly sufficient to show that the x component converges exponentially to the origin. In view of relations (16)-(18) in Claim 1, there exist $\epsilon_0 > 0$, $\delta_0 > 0$, and a positive real number $\tau < 1$ such that, for $|x_0| \leq \delta_0$ and $\epsilon \leq \epsilon_0$

$$|x_\epsilon((k+1)T, t_0, x_0, y_0)| \leq \tau |x_\epsilon(kT, t_0, x_0, y_0)| \,, \qquad \forall k > k_0 \,. \tag{20}$$

This already establishes that the sequence $\{x_\epsilon(kT, t_0, x_0, y_0)\}_{k \in \mathbb{N}}$ converges exponentially to zero, uniformly with respect to the initial conditions (t_0, x_0, y_0).
In order to infer uniform exponential convergence to zero of $x_\epsilon(t, t_0, x_0, y_0)$, it is thus sufficient to show that

$$|x_\epsilon(kT + s, t_0, x_0, y_0)| \leq K |x_\epsilon(kT, t_0, x_0, y_0)|^\eta \qquad , \ \forall s \in [0, T) \ , \ \forall k > k_0 \,, \tag{21}$$

for some positive constants K and η independent of t_0, x_0, y_0. Since u is T-periodic

$$\begin{aligned} x_\epsilon(kT + s, t_0, x_0, y_0) &= x_\epsilon(s, 0, x_\epsilon(kT, t_0, x_0, y_0), y_\epsilon(kT, t_0, x_0, y_0)) \\ &= x_\epsilon(s, 0, x_\epsilon(kT, t_0, x_0, y_0), x_\epsilon(kT, t_0, x_0, y_0)) \,. \end{aligned}$$

Therefore, (21) is equivalent to

$$|x_\epsilon(s, 0, x_0, x_0)| \leq K |x_0|^\eta \qquad , \ \forall s \in [0, T) \ . \tag{22}$$

From Assumption 1, and the continuity of the vector fields f_i, and h_i, the above inequality follows by the classical Gronwall lemma.

ii) uniform stability of the origin
We distinguish two cases, according to whether t_0 is, or is not, a multiple of T.

case 1: t_0 is not a multiple of T.
Then there exists $k_0 \in \mathbb{N}$ such that $k_0 T < t_0 < (k_0 + 1)T$. There also exists an open ball $B_\epsilon \in \mathbb{R}^n$ centered on 0 such that the function $(t, t_0, x_0, y_0) \mapsto x_\epsilon(t, t_0, x_0, y_0)$ is continuous on the set $\{(t, t_0, x, y) \ : \ t_0 \leq t \leq (k_0 + 1)T, t_0 \in (k_0 T, (k_0 + 1)T), x \in B_\epsilon, y \in B_\epsilon\}$. Therefore, the function ν_ϵ defined by

$$\nu_\epsilon(x_0, y_0) \triangleq \sup_{t_0 \in (k_0 T, (k_0+1)T)} \ \sup_{t \in [t_0, (k_0+1)T)} |x_\epsilon(t, t_0, x_0, y_0)| \tag{23}$$

is itself continuous on $B_\epsilon \times B_\epsilon$. Note that $\nu_\epsilon(0, 0) = 0$, since $x_\epsilon(t, t_0, 0, 0) = 0, \forall t \geq t_0$, and that $\nu(x_0, y_0) \geq |x_0|$. Furthermore, there exists $\delta_\epsilon > 0$ such that $\nu_\epsilon(x_0, y_0) < min(1, r_{B_\epsilon}), \forall (x_0, y_0) \ : \ |x_0| \leq \delta_\epsilon, |y_0| \leq \delta_\epsilon$, with r_{B_ϵ} denoting the radius of B_ϵ. Recall also that

$$|y_\epsilon(t, t_0, x_0, y_0)| = |y_0| \quad , \ \forall t_0 \in (k_0 T, (k_0 + 1)T) \ , \ \forall t \in [t_0, (k_0 + 1)T) \ . \tag{24}$$

Now, since by (20) and (21)

$$|x_\epsilon(t, t_0, x_0, y_0)| \leq K |x_\epsilon((k_0 + 1)T, t_0, x_0, y_0)|^\eta \quad , \ \forall t \geq (k_0 + 1)T \ ,$$

and

$$|y_\epsilon(t, t_0, x_0, y_0)| \leq |x_\epsilon((k_0 + 1)T, t_0, x_0, y_0)| \quad , \ \forall t \geq (k_0 + 1)T \ ,$$

one deduces from (23) and (24) (using also the fact that $|(x,y)| \leq |x| + |y|$) that

$$\begin{array}{l} |x_\epsilon(t,t_0,x_0,y_0), y_\epsilon(t,t_0,x_0,y_0)| \leq K\nu_\epsilon(x_0,y_0)^\eta + max(\nu_\epsilon(x_0,y_0), |y_0|) \ , \\ \forall t \geq t_0, \ \forall (x_0,y_0) : \ |x_0| \leq \delta_\epsilon, \ |y_0| \leq \delta_\epsilon \ . \end{array} \quad (25)$$

case 2: t_0 is a multiple of T (i.e. $t_0 = k_0 T$).
Then $y(0) = x_0$, and one easily obtains in this case

$$|x_\epsilon(t,t_0,x_0,y_0), y_\epsilon(t,t_0,x_0,y_0)| \leq K|x_0|^\eta + |x_0| \ , \quad \forall t \geq t_0, \ \forall (x_0,y_0) \ : \ |x_0| \leq \delta_\epsilon, \ |y_0| \leq \delta_\epsilon \ . \quad (26)$$

The comparison of the right-hand sides of inequalities (25) and (26) shows that (25) holds in fact for every value of t_0.

∎

3 Control design

This section addresses the problem of constructing explicit controllers that meet the conditions of Theorem 1. Such a controller has to belong to the set of exponential stabilizers for the extended system $(\bar{S}_0)$. A common approach, for obtaining of continuous (time-varying) feedback control laws which exponentially stabilize a driftless control system such as (S_0), consists in considering a nilpotent *homogeneous approximation* of this system and finding a control which i) asymptotically stabilizes the origin of this simpler system , and ii) makes the corresponding controlled system homogeneous *of degree zero*. This approach is justified by the fact that any time-varying continuous feedback endowed with these two properties is a local exponential stabilizer for the original system. However, as pointed out in the paper's introduction and shown is [12], such a feedback cannot be a robust stabilizer in the sense that we are considering here. Nonetheless, we will show below that the same type of approach, applied the the extended system $(\bar{S}_0)$, instead of the original system (S_0), provides a systematic way of constructing explicit and robust time-periodic feedbacks $u(y,t)$. More precisely, we will show that, given an adequate homogeneous approximation of (S_0), the set of continuous time-periodic feedbacks $u(x,y,t)$ which exponentially stabilize the origin of the corresponding extended system contains a subset of robust stabilizers, and we will provide a systematic procedure for constructing elements that belong to this subset.

We have chosen to decompose the developments yielding to this result in three steps, one subsection per step. Step 1 reviews basic definitions and facts about homogeneity and homogeneous approximations, prior to pointing out an additional sufficient condition, stated in Theorem 2, under which a continuous feedback $u(y,t)$ which robustly exponentially stabilizes the origin of a homogeneous approximation of $(\bar{S}_0)$, by application of Theorem 1, also robustly exponentially stabilizes the origin of $(\bar{S}_0)$. Step 2 provides the expression of a control solution when (S_0) can be approximated by a *chained* system. The proof of stability and robustness just consists in verifying that this control satisfies the four conditions stated in Theorems 1 and 2. The proposed control law is quite simple, and it applies to any dimension. Note also that it is different from the one proposed in [1] for a chained system of dimension four. Finally, step 3 describes a general control design procedure which applies to any driftless system (S_0). The procedure takes advantage of known techniques for nilpotent control systems based on the use of oscillatory open-loop controls in order to achieve net motion in any direction of the state space. Unfortunately (and unavoidably), the procedure also inherits the complexity of the abovementioned techniques, itself directly related to the process of selecting the "right" frequencies which facilitate motion monitoring in the state space. Unsurprisingly, the selection of these frequencies gets all the more involved that controllability of the system relies on high-order Lie brackets of the control vector fields. One can also verify that the proposed general procedure yields control expressions which are more complicated than the one proposed in step 2, in the specific case when the system (S_0) can be approximated by a chained system. This is a good illustration of the fact that Theorems 1 and 2 can be used in various ways for robust control design purposes and that simpler solutions can be obtained by better exploiting the inner structure of the system. For instance, the control solution of step 2 takes advantage of non-uniqueness of dilations which can be associated with chained systems, while the general design procedure of step 3 is carried out only for the so-called *standard dilation*, the interest of which is that it can be systematically determined from the control filtration associated with the system. Apart from this, the two solutions have many features in common.

3.1 Homogeneous approximations and robust asymptotic stability

Given $\lambda > 0$ and a *weight vector* $r = (r_1, \ldots, r_n)$ $(r_i > 0\ \forall i)$, a *dilation* δ_λ^r is a map from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined by:

$$\delta_\lambda^r(z_1, \ldots, z_n) = (\lambda^{r_1} z_1, \ldots, \lambda^{r_n} z_n)\,.$$

A function $f \in \mathcal{C}^0(\mathbb{R}^n; \mathbb{R})$ is *homogeneous of degree l with respect to the family of dilations δ_λ^r $(\lambda > 0)$*, or, more concisely, *δ^r-homogeneous of degree l*, if

$$\forall \lambda > 0, \quad f(\delta_\lambda^r(z)) = \lambda^l f(z)\,.$$

A *δ^r-homogeneous norm* can be defined as a positive definite function of x, δ^r-homogeneous of degree one. Although this is not a "true" norm when the weight coefficients are not all equal, it still provides a means of "measuring" the size of x.
A continuous vector field on $\mathbb{R}^n$ is *δ^r-homogeneous of degree d* if, for all $i = 1, \ldots, n$, the function $x \longmapsto X_i(x)$ is δ^r-homogeneous of degree $r_i + d$. According to these definitions, homogeneity is coordinate dependent, however it is possible to define the above concepts in a coordinate independent framework [7, 18].

The following property is used extensively in the sequel. Given a family of dilations δ_λ^r $(\lambda > 0)$, a smooth function f and a smooth vector field X, δ^r-homogeneous of degree $\deg(f)$ and $\deg(X)$ respectively, the function Xf is δ^r-homogeneous of degree

$$\deg(Xf) = \deg(X) + \deg(f)\,.$$

Finally, we say that the system

$$\dot{z} = \sum_{i=1}^{m} b_i(z) u_i \tag{27}$$

is a *δ^r-homogeneous approximation* of (S_0) if:

1. the change of coordinates $\phi:\ x \longmapsto z$ transforms (S_0) into

$$\dot{z} = \sum_{i=1}^{m} \left(b_i(z) + g_i(z)\right) u_i\,, \tag{28}$$

   where b_i is δ^r-homogeneous of some degree $d_i < 0$, and g_i denotes higher-order terms, i.e. such that $g_{i,j}$ (the j-th component of g_i) satisfies

$$g_{i,j} = o(\rho^{r_j + d_i})\,, \quad (j = 1, \ldots, n)\,. \tag{29}$$

   where ρ is a δ^r-homogeneous norm;

2. the system (27) is controllable.

Hermes [5], and Stefani [20] have shown that any driftless system (S_0) satisfying the LARC (Lie Algebra Rank Condition) at the origin (2) has a homogeneous approximation. Homogeneous approximations of controllable driftless systems are not unique in general. Explicit construction of such approximations requires to find both a weight vector and a change of coordinates for which properties 1 and 2 above are fulfilled. A constructive procedure can be found in [20, 21] (see also [5] but with a less explicit change of coordinates).

The control design procedure presented in step 3 will make use of a particular homogeneous approximation of (S_0) —one for which the vector fields b_i are homogeneous of the same degree -1. Let us briefly recall some features of this approximation.

Consider the *control filtration*, $\mathcal{F}$ of Lie$(f_1, \dots, f_m)$ defined as $\mathcal{F} \triangleq (\mathcal{F}_j)_{j\geq 0}$ with

$$\begin{aligned}
\mathcal{F}_0 &\triangleq \{0\} \\
\mathcal{F}_1 &\triangleq \text{span}\{f_1, \dots, f_m\} \\
\mathcal{F}_2 &\triangleq \text{span}\{f_1, \dots, f_m, [f_1, f_2], \dots, [f_1, f_m], \dots [f_{m-1}, f_m]\} \\
&\vdots \\
\mathcal{F}_k &\triangleq \text{span}\{\text{all Lie brackets of the } f_i's \text{ of length } \ell \leq k\} \\
&\vdots
\end{aligned}$$

Denote also

$$F_k(0) \triangleq \text{span}\{f(0) : f \in \mathcal{F}_k\},$$

and $n_k \triangleq \dim F_k(0)$. Then, by the LARC at the origin, there exists a smallest integer P such that

$$0 = n_0 < n_1 \leq n_2 \leq \dots \leq n_{P-1} < n_P = n.$$

Now, define the weight vector r according to

$$r_j \triangleq p \quad \text{for } n_{p-1} + 1 \leq j \leq n_p \qquad (p = 1, \dots, P). \tag{30}$$

Note that the sequence $r_1, r_2, \dots, r_n$ is increasing, i.e. $1 = r_1 \leq r_2 \leq \dots \leq r_n = P$. The results given in [5, 20] imply

Proposition 2 *[5, 20] There exists a δ^r-homogeneous approximation (27) of system (S_0) with r given by (30). Furthermore, every control vector field of the approximating system is δ^r-homogeneous of degree -1.*

We refer the reader to the references cited above for the construction of a change of coordinates ϕ that transforms (S_0) into (28). Let us remark that the control vector fields b_i of the approximating homogeneous system (27) are polynomials in the z coordinates, and that $\{b_1, \dots, b_m\}$ forms a nilpotent set of vector fields —more precisely, any Lie bracket of the b_i's of length strictly larger than P is identically zero.

Since homogeneous approximations are nilpotent, for any time-function u, the Chen-Fliess series associated with any such an approximation only involves a finite number of terms. This property is very useful when trying to derive exponential stabilizers for the homogeneous approximation of a given system. Of course, such controllers are of interest only if they are also exponential stabilizers for the original system. The following theorem points out sufficient conditions on the control law to ensure that such is the case.

Theorem 2 *Consider a δ^r-homogeneous approximation (27) of (S_0), with $d_i \triangleq deg(b_i)$ $(i = 1, \dots, m)$, and a control function $u \in \mathcal{C}^0(U \times [0,T]; \mathbb{R}^m)$ such that the three assumptions in Theorem 1 are verified for this approximating system. Assume furthermore that the following assumption, which is a stronger version of the third assumption in Theorem 1, is also verified for the approximating system:*

3-bis. for any multi-index $I = (i_1, \dots, i_{|I|})$ of length $|I| \leq 1/\alpha$,

$$\int_0^T u_I(z) = \sum_{k: r_k \geq \|I\|} a_{I,k} z_k + o(z), \tag{31}$$

where $\|I\| \triangleq -\sum_{j=1}^{|I|} d_{i_j}$, and the $a_{I,k}$'s are some scalars.

Then, the three assumptions of Theorem 1 are verified for the system (28).

Note that it is not required that all d_i's be equal, as it occurs when the approximation is obtained by using the dilation defined by (30).

Proof: Assumptions 1 and 3 of Theorem 1 are obviously verified for the system (28), since they only involve the control law and do not depend on the control system. Hence, we only need to take care of Assumption 2 and show that the solution of

$$\dot{z} = \sum_{i=1}^{m} (b_i(z) + g_i(z)) u_i(z_0, t) , \qquad z(t_0) = z_0 , \tag{32}$$

satisfies

$$z(T) = \bar{A} z_0 + o(z_0) \tag{33}$$

for some discrete-stable matrix $\bar{A}$. Let us first introduce some notations. Without loss of generality, we assume that the variables z_i are ordered by increasing weight, i.e.

$$r_1 \leq r_2 \leq \ldots \leq r_n$$

and decompose z as $z = (z^1, \ldots, z^P)$, where each z^p $(1 \leq p \leq P)$ is the sub-vector of z whose components have same weight r^p $(r_1 \leq r^p \leq r_n)$ with

$$r_1 = r^1 < r^2 < \ldots < r^P = r_n .$$

In a similar way, a map f from $\mathbb{R}^n$ to $\mathbb{R}^n$ can be decomposed as $f = (f^1, \ldots, f^P)$.

We may now proceed with the proof. The solution of (32) can be expanded as

$$z(t) = z_0 + \sum_I ((b+g)_I \, id)(z_0) \int_0^t u_I(z_0) .$$

By Proposition 1, the series in the right-hand side of this equality is uniformly convergent w.r.t. z_0 $(|z_0| \leq \delta)$ and $t \in [0, T]$. The above expression may be rewritten as

$$z(t) = z_0 + \sum_I (b_I \, id)(z_0) \int_0^t u_I(z_0) + \sum_I (d_I^g \, id)(z_0) \int_0^t u_I(z_0) . \tag{34}$$

Here, $d_I^g \triangleq d_{i_1}^g \cdots d_{i_k}^g$ (for $I = (i_1, \ldots, i_k)$), where each d_i^g belongs to $\{b_i, g_i\}$, and the product d_I^g contains at least one of the g_i's. Note that the first sum in the right-hand side involves a finite number of terms because $\{b_1, \ldots, b_m\}$ is a nilpotent set of vector fields. As a consequence, the series defined by the second sum is uniformly convergent w.r.t. z_0 and t. Each of these two sums is now considered separately.

Since Assumption 2 in Theorem 1 is verified for the approximating system (27), we have

$$z_0 + \sum_I (b_I \, id)(z_0) \int_0^T u_I(z_0) = A z_0 + o(z_0) . \tag{35}$$

where A is a discrete-stable matrix. We claim that the matrix A is necessarily block upper-triangular in the sense that

$$A z_0 = \begin{pmatrix} A_{11} & \star & \star \\ & \ddots & \star \\ 0 & & A_{PP} \end{pmatrix} \begin{pmatrix} z_0^1 \\ \vdots \\ z_0^P \end{pmatrix} , \tag{36}$$

In order to prove this assertion, it is clearly sufficient to show that, for $p = 1, \dots, P$,

$$\sum_I (b_I\, id)^p(z_0) \int_0^T u_I(z_0) = \sum_{q\geq p} \tilde{A}_{p,q} z_0^q + o(z_0) \tag{37}$$

for some matrices $\tilde{A}_{p,q}$. Let us rewrite the sum in the left-hand side of (37) as

$$\sum_{\|I\|<r^p\,,\,|I|\leq\frac{1}{\alpha}} (b_I\, id)^p \int_0^T u_I + \sum_{\|I\|\geq r^p\,,\,|I|\leq\frac{1}{\alpha}} (b_I\, id)^p \int_0^T u_I + \sum_{|I|>\frac{1}{\alpha}} (b_I\, id)^p \int_0^T u_I \tag{38}$$

where the argument z_0 is omitted for the sake of conciseness.

We first note that the last sum in (38) is a $o(z_0)$ because, from Assumption 1, each $\int_0^T u_I$ is itself a $o(z_0)$ when $|I| > 1/\alpha$.

From Assumption 3-bis in Theorem 2, all iterated integrals in the second sum are of the form

$$\sum_{k:r_k\geq r^p} a_{I,k} z_{0,k} + o(z_0)\,.$$

which may also be written as

$$\sum_{p\leq q\leq P} \tilde{a}_{I,q} z_0^q + o(z_0)\,.$$

since any $z_{0,k}$ whose weight r_k is greater or equal to r^p has to be an element of some z_0^q with $q \geq p$. This clearly implies that the second sum in (38) can be written as the right-hand side of (37).

Let us finally consider the first sum in (38). Since $(b_I\, id)^p(z_0)$ is just $b_I z^p$ evaluated at z_0, and since each component of z^p is homogeneous of degree r^p, it follows that each component of $(b_I\, id)^p$ is a function homogeneous of *positive* degree $r^p - \|I\|$. Therefore, each $(b_I\, id)^p$ vanishes at the origin and, since it is a smooth function, there exists $K_I > 0$ such that $|(b_I\, id)^p(z_0)| \leq K_I|z_0|$. This inequality, combined with Assumption 1 which tells us that $|\int_0^T u_I(z_0)| = o(|z_0|^{\beta_I})$ for some $\beta_I > 0$, implies that the first sum in (38) is a $o(z_0)$. Therefore, relation (37) holds for every $p = 1, \dots, P$ and, subsequently, the matrix A is block upper-triangular. Moreover, A being a discrete-stable matrix, each matrix A_{pp} on the block diagonal is necessarily a discrete-stable matrix itself.

Let us now show that

$$\sum_I (d_I^g\, id)(z_0) \int_0^T u_I(z_0) = Cz_0 + o(z_0) \tag{39}$$

where C is a block upper-triangular matrix with zeroes on the block diagonal , i.e.

$$Cz_0 = \begin{pmatrix} 0 & \star & \star \\ \vdots & \ddots & \star \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} z_0^1 \\ \vdots \\ z_0^P \end{pmatrix} + o(z_0)\,. \tag{40}$$

To this purpose, we just need to show that, for $p = 1, \dots, P$,

$$\sum_I (d_I^g\, id)^p(z_0) \int_0^T u_I(z_0) = \sum_{q>p} C_{p,q} z_0^q + o(z_0) \tag{41}$$

for some matrices $C_{p,q}$. Let us again decompose the sum in the left-hand side of (41) as

$$\sum_{\|I\|\leq r^p,|I|\leq 1/\alpha} (d_I^g\, id)^p \int_0^T u_I + \sum_{\|I\|>r^p,|I|\leq 1/\alpha} (d_I^g\, id)^p \int_0^T u_I + \sum_{|I|>1/\alpha} (d_I^g\, id)^p \int_0^T u_I\,. \tag{42}$$

We start with the third sum in (42), and define

$$\tilde{u}_i(z_0,t) \triangleq \frac{u_i(z_0,t)}{|z_0|^{\alpha-\sigma}} \qquad (i=1,\dots,m)$$

with $\sigma > 0$ small enough so that, by Assumption 1 of Theorem 1, $\tilde{u}_i$ is continuous. Then,

$$\sum_{|I|>1/\alpha} (d_I^g\, id)^p(z_0) \int_0^T u_I(z_0) = \sum_{|I|>1/\alpha} |z_0|^{|I|(\alpha-\sigma)} (d_I^g\, id)(z_0) \int_0^T \tilde{u}_I(z_0)\,.$$

Choosing σ small enough such that the inequality $|I|(\alpha-\sigma) \geq 1+\mu_0 > 1$ holds for every I such that $|I| > 1/\alpha$, and using the fact that, from Proposition 1, the series

$$\sum_{|I|>1/\alpha} (d_I^g id)(z_0) \int_0^T \tilde{u}_I(z_0)$$

is uniformly absolutely convergent for z_0 small enough, we obtain (provided that $|z_0| < 1$)

$$|\sum_{|I|>1/\alpha} (d_I^g\, id)^p(z_0) \int_0^T u_I(z_0)| \leq |z_0|^{1+\mu_0}\ S(z_0) \tag{43}$$

with S a continuous function. This establishes that the third sum in (42) is a $o(z_0)$.

Let us now consider the second sum in (42). From Assumption 3-bis, and as pointed out before in the proof (with the only difference that $||I||$ is now taken *strictly* greater than r^p), all iterated integrals in this sum are of the form

$$\sum_{q>p} \tilde{a}_{I,q} z_0^q + o(z_0)\,.$$

This implies that the second sum in (42) can be written as the right-hand side of (41).

Let us finally consider the first sum in (42). By definition of the product d_I^g, there is at least one term in this product which belongs to $\{g_1,\dots,g_m\}$. Now, since $g_{i,j} = o(\rho^{r_j+d_i})$, for $j=1,\dots,n$ (relation (29)), the Taylor expansion of g_i at the origin gives a sum of vector fields homogeneous of degree strictly larger than d_i. This in turn implies that each d_I^g is a sum of differential operators of degree strictly larger than $-\|I\|$, and that every component of $(d_I^g\, id)^p$ is a sum of homogeneous functions of degree strictly larger than $(r^p - \|I\|)$. Since $(r^p - \|I\|) \geq 0$, this degree is thus strictly positive. Therefore, every $(d_I^g\, id)^p$ vanishes at the origin and, since it is also a smooth function, there exists $K_I > 0$ such that $|(d_I^g\, id)^p(z_0)| \leq K_I |z_0|$. This inequality, combined with Assumption 1 which implies that $|\int_0^T u_I(z_0)|$ tends to zero when z_0 tends to zero, implies that the first sum in (42) is a $o(z_0)$. We have thus proved that (41) holds for any $p=1,\dots,P$, and, subsequently, that relations (39) and (40) also hold. It follows from (34), (35), and (39) that relation (33) is true with $\bar{A} = A + C$, a discrete-stable matrix. Therefore Assumption 2 is verified for the system (28), and this concludes the proof of Theorem 2. ∎

3.2 Robust exponential stabilizers for chained systems

Consider the following n-dimensional chained system

$$(S_0) \quad \begin{cases} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= u_1 x_2 \\ &\vdots \\ \dot{x}_n &= u_1 x_{n-1}\,. \end{cases} \tag{44}$$

The next proposition points out a set of robust exponential stabilizers for this type of system.

Proposition 3 *With the control function $u \in \mathcal{C}^0(\mathbb{R}^n \times [0,T];\mathbb{R}^2)$ defined by*

$$\left\{\begin{array}{lcl} u_1(x,t) & = & \frac{1}{T}[(k_1-1)x_1 + 2\pi\rho_q(x)\sin(\omega t)] \\ u_2(x,t) & = & \frac{1}{T}[(k_2-1)x_2 + \sum_{i=3}^{n} 2^{i-2}(i-2)!(k_i-1)\frac{x_i}{\rho_q^{i-2}(x)}\cos((i-2)\omega t)\,, \end{array}\right. \tag{45}$$

with

$$\begin{array}{l} T = 2\pi/\omega \quad (\omega \neq 0) \\ \rho_q(x) = \sum_{j=2}^{n} a_j|x_j|^{\frac{1}{q+j-2}}\,, \qquad (q \geq n-2\,,\ a_j > 0) \\ |k_i| < 1\,, \quad \forall i = 1,\dots,n\,, \end{array} \tag{46}$$

the three assumptions in Theorem 1, and the extra assumption in Theorem 2, are verified for the system (44).

Proof: We give the proof for $\omega = 1$ ($\Rightarrow T = 2\pi$)—any other value of ω being taken care of by introducing the time-scaling $t \longmapsto \omega t$. Throughout the proof, the control vector fields associated with the chained system (44) are denoted as b_1 and b_2, i.e.

$$\begin{array}{c} b_1(x) = (1,0,x_2,\dots,x_{n-1})' \\ b_2(x) = (0,1,0,\dots,0)'\,. \end{array}$$

For any $q > 0$, and corresponding dilation $\delta^{r(q)}$ such that $r(q) = (1,q,q+1,\dots,q+n-2)$, the v.f. b_1 and b_2 are $\delta^{r(q)}$-homogeneous of degree -1 and $-q$ respectively. In view of (45), $u_1(x,t)$ and $u_2(x,t)$ are $\delta^{r(q)}$-homogeneous of degree 1 and q respectively. Therefore, since q is positive, Assumption 1 of Theorem 1 is verified with $\alpha = \frac{1}{\max_i r_{q,i}} = \frac{1}{q+n-2}$.

Let us now check that Assumption 2 is verified. The solution $x(.)$ of (13) on the time-interval $[0,T]$ is given by

$$x(t) = x_0 + \sum_I (b_I\, id)(x_0) \int_0^t u_I(x_0) \qquad (t \in [0,T])\,. \tag{47}$$

It is simple to show by induction that

$$(\underbrace{b_1 \cdots b_1}_{k} id)(x) = (0,\dots,0,x_2,\dots,x_{n-k})' \qquad (k \geq 2)\,, \tag{48}$$

which implies

$$(b_2 \underbrace{b_1 \cdots b_1}_{k} id)(x) = (\underbrace{0,\dots,0}_{k+1},1,0,\dots,0)' \qquad (k \geq 1)\,, \tag{49}$$

and, subsequently

$$(b_i b_2 b_1 \cdots b_1 id) \equiv 0 \qquad \forall i = 1,2\,.$$

Hence, the only multi-indices for which $b_I\,id$ is not identically zero are those of the form $I=(1,\dots,1)$ or $I=(2,1,\dots,1)$, and one obtains from (47), (48), and (49),

$$\begin{aligned}
x(T) &= x_0 + \sum_{k=1}^{n-2} (\underbrace{b_1\cdots b_1}_{k}\,id)(x_0)\int_0^T u_{(1,\dots,1)}(x_0) + \sum_{k=0}^{n-2}(b_2\underbrace{b_1\cdots b_1}_{k}\,id)(x_0)\int_0^T u_{(2,1,\dots,1)}(x_0)\\
&= x_0 + b_1(x_0)\int_0^T u_1(x_0) + \sum_{k=0}^{n-2}(b_2\underbrace{b_1\cdots b_1}_{k}\,id)(x_0)\int_0^T u_{(2,1,\dots,1)}(x_0) + o(x_0)\\
&= x_0 + \begin{pmatrix} 1 & 0 & 0 & \dots & 0\\ 0 & 1 & 0 & \dots & \vdots\\ x_{0,2} & 0 & 1 & & \vdots\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ x_{0,n-2} & 0 & \dots & 0 & 1\end{pmatrix}\begin{pmatrix}\int_0^T u_1(x_0)\\ \int_0^T u_2(x_0)\\ \int_0^T u_{(2,1)}(x_0)\\ \vdots\\ \int_0^T u_{(2,1,\dots,1)}(x_0)\end{pmatrix} + o(x_0)\,,
\end{aligned}\tag{50}$$

Therefore

$$x(T) = x_0 + \begin{pmatrix}\int_0^T u_1(x_0)\\ \int_0^T u_2(x_0)\\ \int_0^T u_{(2,1)}(x_0)\\ \vdots\\ \int_0^T u_{(2,1,\dots,1)}(x_0)\end{pmatrix} + o(x_0)\,. \tag{51}$$

Let us now calculate the iterated integral involved in the right-hand side of equality (51). First, we have

$$\int_0^T u_1(x_0) = (k_1-1)x_{0,1}\,,\qquad \int_0^T u_2(x_0) = (k_2-1)x_{0,2}\,. \tag{52}$$

Calculation of the other integrals makes use of the following lemma.

Lemma 1 *For any k and i in $\mathbb{N}-\{0\}$, and any $p\in\{1,\dots,k\}$,*

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{t_{k+1}}\cdots\int_0^{t_2}\sin t_{k+1}\cdots\sin t_1\,dt_1\cdots dt_{k+1} = 0\\
&\int_0^{2\pi}\int_0^{t_{k+1}}\cdots\int_0^{t_2}\sin t_{k+1}\cdots\sin t_{p+1}\cos it_p\sin t_{p-1}\cdots\sin t_1\,dt_1\cdots dt_{k+1} =\\
&\qquad\qquad\begin{cases} 0 & \text{if } i>k\\ \frac{2\pi}{2^k k!} & \text{if } i=k \text{ and } p=1\end{cases}
\end{aligned}\tag{53}$$

(Proof in Appendix)

From (45)

$$\begin{aligned}
\int_0^T u_{(\underbrace{2,1,\dots,1}_{k+1})}(x_0) &= \sum_{i=3}^{n}\frac{2^{i-2}(i-2)!}{T}\,\frac{(k_i-1)x_{0,i}}{\rho_q^{i-(k+2)}(x_0)}\int_0^T\int_0^{t_{k+1}}\cdots\int_0^{t_2}\sin t_{k+1}\cdots\\
&\quad \sin t_2\cos(i-2)t_1\,dt_1\dots dt_{k+1} + o(x_0)
\end{aligned}$$

Using the fact that $\rho_q(x) = O(|x|^{\frac{1}{q+n-2}})$

$$\begin{aligned}
\int_0^T u_{(\underbrace{2,1,\dots,1}_{k+1})}(x_0) &= \sum_{i=k+2}^{n}\frac{2^{i-2}(i-2)!}{T}\,\frac{(k_i-1)x_{0,i}}{\rho_q^{i-(k+2)}(x_0)}\int_0^T\int_0^{t_{k+1}}\cdots\int_0^{t_2}\sin t_{k+1}\cdots\\
&\quad \sin t_2\cos(i-2)t_1\,dt_1\dots dt_{k+1} + o(x_0)
\end{aligned}$$

and, by application of Lemma 1

$$\int_0^T u_{\underbrace{(2,1,\dots,1)}_{k+1}}(x_0) = (k_{k+2}-1)x_{0,k+2} + o(x_0) \tag{54}$$

Using (52) and (54) in (51), we obtain

$$x(T) = Ax_0 + o(x_0) \tag{55}$$

with $A = diag\{k_1, k_2, \dots, k_n\}$, a discrete-stable matrix since $k_i \in (-1,1)$ $(i = 1, \dots, n)$.

There remains to show that Assumptions 3 of Theorem 1 and 3-bis of Theorem 2 are verified. It is in fact sufficient to show that the stronger latter assumption holds. To this purpose, we must show that (31) holds for every possible I.

Let us first consider the case where $I = (1, \dots, 1)$. If $|I| = 1$, then

$$\int_0^T u_I(x) = \int_0^T u_1(x) = (k_1-1)x_1$$

and (31) obviously holds. If $|I| > 1$, then

$$\begin{aligned}\int_0^T u_I(x) &= \rho_q^{|I|}(x)\int_0^T\int_0^{t_{|I|}}\dots\int_0^{t_2}\sin t_{|I|}\dots\sin t_1\,dt_{|I|}\dots dt_1 + o(x)\\ &= o(x)\,,\end{aligned}$$

where the last equality results from Lemma 1. Therefore, (31) holds for every I which does not contain the index 2.

Assume now that I contains the index 2 twice at least. If $I = (2,2)$, then simple calculation of $\int_0^T u_I(x)$ yields

$$\int_0^T u_I(x) = o(x)\,. \tag{56}$$

If $|I| > 2$, the iterated integral is $\delta^{r(q)}$-homogeneous of degree strictly larger than $2q$. Since $q \geq n-2$ (see (46)) and $r_i(q) \leq q+n-2$ $(i = 1, \dots, n)$, this degree is larger than the degree of homogeneity of each x_i, so that (56) also holds in this case.

Let us finally consider the case where I contains the index 2 exactly once.
If $I = (2, 1, \dots, 1)$, the satisfaction of (31) follows from (54).
If the index 2 is not in the first entry, i.e. I is of the form

$$(\underbrace{1,\dots,1}_{p},2,\underbrace{1,\dots,1}_{k-p}) \qquad (p>0)\,,$$

then $\|I\| = k+q$ and

$$\begin{aligned}\int_0^T u_I(x) = {} & \sum_{i=3}^{n}\frac{2^{i-2}(i-2)!}{T}\frac{(k_i-1)x_i}{\rho_q^{i-(k+2)}(x)}\int_0^T\int_0^{t_{k+1}}\dots\int_0^{t_2}\sin t_{k+1}\dots\sin t_{p+2}\cos(i-2)t_{p+1}\\ & \sin t_p\dots\sin t_1\,dt_{k+1}\dots dt_1 + o(x)\\ = {} & \sum_{i=k+2}^{n}\frac{2^{i-2}(i-2)!}{T}\frac{(k_i-1)x_i}{\rho_q^{i-(k+2)}}\int_0^T\int_0^{t_{k+1}}\dots\int_0^{t_2}\sin t_{k+1}\dots\sin t_{p+2}\cos(i-2)t_{p+1}\\ & \sin t_p\dots\sin t_1\,dt_{k+1}\dots dt_1 + o(x)\\ = {} & (k_{k+2}-1)x_{k+2}\int_0^T\int_0^{t_{k+1}}\dots\int_0^{t_2}\sin t_{k+1}\dots\sin t_{p+2}\cos(i-2)t_{p+1}\\ & \sin t_p\dots\sin t_1\,dt_{k+1}\dots dt_1 + o(x)\\ + {} & \sum_{i=k+3}^{n}\frac{2^{i-2}(i-2)!}{T}\frac{(k_i-1)x_i}{\rho_q^{i-(k+2)}(x)}\int_0^T\int_0^{t_{k+1}}\dots\int_0^{t_2}\sin t_{k+1}\dots\sin t_{p+2}\cos(i-2)t_{p+1}\\ & \sin t_p\dots\sin t_1\,dt_{k+1}\dots dt_1 + o(x)\end{aligned}$$

Since the weight of x_{k+2} is equal to $k+q$ $(=\|I\|)$, the first term in the right-hand side of the last equality is one of the linear terms involved in the right-hand side of relation (31), whereas all other iterated integrals are equal to zero by application of Lemma 1.
We have thus shown that relation (31) holds for every possible I, and this concludes the proof of Proposition 3. ∎

3.3 A general control design procedure

We present in this section a general algorithm to construct robust and exponential stabilizers for (S_0). The proposed controllers are built from a homogeneous approximation (27) of (S_0). The algorithm uses previous results by Sussmann and Liu [22], and Liu [11]. It is also much related to the algorithm developed in [15] for the construction of continuous time-periodic feedbacks $u(x,t)$ which exponentially stabilize the origin of a driftless system (S_0) but present the shortcoming of not being endowed with the type of robustness here considered.

To simplify the exposition of the algorithm, we assume that (S_0) is given in the form

$$\dot{x} = \sum_{i=1}^{m} (b_i(x) + g_i(x))u_i\,, \tag{57}$$

where

$$\dot{x} = \sum_{i=1}^{m} b_i(x)u_i \tag{58}$$

is a homogeneous approximation[1] of (S_0), and the g_i's denote higher-order terms.
We also assume that the dilation δ^r associated with this approximating system is the *standard* dilation, with r defined as in (30). In view of Proposition 2, we thus have

$$\begin{cases} 1 = r_1 \leq r_2 \leq \ldots \leq r_n = P \\ \deg(b_i) = -1 \qquad (i = 1, \ldots, m)\,, \end{cases} \tag{59}$$

and the state vector x can be written as

$$x = (x^1, \ldots, x^P)$$

with x^p $(p = 1, \ldots, P)$ denoting the sub-vector of x whose components have same weight p. From (30), x^p has dimension $n_p - n_{p-1}$ —note that this dimension might be zero for some values of p (those for which $r_i \neq p$, $\forall i = 1, \ldots, n$).

The algorithm consists of the three steps described below.

Step 1. Select n vector fields $\tilde{b}_j$ $(j = 1, \ldots, n)$, obtained as Lie brackets of length $\ell(j)$ of the control vector fields b_i, and such that the matrix

$$\tilde{B}(x) \triangleq \left(\tilde{b}_1(x), \ldots, \tilde{b}_n(x)\right)$$

is nonsingular at $x = 0$. Existence of such vector fields is guaranteed by the controllability of system (58). We assume that the $\tilde{b}_j$'s are ordered by "increasing length", i.e.,

$$\ell(1) \leq \ldots \leq \ell(n)\,. \tag{60}$$

This is equivalent to ordering them by "decreasing degree of homogeneity" since, by (59),

$$\deg(\tilde{b}_j) = -\ell(j)\,.$$

From this construction, we deduce

[1] Here we use the same symbol for the state vectors of both systems (S_0) and (57), however, it should be clear from Section 3.1 that in general, a change of coordinates is necessary to transform one system into the other.

Lemma 2 *The matrix $\tilde{B}(x)$ is block lower triangular, i.e.*

$$\tilde{B}(x) = \left(\tilde{B}^1(x), \dots, \tilde{B}^P(x)\right) = \begin{pmatrix} \tilde{B}^{11} & 0 & \dots & 0 \\ \tilde{B}^{21}(x) & \tilde{B}^{22} & & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \tilde{B}^{P1}(x) & \dots & \dots & \tilde{B}^{PP} \end{pmatrix} \tag{61}$$

where each sub-matrix $\tilde{B}^p(x)$ $(p = 1, \dots, P)$, of dimension $n \times (n_p - n_{p-1})$, corresponds to the vector fields $\tilde{b}_j$ of equal length $\ell(j) = p$. Furthermore, each $\tilde{B}^{pp}$ is nonsingular, and $\tilde{B}^{qp}(0) = 0$ for $q > p$.

Proof: The lower triangular structure of $\tilde{B}$, the fact that $\tilde{B}^{pp}$ is constant, and the equalities $\tilde{B}^{qp}(0) = 0$ $(q > p)$, are immediate consequences of the homogeneity of the $\tilde{b}_j$'s and the ordering of the sub-vectors x^p by increasing weight, once having recalled that a smooth homogeneous function is identically zero if its degree of homogeneity is strictly negative, is constant if its degree of homogeneity is zero, and vanishes at the origin if its degree of homogeneity is strictly positive. Finally, each $\tilde{B}^{pp}$ is necessarily nonsingular because $\tilde{B}(0)$ is nonsingular by construction. ∎

Step 2. Choose discrete-stable matrices A^p $(p = 1, \dots P)$, with $dim(A^p) = (n_p - n_{p-1}) \times (n_p - n_{p-1})$, and define the linear map a from $\mathbb{R}^n$ to $\mathbb{R}^n$ as follows

$$a(x) = \tilde{B}^{-1}(0)(A - I)x\,, \tag{62}$$

with

$$A = \begin{pmatrix} A^1 & & 0 \\ & \ddots & \\ 0 & & A^P \end{pmatrix}.$$

This construction yields

Lemma 3

1. $x_0 + \tilde{B}(x_0)a(x_0) = Ax_0 + o(x_0)$,
2. *each component a_j of a is a δ^r-homogeneous function of degree $\ell(j)$.*

Proof: Point 1 is a direct consequence of the definition of a. Indeed, $\tilde{B}(x_0) = \tilde{B}(0) + O(x_0)$ since $\tilde{B}$ is a smooth —matrix-valued— function. Therefore, using the fact that a is linear

$$x_0 + \tilde{B}(x_0)a(x_0) = x_0 + \tilde{B}(0)a(x_0) + o(x_0) = Ax_0 + o(x_0)\,.$$

For the proof of Point 2, let us decompose a —with obvious notations— as $a = (a^1, \dots, a^P)$. Then,

$$a^p(x) = (\tilde{B}^{pp})^{-1}(A^p - I)x^p\,. \tag{63}$$

Therefore, each component of the sub-vector a^p is a δ^r-homogeneous function of degree p. This is equivalent to the fact that each a_j is δ^r-homogeneous of degree $\ell(j)$, since $\ell(j) = p$ for every j such that x_j is a component of x^p. ∎

Step 3. This is the last and main step of the algorithm. The objective is to find a set of functions u_i, depending on both x and t, such that

$$\begin{aligned} x_0 + \sum_I (b_I\, id)(x_0) \int_0^T u_I(x_0) &= x_0 + \tilde{B}(x_0)a(x_0) + o(x_0) \\ &= Ax_0 + o(x_0) \quad (\textit{by Lemma } 3) \end{aligned}$$

for some $T > 0$, in order to have Assumption 2 in Theorem 1 verified. The proposed construction of these functions strongly relies on Sussmann and Liu's results. Prior to addressing the construction itself, let us recall some definitions and properties.

Definition 1 *[22, 11] Let Ω be a finite subset of $\mathbb{R}$ and $|\Omega|$ denote the number of elements of Ω. The set Ω is said to be "Minimally Canceling" (in short, MC) if and only if :*

i) $\sum_{\omega\in\Omega} \omega = 0$

ii) this is the only zero sum with at most $|\Omega|$ terms taken in Ω with possible repetitions:

$$\left.\begin{array}{l} \sum_{\omega\in\Omega} \lambda_\omega \omega = 0 \\ (\lambda_\omega)_{\omega\in\Omega} \in \mathbb{Z}^{|\Omega|} \\ \sum_{\omega\in\Omega} |\lambda_\omega| \leq |\Omega| \end{array}\right\} \Longrightarrow \left\{\begin{array}{rcl} (\lambda_\omega)_{\omega\in\Omega} & = & (0,\dots,0) \\ & or & (1,\dots,1) \\ & or & (-1,\dots,-1) \end{array}\right. \tag{64}$$

Definition 2 *[22, 11] Let $(\Omega^\xi)_{\xi\in E}$ be a finite family of finite subsets Ω_ξ of $\mathbb{R}$. The family $(\Omega^\xi)_{\xi\in E}$ is said to be "independent with respect to p" if and only if :*

$$\left.\begin{array}{l} \sum_{\xi\in E}\sum_{\omega\in\Omega^\xi} \lambda_\omega \omega = 0 \\ (\lambda_\omega)_{\omega\in\Omega^\xi,\xi\in E} \in \mathbb{Z}^{\Sigma|\Omega^\xi|} \\ \sum_{\xi\in E}\sum_{\omega\in\Omega^\xi} |\lambda_\omega| \leq p \end{array}\right\} \Longrightarrow \sum_{\omega\in\Omega^\xi} \lambda_\omega \omega = 0 \quad \forall \xi \in E \tag{65}$$

The interest of MC sets in our context comes mainly from the following two results. The first one was proved by Kurzweil and Jarnik [9] (see also [11]). The second result is deduced from the former, after standard computations.

Proposition 4 *[9] Let $f_1, \dots f_l$ be smooth vector fields, and α_i $(i = 1, \dots, l)$ be T-periodic functions such that*

$$|I| \leq l \Longrightarrow \int_0^T \alpha_I = 0\,.$$

Then,

$$\sum_{\substack{I=(\sigma(1),\dots,\sigma(l))\\ \sigma\in\mathfrak{S}(l)}} f_I \int_0^T \alpha_I = \frac{1}{l} \sum_{\substack{I=(\sigma(1),\dots,\sigma(l))\\ \sigma\in\mathfrak{S}(l)}} [f_{\sigma(1)}, [f_{\sigma(2)}, [\dots, f_{\sigma(l)}]\dots]] \int_0^T \alpha_I\,,$$

with $\mathfrak{S}(l)$ denoting the group of permutation of l elements.

Corollary 1 *Let $f_1, \dots f_l$ be smooth vector fields, and $\Omega = \{\omega_1, \dots, \omega_l\}$ be a MC set such that the ω_j's have a common divisor T. Define also*

$$\alpha_i(t) = \left\{\begin{array}{ll} \cos\omega_i t & (i = 1) \\ \sin\omega_i t & (i = 2, \dots, l)\,. \end{array}\right. \tag{66}$$

Then, for any $I = (\sigma(1), \dots, \sigma(l))$ $(\sigma \in \mathfrak{S}(l))$

$$\int_0^T \alpha_I = \frac{(-1)^{l-1}T}{2^{l-1}} \frac{1}{\omega_{i_1}(\omega_{i_1} + \omega_{i_2}) \cdots (\omega_{i_1} + \ldots + \omega_{i_{l-1}})}$$

so that, by Proposition 4,

$$\begin{aligned}&\sum_{\substack{I = (\sigma(1), \dots, \sigma(l)) \\ \sigma \in \mathfrak{S}(l)}} f_I \int_0^T \alpha_I \\ &= \frac{1}{l} \sum_{\substack{I = (\sigma(1), \dots, \sigma(l)) \\ \sigma \in \mathfrak{S}(l)}} \frac{(-1)^{l-1}T}{2^{l-1}} \frac{[f_{\sigma(1)}, [f_{\sigma(2)}, [\dots, f_{\sigma(l)}] \dots]]}{\omega_{\sigma(1)}(\omega_{\sigma(1)} + \omega_{\sigma(2)}) \cdots (\omega_{\sigma(1)} + \ldots + \omega_{\sigma(l-1)})}.\end{aligned} \tag{67}$$

Given a set of vector fields $f_1, \dots, f_m$, since any Lie bracket of the f_i's can be expressed as a linear combination of brackets of the form $[f_{i_1}, [f_{i_2}, [\dots, f_{i_l}] \dots]]$, one might hope, in view of the previous equality, that any Lie bracket can also be expressed as a linear combination of vector fields in the form $\sum_I f_I \int_0^T \alpha_I$. This happens to be the case, as proved by Liu in [11]. More precisely,

Proposition 5
Let:

- $f_1, \dots f_m$ *be smooth vector fields, and* $\tilde{f}_j$ $(j = 1, \dots, n)$ *be vector fields obtained as Lie brackets of length* $\ell(j)$ *of the* f_i*'s.*
- $(F_k)_{k=1,\dots,K}$ *denote a partition of* $\{\tilde{f}_1, \dots, \tilde{f}_n\}$ *in* homogeneous components, *i.e.,* F_k *contains all* $\tilde{f}_j$*'s obtained as Lie brackets of* $l(k)$ *vector fields* $f_{\tau_1^k}, \dots, f_{\tau_{l(k)}^k}$, *with i)* τ_i^k *not necessarily different from* τ_j^k *when* $i \neq j$ *(so that two different symbols may denote the same vector field), and ii) each of the symbols* $f_{\tau_i^k}$ $(i = 1, \dots, l(k))$ *appearing exactly once in the Lie bracket (so that* $l(k)$ *is also the length of the Lie bracket).*

Then, for every $k \in \{1, \dots, K\}$, *there exist an integer* $C(k)$ *and MC sets* $\Omega^{k,c} = \{\omega_1^{k,c}, \dots, \omega_{l(k)}^{k,c}\}_{c=1,\dots,C(k)}$ *such that*

- *the family of sets* $(\Omega^{k,c})_{c=1,\dots,C(k)}^{k=1,\dots,K}$ *is independent w.r.t.* $\max_{k \in \{1,\dots,K\}} l(k)$,
- *all elements in these sets have a common divisor* T (> 0),
- $$\tilde{f}_j \in F_k \;\Rightarrow\; \tilde{f}_j = \sum_{c=1}^{C(k)} \mu_j^{k,c} \sum_{\substack{I = (\tau_{\sigma(1)}^k, \dots, \tau_{\sigma(l(k))}^k) \\ \sigma \in \mathfrak{S}(l(k))}} f_I \int_0^T \alpha_I^{k,c} \tag{68}$$

  with

  $$\alpha_{\tau_i^k}^{k,c}(t) = \begin{cases} \cos \omega_i^{k,c} t & (i = 1) \\ \sin \omega_i^{k,c} t & (i = 2, \dots, l(k)) \end{cases} \tag{69}$$

  and $\mu_j^{k,c}$ *denoting some scalar (the value of which depends on the elements of* $\Omega^{k,c}$*).*

Let us remark that the possibility of choosing the sets $\Omega^{k,c}$ with a common divisor T is not explicitly proved in [11]. However, the proof clearly shows that the set of possible sets $\Omega^{k,c}$ (seen as a subset of $\mathbb{R}^{\sum_k l(k)C(k)}$) is open. Therefore, each element of $\Omega^{k,c}$ can be chosen rational. The main part of Liu's algorithm consists in showing how to find the MC sets $\Omega^{k,c}$ so that (68) can be satisfied. Although we will make use of the result obtained by application of this (rather involved) algorithm, we just refer the reader interested in the details of the algorithm itself to [11].

We can now return to the control design procedure. Let us first first partition the set $\{\tilde{b}_1, \dots, \tilde{b}_n\}$ into homogeneous components B_k ($k = 1, \dots, K$), with $l(k)$ denoting the length of the generating Lie bracket associated with each $\tilde{f}_j \in B_k$. Note that all vector fields in B_k are δ^r-homogeneous of the same degree $-l(k)$. Now, in view of the above proposition, and by applying Liu's algorithm, one can determine a family of MC sets $\Omega^{k,c} = \{\omega_1^{k,c}, \dots, \omega_{l(k)}^{k,c}\}$ ($k = 1, \dots, K$; $c = 1, \dots C(k)$), independent w.r.t. $P = \max_{k\in\{1,\dots,K\}} l(k)$ ($= \max_{j\in\{1,\dots,n\}} \ell(j)$), and with a common divisor T, such that for any $k \in \{1, \dots, K\}$ and $\tilde{b}_j \in B_k$,

$$\tilde{b}_j = \sum_{c=1}^{C(k)} \mu_j^{k,c} \sum_{\substack{I = (\tau^k_{\sigma(1)}, \dots, \tau^k_{\sigma(l(k))}) \\ \sigma \in \mathfrak{S}(l(k))}} b_I \int_0^T \alpha_I^{k,c} \tag{70}$$

with the functions $\alpha_{\tau_i^k}^{k,c}$ chosen as in (69).

To each function $\alpha_{\tau_i^k}^{k,c}$, we associate a state dependent function $v_{\tau_i^k}^{k,c}$ defined as follows

$$v_{\tau_i^k}^{k,c}(x) = \begin{cases} \sum_{j:\tilde{b}_j\in\{B_k\}} \mu_j^{k,c} \rho_k(x^{l(k)})^{1-l(k)} a_j(x) & (i = 1) \\ \rho_k(x^{l(k)}) & (i = 2, \dots, l(k)) \end{cases} \tag{71}$$

with, for example,

$$\rho_k(x^{l(k)}) = |x^{l(k)}|^{1/l(k)} . \tag{72}$$

Note that,

$$\prod_{i=1}^{l(k)} v_{\tau_i^k}^{k,c}(x) = \sum_{j:\tilde{b}_j\in B_k} \mu_j^{k,c} a_j(x) , \tag{73}$$

so that, in view of (70),

$$\begin{aligned} \sum_{k=1}^{K} \sum_{c=1}^{C(k)} \sum_{\substack{I = (\tau^k_{\sigma(1)}, \dots, \tau^k_{\sigma(l(k))}) \\ \sigma \in \mathfrak{S}(l(k))}} b_I\, id(x) \int_0^T (\alpha v)_I{}^{k,c}(x) &= \sum_{k=1}^{K} \sum_{j:\tilde{b}_j\in\{B_k\}} \tilde{b}_j(x) a_j(x) \\ &= \sum_{j=1}^{n} \tilde{b}_j(x) a_j(x) \\ &= \tilde{B}(x) a(x) . \end{aligned} \tag{74}$$

The following result concludes the construction.

Proposition 6 *With the function $u \in C^0(\mathbb{R}^n \times [0,T]; \mathbb{R}^m)$ defined by*

$$u_i(x,t) = \sum_{k=1}^{K} \sum_{c=1}^{C(k)} \sum_{p:\tau_p^k = i} \alpha_{\tau_p^k}^{k,c}(t) v_{\tau_p^k}^{k,c}(x) \tag{75}$$

the three assumptions in Theorem 1, and Assumption 3-bis in Theorem 2, are verified for the system (58).

Proof:
Verification of Assumption 1
Obvious, after remarking that each u_i is δ^r-homogeneous of degree 1 because, from (71) and Lemma 3, each $v^{k,c}_{\tau^k_i}$ is δ^r-homogeneous of degree 1.

Verification of Assumption 2
Since every τ^k_p belongs to $\{1,\ldots,m\}$,

$$\sum_{i=1}^{m}\sum_{p:\tau^k_p=i} b_i\alpha^{k,c}_{\tau^k_p}v^{k,c}_{\tau^k_p} = \sum_{s=1}^{l(k)} b_{\tau^k_s}\alpha^{k,c}_{\tau^k_s}v^{k,c}_{\tau^k_s}\,,$$

so that, by (75),

$$\sum_{i=1}^{m} b_i(x)u_i(x_0,t) = \sum_{k=1}^{K}\sum_{c=1}^{C(k)}\sum_{s=1}^{l(k)} b_{\tau^k_s}(x)\alpha^{k,c}_{\tau^k_s}(t)v^{k,c}_{\tau^k_s}(x_0)\,. \tag{76}$$

To simplify the notations, let us introduce the following onto map from $U\subset(\mathbb{N}-\{0\})\times(\mathbb{N}-\{0\})$ to $V\subset(\mathbb{N}-\{0\})$

$$\boldsymbol{q}\ :\ (k,c)\ \longmapsto\ q = c+\sum_{i=0}^{k-1}C(i)\quad (with\ C(0)=0) \tag{77}$$

and the inverse map

$$\boldsymbol{q}^{-1}\ :\ q\in\{\sum_{i=0}^{k-1}C(i)+1,\ldots,\sum_{i=0}^{k}C(i)\}\ \longmapsto\ (\boldsymbol{k}(q),\boldsymbol{c}(q))\triangleq(k,q-\sum_{i=0}^{k-1}C(i)) \tag{78}$$

We can then rewrite (76) as

$$\sum_{i=1}^{m} b_i(x)u_i(x_0,t) = \sum_{q=1}^{Q}\sum_{s=1}^{S(q)} X^q_s(x)\gamma^q_s(x_0,t) \tag{79}$$

with

$$\begin{aligned}\gamma^q_s &\triangleq \alpha^q_s v^q_s\\ \alpha^q_s &\triangleq \alpha^{\boldsymbol{k}(q),\boldsymbol{c}(q)}_{\tau_s^{\boldsymbol{k}(q)}}\\ v^q_s &\triangleq v^{\boldsymbol{k}(q),\boldsymbol{c}(q)}_{\tau_s^{\boldsymbol{k}(q)}}\\ X^q_s &\triangleq b_{\tau_s^{\boldsymbol{k}(q)}}\\ S(q) &\triangleq l(\boldsymbol{k}(q)))\\ Q &\triangleq \textstyle\sum_{k=1}^{K}C(k)\ .\end{aligned} \tag{80}$$

As a consequence, the solution at time T of $\dot{x}=\sum_{i=1}^{m} b_i(x)u_i(x_0,t)$, with $x(0)=x_0$, is given by

$$x(T) = x_0+\sum_{i=1}^{P}\sum_{q_1,\ldots,q_i=1}^{Q}\sum_{s_1=1}^{S(q_1))}\cdots\sum_{s_i=1}^{S(q_i)}(X^{q_1}_{s_1}\ldots X^{q_i}_{s_i}\,id)(x_0)\int_0^T\int_0^{t_i}\cdots\int_0^{t_1}\gamma^{q_i}_{s_i}(x_0,t_i)\ldots\gamma^{q_1}_{s_1}(x_0,t_1)\,dt_1\ldots dt_i\,. \tag{81}$$

Let us recall a property easily obtained by standard computation.

Lemma 4 *Assume that*

$$\int_0^T \int_0^{t_k} \cdots \int_0^{t_1} g_k(\omega_k t_k) \dots g_1(\omega_1 t_1)\, dt_1 \dots dt_k \neq 0 \tag{82}$$

where each g_i is either the sin *or the* cos *function, and the ω_k's have T as common factor. Then, there exist $\lambda_1, \dots, \lambda_k \in \{0, 1, -1\}$, with $\lambda_\beta \neq 0$ for some $\beta \in \{1, \dots, k\}$, such that*

$$\sum_{p=1}^{k} \lambda_p \omega_p = 0\,.$$

This property, combined with those associated with MC sets, is used to establish the following result

Lemma 5 *The iterated integral*

$$\int_0^T \int_0^{t_i} \cdots \int_0^{t_1} \gamma_{s_i}^{q_i}(x_0, t_i) \dots \gamma_{s_1}^{q_1}(x_0, t_1)\, dt_1 \dots dt_i \tag{83}$$

is possibly not a $o(x_0)$ only if $q_1 = q_2 = \cdots = q_i \triangleq q$ and $(s_1, \dots, s_i) = (\sigma(1), \dots, \sigma(S(q))$ for some permutation $\sigma \in \mathfrak{S}(S(q))$.

Proof of Lemma 5
From the definition of γ_s^q in (80), the iterated integral (83) may also be written as

$$v_{s_i}^{q_i}(x_0) \dots v_{s_1}^{q_1}(x_0) \int_0^T \int_0^{t_i} \cdots \int_0^{t_1} \alpha_{s_i}^{q_i}(t_i) \dots \alpha_{s_1}^{q_1}(t_1)\, dt_1 \dots dt_i\,. \tag{84}$$

From (69), (80), and Lemma 4, the iterated integral in the above expression is different from zero only if there exist $\lambda_1, \dots, \lambda_i \in \{0, 1, -1\}$, with $\lambda_\beta \neq 0$ for some $\beta \in \{1, \dots, i\}$, such that

$$\sum_{p=1}^{i} \lambda_p \omega_{s_p}^{q_p} = 0\,. \tag{85}$$

Let us first assume that there exists $p \in \{1, \dots, i\}$ such that $q_p \neq q_\beta$.
Since the sets $\Omega^q = \Omega^{k,c}$ are independent with respect to P, the above equality then implies, in particular, that

$$\sum_{p:q_p = q_\beta} \lambda_p \omega_{s_p}^{q_\beta} = 0\,.$$

Now, there are two possible cases.

case 1: *cardinal*$\{p : q_p = q_\beta\} \leq S(q_\beta)$ $(= l(\boldsymbol{k}(q_\beta)))$, so that

$$\sum_{p:q_p = q_\beta} |\lambda_p| \leq |\Omega^{q_\beta}|\,. \tag{86}$$

From the fact that the set Ω^{q_β} is minimally cancelling, we deduce that every element $\omega_s^{q_\beta}$ $(s = 1, \dots, S(q_\beta))$ of this set appears in the sum (85) exactly once. Therefore

$$\alpha_{s_i}^{q_i} \dots \alpha_{s_1}^{q_1} = \prod_{s=1}^{S(q_\beta)} \alpha_s^{q_\beta} \prod_{p:q_p \neq q_\beta} \alpha_{s_p}^{q_p}\,.$$

and also

$$v_{s_i}^{q_i}\dots v_{s_1}^{q_1}=\prod_{s=1}^{S(q_\beta)} v_s^{q_\beta}\prod_{p:q_p\neq q_\beta} v_{s_p}^{q_\beta}=\left(\sum_{j:\tilde b_j\in B_{g^{-1}(q_\beta)1}}\mu_j^{q_\beta}a_j\right)\left(\prod_{p:q_p\neq q_\beta} v_{s_p}^{q_p}\right),$$

where the last equality comes from (73) and (80). From (62), a is a smooth function which vanishes at the origin, whereas $v_{s_p}^{q_p}(x)=O(|x|^{\frac{1}{S(q_p)}})$. Therefore, the integral (84) is a $o(x_0)$ as announced.

case 2: $cardinal\{p:q_p=q_\beta\}>S(q_\beta)$ —i.e. (84) involves more than $S(q_\beta)$ terms $v_{s_i}^{q_\beta}$. Then, since each $v_\tau^{k,c}$ is a function of the sub-vector $x^{l(k)}$, with $l(k)=S(q)$, and is homogeneous of degree one

$$\prod_{p=1}^{i} v_{s_p}^{q_p}(x)=\left(\prod_{p:q_p=q_\beta} v_{s_p}^{q_\beta}(x)\right)\left(\prod_{p:q_p\neq q_\beta} v_{s_p}^{q_p}(x)\right)=h(x^{S(q_\beta)})\prod_{p:q_p\neq q_\beta} v_{s_p}^{q_p}(x)$$

where h is δ^r-homogeneous of degree strictly larger than $S(q_\beta)$. Therefore, in a neighborhood of the origin, $|h(x^{S(q_\beta)})|\leq K|x^{S(q_\beta)}|$ for some constant K and the iterated integral (84) is a $o(x_0)$ in this case too.

Now consider the case where all q_i's are equal, but $(s_1,\dots,s_i)\neq(\sigma(1),\dots,\sigma(S(q_\beta)))$. If $i<S(q_\beta)$, or $i=S(q_\beta)$ so that there exist $p_1,p_2\in\{1,\dots,i\})$ such that $s_{p_1}=s_{p_2}$, we directly deduce from Lemma 4 and from the fact that every set $\Omega^{k,c}$ is MC, that the integral in (84) is equal to zero. If $i>S(q_\beta)$, then we deduce from (71), using similar arguments as previously, that this integral is a $o(x_0)$.
This concludes the proof of Lemma 5. ∎

In view of Lemma 5, (81) simplifies into

$$\begin{aligned}
x(T) &= x_0+\sum_{q=1}^{Q}\sum_{\sigma\in\mathfrak{S}(S(q))}(X_{\sigma(1)}^q\dots X_{\sigma(S(q))}^q\,id)(x_0)\int_0^T\int_0^{t_{S(q)}}\dots\int_0^{t_1}\gamma_{\sigma(S(q))}^q(x_0,t_{S(q)})\dots\\
&\qquad \dots\gamma_{\sigma(1)}^q(x_0,t_1)\,dt_1\dots dt_{S(q)}+o(x_0)\\
&= x_0+\sum_{k=1}^{K}\sum_{c=1}^{C(j)}\sum_{\substack{I=(\tau_{\sigma(1)}^k,\dots,\tau_{\sigma(l(k))}^k)\\ \sigma\in\mathfrak{S}(l(k))}}(b_I\,id)(x_0)\int_0^T\alpha_I^{k,c}v_I^{k,c}(x_0)\\
&= x_0+\tilde B(x_0)a(x_0)+o(x_0)\\
&= Ax_0+o(x_0)\,,
\end{aligned}\tag{87}$$

where the second and third equalities come from (80) and (74), and the fourth from Lemma 3.
Assumption 2 is thus verified.

Verification of Assumption 3-bis
Since all vector fields b_i are δ^r-homogeneous of degree one, so that $\|I\|=|I|$, it is sufficient to show that

$$\int_0^I u_I(x)=a_Ix^{|I|}+o(x)\ . \tag{88}$$

From (75) and (80),

$$u_i(x,t)=\sum_{q=1}^{Q}\sum_{s:\tau_s^{k(q)}=i}\gamma_s^q(x,t)\,.$$

Therefore, for any I, $\int_0^T u_I(x)$ is a sum of terms of the form

$$\int_0^T \int_0^{t_{|I|}} \cdots \int_0^{t_2} \gamma_{s_{|I|}}^{q_{|I|}}(x,t)\dots\gamma_{s_1}^{q_1}(x,t)\,dt_1\dots dt_{|I|}\,, \tag{89}$$

for some multi-indices $(q_1,\dots,q_{|I|})$, and $(s_1,\dots,s_{|I|})$. From Lemma 5, we only need to consider the case where $q_1 = \dots = q_{|I|} \stackrel{\Delta}{=} q$ and $(s_1,\dots,s_{|I|}) = (\sigma(1),\dots,\sigma(|I|))$ with $|I| = S(q)$, since otherwise the expression in (89) is known to be a $o(x)$. In this case, (89) rewrites as

$$\begin{aligned}
&\prod_{i=1}^{S(q)} v_i^q(x) \int_0^T \int_0^{t_{S(q)}} \cdots \int_0^{t_2} \alpha^q_{s_{S(q)}}\dots\alpha^q_{s_1}\,dt_1\dots dt_{S(q)}\\
&= \sum_{j:\tilde{b}_j\in B_{\mathbf{k}(q)}} \mu_j^q a_j(x) \int_0^T \int_0^{t_{S(q)}} \cdots \int_0^{t_2} \alpha^q_{s_{S(q)}}\dots\alpha^q_{s_1}\,dt_1\dots dt_{S(q)}\\
&= h(x^{S(q)}) \int_0^T \int_0^{t_{S(q)}} \cdots \int_0^{t_2} \alpha^q_{s_{S(q)}}\dots\alpha^q_{s_1}\,dt_1\dots dt_{S(q)}\,,
\end{aligned} \tag{90}$$

where h is a linear map. Since $S(q) = |I|$, the last equality shows that the expression in (89) is linear in $x^{|I|}$ and, subsequently, that $\int_0^T u_I(x)$ is as announced in (88). ∎

4 Final remarks

We conclude the present study with a few general remarks. The first one concerns the assumption of analycity which has been made on the control vector fields of the system (S_0). In fact, the main results of the study remain valid when the control vector fields are smooth only, or even of class $\mathcal{C}^k$ with k large enough depending on the structure of the system's Control Lie Algebra. The proofs can be carried out in the same manner except for mild complications which arise in particular from using a finite expansion of the control system's solutions instead of the infinite Chen-Fliess expansion. Such a finite expansion can be derived in the same way as a Taylor expansion with integral remainder is obtained for a smooth function.

The second remark concerns possible applications of Theorem 1 in order to construct robust exponential stabilizers. In section 3, this result was combined with the use of sinusoidal functions of time in the control expression. However, there is no obligation for the control law to depend on time in this manner. For instance, when the system (S_0) is known to be differentially flat [4], adequate control functions can be obtained by considering specifically tailored flatness-based solutions to the open-loop steering problem, as done for example in [1] in the case of chained systems. Is is worth mentioning at this point that, although the control design approach and robustness analysis in [1] are very different from the ones developed in the present paper, the specific conditions imposed in this reference on the control law imply that the assumptions of Theorem 1 are verified. This suggests that these assumptions are not unduly strong and also illustrates the fact that the domain of application of Theorem 1 extends to different control design techniques.

How does the general control design procedure described in section 3.3 compare with the related one developed in [15] for the design of exponentially stabilizing continuous time-periodic feedbacks. Besides the fact, already pointed out before, that the latter method fails to produce controls which are robust in the sense considered in the present paper, the number of calculations required to synthesize the control law is also generally much higher and the resulting control expression significantly more complicated (because of state dependent terms which only vanish when the state is constant). Periodic dependency with respect to time also involves high frequencies resulting in highly oscillatory trajectories (a feature rarely desirable when dealing with mechanical systems), whereas the construction here proposed allows for choosing the control frequencies independently of the asymptotic rate of convergence.

Nonetheless, we are also aware that the hybrid open-loop/feedback controls here considered carry their own limitations the importance of which remains to be evaluated in future studies. For instance, just to cite a slightly uncommon issue, robustness with respect to modeling errors has been obtained under the assumption that the updating period of the control, i.e. the time interval during which the control is applied in open-loop fashion, is an exact multiple of the periods of the time functions involved in the control law. It is possible to show (this is beyond the scope of this study) that the slightest violation of this assumption, while unavoidable in practice for reasons that anyone having control implementation in mind will easily figure out, almost invariably results in the loss of stability of the origin of the controlled system. This means that the control is not robust with respect to the imperfect verification of this assumption. Although the source of this robustness problem is little connected with the modeling of the control system itself, its practical consequences should probably not be disregarded when attempting to address the complex and delicate issue of comparing different control techniques.

Appendix

Proof of Claim 1

Let us first consider the issue of existence of the solutions of system (15). This system can equivalently be written as a system in $\mathbb{R}^{n+1}$ with (x,ϵ) as state, and $(u, u_{m+1}\equiv 0)$ as control:

$$\begin{cases} \dot{x} = \sum_{i=1}^{m} (f_i(x) + h_i(\epsilon,x))\, u_i(x_0,t) \\ \dot{\epsilon} = u_{m+1} \triangleq 0\,. \end{cases}$$

By applying Proposition 1 to this system, one deduces that for any compact set $S\times[-\epsilon_1,\epsilon_1]$, $S\subset U$, there exists $\mu>0$ such that, if (10) and (11) are satisfied, the solution of (15) is defined on $[0,T]$ and can be expanded in the form of a Chen-Fliess series. Using Assumption 1 of Theorem 1, which implies that $|u(x,t)|$ tends to zero as $|x|$ tends to zero, one also deduces that for some positive δ, (10) and (11) are satisfied if $|x_0|\le\delta$. Existence (and uniqueness) of the solutions of (15) is therefore guaranteed, and these solutions can be expanded, on $[0,T]$, as

$$x(t) = x_0 + \sum_I ((f+h_\epsilon)_I\, id)(x_0) \int_0^t u_I(x_0)\,,$$

with $h_{\epsilon,i}(.) \triangleq h_i(\epsilon,.)$. We may rewrite this equality as

$$x(t) = x_0 + \sum_I (f_I\, id)(x_0) \int_0^t u_I(x_0) + \sum_I (d_I^{h_\epsilon}\, id)(x_0) \int_0^t u_I(x_0)\,. \tag{91}$$

Here, $d_I^{h_\epsilon} = d_{i_1}\cdots d_{i_k}$ (for $I=(i_1,\dots,i_k)$), with d_i taken in $\{f_i, h_{\epsilon,i}\}$ and the product $d_I^{h_\epsilon}$ involving at least one $h_{\epsilon,i}$. Note that each series in (91) is convergent uniformly w.r.t. x_0 ($|x_0|\le\delta'$, δ' possibly smaller that δ), ϵ ($|\epsilon|\le\epsilon_1$), and $t\in[0,T]$. This simply results from the existence (previously established) of $x(t)$, for $t\in[0,T]$, and the fact that the first series is convergent since

$$x_0 + \sum_I (f_I\, id)(x_0) \int_0^t u_I(x_0) \tag{92}$$

is precisely the Chen-Fliess expansion associated with (S_0).
Moreover, by Assumption 2 of Theorem 1

$$x_0 + \sum_I (f_I\, id)(x_0) \int_0^T u_I(x_0) = Ax_0 + o(x_0)\,. \tag{93}$$

Let us now define $\beta(\epsilon, x_0)$ and $\gamma(\epsilon, x_0)$ as follows

$$\beta(\epsilon, x_0) \triangleq \sum_{|I| \leq 1/\alpha} (d_I^{h_\epsilon}\, id)(x_0) \int_0^T u_I(x_0)\,, \quad \gamma(\epsilon, x_0) \triangleq \sum_{|I| > 1/\alpha} (d_I^{h_\epsilon}\, id)(x_0) \int_0^T u_I(x_0)\,. \tag{94}$$

so that

$$\sum_I (d_I^h\, id)(x_0) \int_0^T u_I(x_0) = \beta(\epsilon, x_0) + \gamma(\epsilon, x_0)\,, \tag{95}$$

From Assumption 1 and 3 of Theorem 1, each iterated integral involved in (94) satisfies, in the neighborhood of $x_0 = 0$,

$$|\int_0^T u_I(x_0)| \leq K|x_0|$$

for some positive constant K. This implies, in particular, that

$$\frac{|\beta(\epsilon, x_0)|}{|x_0|} \leq K \sum_{|I| \leq 1/\alpha} |(d_I^{h_\epsilon} id)(x_0)|\,.$$

Recalling that each product $d_I^{h_\epsilon}$ contains at least one $h_{\epsilon,i}$ and that $h_i(0, x) = 0,\ \forall x \in \mathbb{R}^n$, one deduces that every function $(\epsilon, x_0) \mapsto d_I^{h_\epsilon}(x_0)$ involved in right-hand side of the above inequality is continuous w.r.t x_0 and ϵ, and vanishes at $\epsilon = 0$. Therefore, for δ small enough, and using the fact that the number of multi-indices I such that $|I| < 1/\alpha$ is finite

$$\sup_{|x_0| \leq \delta} \frac{|\beta(\epsilon, x_0)|}{|x_0|} \longrightarrow 0 \text{ as } \epsilon \longrightarrow 0\,.$$

This establishes (17). Let us now define

$$\tilde{u}_i(x_0, t) \triangleq \frac{u_i(x_0, t)}{|x_0|^{\alpha - \sigma}} \qquad (i = 1, \ldots, m)$$

with $\sigma > 0$ so that, by Assumption 1 of Theorem 1, $\tilde{u}_i$ is continuous. Then,

$$\gamma(\epsilon, x_0) = \sum_{|I| > 1/\alpha} |x_0|^{|I|(\alpha - \sigma)} (d_I^{h_\epsilon}\, id)(x_0) \int_0^T \tilde{u}_I(x_0)\,.$$

Choosing σ small enough so that $|I|(\alpha - \sigma) \geq 1 + \mu_0 > 1$ for every I such that $|I| > 1/\alpha$, and since, from Proposition 1, the series

$$\sum_{|I| > 1/\alpha} (d_I^{h_\epsilon} id)(x_0) \int_0^T \tilde{u}_I(x_0)$$

is uniformly absolutely convergent for x_0 small enough, one obtains (provided that $|x_0| < 1$)

$$|\gamma(\epsilon, x_0)| \leq |x_0|^{1+\mu_0}\ S(\epsilon, x_0) \tag{96}$$

with

$$S(\epsilon, x_0) \triangleq \sum_{|I| > 1/\alpha} |(d_I^{h_\epsilon} id)(x_0)|\ |\int_0^T \tilde{u}_I(x_0)|\,,$$

a positive continuous function. Relation (18) directly follows from this inequality, and this concludes the proof of Claim 1. ∎

Proof of Lemma 1

We shall use the following relation, the proof of which is easily worked out by induction on k:

$$\int_0^t \int_0^{t_k} \dots \int_0^{t_2} \sin t_k \dots \sin t_1 \, dt_1 \dots dt_k = \sum_{j=1}^{k} c_{k,j}(1-\cos jt), \quad c_{k,k} = \frac{(-1)^{k-1}}{2^{k-1}k!}. \tag{97}$$

From this, we readily obtain the first equation in (53). We also deduce that

$$\begin{aligned} &\int_0^T \int_0^{t_{k+1}} \dots \int_0^{t_2} \cos it_{k+1} \sin t_k \dots \sin t_1 \, dt_1 \dots dt_{k+1} = \\ &\left\{ \begin{array}{ll} 0 & \text{if } i > k \\ \frac{(-1)^k}{2^k k!} T & \text{if } i = k \,. \end{array} \right. \end{aligned} \tag{98}$$

Now, we claim that for any $i \geq k$,

$$\begin{aligned} &\int_0^T \int_0^{t_{k+1}} \dots \int_0^{t_2} \sin t_{k+1} \dots \sin t_2 \cos it_1 \, dt_1 \dots dt_{k+1} = \\ (-1)^k &\int_0^T \int_0^{t_{k+1}} \dots \int_0^{t_2} \cos it_{k+1} \sin t_k \dots \sin t_1 \, dt_1 \dots dt_{k+1} \,. \end{aligned} \tag{99}$$

To show this, we view the first integral as a multiple integral on $\mathbb{R}^{k+1}$ (on the domain $\{(t_1, \dots, t_{k+1}) \in \mathbb{R}^{k+1} : 0 \leq t_1 \leq \dots \leq t_{k+1} \leq T\}$, to be more precise) so that this integral can also be written as

$$\int_0^T \int_{t_1}^T \dots \int_{t_k}^T \cos it_1 \sin t_2 \dots \sin t_{k+1} \, dt_1 \dots dt_{k+1} \,.$$

Setting $\tau_i \triangleq t_{k+2-i}$ $(i = 1, \dots, k+1)$, this gives

$$\begin{aligned} &\int_0^T \int_{\tau_{k+1}}^T \dots \int_{\tau_2}^T \cos i\tau_{k+1} \sin \tau_k \dots \sin \tau_1 \, d\tau_1 \dots d\tau_{k+1} = \\ \int_0^T &\left(\int_0^T - \int_0^{\tau_{k+1}} \right) \dots \left(\int_0^T - \int_0^{\tau_2} \right) \cos i\tau_{k+1} \sin \tau_k \dots \sin \tau_1 \, d\tau_1 \dots d\tau_{k+1} \,. \end{aligned}$$

Using (97), this last term simplifies into

$$(-1)^k \int_0^T \int_0^{\tau_{k+1}} \dots \int_0^{\tau_2} \cos i\tau_{k+1} \sin \tau_k \dots \sin \tau_1 \, d\tau_1 \dots d\tau_{k+1} \,,$$

and (99) follows. Finally, (53) follows directly from (98) and (99). ∎

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