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Irregularity of optimal trajectories in a control problem for a car-like robot *

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Abstract: We study the problem to find (a) shortest plane curve(s) joining two given points with given tangent angles and curvatures. The tangent angle and the curvature of the path are continuous and the derivative of the curvature is bounded by 2.

At a regular (i.e. of the class C^3) point such a curve must be locally a piece of a clothoid or a line segment (up to isometry a clothoid is given by Fresnel's integrals $x(t) = \int_0^t \cos \tau^2 d\tau$, $y(t) = \int_0^t \sin \tau^2 d\tau$).

We prove that if the distance between the initial and final points is greater than $320\sqrt{\pi}$, then a generic shortest curve contains infinitely many switching points.

Key-words: car-like robot, (sub)optimal path, clothoid, Maximum Principle of Pontryagin

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Irrégularité des trajectoires optimales dans le problème de contrôle optimale pour un robot mobile

Résumé : Nous considérons le problème de trouver les trajectoires les plus courtes joignant deux points dans $\mathbf{R}^2$, la dérivée de la courbure étant bornée par 2, les tangentes et les courbures du départ et de l'arrivée étant données, la tangente et la courbure de la trajectoire étant continues.

Aux points de continuité de la fonction contrôle la trajectoire optimale est un arc de clothoïde (à isométrie près une clothoïde est définie par les intégrales de Fresnel $x(t) = \int_0^t \cos \tau^2 d\tau$, $y(t) = \int_0^t \sin \tau^2 d\tau$) ou un segment de droite.

Si la distance entre les positions initiales et finales est plus grande que $320\sqrt{\pi}$, on prouve que les trajectoires optimales génériques sont irrégulières (i.e. la fonction contrôle a une infinité de points de discontinuité).

Mots-clés : robot mobile, chemin (sous)optimal, clothoïde, principe du maximum de Pontryagine

1 Introduction

We consider the problem to find the shortest path connecting two given points of $\mathbf{R^2}$ with given initial and final tangent angles and curvatures. The tangent angle and the curvature vary continuously, the speed of changing the curvature is bounded by 2 (we denote this derivative of the curvature by $u(t)$). The real background of the problem is to find the shortest paths for a car-like robot to go from one given point to another with the above mentioned initial and final conditions. One can turn the wheels of a car with a bounded speed. Hence, the speed of changing the curvature of the path of a real car is bounded.

This and similar problems have been the object of several efforts recently. Dubins in [8] considers the problem of constructing the optimal trajectory between two given points with given tangent angles and with bounded curvature (cusps are not allowed). He proves that there exists a unique optimal trajectory which is a concatenation of at most three pieces; every piece is either a straight line segment or an arc of a circle of fixed radius. The same model is considered by Cockayne and Hall in [7] but from another point of view: they provide the classes of trajectories by which a moving "oriented point" can reach a given point in a given direction and they obtain the set of all the points reachable at a fixed time.

Reeds and Shepp in [17] solve a similar problem, when cusps are allowed. They obtain the list of all possible optimal trajectories. This list contains forty eight types of trajectories. Each of them is a finite concatenation of pieces each of which is either a straight line or an arc of a circle.

Laumond and Souères in [14] obtain a complete synthesis for the Reeds-Shepp model in the case without obstacles.

A complete synthesis for the Dubins model in the case without obstacles is obtained by Boissonnat, Bui, Laumond and Souères (1994, see [4] and [5]).

All these authors use very particular methods in their proofs. It seems very difficult to generalize them. That is why the same problem is solved by Sussman and Tang in [18] and by Boissonnat, Cérézo and Leblond in [1] by means of simpler arguments based on the Maximum Principle of Pontryagin.

Using these arguments allows to treat more difficult models as the one considered in this paper. Here we consider a similar problem but now with a bounded derivative of the curvature (cusps are not allowed).

The same problem is considered in [3] by Boissonnat, Cérézo and Leblond. After applying the Maximum Principle of Pontryagin they obtain the following result: any extremal path is a C^2 concatenation of line segments in one and the same direction ($u(t) \equiv 0$) and of arcs of clothoid ($u(t) = \pm 2$), all of finite length. They study the possible variants of concatenation of arcs of clothoid and line segments and obtain that if an extremal path contains but is not reduced to a line segment, then it contains an infinite number of concatenated arcs of clothoids which accumulate towards each endpoint of the segment which is a switching point.

Thus, in the generic case, an optimal path can have at most a finite number of switching points only if it is a finite concatenation of arcs of clothoid ($u(t) = \pm 2$). Therefore to solve the problem of the irregularity of an optimal path in the generic case we consider extremal trajectories such that they contain a finite number of concatenated arcs of clothoids.

In the paper we obtain the following result: if the distance between the initial and the final points is greater than $320\sqrt{\pi}$, then, in the generic case, optimal paths have an infinite number of switching points. We prove this by showing that a path which is a finite concatenation of arcs of clothoids can be shortened while preserving the initial and final conditions, the continuity of the tangent angle and curvature and the boundedness of the curvature's derivative.

In Section 2 we consider the theoretical aspect of the problem, using the Maximum Principle of Pontryagin, and we formulate the main result of the paper in Theorem 2.7. In Section 3 we give the general outline of the solution of the problem. In this section we explain the main idea of the proof and we give a plan of Sections 4–11.

2 Statement of the problem, existence of an optimal solution and application of the Maximum Principle of Pontryagin to this problem

We study the shortest path on the plane joining two given points with given tangent angles and curvatures along which the derivative of the curvature remains bounded. We solve this problem in the class of all paths which are a C^2 concatenation of a finite number of open C^3 arcs of finite length. The tangent angle $\alpha(t)$ between the axis Ox and the tangent-vector to the path is a continuous and piecewise C^2 function, the curvature $\kappa(t)$ is a continuous and piecewise C^1 function.

We have the following system (from now on we denote "d/dt" by "$\cdot$"):

$$\dot{X}(t) = \begin{cases} \dot{x}(t) = \cos\alpha(t) \\ \dot{y}(t) = \sin\alpha(t) \\ \dot{\alpha}(t) = \kappa(t) \\ \dot{\kappa}(t) = u(t) \qquad |u(t)| \leq B \end{cases} \tag{1}$$

with the following initial and final conditions:

$$X(0) = (x^0, y^0, \alpha^0, \kappa^0)\ , \qquad X(T) = (x^T, y^T, \alpha^T, \kappa^T)\ . \tag{2}$$

Here $x(t)$ and $y(t)$ are the coordinates of some point in $\mathbf{R}^2$, $\kappa(t)$ is its curvature, $\alpha(t)$ is the angle between its tangent vector and the axis Ox.

A suitable changing of variables (homothety in x, y and t) allows us to consider only the case $B = 2$. From now on we set $B = 2$ (not $B = 1$ – which is more classical for optimal control problems) because it simplifies calculations. So, we consider the following system:

$$\dot{X}(t) = \begin{cases} \dot{x}(t) = \cos\alpha(t) \\ \dot{y}(t) = \sin\alpha(t) \\ \dot{\alpha}(t) = \kappa(t) \\ \dot{\kappa}(t) = u(t) \qquad |u(t)| \leq 2 \end{cases} \tag{3}$$

with initial and final conditions (2).

We control the derivative of the curvature by the control function u. The control function u is a measurable, real-valued function and $u \in U$, where $U = [-2, 2]$. We want to find $X(t)$ such that the associated control function $u(t)$ should minimize the length of the path

$$J(u) = T = \int_0^T dt \ . \tag{4}$$

Here the variable t is the arc length but it will be called the time because the point moves with a constant speed 1, that is why this "minimum length problem" is also a "minimum time problem".

The controllability of system (3), (2) and the existence of an optimal solution to the problem (2)–(4) is proved in [3].

To obtain necessary conditions for the control function $u(t)$ and for the trajectory ($x(t)$, $y(t)$, $\alpha(t)$, $\kappa(t)$) to be optimal we can apply the Maximum Principle of Pontryagin (see the details in [3]).

If we denote by $\Psi(t) = (\psi_0, \psi_1, \psi_2, \psi_3, \psi_4)$ the vecteur of "dual" variables, then for the Hamiltinian H we have the following formula:

$$H(X, \Psi, u) = \psi_0 + \psi_1 \cos\alpha + \psi_2 \sin\alpha + \psi_3 \kappa + \psi_4 u, \quad \text{for every } t \in [0, T]. \tag{5}$$

We have the following adjoint system (for every $t \in [0, T]$):

$$\dot{\Psi}(t) = \begin{cases} \dot{\psi}_0(t) = 0 \\ \dot{\psi}_1(t) = 0 \\ \dot{\psi}_2(t) = 0 \\ \dot{\psi}_3(t) = \psi_1(t) \sin\alpha(t) - \psi_2(t) \cos\alpha(t) \\ \dot{\psi}_4(t) = -\psi_3(t) \end{cases} \tag{6}$$

A measurable control function u and the associated trajectory of (1) satisfying all conditions of the Maximum Principle of Pontryagin (see [6], th.5.1i, [16], Chapter 1, th.1 and [3] Section 3.1) will be called *extremal control* and *extremal trajectory*. A point $X(t_p)$ of an extremal trajectory will be called a *switching point* if at $t = t_p$ the control function $u(t)$ has a discontinuity. An extremal trajectory is called *regular extremal trajectory* if it has a finite number of switching points.

After applying the Maximum Principle of Pontryagin we obtain the following result (see [3]):

Lemma 2.1 *Any regular extremal path is a C^2 concatenation of line segments in one and the same direction ($u = 0$) and of arcs of clothoids ($u = \pm 2$), all of finite length.*

A *clothoid* is a curve along which the curvature $\kappa(t)$ depends linearly on the arc length t and varies continuously from $-\infty$ to $+\infty$. In our case we consider only clothoids which satisfy the following equation (see Lemma 2.1):

$$\kappa(t) = \pm 2t \ , \qquad t \in (-\infty, +\infty) \ . \tag{7}$$

We can also define the clothoid by its parametrized form (setting $x(0) = y(0) = 0$, $\alpha(0) = 0$, $\kappa(0) = 0$)

$$\begin{cases} x(t) = \int_0^t \cos \tau^2 d\tau \\ y(t) = \pm \int_0^t \sin \tau^2 d\tau \end{cases}$$

The two possible choices of the sign correspond to the two possible orientations of the clothoid. Here t is the natural parameter and the curvature at the point $(x(t), y(t))$ equals $\pm 2t$. A clothoid is a curve symmetric with respect to its point of zero curvature. The part of the clothoid corresponding to $t \in [0, +\infty)$ or to $t \in (-\infty, 0]$ is called a *half-clothoid* (see an example of a half-clothoid on Figure 1).

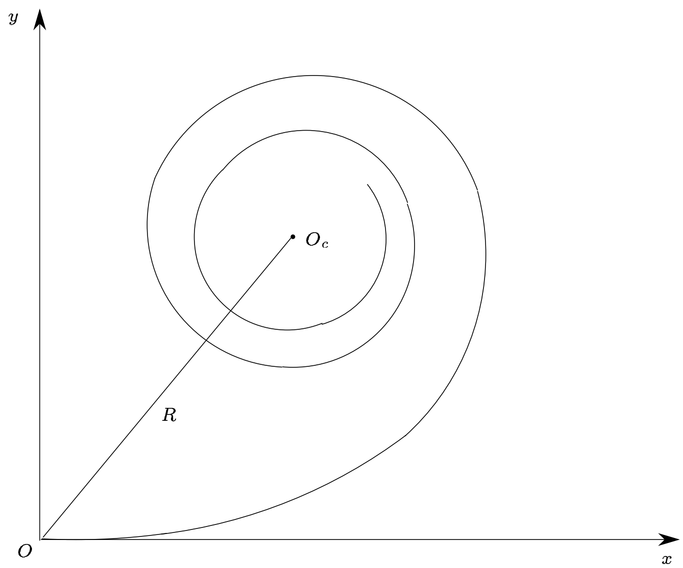


Figure 1

Define as "the centre of the half-clothoid" the point O_c with coordinates (x_{O_c}, y_{O_c}) defined as follows:

$$\begin{cases} x_{O_c} = \int_0^\infty \cos \tau^2 d\tau = \sqrt{\pi}/(2\sqrt{2}) \\ y_{O_c} = \int_0^\infty \sin \tau^2 d\tau = \sqrt{\pi}/(2\sqrt{2}) \end{cases}$$

Denote by R the length of the line segment $|OO_c|$. Then

$$R = \sqrt{\pi}/2 \ . \tag{8}$$

From now on we use the following notations for the arcs of clothoid and for the line segments:

1) "Cl^+" – an arc of clothoid with $u(t) \equiv B$, $\psi_4(t) < 0$,
2) "Cl^-" – an arc of clothoid with $u(t) \equiv -B$, $\psi_4(t) > 0$,
3) "S^φ" – a line segment of the direction φ ($u(t) \equiv 0$, $\psi_4(t) \equiv 0$),
4) "." – a switching point.

Proposition 2.2 *At any switching point ($Cl.Cl$, $Cl.S^\varphi$ or $S^\varphi.Cl$) one has $\psi_4(t) = 0$.*

Proof

On S^φ the continuous function $\psi_4(t) \equiv 0$, hence, $\psi_4(t) = 0$ at any switching point $Cl.S^\varphi$ or $S^\varphi.Cl$. At a switching point $Cl.Cl$ the signs of $u(t)$ and $\psi_4(t)$ change, hence, $\psi_4(t) = 0$ at this point. The proposition is proved. □

Proposition 2.3 *The expression $\psi_3(t)-\psi_1(t)y(t)+\psi_2(t)x(t)$ is constant along any extremal path.*

Proof

Calculate $\frac{d}{dt}[\psi_3(t) - \psi_1(t)y(t) + \psi_2(t)x(t)]$:

$$\frac{d}{dt}[\psi_3(t) - \psi_1(t)y(t) + \psi_2(t)x(t)] = \dot\psi_3(t) - \dot\psi_1(t)y(t) - \psi_1(t)\dot y(t) + \dot\psi_2(t)x(t) + \psi_2(t)\dot x(t)\ .$$

Hence, using (6) and (3), we obtain

$$\frac{d}{dt}[\psi_3(t) - \psi_1(t)y(t) + \psi_2(t)x(t)] = 0\ . \tag{9}$$

Thus, it follows from (9) that there is some constant $c_0 \in \mathbf{R}$ such that $\psi_3(t)-\psi_1(t)y(t)+\psi_2(t)x(t) = c_0$.

The proposition is proved. □

Lemma 2.4 *For every extremal path there is a coordinate system Oxy such that in this coordinate system the mean values of the y coordinate on any interval between two consecutive switching points are equal to zero.*

Proof

We consider some extremal trajectory. It follows from Proposition 2.3 that $\psi_3(t) = \psi_1(t)y(t)-\psi_2(t)x(t)-c_0$ along any extremal path. Hence, we can rotate the given coordinate system Oxy to some angle $\tilde\alpha$ such that in the new coordinate system $\psi_3(t) = y(t)$ along the extremal path under consideration.

Thus, using (6) we obtain

$$\dot\psi_4(t) = -y(t) \tag{10}$$

This equation holds along the extremal path. Consider some interval $[t_1, t_2] \in [0, T]$ between two consecutive switching points. Then it follows from (10) that

$$\psi_4(t_2) = -\int_{t_1}^{t_2} y(t) + \psi_4(t_1) \ . \tag{11}$$

But $\psi_4(t_1) = \psi_4(t_2) = 0$ (it follows from Proposition 2.2). Hence, we obtain $\int_{t_1}^{t_2} y(t) = 0$ for any interval between two consecutive switching points, i.e. the mean values of the y coordinate on any interval between two consecutive switching points are equal to zero.

The lemma is proved. □

In [3] Boissonnat, Cérézo and Leblond study the following problem: how are the arcs of clothoid and the line segments arranged along an extremal path? They obtain the following result:

Theorem 2.5 *If an extremal path contains but is not reduced to a line segment, then it contains an infinite number of concatenated arcs of clothoid which accumulate towards each endpoint of the segment which is a switching point.*

Remark: We can easily reprove Theorem 2.5 using Lemma 2.4. Really, we suppose that some regular extremal path contains but is not reduced to a line segment. So, for this path we choose a coordinate system Oxy such that in this coordinate system the mean values of the y coordinate on any interval between two consecutive switching points are equal to zero. In this coordinate system $\psi_3(t) = y(t)$ along the extremal path under consideration, i.e. $\psi_1(t) \equiv 1$, $\psi_2(t) \equiv 0$ (hence, $\lambda = 1$, $\varphi = 0$). Thus, along the line segment S^{φ} we have $y(t) \equiv 0$ (we denote this interval by $[t_1, t_2]$). But along any neighbour piece of clothoid (which corresponds, for example, to the interval $[t_2, t_3]$), we can't have $\int_{t_2}^{t_3} y(t)dt = 0$ (because $y(t_2) = 0$, $\kappa(t_2) = 0$ and $k(t)$ is either positive, or negative on the interval $(t_2, t_3]$; hence, $y(t)$ is either positive, or negative on the interval $(t_2, t_3]$) – a contradiction with Lemma 2.4.

Thus, in the generic case an optimal path can have at most a finite number of switching points only if it is a finite concatenation of arcs of clothoids. Therefore in the present paper we consider extremal trajectories such that they contain a finite number of concatenated arcs of clothoids and we prove the following theorem:

Theorem 2.6 *If the distance between the initial and the final points is greater than* $320\sqrt{\pi}$*, then, in the generic case, an optimal path can't consist of a finite number of concatenated arcs of clothoid.*

The proof of this theorem is given in Section 11. The reader who doesn't want to go into technical details but will be satisfied with the broad outlines of the proof can read only Section 3 and Section 11.

As a corollary of Theorem 2.5 and Theorem 2.6 we obtain Theorem 2.7 which contains the main result of the paper:

Theorem 2.7 *If the distance between the initial and the final points is greater than* $320\sqrt{\pi}$*, then, in the generic case, optimal paths have an infinite number of switching points.*

3 General outline of the solution of the problem

The main idea of the proof is to consider a class of C^2 and piecewise C^3 paths defined on the interval $[0, T]$, which are finite concatenations of arcs of clothoids defined by the equation $\kappa(t) = \pm 2t$ and which satisfy the initial and final conditions. We describe a procedure allowing to shorten an arbitrary path $\mathcal{P}$ from this class and at the same time to preserve the initial and final conditions and the bound of the derivative of the curvature. Denote by N the number of switching points of $\mathcal{P}$ and denote by t_s $(s = 1, \ldots, N)$ the value of t corresponding to some switching point.

Remark 3.1 *Remind that it follows from Lemma 2.4 that for any extremal path there is some coordinate system Oxy such that in this coordinate system the mean values of the y coordinate on any interval between two consecutive switching points are equal to zero. Hence, without loss of generality from now on we suppose that we consider this coordinate system. Remark that it is defined after the path.*

Consider the graph of the curvature κ as a function of the arc length t of the path $\mathcal{P}$. This graph is a piecewise linear function on $[0, T]$ and all pieces are line segments of the kind $\kappa(t) = \pm 2t + \tilde{\kappa}^0$ (see an example of such a graph on Figure 2).

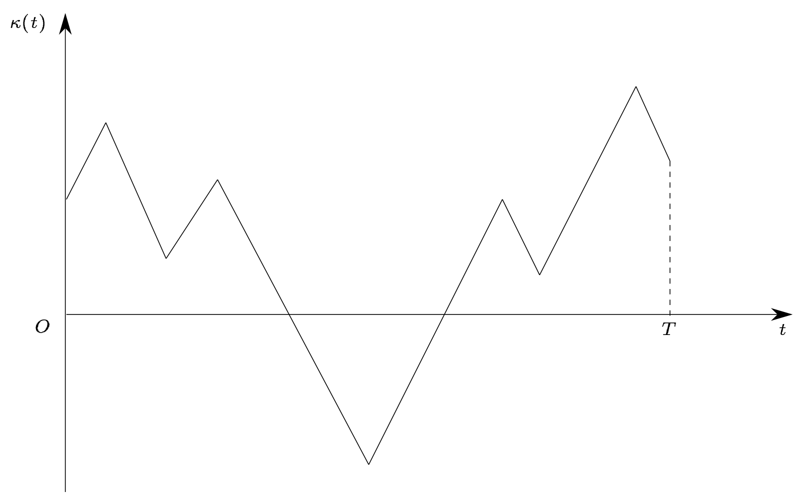


Figure 2

We carry out a modification of the path $\mathcal{P}$ by modifying the graph of its curvature. We want to modify the graph $\kappa(t)$ outside some neighbourhood of the initial point so that at the final point we shall have the same values of the final conditions and that the modified path will be shorter than the initial one.

Denote by

Final Conditions 3.2 *a) the given value κ^T of the curvature;*
b) the given value α^T of the tangent angle;
c) the given value x^T of the coordinate x;
d) the given value y^T of the coordinate y.

Denote by t_n the switching points (with even p for the maxima of the curvature and with odd p for the minima of the curvature). Respectively, denote by κ_n and y_n the curvature and the y-coordinate at the point t_n.

To prove Theorem 2.6 we introduce three different methods of modification. The choice of the method depends of the type of the path $\mathcal{P}$.

The path $\mathcal{P}$ consists of a finite number of arcs of clothoid. Hence, a priori we have 4 posibilities:

1) the generic case ($y_s \neq 0$ for any s);
2) the special case when there exists t_s such that $y_s = 0$, $y'_s \neq 0$;
3) the special case when there exists t_s such that $y_s = 0$, $y'_s = 0$, $y''_s \neq 0$;
4) the special case when there exists t_s such that $y_s = 0$, $y'_s = 0$, $y''_s = 0$, $y'''_s \neq 0$.

The are no other possibilities because the derivative of the curvature of the clothoid isn't equal to zero (it is equal to ± 2). Hence, no line can have a contact of order higher than 3 with a clothoid.

In the case when the mean values of the y-coordinates on every interval except the first and the last one are zero possibility 4) is absent. Really, if some point ($t = t_s$) has y-coordinate with a zero of third order, then this point is an inflexion point of the clothoid with $y(t_s) = y'(t_s) = y''(t_s) = 0$. Hence, the value of the y-coordinate of the path on the interval $[t_s, t_{s+1}]$ is positive (as on Figure 3) or negative, because in any coordinate system such that the inflexion point of the clothoid has y and y' equal to zero any half-clothoid of this clothoid is situated either above or below the axis Ox. So, the mean value of the y-coordinate of the path on the interval $[t_s, t_{s+1}]$ is positive (as on Figure 3) or negative, but it can't be equal to zero – a contradiction. Hence, we'll consider only the first three cases.

Consider, at first, the generic case when $y_s \neq 0$ for any s. In the generic case we must consider two possibilities:

1^0) there exist two indices p and q of the same parity such that $y_p \times y_q < 0$;
2^0) $y_p \times y_q > 0$ for any p and q of the same parity.

The possibility 1^0) is considered in Section 8. Consider now the possibility 2^0). Without loss of generality we can assume that $y_p > 0$ for any even p. In this case there are two subcases:

a) $y_p > 0$ for any p,
b) $y_p > 0$ for any even p and $y_p < 0$ for any odd p.

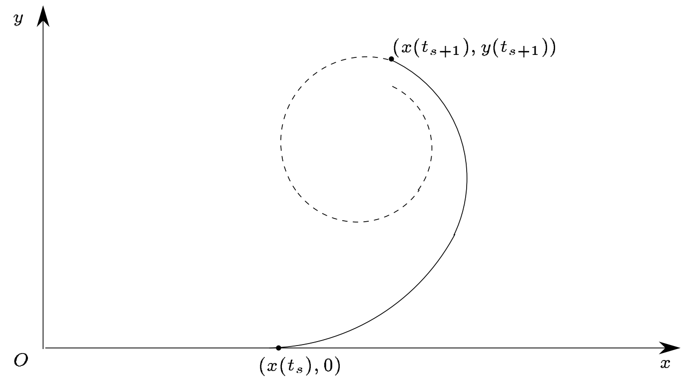


Figure 3

The subcase a) is considered in Subsection B.1.

The subcase b) is considered in Section 10 (the case when $y_p > 0$, $\kappa_p > 0$ for any even p and $y_p < 0$, $\kappa_p < 0$ for any odd p) and in Section 9 (the case when $y_p > 0$ for any even p; $y_p < 0$ for any odd p; and there exists at least one even index p (odd index p) such that $\kappa_p \leq 0$ ($\kappa_p \geq 0$)).

The special case 2) (i.e. the case when there exists t_s such that $y_s = 0$, $y_s' \neq 0$) is considered in Subsection B.2.

The special case 3) (i.e. the case when there exists t_s such that $y_s = 0$, $y_s' = 0$, $y_s'' \neq 0$) is considered in Subsection B.3.

One can see the scheme of all cases on Figure 4.

So there are many cases to consider, but in all these cases one uses only three different methods of modification of the path.

The first method is described in Sections 4–8 and in Appendix B. As a result we obtain a new path with the given initial and final conditions, which is shorter than the initial one and which is a finite concatenation of arcs of clothoids. In this method we use three types of modifications (see modification of type A in Section 5, modification of type B in Section 6 and modification of type C in Section 7).

The second method is introduced in the case when $y_p > 0$ for any even p, $y_p < 0$ for any odd p (with the possible exception of the first and of the last one) and there exists at least one even index p (odd index p) such that $\kappa_p \leq 0$ ($\kappa_p \geq 0$).

We describe this method in Section 9. As a result we obtain a new path with the given initial and final conditions, which is shorter than the initial one. However, it belongs to another class of paths, i.e. it consists of a line segment and of a finite number of arcs of clothoids.

The third method is introduced in the case when $y_p > 0$, $\kappa_p > 0$ for any even p and $y_p < 0$, $\kappa_p < 0$ for any odd p (with the possible exception of the first and of the last one).

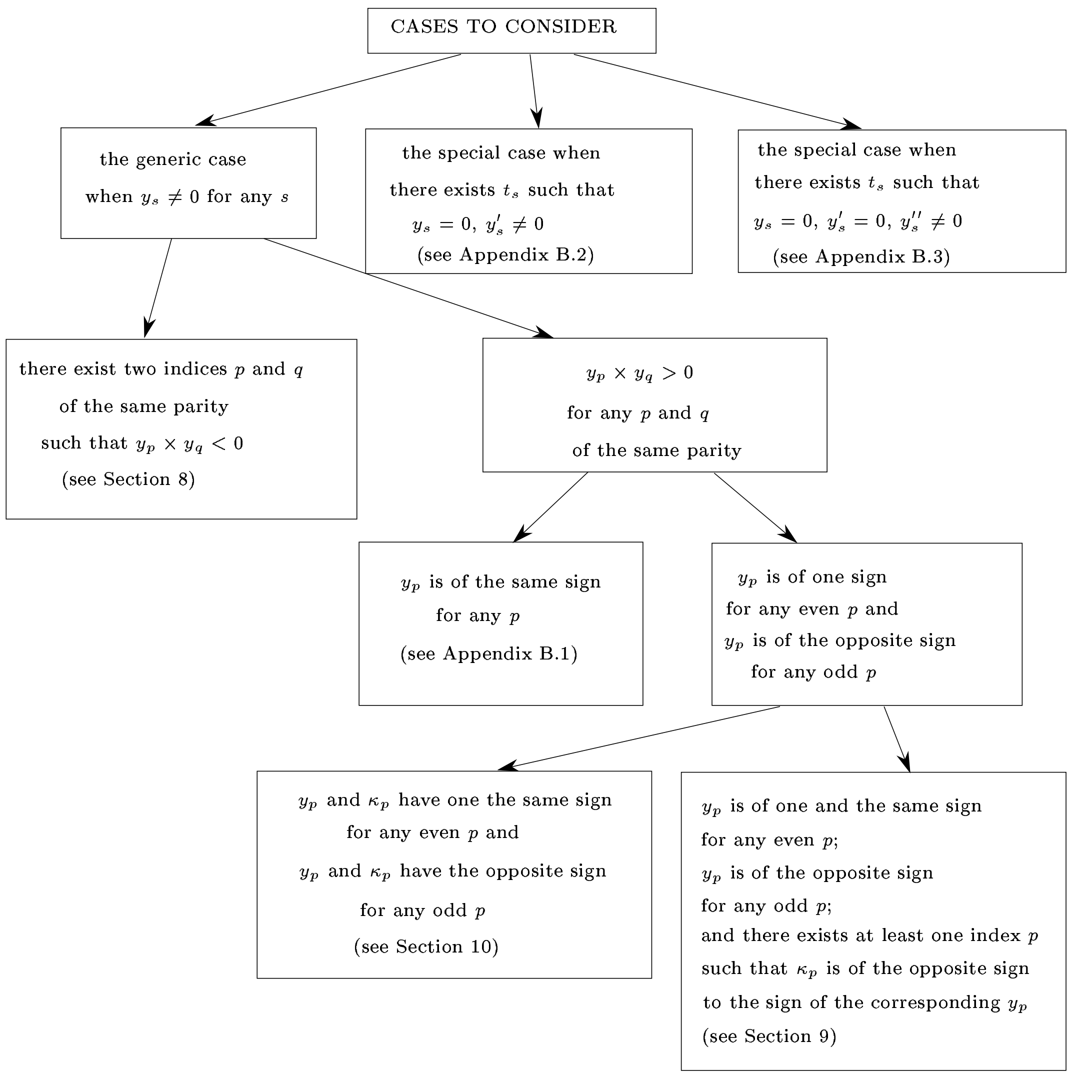

CASES TO CONSIDER
the generic case when $y_s \neq 0$ for any s
the special case when there exists t_s such that $y_s = 0,\ y'_s \neq 0$ (see Appendix B.2)
the special case when there exists t_s such that $y_s = 0,\ y'_s = 0,\ y''_s \neq 0$ (see Appendix B.3)
there exist two indices p and q of the same parity such that $y_p \times y_q < 0$ (see Section 8)
$y_p \times y_q > 0$ for any p and q of the same parity
y_p is of the same sign for any p (see Appendix B.1)
y_p is of one sign for any even p and y_p is of the opposite sign for any odd p
y_p and κ_p have one the same sign for any even p and y_p and κ_p have the opposite sign for any odd p (see Section 10)
y_p is of one and the same sign for any even p; y_p is of the opposite sign for any odd p; and there exists at least one index p such that κ_p is of the opposite sign to the sign of the corresponding y_p (see Section 9)

Figure 4

We describe this method in Section 10. As a result we obtain a new path with the given initial and final conditions, which is shorter than the initial one, but which belongs to another class of paths, i.e. it consists of a line segment and of a finite number of arcs of clothoids.

We prove Theorem 2.6 in Section 11.

Hence, we can conclude that there is no optimal path in the class of paths under consideration.

4 General description of an elementary modification

We consider some path $\mathcal{P}$ and we define an *elementary modification* as follows:

> 1) if we denote by $\mathcal{P}_{i,\varepsilon}$ some modified path, then $\mathcal{P}_{i,\varepsilon}$ contains a finite number of concatenated arcs of clothoid defined by the equation $\kappa(t) = \pm 2t$; it is defined on the interval $[0, T]$ and satisfies the initial conditions,
>
> 2) for $\mathcal{P}_{i,\varepsilon}$ all switching points are the same as the ones of $\mathcal{P}$, except the i-th one which changes from t_i to $t_i + \varepsilon$

(see an example of some such modification on the graph $\kappa(t)$, Figure 5).

On Figure 5 the dotted lines denote the pieces of the new graph. We remark that we can consider positive and negative ε such that $\varepsilon \in (t_{i-1} - t_i, t_{i+1} - t_i)$.

Proposition 4.1 *For any path $\mathcal{P}_{i,\varepsilon}$ obtained from the path $\mathcal{P}$ by means of some elementary modification, the final conditions are analytic functions of ε defined for $\varepsilon \in (t_{i-1} - t_i, t_{i+1} - t_i)$.*

Proof

The coordinates, the tangent angle and the curvature at the final point of the path $\mathcal{P}_{i,\varepsilon}$ corresponding to the value $t_i + \varepsilon$ of the modified path depend analytically on ε because they are expressed by formulas containing only analytic functions of t. For the same reason the coordinates, the tangent angles and the curvatures at the points corresponding to the values $t_{i+1}, t_{i+2}, \ldots, T$ are also some analytic functions of ε.

The proposition is proved. □

5 General description of a modification of type A

We consider some path $\mathcal{P}$ and we define a *modification of type A* as follows:

> 1) we modify the graph $\kappa(t)$ of $\mathcal{P}$ on two different intervals $[t_i, t_{i+1}]$ and $[t_j, t_{j+1}]$ (except the first and the last one) such that if $\kappa(t)$ has some local maximum at the moment t_i, it has some local minimum at the moment t_j and vice versa,
>
> 2) any modification of type A is a superposition of four elementary modifications:

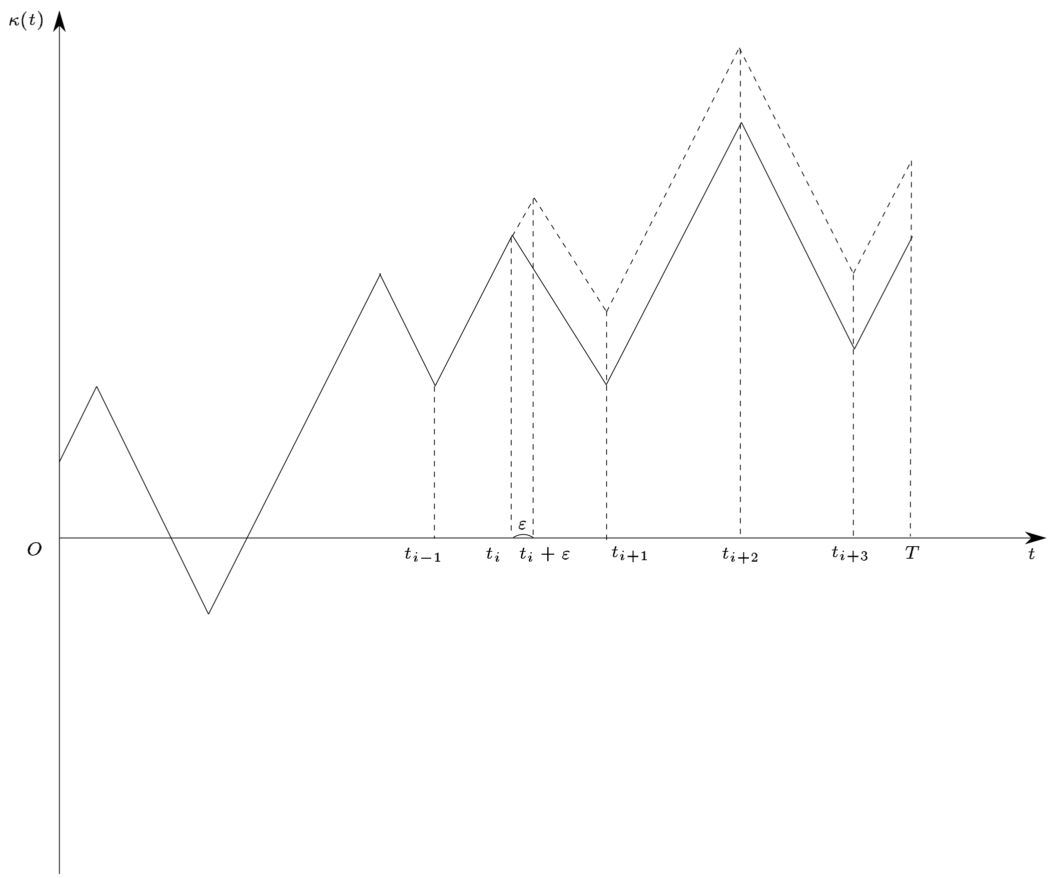


Figure 5

$$t_i \mapsto t_i + \delta_A K_i,$$
$$t_{i+1} \mapsto t_{i+1} + \delta_A K_i,$$
$$t_j \mapsto t_j + \delta_A K_j,$$
$$t_{j+1} \mapsto t_{j+1} + \delta_A K_j.$$

Here we denote by δ_A some small parameter (we can consider positive or negative δ_A) and we denote by K_i, K_j two positive constants such that $\delta_A K_i \in (t_{i-1} - t_i, t_{i+1} - t_i)$, $\delta_A K_j \in (t_{j-1} - t_j, t_{j+1} - t_j)$. We remark that for the modified path (we denote it by $\mathcal{P}_A$) condition a) of Final Conditions 3.2 holds (because by definition $\kappa(T)$ doesn't change, see an example of some such modification on Figure 6).

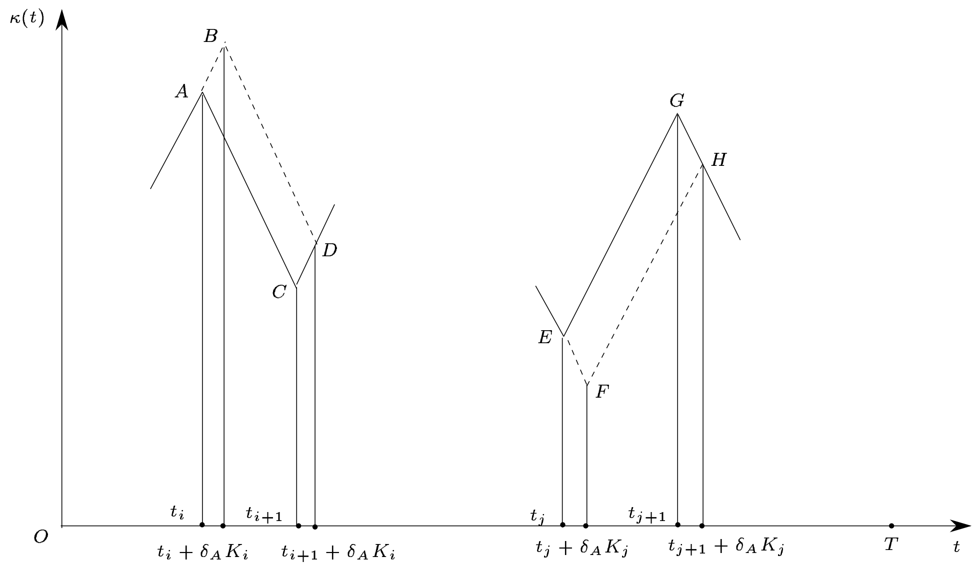


Figure 6

On Figure 6 the dotted lines denote the pieces of the new graph. We denote by $x_A(t)$, $y_A(t)$, $\alpha_A(t)$, $\kappa_A(t)$ the coordinates, the tangent angle and the curvature of a point of $\mathcal{P}_A$ corresponding to t.

Proposition 5.1 *If we want that on the path obtained from the path $\mathcal{P}$ by means of some modification of type A condition b) of Final Condition 3.2 hold, then we must construct this modification by means of constansts K_i, K_j such that equality (12) holds:*

$$K_i(t_{i+1} - t_i) = K_j(t_{j+1} - t_j) \tag{12}$$

(i.e. the area of $ABCD$ is equal to the area of $EFGH$, see Figure 6).

Remark 5.2 *We modify on two intervals ($[t_i, t_{i+1}]$ and $[t_j, t_{j+1}]$) to preserve condition b) of Final Conditions 3.2.*

Proof of Proposition 5.1

The tangent angle of the path $\mathcal{P}_A$ at the final point (we denote it by $\alpha_A(T)$) is equal to $\int_0^T \kappa_A(t)dt$. Hence, if we want that $\alpha_A(T) = \alpha^T$, then the area of $ABCD$ (we denote it by S_{ABCD}) must be equal to the area of $EFGH$ (we denote it by S_{EFGH}). We have

$$S_{ABCD} = 4\delta_A K_i(t_{i+1} - t_i) \ , \qquad S_{EFGH} = 4\delta_A K_j(t_{j+1} - t_j) \ .$$

So, we obtain the following equality:

$$K_i(t_{i+1} - t_i) = K_j(t_{j+1} - t_j) \ .$$

The proposition is proved. □

Proposition 5.3 *For any curve $\mathcal{P}_A$ obtained from the curve $\mathcal{P}$ by means of some modification of type A, the final coordinates $x_A(T)$ and $y_A(T)$ are some analytic functions of δ_A.*

Proposition 5.3 follows from Proposition 4.1 and from the definition of a modification of type A.

Proposition 5.4 *If $\kappa(t)$ of $\mathcal{P}$ has some local maximum at the moment t_i and if we consider a modification of type A such that on the new path $\mathcal{P}_A$ a) and b) of Final Conditions 3.2 hold, then for the coordinates of the final point of the path $\mathcal{P}_A$ we have the following formulas:*

$$\begin{aligned} x_A(T) &= x_T + \delta_A K_x^{(A)} + O(\delta_A^2) \ , \\ y_A(T) &= y_T + \delta_A K_y^{(A)} + O(\delta_A^2) \ , \end{aligned} \tag{13}$$

where

$$K_x^{(A)} = C_A(y(\mu) - y(\nu)) = 0 \ , \qquad K_y^{(A)} = -C_A(x(\vartheta) - x(\chi)) \ , \tag{14}$$

$$C_A = 4K_i(t_{i+1} - t_i) = 4K_j(t_{j+1} - t_j) \ .$$

Here the point μ (the point ν) is a point of the interval $[t_i, t_{i+1}]$ ($[t_j, t_{j+1}]$) such that $y(\mu)$ ($y(\nu)$) is equal to the mean value of the function $y(t)$ on this interval (respectively, $x(\vartheta)$ ($x(\chi)$) is equal to the mean value of the function $x(t)$ on the interval $[t_i, t_{i+1}]$ ($[t_j, t_{j+1}]$)). Remind that $y(\mu) = y(\nu) = 0$ for all the intervals $[t_i, t_{i+1}]$ and $[t_j, t_{j+1}]$ (with the possible exception of the first and of the last one, see Remark 3.1).

Proposition 5.5 *If $\kappa(t)$ of $\mathcal{P}$ has some local minimum at the moment t_i and if we consider a modification of type A such that on the new path $\mathcal{P}_A$ a) and b) of Final Conditions 3.2 hold, then for the coordinates of the final point of the path $\mathcal{P}_A$ we have formulas (13) where*

$$K_x^{(A)} = -C_A(y(\mu) - y(\nu)) = 0 \ , \qquad K_y^{(A)} = C_A(x(\vartheta) - x(\chi))$$

(see the definition of the points μ, ν, ϑ, χ and the definition of the constant C_A in Proposition 5.4).

One proves Proposition 5.5 in the same way as Proposition 5.4.

Proof of Proposition 5.4

By definition we have the following formulas:

$$x_A(T) = \int_0^T \cos\alpha_A(t)dt \ , \qquad y_A(T) = \int_0^T \sin\alpha_A(t)dt \ ,$$

$$x^T = \int_0^T \cos\alpha(t)dt \ , \qquad y^T = \int_0^T \sin\alpha(t)dt \ .$$

Hence, we obtain

$$x_A(T) - x^T = \int_0^T \cos\alpha_A(t)dt - \int_0^T \cos\alpha(t)dt = \int_0^T (\cos\alpha_A(t) - \cos\alpha(t))dt =$$

$$= -2\int_0^T \sin\frac{\alpha_A(t) - \alpha(t)}{2} \sin\frac{\alpha_A(t) + \alpha(t)}{2} dt \ .$$

For the small δ_A we have $\alpha_A(t) - \alpha(t) = O(\delta_A)$, hence,

$$\sin\frac{\alpha_A(t) - \alpha(t)}{2} = \frac{\alpha_A(t) - \alpha(t)}{2} + O(\delta_A^2) \ .$$

Thus,

$$x_A(T) - x^T = -\int_0^T (\alpha_A(t) - \alpha(t)) \sin\frac{\alpha_A(t) + \alpha(t)}{2} dt + O(\delta_A^2) =$$

$$= -\int_0^T (\alpha_A(t) - \alpha(t)) \sin\alpha(t)dt + O(\delta_A^2) = -\int_0^T (\alpha_A(t) - \alpha(t))\dot{y}(t)dt + O(\delta_A^2) \ .$$

Now we integrate by parts and we obtain

$$x_A(T) - x^T = -[(\alpha_A(t) - \alpha(t))y(t)]|_0^T + \int_0^T (\dot{\alpha}_A(t) - \dot{\alpha}(t))y(t)dt + O(\delta_A^2) =$$

$$= 0 + \int_0^T (\kappa_A(t) - \kappa(t))y(t)dt + O(\delta_A^2) =$$

$$= \int_{t_i}^{t_{i+1}} 4\delta_A K_i y(t)dt - \int_{t_j}^{t_{j+1}} 4\delta_A K_j y(t)dt + O(\delta_A^2) =$$

$$= 4\delta_A \left[K_i \int_{t_i}^{t_{i+1}} y(t)dt - K_j \int_{t_j}^{t_{j+1}} y(t)dt \right] + O(\delta_A^2) =$$

$$= 4\delta_A [K_i(t_{i+1} - t_i)y(\mu) - K_j(t_{j+1} - t_j)y(\nu)] + O(\delta_A^2)$$

(here the point μ (the point ν) is a point of the interval $[t_i, t_{i+1}]$ ($[t_j, t_{j+1}]$) such that $y(\mu)$ ($y(\nu)$) is equal to the mean value of the function $y(t)$ on this interval). Remind that $y(\mu) = y(\nu) = 0$ for all the intervals $[t_i, t_{i+1}]$ and $[t_j, t_{j+1}]$ (with the possible exception of the first and of the last one, see Remark 3.1).

Using (12), we obtain:

$$x_A(T) - x^T = 4\delta_A K_i(t_{i+1} - t_i)(y(\mu) - y(\nu)) + O(\delta_A^2) \ .$$

So, we obtain the following formula:

$$x_A(T) = x^T + K_x^{(A)}\delta_A + O(\delta_A^2)$$

where

$$K_x^{(A)} = C_A(y(\mu) - y(\nu)) = 0 \ , \qquad C_A = 4K_i(t_{i+1} - t_i) = 4K_j(t_{j+1} - t_j) \ .$$

We obtain the formula for $y_A(T)$ by analogy.

The proposition is proved. □

6 General description of a modification of type B

We consider some path $\mathcal{P}$ and we define a *modification of type B* as follows:

1) we modify the graph $\kappa(t)$ of $\mathcal{P}$ on the two last intervals (we denote them by $[t_{f-2}, t_{f-1}]$ and $[t_{f-1}, T]$ so that the modified path (we denote it by $\mathcal{P}_B$) should be shorter than the initial one,

2) any modification of type B is a superposition of three elementary modifications:

$$t_{f-2} \mapsto t_{f-2} + \delta_B K_1,$$
$$t_{f-1} \mapsto t_{f-1} + \delta_B K_1 + \delta_B K_2/2,$$
$$T \mapsto T + \delta_B K_2$$

(note that the modification of type B should shorten the path $\mathcal{P}$, therefore we consider only negative δ_B).

Here we denote by δ_B some small parameter and we denote by K_1, K_2 two positive constants such that $\delta_B K_1 \in (t_{f-3} - t_{f-2}, 0)$, $\delta_B K_2 \in (t_{f-1} - T, 0)$. We remark that for the modified path a) of Final Conditions 3.2 holds (because, by definition, $\kappa(T)$ doesn't change, see an example of some such modification of Figure 7).

On this figure the piece ADG belongs to the new graph and the piece $ABEF$ belongs to the old graph. We denote by $x_B(t)$, $y_B(t)$, $\alpha_B(t)$, $\kappa_B(t)$ the coordinates, the tangent angle and the curvature of a point of $\mathcal{P}_B$ corresponding to t and we denote by T_B the final point of the path $\mathcal{P}_B$ (i.e. $T_B = T + \delta_B K_2$).

Proposition 6.1 *If $\kappa(t)$ of $\mathcal{P}$ has some local maximum at the moment t_{f-1} and if we want that on the path obtained from the path $\mathcal{P}$ by means of some modification of type B condition b) of Final Conditions 3.2 holds, then we must construct this modification by means of constants K_1, K_2 verifying:*

$$4K_1(t_{f-1} - t_{f-2}) - K_2|\kappa_{f-1}| + \delta_B\left(K_2(2K_1 - K_2/2)\right) = 0 \qquad (15)$$

(i.e. the area of $ABCD$ is equal to the area of $CGKLFE$, see Figure 7).

Proof

The tangent angle of $\mathcal{P}_B$ at the final point (we denote it by $\alpha_B(T_B)$) is equal to $\int_0^{T_B} \kappa_B(t)dt$. Hence, if we want that $\alpha_B(T_B) = \alpha^T$, then the area of $ABCD$ (we denote it by S_{ABCD}) must be equal to the area of $CGKLFE$ (we denote it by S_{CGKLFE}). We have

$$S_{ABCD} = -4\delta_B K_1(t_{f-1} - t_{f-2} + \delta_B K_2/2) , \qquad S_{CGKLFE} = -|\kappa_{f-1}|\delta_B K_2 - \delta_B^2 K_2^2/2 .$$

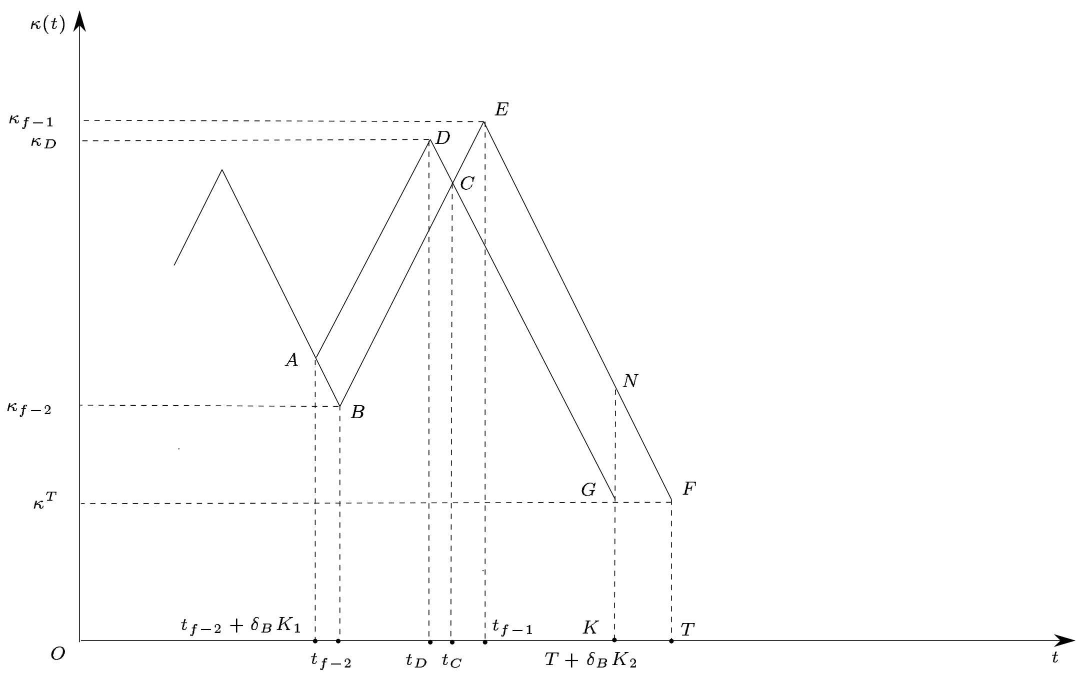
Thus, we obtain the following equality:

$$-4\delta_B K_1(t_{f-1} - t_{f-2} + \delta_B K_2/2) = -|\kappa_{f-1}|\delta_B K_2 - \delta_B^2 K_2^2/2 ,$$

i.e.

$$4K_1(t_{f-1} - t_{f-2}) - K_2|\kappa_{f-1}| + \delta_B\left(K_2(2K_1 - K_2/2)\right) = 0 .$$

The proposition is proved. □



$t_C = t_{f-1} + \delta_B K_2/2$

$t_D = t_{f-1} + \delta_B(K_1 + K_2/2)$

Figure 7

Proposition 6.2 *If $\kappa(t)$ of $\mathcal{P}$ has some local minimum at the moment t_{f-1} (see an example of some such modification on Figure 8) and if we want that on the path obtained from the path $\mathcal{P}$ by means of some modification of type B condition b) of Final Conditions 3.2 hold, then we must construct this modification by means of constants K_1, K_2 verifying:*

$$4K_1(t_{f-1} - t_{f-2}) + K_2(|\kappa^T| - 2(T - t_{f-1})) + \delta_B\,(K_2(2K_1 - K_2/2)) = 0 \qquad (16)$$

(i.e. the area of $ECGN$ is equal to the sum of the area of $ABCD$ and of the area of $KNFT$, see Figure 8).

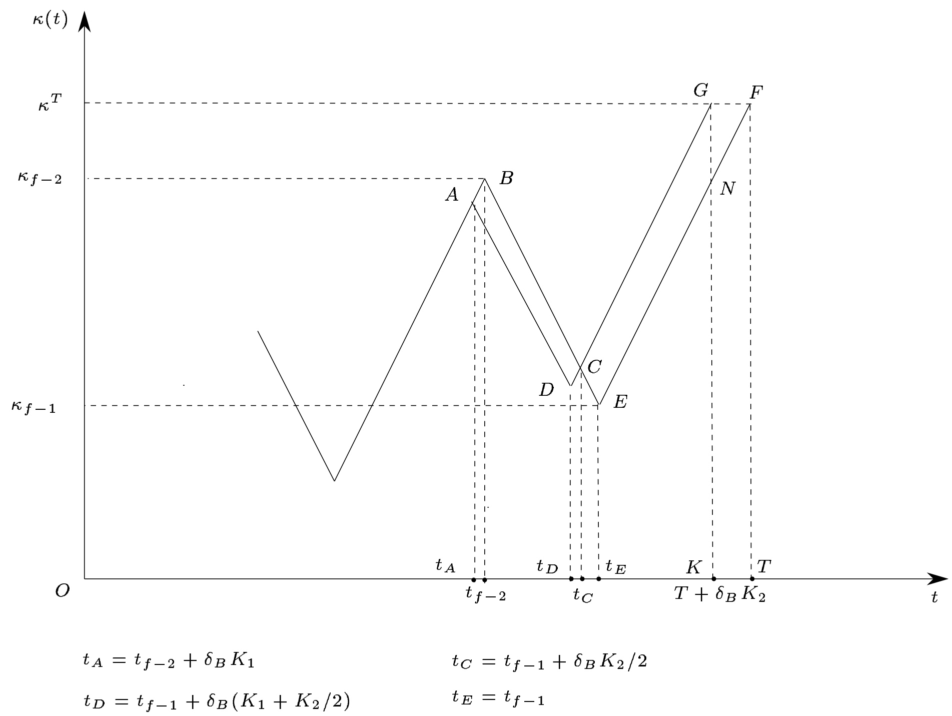


Figure 8

Proof

The tangent angle of $\mathcal{P}_B$ at the final point (we denote it by $\alpha_B(T_B)$) is equal to $\int_0^{T_B} \kappa_B(t)dt$. Hence, if we want that $\alpha_B(T_B) = \alpha^T$, then the area of $ECGN$ (we denote it by S_{ECGN}) must be equal to the sum of the area of $ABCD$ (we denote it by S_{ABCD}) and the area of $KNFT$ (we denote it by S_{KNFT}). We have

$$S_{ABCD} = -4\delta_B K_1(t_{f-1} - t_{f-2} + \delta_B K_2/2)\,, \qquad S_{ECGN} = \delta_B^2 K_2^2/2 - 2\delta_B K_2(T + \delta_B K_2 - t_{f-1})\,,$$

$$S_{KNFT} = -\delta_B K_2|\kappa^T| - \delta_B^2 K_2^2 \ .$$

Thus, we obtain the following equality:

$$-4\delta_B K_1(t_{f-1} - t_{f-2} + \delta_B K_2/2) - \delta_B K_2|\kappa^T| - \delta_B^2 K_2^2 = \delta_B^2 K_2^2/2 - 2\delta_B K_2(T + \delta_B K_2 - t_{f-1}) \ ,$$

i.e.

$$4K_1(t_{f-1} - t_{f-2}) + K_2|\kappa^T| - 2K_2(T - t_{f-1}) + \delta_B\left(K_2(2K_1 - K_2/2)\right) = 0 \ .$$

The proposition is proved. □

Proposition 6.3 *For any path $\mathcal{P}_B$ obtained from $\mathcal{P}$ by means of some modification of type B, the final coordinates $x_B(T_B)$ and $y_B(T_B)$ are some analytic functions of δ_B.*

Proposition 6.3 follows from Proposition 4.1 and from the definition of a modification of type B.

Proposition 6.4 *If $\kappa(t)$ of $\mathcal{P}$ has some local maximum at the moment t_{f-1} and if we consider a modification of type B such that on the new path $\mathcal{P}_B$ the condition b) of Final Conditions 3.2 hold, then for the coordinates of the final point of the path $\mathcal{P}_B$ we have the following formulas:*

$$x_B(T_B) = x^T + \delta_B K_x^{(B)} + O(\delta_B^2) \ ,$$
$$y_B(T_B) = y^T + \delta_B K_y^{(B)} + O(\delta_B^2) \ , \qquad (17)$$

où

$$K_x^{(B)} = -d'_B y(\mu) + d''_B y(\nu) + K_2(y(T_B)|\kappa^T| + \cos\alpha(T_B)) =$$
$$= d''_B y(\nu) + K_2(y(T_B)|\kappa^T| + \cos\alpha(T_B)) \ , \qquad (18)$$
$$K_y^{(B)} = d'_B x(\vartheta) - d''_B x(\chi) + K_2(-x(T_B)|\kappa^T| + \sin\alpha(T_B)) \ ,$$

$$d'_B = 4K_1(t_{f-1} - t_{f-2}) > 0 \ , \qquad d''_B = 2K_2(T - t_{f-1}) > 0 \ .$$

Here the point μ (the point ν) is a point from the interval $[t_{f-2}, t_{f-1}]$ ($[t_{f-1}, T]$) such that $y(\mu)$ ($y(\nu)$) is equal to the mean value of the function $y(t)$ on this interval (respectively, $x(\vartheta)$ ($x(\chi)$) is equal to the mean value of $x(t)$ on the interval $[t_{f-2}, t_{f-1}]$ ($[t_{f-1}, T]$)). Remind that $y(\mu) = 0$ on the interval $[t_{f-2}, t_{f-1}]$ (see Remark 3.1).

Proposition 6.5 *If $\kappa(t)$ of $\mathcal{P}$ has some local minimum at the moment t_{f-1} and if we consider a modification of type B such that on the new path $\mathcal{P}_B$ the condition b) of Final*

Conditions 3.2 holds, then for the coordinates of the final point of the path $\mathcal{P}_B$ we have formulas (17) where:

$$K_x^{(B)} = d'_B y(\mu) - d''_B y(\nu) + K_2(y(T_B)|\kappa^T| + \cos\alpha(T_B)) =$$

$$= -d''_B y(\nu) + K_2(y(T_B)|\kappa^T| + \cos\alpha(T_B)) \ , \qquad (19)$$

$$K_y^{(B)} = -d'_B x(\vartheta) + d''_B x(\chi) + K_2(-x(T_B)|\kappa^T| + \sin\alpha(T_B)) \ ,$$

(see the definition of the points μ, ν, ϑ, χ and the definition of the constants d'_B, d''_B in Proposition 6.4).

See the proofs of Proposition 6.4 (of Proposition 6.5) in Appendix A.1 (A.2 respectively).

Conclusion 6.6 *We can choose a modification of type B such that the modified path is shorter than the initial one and on the modified path conditions a), b) of Final Conditions 3.2 hold (by choosing suitable constants K_1, K_2 – see Proposition 6.1).*

7 General description of a modification of type C

A modification of type C can be considered as some modification of type A but in this case we modify the graph $\kappa(t)$ not on an interval but on a small neighbourhood of a switching point. We can carry out this modification on two small neighbourhoods of two switching points or on a small neighbourhood of some switching point and on some whole interval.

Consider an example of some modification of type C. Modify the graph κ as a function of t (corresponding to a given path $\mathcal{P}$) on a small left half-neighbourhood of the point $t_p + \zeta$ (i.e. on the interval $[t_p - \zeta, t_p]$) and on the interval $[t_l, t_{l+1}]$ (see Figure 9).

On this figure the pieces ADC and FEH belong to the new graph and the pieces ABC and FGH belong to the old one. We denote the obtained path by $\mathcal{P}_C$ and we denote by $x_C(t)$, $y_C(t)$, $\alpha_C(t)$, $\kappa_C(t)$ the coordinates, the tangent angle and the curvature of a point of $\mathcal{P}_C$ corresponding to t.

Remark 7.1 *Note that if we decide that the point $t_p - \zeta$ is a 'double' switching point (i.e. at the point $t_p - \zeta$ we change the vector field $\dot\kappa = 2$ to the vector field $\dot\kappa = -2$ during the time zéro), a modification of type C is some modification of type A and we can use the results obtained in Section 5 paying attention on the sign of δ_C.*

Remark 7.2 *We remark that*

1) in the case when we carry out a modification in the left half-neighbourhood of some local maximum (minimum) of the graph $\kappa(t)$ we consider only positive δ_C and we consider only switching points t_l which are local maxima (minima) of the graph $\kappa(t)$,

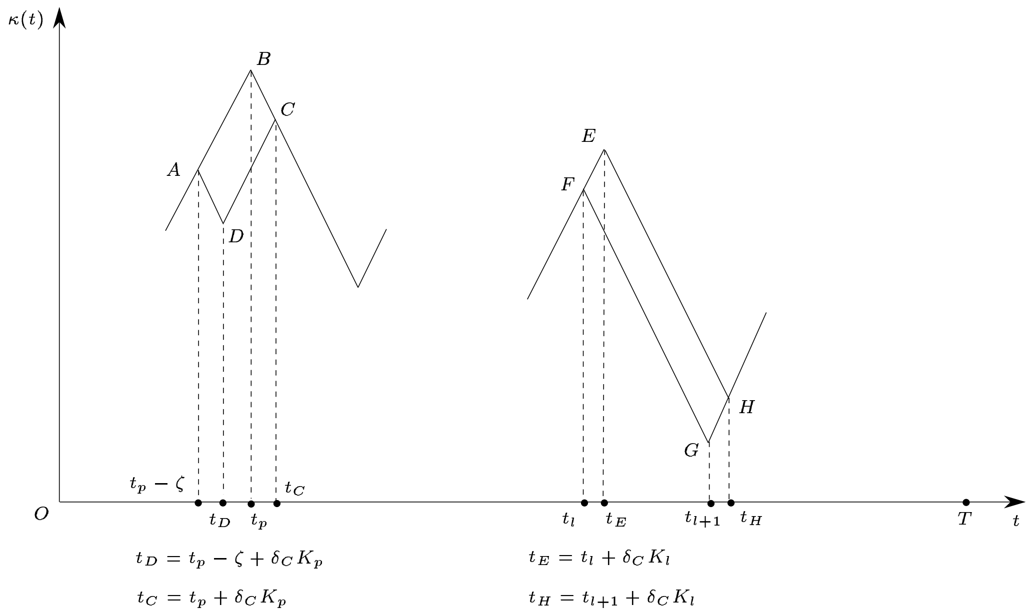

$\kappa(t)$
B
C
A
D
E
F
H
G
$t_p - \zeta$
t_C
O
t_D
t_p
t_l
t_E
t_{l+1}
t_H
T
t
$t_D = t_p - \zeta + \delta_C K_p$
$t_C = t_p + \delta_C K_p$
$t_E = t_l + \delta_C K_l$
$t_H = t_{l+1} + \delta_C K_l$

Figure 9

2) in the case when we carry out a modification in the right half-neighbourhood of some local maximum (minimum) of the graph $\kappa(t)$ we consider only negative δ_C and we consider only switching points t_l which are local minima (maxima) of the graph $\kappa(t)$.

As in Section 5 we obtain the following

Proposition 7.3 *If we want that on the path obtained from the path $\mathcal{P}$ by means of some modification of type C condition b) of Final Conditions 3.2 hold, then we must carry out this modification by means of constants K_p, K_l such that equality (20) holds:*

$$K_p\zeta = K_l(t_{l+1} - t_l) \tag{20}$$

(i.e. the area of $ABCD$ is equal to the area of $EFGH$, see Figure 9).

Proposition 7.4 *For any path $\mathcal{P}_C$ obtained from $\mathcal{P}$ by means of some modification of type C, the final coordinates $x_C(T)$ and $y_C(T)$ are some analytic functions of δ_C.*

Proposition 7.5 *If $\kappa(t)$ of $\mathcal{P}$ has some local maximum at the moment t_p and if we consider a modification of type C such that on the new path $\mathcal{P}_C$ conditions a) and b) of Final Conditions 3.2 hold, then for the coordinates of the final point of the path $\mathcal{P}_C$ we have the folowing formulas:*

$$\begin{aligned} x_C(T) &= x^T + \delta_C K_x^{(C)} + O(\delta_C^2)\ , \\ y_C(T) &= y^T + \delta_C K_y^{(C)} + O(\delta_C^2)\ , \end{aligned} \tag{21}$$

where

$$K_x^{(C)} = -C_C(y(\mu) - y(\nu)) = -C_C y(\mu)\ , \qquad K_y^{(C)} = C_C(x(\vartheta) - x(\chi))\ , \tag{22}$$

$$C_C = 4K_l(t_{l+1} - t_l) = 4K_p\zeta\ .$$

Here $x(\vartheta)$, $y(\mu)$, $(x(\chi)$, $y(\nu))$ are equal to the mean values of the corresponding coordinates on $[t_p - \zeta, t_p]$ $([t_l, t_{l+1}])$. Remind that $y(\nu) = 0$ for all the intervals $[t_l, t_{l+1}]$ (except the first and the last one, see Remark 3.1).

Proposition 7.6 *If $\kappa(t)$ of $\mathcal{P}$ has some local minimum at the moment t_p and if we consider a modification of type C such that on the new path $\mathcal{P}_C$ conditions a) and b) of Final Conditions 3.2 hold, then for the coordinates of the final point of the path $\mathcal{P}_C$ we have formulas (21) where:*

$$K_x^{(C)} = C_C(y(\mu) - y(\nu)) = C_C y(\mu)\ , \qquad K_y^{(C)} = -C_C(x(\vartheta) - x(\chi)) \tag{23}$$

(see the definition of the points μ, ν, ϑ, χ and the definition of the constant C_C dans la Proposition 7.5).

8 Proof of the non-optimality of the path $\mathcal{P}$ – case I

Remind that we denote by t_n switching points (with even n for the maxima of the curvature and with odd n for the minima of the curvature). Respectively, denote by κ_n and y_n the curvature and the y-coordinate at the point t_n.

Definition *Call 'case I' the case when $y_n \neq 0$ for any n and there exist two indices p and q of the same parity such that $y_p \times y_q < 0$.*

Without loss of generality we suppose that t_p and t_q are the switching points corresponding to two local maxima on the graph $\kappa(t)$ of the path $\mathcal{P}$ whose y-coordinates have different signs. Assume, that $y(t_p) > 0$, $y(t_q) < 0$. Hence, there exist small neighbourhoods $[t_p - \zeta, t_p + \zeta]$ and $[t_q - \gamma, t_q + \gamma]$ where the y-coordinates are positive or negative respectively.

Without loss of generality we suppose that the curvature of $\mathcal{P}$ has some local maximum at the moment t_{f-1}.

Now we perform two modifications. At first we modify the graph of $\kappa(t)$ on two intervals $[t_p - \zeta, t_p]$ and $[t_l, t_{l+1}]$ (it's a modification of type C, see Figure 9). We obtain the path $\mathcal{P}_{C1}$ and the graph $\kappa_{C1}(t)$. For $x_{C1}(T)$, $y_{C1}(T)$ we obtain formulas (21), (22).

The second modification is a modification of the graph of $\kappa(t)$ on two intervals $[t_q - \gamma, t_q]$ and $[t_l, t_{l+1}]$ (it's a modification of type C). We obtain the path $\mathcal{P}_{C2}$ and the graph $\kappa_{C2}(t)$. For $x_{C2}(T)$, $y_{C2}(T)$ we obtain formulas of the kind of formulas (21), (22).

Now we modify the given path $\mathcal{P}$ in two different ways and we show (see Lemma 8.1) that one of them shortens the path.

1) We make the first modification which amounts to simultaneously performing three modifications: a modification of type B (on the intervals $[t_{f-2}, t_{f-1}]$ and $[t_{f-1}, T]$), a modification of type A (on the intervals $[t_i, t_{i+1}]$ and $[t_j, t_{j+1}]$) and a modification of type C (on the intervals $[t_p - \zeta, t_p]$ and $[t_l, t_{l+1}]$). We obtain the path $\mathcal{P}_{BAC1}$.

2) The second modification is performed in a similar way but we perform a modification of type C on the intervals $[t_q - \gamma, t_q]$ and $[t_l, t_{l+1}]$ (instead of the intervals $[t_p - \zeta, t_p]$ and $[t_l, t_{l+1}]$). We obtain the path $\mathcal{P}_{BAC2}$.

We remark that all points t_i, t_{i+1}, t_j, t_{j+1}, t_q, t_p, t_l, t_{l+1}, t_{f-2}, t_{f-1}, T are different.

We formulate the fundamental result of this section in Lemma 8.1.

Lemma 8.1 *In case I we can choose three modifications (a modification of type A, a modification of type B and a modification of type C) such that the corresponding path $\mathcal{P}_{BAC1}$ (or $\mathcal{P}_{BAC2}$) should be shorter than the initial one and that for $\mathcal{P}_{BAC1}$ (or $\mathcal{P}_{BAC2}$) all conditions of Final Conditions 3.2 should hold.*

Lemma 8.1 follows from Propositions 8.2, 8.3 and from Lemma 8.4.

Using the results obtained in Propositions 5.1, 6.1 (or 6.2) and 7.3 we obtain:

Proposition 8.2 *If we carry out a modification which amounts to simultaneously performing three modifications (a modification of type B (on the intervals $[t_{f-2}, t_{f-1}]$ and $[t_{f-1}, T]$), a modification of type A (on the intervals $[t_i, t_{i+1}]$ and $[t_j, t_{j+1}]$) and a modification of type*

C (on the intervals $[t_p-\zeta, t_p]$ and $[t_l, t_{l+1}]$)) such that for the corresponding constants K_1, K_2, K_i, K_j, K_p, K_l the following equalities hold:

$$\begin{array}{ll} K_2 = 1 \ , & K_1 = f(\delta_B) \ , \\ K_j = t_{i+1} - t_i \ , & K_i = t_{j+1} - t_j \ , \\ K_l = \zeta \ , & K_p = t_{l+1} - t_l \ , \end{array}$$

(f is defined from formula (15) (or (16))), then for the thus constructed path $\mathcal{P}_{BAC1}$ conditions a) and b) of Final Conditions 3.2 hold.

Evidently, the same statement holds for the path $\mathcal{P}_{BAC2}$.

From now on in this section we consider three modifications such that for the corresponding path $\mathcal{P}_{BAC1}$ ($\mathcal{P}_{BAC2}$) conditions a) and b) of Final Conditions 3.2 hold.

Recall that we denote the coordinates of the final point of the path $\mathcal{P}_B$ by $(x_B(T), y_B(T))$. For the paths $\mathcal{P}_{BA}$ and $\mathcal{P}_{BAC1}$ ($\mathcal{P}_{BAC2}$) we have respectively $(x_{BA}(T), y_{BA}(T))$ and $(x_{BAC1}(T), y_{BAC1}(T))$ $((x_{BAC2}(T), y_{BAC2}(T)))$.

Using formulas (13), (17) and (21) (see Propositions 5.4, 6.4 and 7.5) we obtain:

Proposition 8.3 *The coordinates $(x_{BAC1}(T_B), y_{BAC1}(T_B))$ $((x_{BAC2}(T_B), y_{BAC2}(T_B)))$ are described by analytic functions defined in a small neighbourhood of zero with respect to the small parameters $(\delta_B, \delta_A, \delta_{C1})$ $((\delta_B, \delta_A, \delta_{C2}))$.*

Set

$$\begin{array}{ll} \Delta x_{BAC1} = x_{BAC1}(T_B) - x^T \ , & \Delta y_{BAC1} = y_{BAC1}(T_B) - y^T \ , \\ \Delta x_{BAC2} = x_{BAC2}(T_B) - x^T \ , & \Delta y_{BAC2} = y_{BAC2}(T_B) - y^T \ . \end{array}$$

So

$$\begin{array}{ll} \Delta x_{BAC1} = \Delta x_{BAC1}(\delta_B, \delta_A, \delta_{C1}) \ , & \Delta y_{BAC1} = \Delta y_{BAC1}(\delta_B, \delta_A, \delta_{C1}) \ , \\ \Delta x_{BAC2} = \Delta x_{BAC2}(\delta_B, \delta_A, \delta_{C2}) \ , & \Delta y_{BAC2} = \Delta y_{BAC2}(\delta_B, \delta_A, \delta_{C2}) \ . \end{array}$$

We must prove that we can chose some path (either in the class of paths $\mathcal{P}_{BAC1}$ or in the class of paths $\mathcal{P}_{BAC2}$) such that for this path all conditions of Final Conditions 3.2 hold and which is shorter than the initial path $\mathcal{P}$ (it means that we can express δ_A and δ_{C1} (or δ_A and δ_{C2}) as functions of δ_B verifying system (24) (or system (25)).

$$\left\{ \begin{array}{l} \Delta x_{BAC1}(\delta_B, \delta_A, \delta_{C1}) = 0 \\ \Delta y_{BAC1}(\delta_B, \delta_A, \delta_{C1}) = 0 \end{array} \right. \tag{24}$$

$$\left\{ \begin{array}{l} \Delta x_{BAC2}(\delta_B, \delta_A, \delta_{C2}) = 0 \\ \Delta y_{BAC2}(\delta_B, \delta_A, \delta_{C2}) = 0 \end{array} \right. \tag{25}$$

Lemma 8.4 *In case I one can choose some path (either in the class of paths $\mathcal{P}_{BAC1}$ or in the class of paths $\mathcal{P}_{BAC2}$) such that for this path all conditions of Final Conditions 3.2 hold and which is shorter than the initial path $\mathcal{P}$.*

Proof

At first we study system (24).

In the equation $\Delta x_{BAC1}(\delta_B, \delta_A, \delta_{C1}) = 0$ the term in δ_A equals $K_x^{(A)}\delta_A = 0$ (see formula (14)), the term in δ_{C1} equals $K_x^{(C1)}\delta_{C1} = -C_{C1}\zeta y(\mu)\delta_{C1} \neq 0$ (see formula (22)), the term in δ_B equals $K_x^{(B)}\delta_B$ (see formula (18)). Also in $\Delta x_{BAC1}(\delta_B, \delta_A, \delta_{C1}) = 0$ there are some terms of higher order. We denote them by $O(\delta_A\delta_{C1})$, $O(\delta_B\delta_{C1})$, $O(\delta_A\delta_B)$ and $O(\delta_B^2)$.

In the equation $\Delta y_{BAC1}(\delta_B, \delta_A, \delta_{C1}) = 0$ the term in δ_A equals $K_y^{(A)}\delta_A = -C_A(x(\vartheta) - x(\chi))$ (see formula (14)), the term in δ_{C1} equals $K_y^{(C1)}\delta_{C1}$ (see formula (22)), the term in δ_B equals $K_y^{(B)}\delta_B$ (see formula (18)). Also in $\Delta y_{BAC1}(\delta_B, \delta_A, \delta_{C1}) = 0$ there are some terms of higher order. We denote them by $O(\delta_A\delta_{C1})$, $O(\delta_B\delta_{C1})$, $O(\delta_A\delta_B)$ and $O(\delta_B^2)$.

We remark that in the formula $K_y^{(A)}\delta_A = -C_A(x(\vartheta) - x(\chi))$ the constant C_A is positive and the point ϑ (the point χ) is a point from the interval $[t_i, t_{i+1}]$ ($[t_j, t_{j+1}]$) such that $x(\vartheta)$ ($x(\chi)$) equals the mean value of the function $x(t)$ on this interval. As the distance between the initial and final points is rather great and as the mean value of the function $y(t)$ equals zero for all intervals (with the possible exception of the first and of the last one, see Remark 3.1), then one can always find two intervals $[t_i, t_{i+1}]$ and $[t_j, t_{j+1}]$ such that $x(\vartheta) \neq x(\chi)$. So, $K_y^{(A)}\delta_A \neq 0$.

If $(K_x^{(B)}, K_y^{(B)}) = (0, 0)$, then among the terms of order δ_B^2, $\delta_B^3, \ldots$ there are some terms which don't vanish in at least one equation from system (24), because the path $\mathcal{P}$ consists of several arcs of half-clothoid (not of a line segment).

Thus there exist some p, q and $k \geq 1$ such that $(p, q) \neq (0, 0)$ and that in the first equation $O(\delta_B^2) = p\delta_B^k + O(\delta_B^{k+1})$, in the second equation $O(\delta_B^2) = q\delta_B^k + O(\delta_B^{k+1})$.

Now we study system (25).

In the equation $\Delta x_{BAC2}(\delta_B, \delta_A, \delta_{C2}) = 0$ the term in δ_{C2} equals $K_x^{(C2)}\delta_{C2} = -C_{C2}\gamma y(\mu)\delta_{C2} \neq 0$ (see formula (22)). In the equations $\Delta x_{BAC2}(\delta_B, \delta_A, \delta_{C2}) = 0$ and $\Delta x_{BAC1}(\delta_B, \delta_A, \delta_{C1}) = 0$ the term in δ_A and the term in δ_B are the same because they don't depend on the modifications of type C. The terms of higher order in these two equations are different but we always denote them by $O(\delta_A\delta_{C1})$, $O(\delta_B\delta_{C1})$, $O(\delta_A\delta_B)$ and $O(\delta_B^{k+1})$.

In the equation $\Delta y_{BAC2}(\delta_B, \delta_A, \delta_{C2}) = 0$ the term in δ_{C2} equals $K_y^{(C2)}\delta_{C2}$ (see formula (22)). In the equations $\Delta y_{BAC2}(\delta_B, \delta_A, \delta_{C2}) = 0$ and $\Delta y_{BAC1}(\delta_B, \delta_A, \delta_{C1}) = 0$ the term in δ_A and the term in δ_B are the same because they don't depend on the modifications of type C. The terms of higher order in these two equations are different but we always denote them by $O(\delta_A\delta_{C2})$, $O(\delta_B\delta_{C2})$, $O(\delta_A\delta_B)$ and $O(\delta_B^{k+1})$.

We remark that as the functions participating in the equations are analytic at 0 (i.e. they admit convergent Taylor series expansion with respect to δ_A, δ_B, δ_{C1} (with respect to δ_A, δ_B, δ_{C2})), to find the terms δ_B^k it is sufficient to consider the case when there is no modification of type A and C (i.e. $\delta_A = \delta_C = 0$). Thus, p and q are the same in systems (24) and (25).

So, we can rewrite system (24) (system (25)) as system (26) (as system (27) respectively):

$$\left\{ \begin{array}{r} 0 + K_x^{(C1)}\delta_{C1} + p\delta_B^k + O(\delta_A\delta_{C1}) + O(\delta_B\delta_{C1}) + O(\delta_A\delta_B) + O(\delta_B^{k+1}) = 0 \\ K_y^{(A)}\delta_A + K_y^{(C1)}\delta_{C1} + q\delta_B^k + O(\delta_A\delta_{C1}) + O(\delta_B\delta_{C1}) + O(\delta_A\delta_B) + O(\delta_B^{k+1}) = 0 \end{array} \right. \qquad (26)$$

$$\left\{ \begin{array}{r} 0 + K_x^{(C2)}\delta_{C2} + p\delta_B^k + O(\delta_A\delta_{C2}) + O(\delta_B\delta_{C2}) + O(\delta_A\delta_B) + O(\delta_B^{k+1}) = 0 \\ K_y^{(A)}\delta_A + K_y^{(C2)}\delta_{C2} + q\delta_B^k + O(\delta_A\delta_{C2}) + O(\delta_B\delta_{C2}) + O(\delta_A\delta_B) + O(\delta_B^{k+1}) = 0 \end{array} \right. \qquad (27)$$

Here

$$K_x^{(C1)} = -C_{C1}\zeta y(\mu) < 0 \ , \qquad K_x^{(C2)} = -C_{C2}\gamma y(\kappa) > 0$$

(because $C_{C1} > 0$, $C_{C2} > 0$, $\zeta > 0$, $\gamma > 0$ and $y(\mu) > 0$, $y(\kappa) < 0$, as $y(\mu)$ ($y(\kappa)$ respectively) equals the mean value of the y-coordinate of the path $\mathcal{P}$ on the interval $[t_p - \zeta, t_p]$ (on $[t_q - \gamma, t_q]$ respectively), see the definition of these intervals at the beginning of the section).

I. At first we consider the case $p \neq 0$. In this case we choose what modification of type C we'll carry out in the following way.

Consider system (26). We can apply the implicit function theorem (for the germs of analytic functions) because the Jacobi matrix of system (26)

$$\left(\begin{array}{cc} \frac{\partial(\Delta x_{BAC1})}{\partial\delta_A} & \frac{\partial(\Delta x_{BAC1})}{\partial\delta_{C1}} \\ \frac{\partial(\Delta y_{BAC1})}{\partial\delta_A} & \frac{\partial(\Delta y_{BAC1})}{\partial\delta_{C1}} \end{array} \right) \Bigg|_{\delta_A=\delta_B=\delta_{C1}=0} = \left(\begin{array}{cc} 0 & K_x^{(C1)} \\ K_y^{(A)} & K_y^{(C1)} \end{array} \right)$$

is non-degenerate.

So, there exists a unique pair of Taylor series $\delta_A = \delta_A(\delta_B)$, $\delta_{C1} = \delta_{C1}(\delta_B)$ satisfying system (26) and $\delta_A(0) = 0$, $\delta_{C1}(0) = 0$.

Set

$$\begin{array}{ll} \delta_A = \tilde{\lambda}_1\delta_B^k \ , & \tilde{\lambda}_1 = \lambda_1(1 + o(1)) \ , \\ \delta_{C1} = \tilde{\mu}_1\delta_B^k \ , & \tilde{\mu}_1 = \mu_1(1 + o(1)) \ . \end{array} \qquad (28)$$

It is sufficient to prove that there exist some Taylor series (formal) $\tilde{\lambda}_1(\delta_B)$, $\tilde{\mu}_1(\delta_B)$; the uniqueness of δ_A, δ_{C1} (assured by the implicit function theorem) implies that δ_A, δ_{C1} of (28) is solution of (26).

Using (28) we can rewrite (26) as follows:

$$\left\{ \begin{array}{r} 0 + K_x^{(C1)}\mu_1 + p + \lambda_1\mu_1 O(\delta_B^k) + \mu_1 O(\delta_B) + \lambda_1 O(\delta_B) + O(\delta_B) = 0 \\ K_y^{(A)}\lambda_1 + K_y^{(C1)}\mu_1 + q + \lambda_1\mu_1 O(\delta_B^k) + \mu_1 O(\delta_B) + \lambda_1 O(\delta_B) + O(\delta_B) = 0 \end{array} \right. \qquad (29)$$

We can apply the implicit function theorem to system (29) (here λ_1 and μ_1 are variables) because the Jacobi matrix of this system is non-degenerate:

$$\det \left| \begin{array}{cc} 0 & K_x^{(C1)} \\ K_y^{(A)} & K_y^{(C1)} \end{array} \right| \neq 0 \ .$$

So from (29) we obtain λ_1 and μ_1:

$$\begin{cases} 0 & + & K_x^{(C1)}\mu_1 & + & p & = & 0 \\ K_y^{(A)}\lambda_1 & + & K_y^{(C1)}\mu_1 & + & q & = & 0 \end{cases}$$

i.e.

$$\lambda_1 = \frac{K_y^{(C1)}p - K_x^{(C1)}q}{K_x^{(C1)}K_y^{(A)}} \ , \qquad \mu_1 = -\frac{p}{K_x^{(C1)}} \ .$$

We consider system (27) by analogy, i.e. we search for δ_A and δ_{C2} as functions of δ_B in the following form:

$$\begin{array}{ll} \delta_A = \tilde{\lambda}_2 \delta_B^k \ , & \tilde{\lambda}_2 = \lambda_2(1 + o(1)) \ , \\ \delta_{C2} = \tilde{\mu}_2 \delta_B^k \ , & \tilde{\mu}_2 = \mu_2(1 + o(1)) \ , \end{array} \tag{30}$$

and we obtain

$$\lambda_2 = \frac{K_y^{(C2)}p - K_x^{(C2)}q}{K_x^{(C2)}K_y^{(A)}} \ , \qquad \mu_2 = -\frac{p}{K_x^{(C2)}} \ .$$

Recall that $\delta_{C1} > 0$, $\delta_{C2} > 0$, $\delta_B < 0$. Hence, we must consider four cases.

1) If k (see (28), (30)) is an even number, then $\delta_B^k > 0$ and $\mu_1 > 0$, $\mu_2 > 0$. There are two subcases depending on the sign of p.

a) If $p > 0$, then, using $K_x^{(C1)} < 0$ and $K_x^{(C2)} > 0$, we obtain

$$\mu_1 = -\frac{p}{K_x^{(C1)}} > 0 \ , \qquad \mu_2 = -\frac{p}{K_x^{(C2)}} < 0 \ .$$

Hence, we carry out a modification of type C on the intervals $[t_p - \zeta, t_p]$ and $[t_l, t_{l+1}]$ (see system (26)).

b) If $p < 0$, then, using $K_x^{(C1)} < 0$ and $K_x^{(C2)} > 0$, we obtain

$$\mu_1 = -\frac{p}{K_x^{(C1)}} < 0 \ , \qquad \mu_2 = -\frac{p}{K_x^{(C2)}} > 0 \ .$$

Hence, we carry out a modification of type C on the intervals $[t_q - \gamma, t_q]$ and $[t_l, t_{l+1}]$ (see system (27)).

2) If k is an odd number, then $\delta_B^k < 0$ and $\mu_1 < 0$, $\mu_2 < 0$. There are two subcases depending on the sign of p.

a) If $p > 0$, then, using $K_x^{(C1)} < 0$ and $K_x^{(C2)} > 0$, we obtain

$$\mu_1 = -\frac{p}{K_x^{(C1)}} > 0 \ , \qquad \mu_2 = -\frac{p}{K_x^{(C2)}} < 0 \ .$$

Hence, we carry out a modification of type C on the intervals $[t_q - \gamma, t_q]$ and $[t_l, t_{l+1}]$ (see system (27)).

b) If $p<0$, then, using $K_x^{(C1)}<0$ and $K_x^{(C2)}>0$, we obtain

$$\mu_1=-\frac{p}{K_x^{(C1)}}<0\ ,\quad \mu_2=-\frac{p}{K_x^{(C2)}}>0\ .$$

Hence, we carry out a modification of type C on the intervals $[t_p-\zeta,t_p]$ and $[t_l,t_{l+1}]$ (see system (26)).

II. Consider now the case $p=0$. In this case (consider, for example, system (26)) instead of the variable δ_A we introduce the variable $\delta_A^{(1)}$ in the following way:

$$\delta_A^{(1)}=\frac{q}{K_y^{(A)}}\delta_B^k+\delta_A\ . \tag{31}$$

Then, we can rewrite system (26) as the following system of the variables $\delta_A^{(1)}$, δ_{C1} and δ_B:

$$\left\{\begin{array}{r} 0+K_x^{(C1)}\delta_{C1}+O(\delta_A^{(1)}\delta_{C1})+O(\delta_B\delta_{C1})+O(\delta_A^{(1)}\delta_B)+O(\delta_B^{k+1})=0\\ K_y^{(A)}\delta_A^{(1)}+K_y^{(C1)}\delta_{C1}+O(\delta_A^{(1)}\delta_{C1})+O(\delta_B\delta_{C1})+O(\delta_A^{(1)}\delta_B)+O(\delta_B^{k+1})=0\end{array}\right. \tag{32}$$

The path $\mathcal{P}$ consists of arcs of half-clothoids. Hence, in the first equation of system (32) among the terms of order δ_B^{k+1}, $\delta_B^{k+2},\ldots$ there are some terms which don't vanish. So, using some consecutive substitutions of variables of type (31), we obtain the following system (with $m\neq 0$):

$$\left\{\begin{array}{cccccccccc} 0 & + & K_x^{(C1)}\delta_{C1} & + & \ldots & + & m\delta_B^{k+S} & + & O(\delta_B^{k+S+1})= & 0\\ K_y^{(A)}\delta_A^{(S)} & + & K_y^{(C1)}\delta_{C1} & + & \ldots & + & n\delta_B^{k+S} & + & O(\delta_B^{k+S+1})= & 0\end{array}\right. \tag{33}$$

and we can study this system as in the previous case (i.e. the sign of m defines what modification of type C we'll carry out).

In system (33) we denote by dots all terms of higher order except the terms of order δ_B^{k+S+1}.

The substitutions (31) which are done up to this moment don't depend on the choice between systems (26) and (27), hence one can first find out (after a finite number of such substitutions) which system is to be chosen and then solve this system; it will provide the existense of $\delta_A^{(S)}(\delta_B)$, $\delta_{C1}(\delta_B)$ (or $\delta_A^{(S)}(\delta_B)$, $\delta_{C2}(\delta_B)$) with the right sign for small δ_B.

The lemma is proved. □

The method introduced in case I can be used in several cases (see Appendix B).

9 Proof of the non-optimality of the path $\mathcal{P}$ – case II

Definition *Call 'case II' the case when $y_p>0$ for any even p, $y_p<0$ for any odd p and there exists at least one even index p such that $\kappa_p\leq 0$ or at least one odd index p such that $\kappa_p\geq 0$.*

Without loss of generality we suppose that there exists one local minimum with non-negative curvature (see Figure 10).

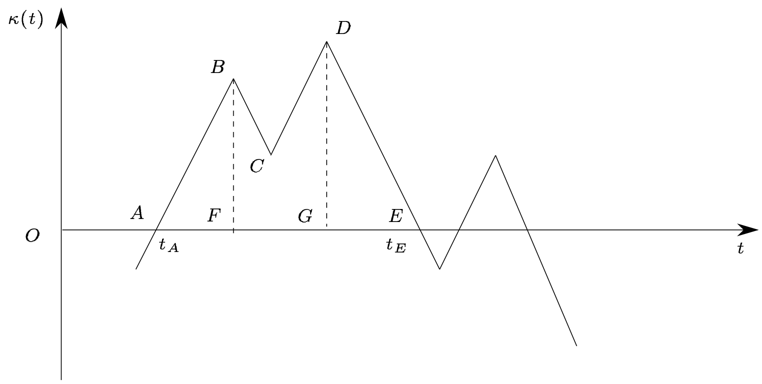


Figure 10

It follows from the definition of case II that there are many possible subcases. We divide all possible cases in the following way:

1) the case when the path $\mathcal{P}$ has at most one point of zero curvature (call it 'subcase A'),

2) the case when

1^0 the path $\mathcal{P}$ has more than one point of zero curvature,

2^0 there exists at least one piece of the path $\mathcal{P}$ between two consecutive points of zero curvature such that there are three switching points belonging to this piece and the tangent angle makes a turn of at least 2π on this piece

(call it 'subcase B'),

3) the case when

1^0 the path $\mathcal{P}$ has more than one point of zero curvature,

2^0 there exists at least one piece of the path $\mathcal{P}$ between two consecutive points of zero curvature such that there are three switching points belonging to this piece and the tangent angle makes a turn of less than 2π on this piece

(call it 'subcase C'),

4) the case when

1^0 the path $\mathcal{P}$ has more than one point of zero curvature,

2^0 there exists at least one piece of the path $\mathcal{P}$ between two consecutive points of zero curvature such that there are at least five switching points belonging to this piece

(call it 'subcase D').

Evidently, any path belonging to case II belongs to some of these subcases. We study any case in the corresponding subsection and, summarizing the obtained results, we formulate the following lemma:

Lemma 9.1 *In case II if the distance between the initial and final points of the path $\mathcal{P}$ is greater than $320\sqrt{\pi}$, then we can modify $\mathcal{P}$ so that the obtained path should be shorter than $\mathcal{P}$ and should satisfy the initial and final conditions. Hence, the path $\mathcal{P}$ isn't optimal.*

Lemma 9.1 follows from Lemmas 9.2, 9.3, 9.12, 9.16 (see the description of these lemmas below).

Plan of Section 9

1) At first we consider the subcase A and we obtain that if the distance between the initial and final points of the path $\mathcal{P}$ is greater than $25.25\sqrt{\pi}$, then, $\mathcal{P}$ isn't optimal (see Lemma 9.2, Subsection 9.1).

2) Then we consider the subcase B and we obtain that if the distance between the initial and final points of the path $\mathcal{P}$ is greater than $90.5\sqrt{\pi}$, then we can modify $\mathcal{P}$ so that the obtained path should be shorter than $\mathcal{P}$ and should satisfy the initial and final conditions, hence, the path $\mathcal{P}$ isn't optimal (see Lemma 9.3, Subsection 9.2).

3) Then we consider the subcase C and we obtain that if the distance between the initial and final points of the path $\mathcal{P}$ is greater than $320\sqrt{\pi}$, then we can modify $\mathcal{P}$ so that the obtained path should be shorter than $\mathcal{P}$ and should satisfy the initial and final conditions, hence, the path $\mathcal{P}$ isn't optimal (see Lemma 9.12, Subsection 9.8),

4) Finaly we consider the subcase D and we obtain that if the distance between the initial and final points of the path $\mathcal{P}$ is greater than $90.5\sqrt{\pi}$, then we can modify $\mathcal{P}$ so that the obtained path should be shorter than $\mathcal{P}$ and should satisfy the initial and final conditions, hence, the path $\mathcal{P}$ isn't optimal (see Lemma 9.16, Subsection 9.9).

9.1 Proof of the non-optimality of the path $\mathcal{P}$ – subcase A

Definition *Call 'subcase A' the case when the path $\mathcal{P}$ has at most one point of zero curvature.*

Lemma 9.2 *In subcase A if the distance between the initial and final points of the path $\mathcal{P}$ is greater than $25.25\sqrt{\pi}$, then, $\mathcal{P}$ isn't optimal.*

Without loss of generality we suppose that the curvature of the path $\mathcal{P}$ is non-negative.

Proof

Denote by d the distance between the initial and final points of the path $\mathcal{P}$ and by l the length of $\mathcal{P}$. Denote by $P_{\mathcal{P}}$ a point belonging to $\mathcal{P}$ such that the length of the piece of $\mathcal{P}$ between the initial point and the point $P_{\mathcal{P}}$ equals $\kappa^0/2$. Denote by $Q_{\mathcal{P}}$ a point belonging to $\mathcal{P}$ such that the length of the piece of $\mathcal{P}$ between the final point and the point $Q_{\mathcal{P}}$ equals $\kappa^T/2$. Denote by $d_{P_{\mathcal{P}}Q_{\mathcal{P}}}$ (by $l_{P_{\mathcal{P}}Q_{\mathcal{P}}}$) the distance (the length of the piece of the path) between the points $P_{\mathcal{P}}$ and $Q_{\mathcal{P}}$.

Plan of the proof

1) Construct some auxiliary path $\mathcal{S}$.
2) a) Using some property of the suboptimal path, we obtain

$$l_{P_{\mathcal{P}}Q_{\mathcal{P}}} < d_{P_{\mathcal{P}}Q_{\mathcal{P}}} + (6.5 + 4\sqrt{2})\sqrt{\pi}$$

(see (37)).
b) Using some property of the path $\mathcal{P}$, we obtain

$$l_{P_{\mathcal{P}}Q_{\mathcal{P}}} > \frac{5}{3}d_{P_{\mathcal{P}}Q_{\mathcal{P}}} - \frac{10}{3}\sqrt{\pi}$$

(see (41)).
c) So, comparing the results obtained in a) and b), we obtain

$$d_{P_{\mathcal{P}}Q_{\mathcal{P}}} < (14.75 + 6\sqrt{2})\sqrt{\pi}$$

(see (42)), and as

$$d < d_{P_{\mathcal{P}}Q_{\mathcal{P}}} + 2\sqrt{\pi}$$

(see (36)), then

$$d < (16.75 + 6\sqrt{2})\sqrt{\pi} \approx 25.23\sqrt{\pi} < 25.25\sqrt{\pi}$$

(see (43)).
Hence, we conclude that if $d > 25.25\sqrt{\pi}$, then

$$d_{P_{\mathcal{P}}Q_{\mathcal{P}}} > (14.75 + 6\sqrt{2})\sqrt{\pi}$$

(see (44)), this is a contradiction with (42) which follows from the fact that we suppose that the path $\mathcal{P}$ is optimal (i.e. l is at most the length of some suboptimal path, i.e. l is at most the sum of d and some constant).
So, the path $\mathcal{P}$ isn't optimal. Hence, the lemma is proved.

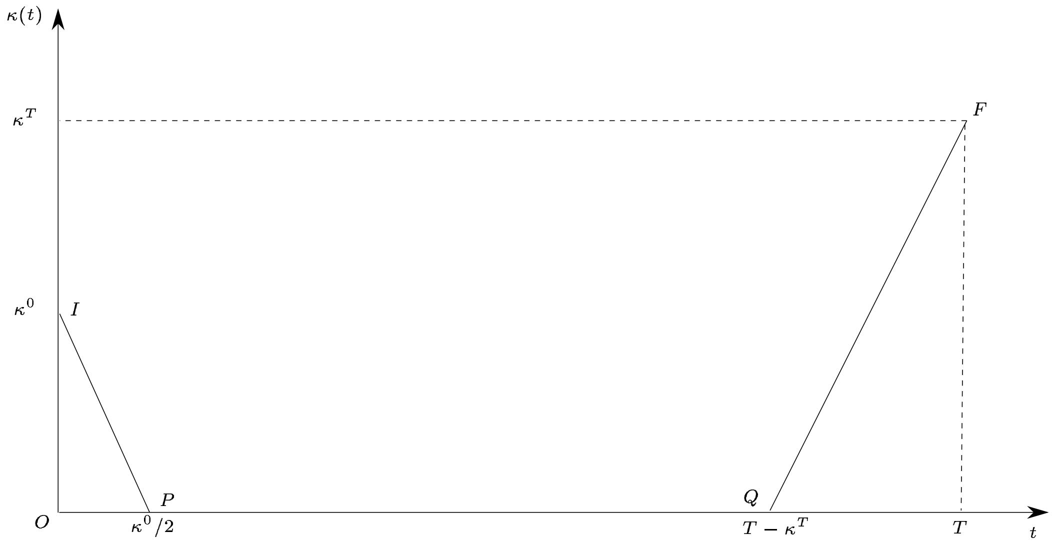


Figure 11

1) Construct some auxiliary path $\mathcal{S}$ as follows.

Construct, at first, the arc of half-clothoid corresponding to $\kappa(t) = -2t + \kappa^0$ on the interval $[0, \kappa^0/2]$; construct the arc of half-clothoid corresponding to $\kappa(t) = 2(t-(T-\kappa^T/2))$ on the interval $[T-\kappa^T/2, T]$ (see Figure 11: there are the segments IP and FQ).

Now consider the points P and Q as two points on the plane with fixed tangent angles and with zero curvature. Construct some suboptimal path joining these points as in [10], [12], [13]. We give below the general idea of the construction of some suboptimal path connecting the points P and Q. The justification of this construction is given in [10], [13].

We construct the path from P to Q by means of the graph of the curvature as a function of the path length (see an example of such graph on Figure 12). The graph of the curvature is a continuous piecewise-linear function (any piece is of type $\kappa = \pm 2t + \kappa_{**}$ or $\kappa = 0$). The piece of the graph between the points V and W corresponds to the line segment, the other pieces correspond to arcs of a half-clothoid.

Here ξ', ξ'' are the lengths of the path and they can be considered as two parameters.

To construct some path from P to Q we vary ξ' and ξ'' on the interval $[0, 2\sqrt{\pi}]$ so that the tangent lines at the points V and W should be parallel (i.e. ξ'' should be some function of ξ') and the tangent vectors at the points V and W should have opposite directions. If ξ' (or if ξ'') varies on the interval $[0, 2\sqrt{\pi}]$, then the tangent angle α_V (or α_W) at the point V (or at the point W) takes continuously all values from $[\alpha_P, \alpha_P + 2\pi]$ or $[\alpha_P - 2\pi, \alpha_P]$ (from

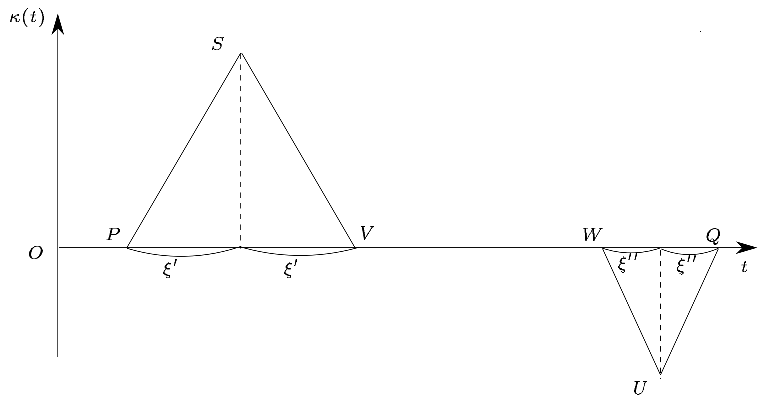


Figure 12

$[\alpha_Q - 2\pi, \alpha_Q]$ or $[\alpha_Q, \alpha_Q + 2\pi]$) (the choice of interval depends on the sign of κ_S (on the sign of κ_U respectively)).

For $\alpha_V = \pi/2$, $\alpha_W = -\pi/2$ and for $\alpha_V = -\pi/2$, $\alpha_W = \pi/2$ the angles between the tangent line at the point V and the vector VW have different signs. Thus, varying ξ' and ξ'' on the interval $[0, 2\sqrt{\pi}]$, we obtain that for some values of ξ', ξ'' this angle becomes equal to zero. Thus, we obtain the desired path from P to Q.

See the graph of the curvature of the constructed path $\mathcal{S}$ on Figure 13.

The thus constructed path $\mathcal{S}$ consists of at most 6 arcs of half-clothoid (because some arcs can degenerate in a point) and of at most one line segment (because this line segment can degenerate in a point).

2) Denote by d_{PQ} the distance between the points P and Q of the path $\mathcal{S}$ and by l_{PQ} the length of the piece of $\mathcal{S}$ between the points P and Q.

a) The maximal distance between the points I and P (F and Q) is smaller than $3R/2 = 3\sqrt{\pi}/4$ (as the maximal distance between two points of a half-clothoid is smaller than $3R/2 = 3\sqrt{\pi}/4$ – see Proposition 5.3 of [13]). Hence,

$$d_{PQ} \leq d + 2 \times 3\sqrt{\pi}/4 = d + 3\sqrt{\pi}/2 \ . \tag{34}$$

As the piece of $\mathcal{S}$ between the points P and Q is some suboptimal path, then, it follows from Proposition 8.1 of [11] that for l_{PQ} we have the following estimation:

$$l_{PQ} \leq d_{PQ} + (3 + 4\sqrt{2})\sqrt{\pi} \ .$$

As $l_{P_\mathcal{P} Q_\mathcal{P}} = l_{PQ}$, so, using (34), we obtain

$$l_{P_\mathcal{P} Q_\mathcal{P}} = l_{PQ} \leq d_{PQ} + (3 + 4\sqrt{2})\sqrt{\pi} \leq$$

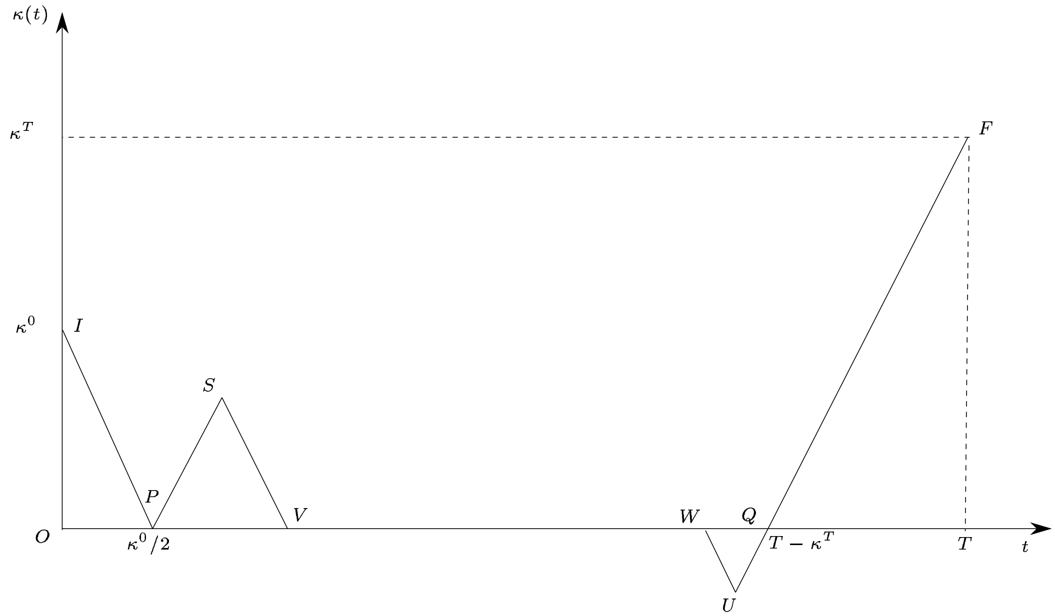


Figure 13

$$\leq d + 3\sqrt{\pi}/2 + (3 + 4\sqrt{2})\sqrt{\pi} = d + (4.5 + 4\sqrt{2})\sqrt{\pi} \ . \tag{35}$$

It follows from Proposition C.2 (see Appendix C) that the distance between the initial point of $\mathcal{P}$ and the point $P_{\mathcal{P}}$ (between the final point of $\mathcal{P}$ and the point $Q_{\mathcal{P}}$ respectively) is smaller than $\sqrt{\pi}$. Hence, for the minimal value of $d_{P_{\mathcal{P}}Q_{\mathcal{P}}}$ we have the following estimation:

$$d_{P_{\mathcal{P}}Q_{\mathcal{P}}} > d - (\sqrt{\pi} + \sqrt{\pi}) = d - 2\sqrt{\pi} \ ,$$

i.e.

$$d < d_{P_{\mathcal{P}}Q_{\mathcal{P}}} + 2\sqrt{\pi} \ . \tag{36}$$

Now, the following inequality follows from (35) and (36):

$$l_{P_{\mathcal{P}}Q_{\mathcal{P}}} \leq d + (4.5 + 4\sqrt{2})\sqrt{\pi} <$$

$$< (d_{P_{\mathcal{P}}Q_{\mathcal{P}}} + 2\sqrt{\pi}) + (4.5 + 4\sqrt{2})\sqrt{\pi} = d_{P_{\mathcal{P}}Q_{\mathcal{P}}} + (6.5 + 4\sqrt{2})\sqrt{\pi} \ . \tag{37}$$

Remark that we have obtained inequality (37) using some property of the suboptimal path (namely, the fact that the difference between the length of the suboptimal path and the distance between the initial and final points is at most $(3 + 4\sqrt{2})\sqrt{\pi}$).

b) Denote by L the first point belonging to $P_{\mathcal{P}}Q_{\mathcal{P}}$ such that the tangent angle at this point equals zero (modulo 2π). Denote by K the last point belonging to $P_{\mathcal{P}}Q_{\mathcal{P}}$ such that the tangent angle at this point equals zero (modulo 2π). Denote by L_{pr} (by K_{pr}) the projection of the point L (of the point K) on the axis Ox and denote by $d_{L_{pr}K_{pr}}$ the distance between these projections.

Apply the result of Proposition C.3 (see Appendix C) to the piece $\widehat{LK}$ of $\mathcal{P}$ and obtain the following inequality:

$$l_{LK} > \frac{5}{3} d_{L_{pr}K_{pr}} \ . \tag{38}$$

But $l_{P_{\mathcal{P}}Q_{\mathcal{P}}} \geq l_{LK}$. Hence, the following inequality follows from (38):

$$l_{P_{\mathcal{P}}Q_{\mathcal{P}}} > \frac{5}{3} d_{L_{pr}K_{pr}} \ . \tag{39}$$

We have

$$d_{L_{pr}K_{pr}} > d_{P_{\mathcal{P}}Q_{\mathcal{P}}} - 2\sqrt{\pi} \tag{40}$$

(because it follows from the definition of the points L and K that

$$d_{P_{\mathcal{P}}Q_{\mathcal{P}}} < d_{L_{pr}K_{pr}} + 4R = d_{L_{pr}K_{pr}} + 4 \times \sqrt{\pi}/2 = d_{L_{pr}K_{pr}} + 2\sqrt{\pi} \ ).$$

So, from (39) and (40) we obtain

$$l_{P_{\mathcal{P}}Q_{\mathcal{P}}} > \frac{5}{3} d_{P_{\mathcal{P}}Q_{\mathcal{P}}} - \frac{10}{3}\sqrt{\pi} \ . \tag{41}$$

Remark that we obtain inequality (41) using some property of the path $\mathcal{P}$ (more precisely, the fact that the length of the piece $\widehat{LK}$ of $\mathcal{P}$ is greater than $\frac{5}{3}$ of the distance between the points L and K).

c) Now, using (37) and (41), we obtain

$$\frac{5}{3}d_{P_\mathcal{P} Q_\mathcal{P}} - \frac{10}{3}\sqrt{\pi} < d_{P_\mathcal{P} Q_\mathcal{P}} + (6.5 + 4\sqrt{2})\sqrt{\pi} \ ,$$

i.e.

$$d_{P_\mathcal{P} Q_\mathcal{P}} < (14.75 + 6\sqrt{2})\sqrt{\pi} \ . \tag{42}$$

So, using inequalities (36) and (42), we obtain

$$d < d_{P_\mathcal{P} Q_\mathcal{P}} + 2\sqrt{\pi} < (16.75 + 6\sqrt{2})\sqrt{\pi} \ . \tag{43}$$

Thus, if the distance d satisfies inequality (43), then $l_{P_\mathcal{P} Q_\mathcal{P}}$ satisfies inequalities (37) and (41) and there is no contradiction. But if

$$d \geq (16.75 + 6\sqrt{2})\sqrt{\pi} \approx (16.75 + 6 \times 1.41421)\sqrt{\pi} \approx 25.2353\sqrt{\pi} \ ,$$

i.e. if $d > 25.25\sqrt{\pi}$, then, it follows from (36) that

$$d_{P_\mathcal{P} Q_\mathcal{P}} > (14.75 + 6\sqrt{2})\sqrt{\pi} \ . \tag{44}$$

This is a contradiction with (42). This contradiction comes from the fact that we suppose that the path $\mathcal{P}$ is optimal (i.e. l is at most the length of some suboptimal path).

Hence, if $d > 25.25\sqrt{\pi}$, then the path $\mathcal{P}$ isn't optimal.

The lemma is proved. □

9.2 Proof of the non-optimality of the path $\mathcal{P}$ – subcase B

Definition *Call 'subcase B' the case when*

1^0 *the path $\mathcal{P}$ has more than one point of zero curvature,*

2^0 *there exists at least one piece of the path $\mathcal{P}$ between two consecutive points of zero curvature such that there are three switching points belonging to this piece and the tangent angle makes a turn of at least 2π on this piece*

Lemma 9.3 *In subcase B if the distance between the initial and final points of the path $\mathcal{P}$ is greater than $90.5\sqrt{\pi}$, then we can modify $\mathcal{P}$ so than the obtained path $\widetilde{\mathcal{P}}$ should be shorter than $\mathcal{P}$ and that it should satisfy the initial and final conditions, hence, the path $\mathcal{P}$ isn't optimal.*

See the proof of Lemma 9.3 in Subsection 9.7.

Consider, at first, some lemma important for the demonstration of Lemma 9.3.

Lemma 9.4 *In the case when*

1^0 *the path $\mathcal{P}$ has more than one point of zero curvature,*

2^0 *there is no couple of points of zero curvature such that the distance between these points is greater than $40\sqrt{\pi}$,*

3^0 *the distance between the initial and final points of the path $\mathcal{P}$ is greater than $90.5\sqrt{\pi}$,*

then, $\mathcal{P}$ isn't optimal.

Proof

Prove Lemma 9.4, using Lemma 9.2.

Denote by I (by F) the initial point (the final point) of $\mathcal{P}$ and denote by A (by E) the first (the last) point of zero curvature.

If the distance between the initial and final points of the path $\mathcal{P}$ is greater than $90.5\sqrt{\pi}$ and if the distance between the points A and E is at most $40\sqrt{\pi}$, then the sum of the distances between the points I and A and between the points E and F is greater than $50.5\sqrt{\pi}$. Hence, at least one among these distances is greater than $25.25\sqrt{\pi}$. Thus, it follows from Lemma 9.2, that the corresponding piece of $\mathcal{P}$ isn't optimal. Hence, the path $\mathcal{P}$ isn't optimal.

The lemma is proved. □

We consider only the part of the path $\mathcal{P}$ from the initial point to the first point of zero curvature (if $\kappa^0 \neq 0$) and the part of the path $\mathcal{P}$ from the last point of zero curvature to the final point (if $\kappa^T \neq 0$). So, the initial and final points of the new path (we denote it by $\mathcal{P}_d$) are points of zero curvature. Thus the graph of the curvature κ as the function of t for the path $\mathcal{P}_d$ is of the kind of the graph shown on Figure 14.

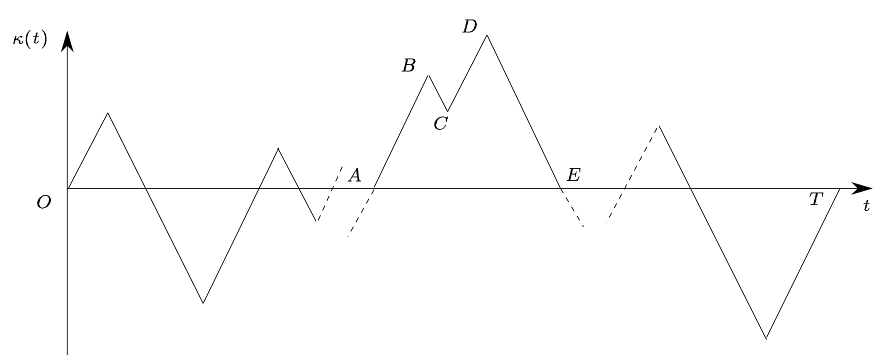


Figure 14

Without loss of generality we suppose that the local minimum of the graph $\kappa(t)$ is situated outside some small neighbourhood of the initial point and outside some small neighbourhood of the final point of the path $\mathcal{P}_d$ because if not, then we can consider only the part of the path $\mathcal{P}_d$ outside these neighbourhoods and it is case III considered in Section 10. We define these two small neighbourhoods as two circles with centres at the points O and T respectively and of radius $\sqrt{\pi}$.

We denote by $\mathcal{P}_{opt}$ an optimal path. We denote by $\mathcal{P}_{min}$ a path such that it isn't longer than $\mathcal{P}_{opt}$ and that it satisfies all initial and final conditions (but it may not satisfy the condition of continuity of variables). Thus,

$$|\mathcal{P}_{opt}| \geq |\mathcal{P}_{min}| \ .$$

General idea and plan of the proof of the non-optimality of the path $\mathcal{P}_d$.

1. The general idea is to modify the path $\mathcal{P}_d$ so that the new path (we denote it by $\widetilde{\mathcal{P}}$) should be shorter than the path $\mathcal{P}_d$. One can see an example of such a modification on Figure 15 (the path $\mathcal{P}_d$ is marked by the points O, R, P, S, Q, Z, V and T, the path $\widetilde{\mathcal{P}}$ is marked by the points O, R, X, Y, V and T; we denote the line segment between the points X and Y by l).

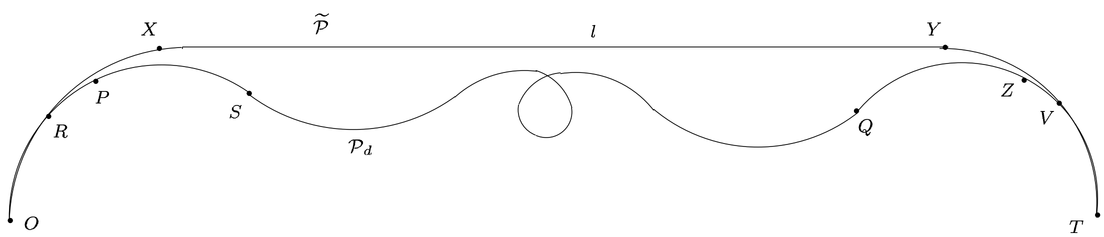


Figure 15

2. We prove the inequality $|\widetilde{\mathcal{P}}| < |\mathcal{P}_d|$ in three stages:

a) at first we compare the lengths of the paths $\mathcal{P}_d$ and $\mathcal{P}_{min}$ – we obtain the following inequality:

$$|\mathcal{P}_d| - |\mathcal{P}_{min}| > 1.139456743$$

(see Lemma 9.8),

b) then we compare the lengths of the paths $\widetilde{\mathcal{P}}$ and $\mathcal{P}_{min}$ – we obtain the following inequality:

$$|\widetilde{\mathcal{P}}| - |\mathcal{P}_{min}| > 1.050758327$$

(see Lemma 9.9),

c) then we compare (using the results obtained in Lemmas 9.8, 9.9) the lengths of the paths $\widetilde{\mathcal{P}}$ and $\mathcal{P}_d$ – we obtain the desired inequality:

$$|\widetilde{\mathcal{P}}| < |\mathcal{P}_d|$$

(see Lemma 9.3).

Construction of some path $\widetilde{\mathcal{P}}$.

We show how one can construct some path $\widetilde{\mathcal{P}}$ from the initial point of $\mathcal{P}_d$ (we denote it by O) to the final point of $\mathcal{P}_d$ (we denote it by T) with four switching points ($\widetilde{\mathcal{P}}$ is the concatenation of four arcs of a half-clothoid and of a line segment; along $\widetilde{\mathcal{P}}$ the tangent angle and the curvature are continuous).

We construct $\widetilde{\mathcal{P}}$ by means of the graph of the curvature as a function of the path length (see an example of such graph on Figure 16). The graph of the curvature is a continuous piecewise-linear function (any piece is of type $\kappa = \pm 2t + \kappa_{**}$ or $\kappa = 0$). The piece of the graph between the points X and Y corresponds to the line segment, the other pieces correspond to arcs of a half-clothoid (remind that the initial and final points of $\mathcal{P}_d$ are the points of zero curvature).

Here ξ', ξ'' are the lengths of the path and they can be considered as two parameters.

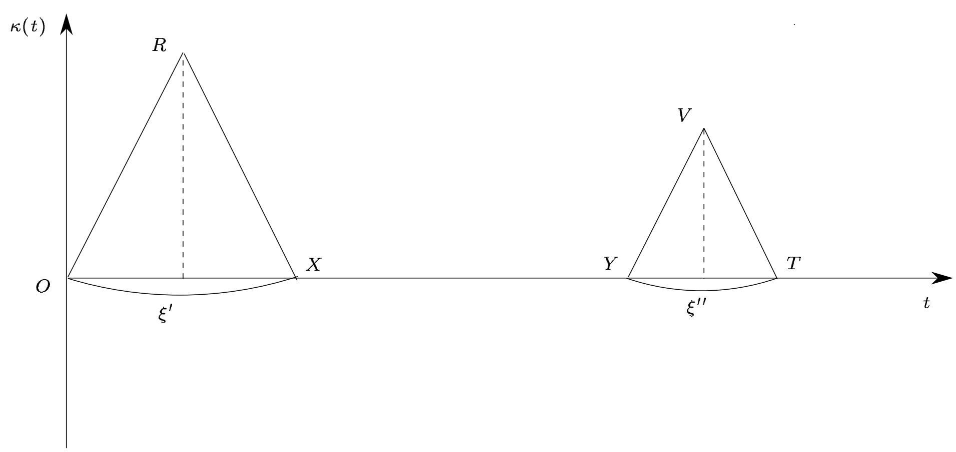


Figure 16

We consider the piece of the path $\widetilde{\mathcal{P}}$ from the initial point O to some point X (the point X of the path $\widetilde{\mathcal{P}}$ corresponds to the point X of the graph of the curvature).

Increasing ξ' monotonously we obtain the increasing of the absolute value of the tangent angle α at the point X (we denote it by α_X), because the curvature doesn't change sign on $[0, \xi']$ and the angle $\alpha_X - \alpha_O$ is the integral of the curvature on the interval:

$$\alpha_X - \alpha_O = \int_0^{\xi'} \kappa(t)dt \ .$$

The absolute value of $\alpha_X - \alpha_O$ is equal to the area of the triangle ORX, i.e. to ${\xi'}^2/2$. If we want that $|\alpha_X - \alpha_O| \leq 2\pi$, then, ${\xi'}^2 \leq 4\pi$ and $\xi' \leq 2\sqrt{\pi}$.

Hence, if ξ' varies on $[0, 2\sqrt{\pi}]$, then the tangent angle α_X at the point X takes continuously all values from $[\alpha_O, \alpha_O + 2\pi]$ or $[\alpha_O - 2\pi, \alpha_O]$ (the choice of the interval depends on the sign of κ_R).

To construct some path from the point O to the point T we vary ξ' and ξ'' on the interval $[0, 2\sqrt{\pi}]$ so that the tangent lines at the points X and Y should be parallel (i.e. ξ'' should be some function of ξ') and the tangent vectors at the points X and Y should have the opposite directions. Remind that if ξ' (or if ξ'') varies on the interval $[0, 2\sqrt{\pi}]$, then the tangent angle α_X (or α_Y) at the point X (or at the point Y) takes continuously all values from $[\alpha_O, \alpha_O + 2\pi]$ or $[\alpha_O - 2\pi, \alpha_O]$ (from $[\alpha_T - 2\pi, \alpha_T]$ or $[\alpha_T, \alpha_T + 2\pi]$) (the choice of interval depends on the sign of κ_R (on the sign of κ_V respectively)).

For $\alpha_X = \pi/2$, $\alpha_Y = -\pi/2$ and for $\alpha_X = -\pi/2$, $\alpha_Y = \pi/2$ the angles between the tangent line at the point X and the vector XY have different signs. Thus, varying ξ' and ξ'' on the interval $[0, 2\sqrt{\pi}]$, we obtain that for some values of ξ', ξ'' this angle becomes equal to zero. Thus, we obtain the desired path $\widetilde{\mathcal{P}}$. The constructed path $\widetilde{\mathcal{P}}$ satisfies all requirements.

Generally, if we construct some path $\widetilde{\mathcal{P}}$ by this method, there exist four possibilities. These possibilities correspond to the 4 possible choices of sign of the curvature $\kappa(t)$ on the intervals $(0, \xi')$, $(T - \xi'', T)$ (see Figure 17).

We denote by V_O (by V_T) the tangent vector at the point O (at the point T).

Among these 4 possibilities we choose a modification such that the constructed path $\widetilde{\mathcal{P}}$ should be the shortest. From now on when we say "the constructed path $\widetilde{\mathcal{P}}$", it means that we have chosen the best modification among these 4 possibilities (i.e. that we have constructed the shortest path $\widetilde{\mathcal{P}}$).

9.3 Subcase B: some remarks on the aspect of the path $\mathcal{P}_d$.

Remind that it follows from the definition of subcase B that the tangent angle of the path $\mathcal{P}_d$ between the points A and E makes a turn of at least 2π (i.e. the piece of the path $\mathcal{P}_d$ between the points A and E is of the kind of the path shown on Figure 18). Hence, the area of $ABCDE$ (we denote it by S_{ABCDE}, see Figure 10) is at least 2π.

We denote by ρ the maximal value of the curvature on the interval $[t_A, t_E]$ (i.e. ρ is equal either to $|BF|$, or to $|DG|$). We have $S_{ABCDE} \geq 2\pi$. We consider only the case $2\pi \leq S_{ABCDE} < 4\pi$, because if we prove that for $S_{ABCDE} < 4\pi$ (i.e. in the case when the tangent angle of the path $\mathcal{P}_d$ between the points A and E makes a turn of at most 4π) the path $\widetilde{\mathcal{P}}$ is shorter than the path $\mathcal{P}_d$; evidently, it proves the statement also for the case when $S_{ABCDE} \geq 4\pi$ (see Remark 9.7).

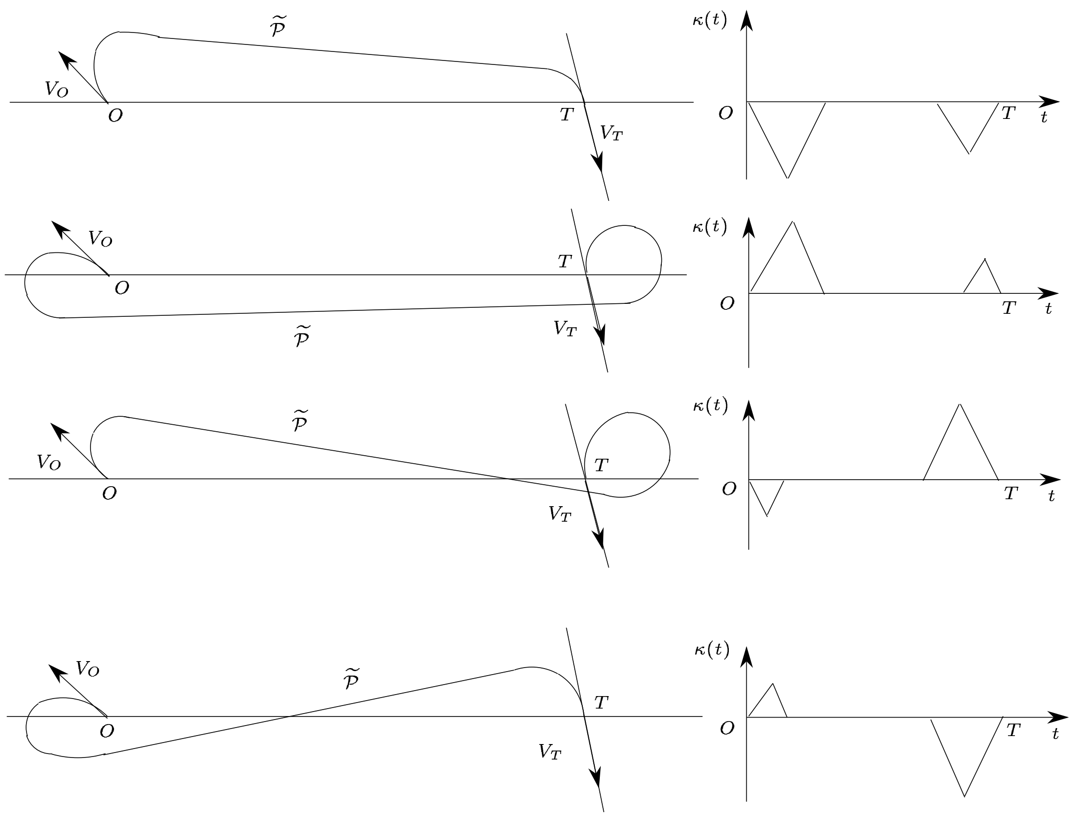


Figure 17

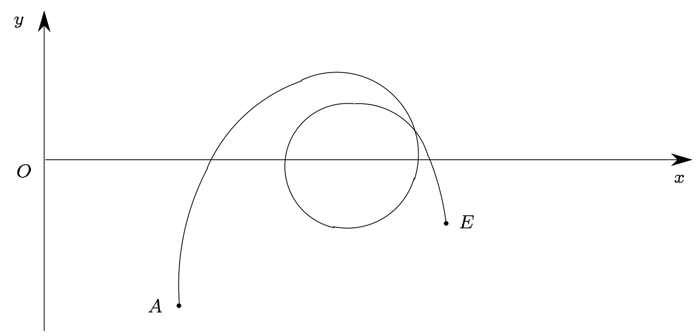


Figure 18

Proposition 9.5 *For ρ we have the following inequalities:*

$$2\sqrt{\pi} \leq \rho < 2\sqrt{2\pi} \ . \tag{45}$$

Proof

For S_{ABCDE} we have the following inequalies:

$$2\pi \leq S_{ABCDE} < 4\pi \ .$$

We obtain the maximal possible ρ if $S_{ABCDE} = \rho^2/2$. Hence,

$$2\pi \leq \rho^2/2 < 4\pi \ , \qquad \text{i.e.} \quad 2\sqrt{\pi} \leq \rho < 2\sqrt{2\pi} \ .$$

The proposition is proved. □

We consider now the piece of the path $\mathcal{P}_d$ from the point A to the point E (see Figure 10). On Figure 19 we denote by V_M, (by V_K, by V_N, by V_L, by V_J) the tangent vector at the point M (at the point K, at the point N, at the point L, at the point J). We denote by M the first point belonging to $\mathcal{P}_d$ such that V_M is perpendicular to the straight line connecting the initial and the final points of $\mathcal{P}_d$ (we denote it by p), we denote by N the first point belonging to $\mathcal{P}_d$ such that the vectors V_M and V_N should have opposite directions. We denote by K a point belonging to $\mathcal{P}_d$ such that $K \in \widehat{MN}$ and V_K is parallel to p. We denote by Q the first point (after the point M) belonging to $\mathcal{P}_d$ and to the straight line m.

Set $R_\rho = 1/\rho$. We consider the tangent circle at the point K (at the point N); its radius is equal to R_ρ and we denote its centre by O_K (by O_N). We denote these circles by $\mathcal{C}_{O_K}$, $\mathcal{C}_{O_N}$. We denote by H a point belonging to the arc $|\widehat{NQ}|$ such that $|\widehat{NH}| = |\widehat{NW}| = \pi R_\rho/2$. We denote by V the projection of the point H on the line segment NO_N. The line segment

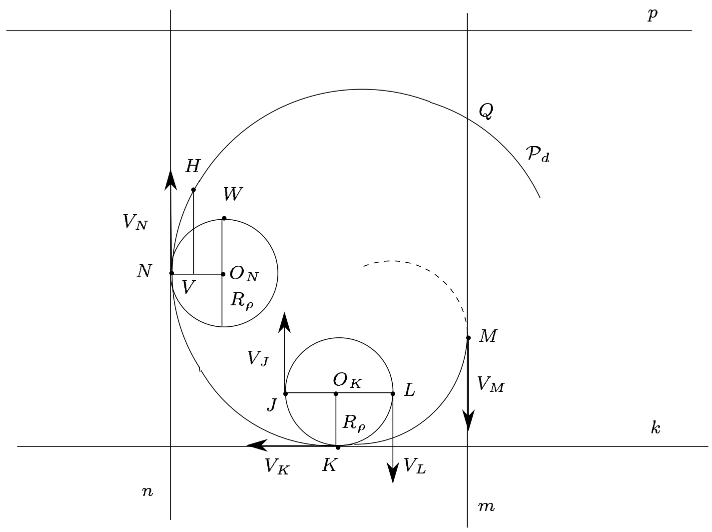


Figure 19

JL is parallel to the straight line p and the line segments WO_N, HV are parallel to the straight line n.

Lemma 9.6 *The length of the piece of $\mathcal{P}_d$ between the points M and Q is greater than $(3\pi/2+1)/(2\sqrt{2\pi})$, i.e.*

$$|\widehat{MQ}| > (3\pi/2+1)/(2\sqrt{2\pi}) \approx 1.139456743 \; . \tag{46}$$

Proof

A tangent angle to the path is equal to the integral of the curvature on the corresponding arc. Hence, as $\overrightarrow{V_M}=\overrightarrow{V_L}$ and as the curvature at every point of the arc $\widehat{MK}$ is at most ρ and the curvature at every point of the arc $\widehat{LK}$ is equal to ρ, then, $|\widehat{MK}| > |\widehat{LK}| = \pi R_\rho/2$. By analogy we obtain that $|\widehat{KN}| > |\widehat{KJ}| = \pi R_\rho/2$. So,

$$|\widehat{MKN}| > \pi R_\rho \; . \tag{47}$$

As $|\widehat{NH}| = |\widehat{NW}| = \pi R_\rho/2$ and as the curvature at every point of the arc $\widehat{NQ}$ is at most ρ (the curvature at every point of the circle $\mathcal{C}_{O_N}$ is equal to ρ), so, V is situated to the left with respect to the point O_N. We remark that the distance between the point O_N and the straight line m (we denote it by $\mathrm{dist}(O_N, m)$) is greater than the distance between the point O_K and the straight line m. Hence, $\mathrm{dist}(O_N, m) > R_\rho$.

So,

$$|\widehat{NQ}| = |\widehat{NH}| + |\widehat{HQ}| > \pi R_\rho/2 + \mathrm{dist}(O_N, m) > (\pi/2 + 1)R_\rho \ . \tag{48}$$

It follows from inequalities (47) and (48) that:

$$|\widehat{MQ}| = |\widehat{MN}| + |\widehat{NQ}| > \pi R_\rho + (\pi/2 + 1)R_\rho = (3\pi/2 + 1)R_\rho \ .$$

As $R_\rho = 1/\rho$ and $\rho < 2\sqrt{2\pi}$ (see Proposition 9.5, formula (45)), so, $R_\rho > 1/(2\sqrt{2\pi})$ and for $|\widehat{MQ}|$ we have the following estimation:

$$|\widehat{MQ}| > (3\pi/2 + 1)R_\rho > (3\pi/2 + 1)/(2\sqrt{2\pi}) \approx 1.139456743 \ .$$

The lemma is proved. □

Remark 9.7 *If we prove that in the case when $2\pi \le S_{ABCDE} < 4\pi$ the path $\widetilde{\mathcal{P}}$ is shorter than $\mathcal{P}_d$, it will prove the statement also for the case when $S_{ABCDE} \ge 4\pi$.*

Really, in the case $S_{ABCDE} \ge 4\pi$ we can choose a moment $t_H \in (t_A, t_E)$ such that $2\pi \le S_{ABCWH} < 4\pi$ (see Figure 20), i.e. the tangent angle of the path $\mathcal{P}_d$ between the points A and W makes a turn of at most 4π. Hence, considering the piece of the path $\mathcal{P}_d$ from the point A to the point W, we obtain Proposition 9.5 and Lemma 9.6 for the new ρ (now we denote by ρ the maximal value of the curvature on the interval $[t_A, t_H]$). So, in the case $S_{ABCDE} \ge 4\pi$ we have obtained Proposition 9.5 and Lemma 9.6 utilised in the proof of the inequality $|\widetilde{\mathcal{P}}| < |\mathcal{P}_d|$.

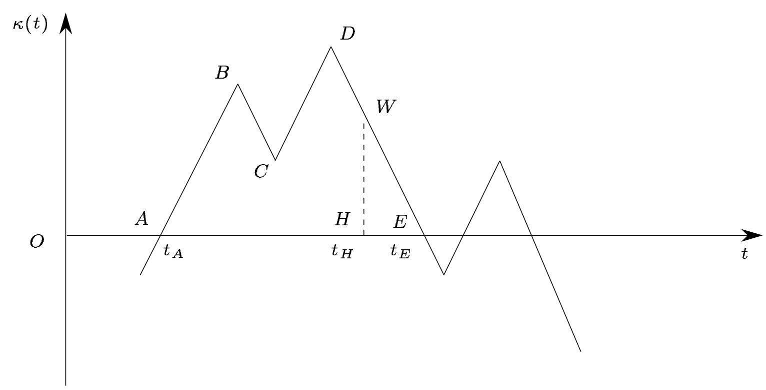


Figure 20

9.4 Subcase B: aspect of some path $\mathcal{P}_{min}$.

The type of a curve $\mathcal{P}_{min}$ depends on the constructed path $\widetilde{\mathcal{P}}$. So, without loss of generality, we consider the following case (see Figure 21).

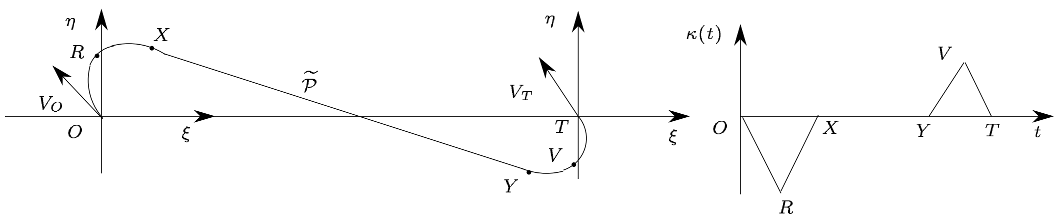


Figure 21

We consider two coordinate systems – $O\xi\eta$ and $T\xi\eta$ – such that the axis $O\xi$ ($T\xi$) and the vector OT should have the same directions (see Figure 22).

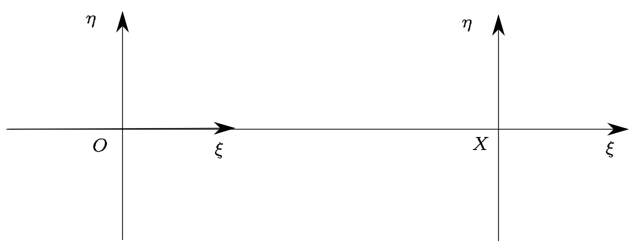


Figure 22

We denote by α_O (by α_T) the tangent angle at the point O (at the point T) in the coordinate system $O\xi\eta$ (in the coordinate system $T\xi\eta$).

If $\alpha_O = 0$ and $\alpha_T = 0$, then it is evident that the optimal path is a line segment p between the points O and T.

If $\alpha_O \in [0, \pi]$ and $\alpha_T \in [0, \pi]$ (except the case when $\alpha_O = 0$ and $\alpha_T = 0$), then an optimal path is longer than the distance between the points O and T. We construct a path $\mathcal{P}_{min}$ by means of its graph of the curvature (it is of the kind of the graph shown on Figure 23).

On this figure the point I (corresponding to $t = t^*$) is the first point belonging to the arc OI of a half-clothoid such that the tangent line at the point I becomes parallel to the straight line p. Respectively, the point W (corresponding to $t = T - t^{**}$) is the first point belonging to the arc TW of a half-clothoid such that the tangent line at the point W becomes parallel to the straight line p (if we construct the arc TW of a half-clothoid from the point T to

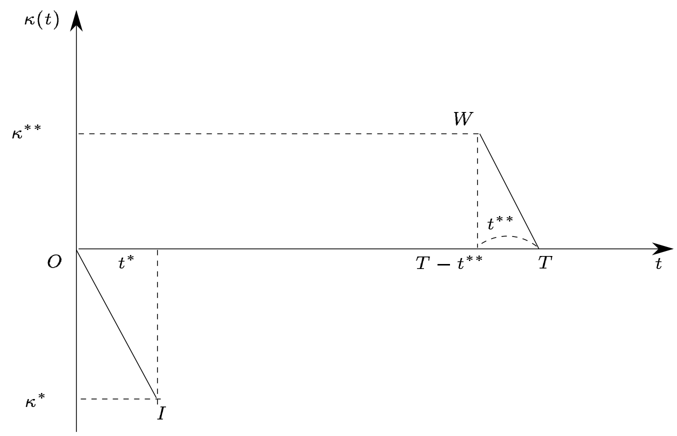


Figure 23

the point W). Thus, the tangent lines at the points I and W are parallel but they may not coincide. In this case there is a point of discontinuity of the variables x and y (in general, see Figure 24). The points I and W are always points of discontinuity of the curvature.

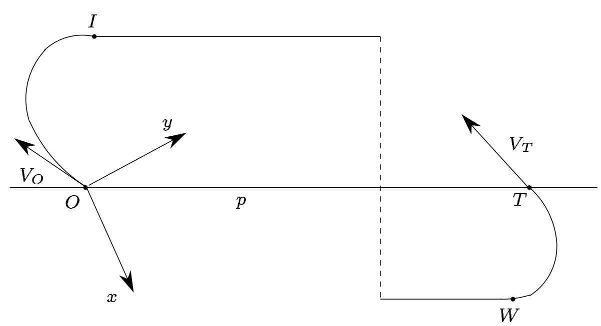


Figure 24

9.5 Subcase B: the comparison of the lengths of $\mathcal{P}_d$ and $\mathcal{P}_{min}$

Lemma 9.8 *The path $\mathcal{P}_d$ is longer than the path $\mathcal{P}_{min}$ by no less than* 1.139456743, *i.e.*

$$|\mathcal{P}_d| - |\mathcal{P}_{min}| > 1.139456743 \ . \tag{49}$$

Proof

We denote by Q a point belonging to $\mathcal{P}_d$ such that $|\widehat{OQ}| = |\widehat{OI}|$ (see Figure 25). Respectively, we denote by Z a point belonging to $\mathcal{P}_d$ such that $|\widehat{TZ}| = |\widehat{TW}|$. We denote by Q_{pr} (by I_{pr}, by W_{pr}, by Z_{pr}) the projection of the point Q (of I, of W, of Z) on the straight line p.

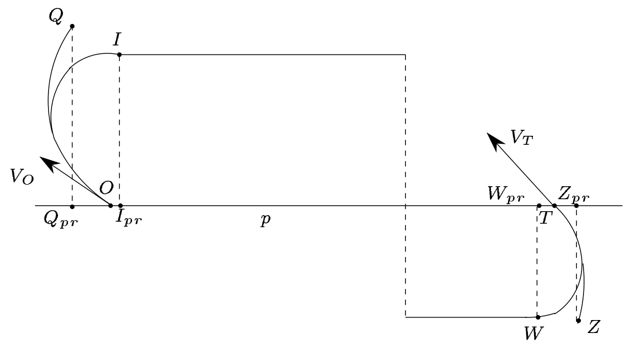


Figure 25

As the derivative of the curvature of an optimal path is at most 2 and as the arc OI is an arc of a half-clothoid with $\kappa(t) = 2$, then either Q coincides with I, or Q is situated to the left with respect to I.

Respectively, either Z coincides with W, or Z is situated to the right with respect to W. Hence,

$$|Q_{pr}Z_{pr}| - |I_{pr}W_{pr}| \geq 0 \ . \tag{50}$$

It follows from Lemma 9.6 that $\mathcal{P}_d$ has a "lace" of length at least 1.139456743 (we denote this length by l_{lace}). Here we use the word "lace" in order to define a piece of the path $\mathcal{P}_d$ of "useless length": when a point goes through the path $\mathcal{P}_d$, the length of this "lace" is useless because the projections of the initial and final points of this lace on the straight line lying the initial and final points of the path $\mathcal{P}_d$ coincide, so, the projection of the point of the path $\mathcal{P}_d$ on this straight line has not avanced to the final point when the point has gone through this lace.

This lace is situated between the points Q and Z because we suppose that the local minimum of the graph $\kappa(t)$ is situated outside the two circles with centres at the points O and T respectively and of radius $\sqrt{\pi}$ ($|\widehat{OI}| \leq \sqrt{\pi}$, $|\widehat{TW}| \leq \sqrt{\pi}$).

We have the following equalities:

$$|\mathcal{P}_d| = |\widehat{OQ}| + |\widehat{QZ}| + |\widehat{TZ}| \ , \qquad |\mathcal{P}_{min}| = |\widehat{OI}| + |I_{pr}W_{pr}| + |\widehat{WT}| \ .$$

Hence,

$$|\mathcal{P}_d| - |\mathcal{P}_{min}| = (|\widehat{OQ}| - |\widehat{OI}|) + (|\widehat{TZ}| - |\widehat{TW}|) + (|\widehat{QZ}| - |I_{pr}W_{pr}|) = |\widehat{QZ}| - |I_{pr}W_{pr}| \ .$$

Using inequality (50), we obtain

$$|\mathcal{P}_d| - |\mathcal{P}_{min}| \geq |Q_{pr}Z_{pr}| + l_{lace} - |I_{pr}W_{pr}| \geq l_{lace} > 1.139456743 \ .$$

The lemma is proved. □

9.6 Subcase B: the comparison of the lengths of $\widetilde{\mathcal{P}}$ and $\mathcal{P}_{min}$

Lemma 9.9 *If the distance between the initial and final points is greater than* $40\sqrt{\pi}$*, then the path* $\widetilde{\mathcal{P}}$ *is longer than the path* $\mathcal{P}_{min}$ *by no more than* 1.050758327*, i.e.*

$$|\widetilde{\mathcal{P}}| - |\mathcal{P}_{min}| < 1.050758327 \ . \tag{51}$$

To prove this lemma we need some auxiliary propositions (more precisely, Propositions 9.10 and 9.11; see the proof of Lemma 9.9 at the end of Subsection 9.6).

1) At first we compare the lengths of $\mathcal{P}_{min}$ and $\widetilde{\mathcal{P}}$ in some neighbourhoods of the initial and final points. Without loss of generality we consider only some neighbourhood of the initial point.

We consider the arc $\widehat{ORI}$ of the path $\mathcal{P}_{min}$ and two arcs $\widehat{OR}$ and $\widehat{RX}$ of the path $\widetilde{\mathcal{P}}$. We suppose that the tangent line at the point X is parallel to the straight line p (see Figure 26). Remind that the tangent line at the point I is parallel to p. We denote by H a point belonging to the tangent line at the point I and such that $|\widehat{ORI}| + |HI| = |\widehat{ORX}|$.

Now we can define more precisely the neighbourhood of the initial point O: we compare the piece of $\widetilde{\mathcal{P}}$ consisting of two arcs of half-clothoid (i.e. the arcs $\widehat{OR}$ and $\widehat{RX}$) and the piece of $\mathcal{P}_{min}$ consisting of an arc of half-clothoid (i.e. the arc $\widehat{ORI}$) and a line segment IH.

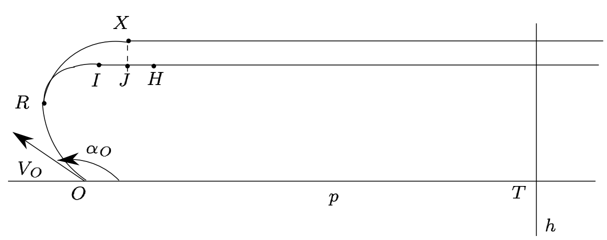


Figure 26

We denote by J the projection of the point X on the tangent line at the point I and we denote by h the straight line passing through the point T and perpendicular to the straight line p.

If J is situated to the right with respect to H, then the distance between X and the straight line h is smaller than the distance between H and the straight line h, i.e. we can say that in the neighbourhood of the point O the path $\widetilde{\mathcal{P}}$ is shorter than the path $\mathcal{P}_{min}$.

If not (see this situation on Figure 26), we can't say that in the neighbourhood of the point O the path $\widetilde{\mathcal{P}}$ is shorter than the path $\mathcal{P}_{min}$, but we can estimate the maximal length of the segment HJ (see Proposition 9.10).

Proposition 9.10 *For* $\max_{\alpha_O \in [0,\pi]} |HJ|$ *we have the following estimation:*

$$\max_{\alpha_O \in [0,\pi]} |HJ| < \sqrt{\pi}/(3\sqrt{2}) \approx 0.4177713791 \ . \qquad (52)$$

See the proof of Proposition 9.10 in Appendix D.1.

2) Now we compare the lengths of $\mathcal{P}_{min}$ and $\widetilde{\mathcal{P}}$ outside the neighbourhoods of the initial and final points (see the definitions of these neighbourhoods in **1**)).

The path $\widetilde{\mathcal{P}}$ outside the neighbourhoods of the initial and final points consists of the line segment XY (remind that we denote it by l). The path $\mathcal{P}_{min}$ outside the neighbourhoods of the initial and final points consists of a line segment IW (remind that there can exist a point of discontinuity of the variables x and y between the points I and W). We denote by l_{min} the line segment IW.

We want to estimate $\max_{\substack{\alpha_O \in [0,\pi] \\ \alpha_T \in [0,\pi]}} (|l| - |l_{min}|)$.

We consider the case when the distance between the points O and T is greater than $40\sqrt{\pi}$ (the choice of this bound on the distance is explained in Lemma 9.3).

Proposition 9.11 *If the distance between the points O and T is greater than $40\sqrt{\pi}$, then for* $\max_{\substack{\alpha_O \in [0,\pi] \\ \alpha_T \in [0,\pi]}} (|l| - |l_{min}|)$ *we have the following estimation:*

$$\max_{\substack{\alpha_O \in [0,\pi] \\ \alpha_T \in [0,\pi]}} (|l| - |l_{min}|) < 0.2152155686 \ . \qquad (53)$$

See the proof of Proposition 9.11 in Appendix D.2.

Proof of Lemma 9.9

Using the results obtained in Propositions 9.10 and 9.11, we obtain the following estimation:

$$\max_{\substack{\alpha_O \in [0,\pi] \\ \alpha_T \in [0,\pi]}} (|\widetilde{\mathcal{P}}| - |\mathcal{P}_{min}|) = 2 \times \max_{\substack{\alpha_O \in [0,\pi] \\ \alpha_T \in [0,\pi]}} |HJ| + \max_{\substack{\alpha_O \in [0,\pi] \\ \alpha_T \in [0,\pi]}} (|l| - |l_{min}|) <$$

$$< 2 \times 0.4177713791 + 0.2152155686 \approx 1.050758327 \ .$$

Hence, $|\widetilde{\mathcal{P}}| - |\mathcal{P}_{min}| < 1.050758327$.
The lemma is proved. □

9.7 Subcase B: the comparison of the lengths of the initial path $\mathcal{P}$ and the modified path $\widetilde{\mathcal{P}}$ (proof of Lemma 9.3)

Remind that we consider only the part of the path $\mathcal{P}$ from the initial point to the first point of zero curvature (if $\kappa^0 \neq 0$) and the part of the path $\mathcal{P}$ from the last point of zero curvature to the final point (if $\kappa^T \neq 0$) and that we denote this path by $\mathcal{P}_d$. It follows from Lemma 9.4 that if the distance between the initial and final points of the path $\mathcal{P}$ is greater than $90.5\sqrt{\pi}$ and if there don't exist two points of zero curvature belonging to the path $\mathcal{P}$ such that the distance between these points is greater than $40\sqrt{\pi}$, then the path $\mathcal{P}$ isn't optimal. So, we must consider the following case: the distance between the initial and final points of the path $\mathcal{P}$ is greater than $90.5\sqrt{\pi}$ and there exist two points of zero curvature belonging to the path $\mathcal{P}$ such that the distance between these points is greater than $40\sqrt{\pi}$.

In this case we can use the result of Lemma 9.9.

It follows from Lemma 9.8 that the path $\mathcal{P}_d$ is longer than the path $\mathcal{P}_{min}$ by no less than 1.139456743, i.e.

$$|\mathcal{P}_d| - |\mathcal{P}_{min}| > 1.139456743 \ .$$

Hence,

$$|\mathcal{P}_d| > |\mathcal{P}_{min}| + 1.139456743 \ . \tag{54}$$

It follows from Lemma 9.9 that the path $\widetilde{\mathcal{P}}$ is longer than the path $\mathcal{P}_{min}$ by no more than 1.050758327, i.e.

$$|\widetilde{\mathcal{P}}| - |\mathcal{P}_{min}| < 1.050758327 \ .$$

Hence,

$$|\mathcal{P}_{min}| > |\widetilde{\mathcal{P}}| - 1.050758327 \ . \tag{55}$$

So, using inequalities (54) and (55), we obtain the desired inequality:

$$|\mathcal{P}_d| > |\mathcal{P}_{min}| + 1.139456743 > |\widetilde{\mathcal{P}}| - 1.050758327 + 1.139456743 =$$

$$= |\widetilde{\mathcal{P}}| + 0.088698416 > |\widetilde{\mathcal{P}}| \ .$$

But $|\mathcal{P}| \geq |\mathcal{P}_d|$. Thus, $|\mathcal{P}| > |\widetilde{\mathcal{P}}|$.
The lemma is proved. □

9.8 Proof of the non-optimality of the path $\mathcal{P}$ – subcase C

Definition *Call 'subcase C' the case when*

1^0 *the path $\mathcal{P}$ has more than one point of zero curvature,*

2^0 *there exists at least one piece of the path $\mathcal{P}$ between two consecutive points of zero curvature such that there are three switching points belonging to this piece and the tangent angle makes a turn of less than 2π on this piece.*

Lemma 9.12 *In subcase C if the distance between the initial and final points of the path $\mathcal{P}$ is greater than $320\sqrt{\pi}$, then we can modify $\mathcal{P}$ so that the obtained path $\widetilde{\mathcal{P}}$ should be shorter than $\mathcal{P}$ and should satisfy the initial and final conditions, hence, the path $\mathcal{P}$ isn't optimal.*

The proof of Lemma 9.12 follows from Propositions 9.13, 9.14 and 9.15.

So, consider the case when the tangent angle makes a turn of less than 2π on the piece $ABCDE$ of the path $\mathcal{P}$ (see the graph of its curvature on Figure 27).

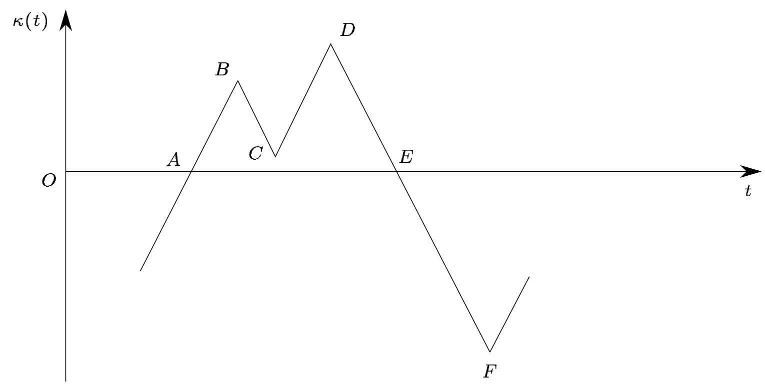


Figure 27

As the mean value of the y-coordinate between two consecutive switching points equals zero, then the piece $CDEF$ is of the kind of the one on Figure 28 or of the one on Figure 29, i.e. with or without vertical tangent line on the arc $\widehat{LF}$.

Denote by L a point belonging to the arc $\widehat{DF}$ such that its tangent line is horizontal. Denote by K (by Q) the intersection point of the arc $\widehat{CD}$ (of the arc $\widehat{DF}$ respectively) and the axis Ox.

Proposition 9.13 *In the case when the distance between the initial and final points of the path $\mathcal{P}$ is greater than $90.5\sqrt{\pi}$ and if on the arc $\widehat{LF}$ there is at least one point with vertical tangent line, then we can modify $\mathcal{P}$ so that the obtained path $\widetilde{\mathcal{P}}$ should be shorter than $\mathcal{P}$ and it should satisfy the initial and final conditions, hence, the path $\mathcal{P}$ isn't optimal.*

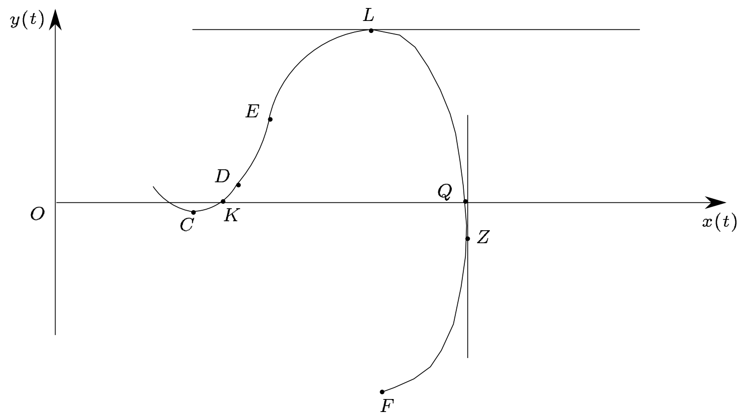


Figure 28

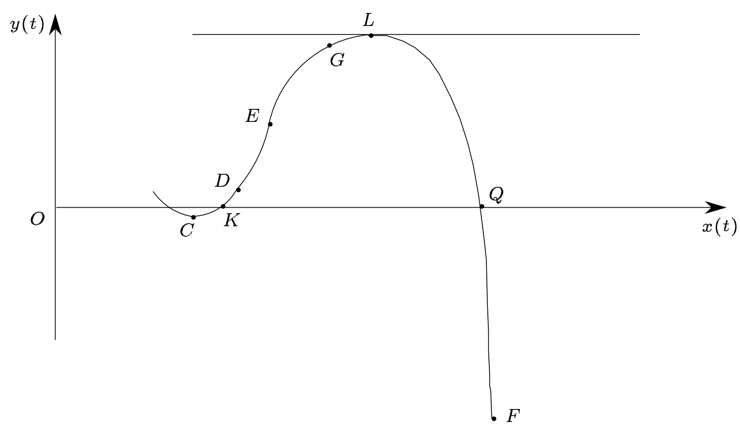


Figure 29

See the proof of Proposition 9.13 in Appendix E.1.

Denote the piece of the path $\mathcal{P}$ from the point E to the final point by $\mathcal{P}_{dr}$.

Proposition 9.14 *In the case when the distance between the initial and final points of the path $\mathcal{P}$ is greater than $90.5\sqrt{\pi}$ and if on the arc $\widehat{LF}$ there is no point with vertical tangent line and if the path $\mathcal{P}_{dr}$ has at least one switching point t_q (here q is some even number, i.e. it corresponds to some maximum of the graph of $\kappa(t)$) with $\kappa(t_q) \leq 0$ or at least one switching point t_p (here p is some odd number, i.e. it corresponds to some minimum of the graph of $\kappa(t)$) with $\kappa(t_p) \geq 0$, then the path $\mathcal{P}$ isn't optimal.*

See the proof of Proposition 9.14 in Appendix E.2.

We have considered the piece of the path $\mathcal{P}$ from the point C to the final point. By the same way we can consider the piece of the path $\mathcal{P}$ from the initial point to the point C and we can obtain two propositions of the type of Propositions 9.13 and 9.14. As a corollary of these four propositions we obtain the following result: we must consider only the case when the curvature changes sign on any interval (except the intervals BC and CD, see Figure 27). This case is considered in Proposition 9.15.

Proposition 9.15 *In the case when the curvature changes sign on any interval (except the intervals BC and CD, see Figure 27) and if the distance between the initial and final points of the path $\mathcal{P}$ is greater than $320\sqrt{\pi}$, then we can modify $\mathcal{P}$ so that the obtained path $\widetilde{\mathcal{P}}$ should be shorter than $\mathcal{P}$ and that it should satisfy the initial and final conditions. Thus the path $\mathcal{P}$ isn't optimal.*

See the proof of Proposition 9.15 in Appendix E.3.

9.9 Proof of the non-optimality of the path $\mathcal{P}$ – subcase D

Definition *Call 'subcase D' the case when*

1^0 the path $\mathcal{P}$ has more than one point of zero curvature,

2^0 there exists at least one piece of the path $\mathcal{P}$ between two consecutive points of zero curvature such that there are at least five switching points belonging to this piece.

Lemma 9.16 *In subcase D if the distance between the initial and final points of the path $\mathcal{P}$ is greater than $90.5\sqrt{\pi}$, then we can modify $\mathcal{P}$ so that the obtained path $\widetilde{\mathcal{P}}$ should be shorter than $\mathcal{P}$ and should satisfy the initial and final conditions, hence, the path $\mathcal{P}$ isn't optimal.*

Proof

Consider the case when there are five switching points between two consecutive points of zero curvature (see Figure 30).

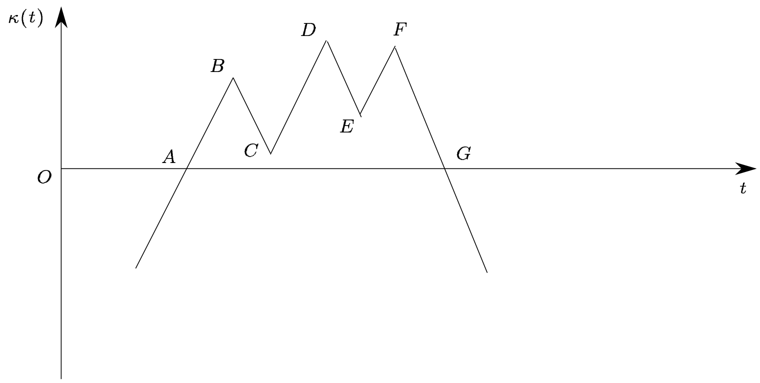


Figure 30

As the mean value of the y-coordinate between two consecutive switching points equals zero (i.e. the piece of the path corresponding to any such interval intersects l'axis Ox) and as the curvature doesn't change sign between the points A and G (see Figure 30), then the tangent angle makes a turn of at least 2π on the piece between the points A and G. Evidently, in the case when there are more than five switching points between two consecutive points of zero curvature, we obtain also a lace. Thus, we have brought subcase D to subcase B (because the crucial point of the proof is the presence of at least one "lace" on $\mathcal{P}$, i.e. the presence of some "useless" length).

The lemma is proved. □

10 Proof of the non-optimality of the path $\mathcal{P}$ – case III

Definition *Call 'case III' the case when* $y_p > 0$, $\kappa_p > 0$ *for any even* p *and* $y_p < 0$, $\kappa_p < 0$ *for any odd* p.

Lemma 10.1 *In case III if the distance between the initial and final points of the path* $\mathcal{P}$ *is greater than* $135.5\sqrt{\pi}$, *then we can modify* $\mathcal{P}$ *so that the obtained path should be shorter than* $\mathcal{P}$ *and that it should satisfy the initial and final conditions.*

Lemma 10.1 follows from Lemmas 10.3 and 10.7.

In order to prove Lemma 10.1 we use other technique than in previous cases (see Sections 4 –9).

We don't consider the part of the path $\mathcal{P}$ from the initial point to the first point of zero curvature (if $\kappa^0 \neq 0$) and the part of the path $\mathcal{P}$ from the last point of zero curvature to the final point (if $\kappa^T \neq 0$). Hence, the initial and final points of the new path (denote it by $\mathcal{P}_d$) are points of zero curvature.

From now on in Section 10 we call *interval of some path* the part of the axis Ot corresponding to the part of this path between two consecutive points of zero curvature (and not the part between two consecutive switching points).

Plan of Section 10 and of the proof of Lemma 10.1

1) At first we give the general idea of the proof (see Subsection 10.1).

2) Then we study the path $\mathcal{P}$ and we obtain that if the distance between the initial and final points of the path $\mathcal{P}$ is greater than $90.5\sqrt{\pi}$ and if there exists some interval $[t_i, t_{i+1}] \subset [O, T]$ such that $|t_{i+1} - t_i| \geq \sqrt{2.926\pi}$, then there exists some lace on this interval and, then we can shorten the given path $\mathcal{P}$ (see Subsection 10.2, Lemmas 10.2, 10.3).

3) Then we consider the paths $\mathcal{P}$ consisting of intervals whose lengths are smaller than $\sqrt{2.926\pi}$, we construct some auxiliary path $\widetilde{\mathcal{P}}_1$ and we formulate the following result: in case III if the path $\mathcal{P}$ consists of intervals whose lengths are smaller than $\sqrt{2.926\pi}$ and if the distance between the initial and final points of $\mathcal{P}$ is greater than $135.5\sqrt{\pi}$, then we can modify $\mathcal{P}$ so that the obtained path should be shorter than $\mathcal{P}$ and that it should satisfy the initial and final conditions (see Subsection 10.3, Lemma 10.7).

4) Then to prove Lemma 10.7 we give the general description of two possible cases (i.e. the case when the paths $\mathcal{P}$ and $\widetilde{\mathcal{P}}_1$ have no lace and the case when the path $\mathcal{P}$ has no lace, but the constructed path $\widetilde{\mathcal{P}}_1$ has at least one lace, see Subsection 10.3).

5) Then we prove Lemma 10.7 for every of these two cases.

We consider the first case in Subsections 10.4, 10.9 and we obtain that if the distance between the initial and final points of the path $\mathcal{P}$ is greater than $135.5\sqrt{\pi}$ and if the paths $\mathcal{P}$ and $\widetilde{\mathcal{P}}_1$ have no lace, then we can modify $\mathcal{P}$ so that the obtained path should be shorter than $\mathcal{P}$ and that it should satisfy the initial and final conditions (see Subsection 10.4, Lemma 10.8).

We consider the second case in Subsections 10.5–10.8 and we obtain that if the distance between the initial and the final points of the path $\mathcal{P}$ is greater than $135.5\sqrt{\pi}$ and if the path $\mathcal{P}$ has no lace, but the path $\widetilde{\mathcal{P}}_1$ has at least one lace, then we can modify $\mathcal{P}$ so that the obtained path should be shorter than $\mathcal{P}$ and that it should satisfy the initial and final conditions (see Subsection 10.5, Lemma 10.9).

6) Thus, Lemma 10.7 follows from Lemmas 10.8 and 10.9; and Lemma 10.1 follows from Lemmas 10.3 and 10.7.

10.1 General idea of a modification of the path $\mathcal{P}$ such that the obtained path satisfies the initial and final conditions and is shorter than the initial one.

Remind that the path $\mathcal{P}$ consists of $N+1$ arcs of half-clothoids, where by N we denote the number of switching points of $\mathcal{P}$. Denote by N_d the number of switching points of $\mathcal{P}_d$. Evidently, $0 \leq N - N_d \leq 2$. Then the graph of the curvature κ as a function of t for the path $\mathcal{P}_d$ should be as shown on Figure 31.

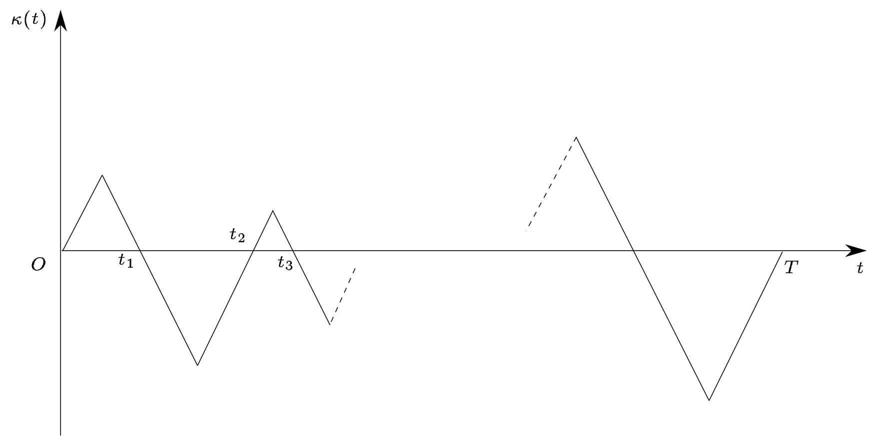


Figure 31

The general idea is to modify some piece of the path $\mathcal{P}_d$ so that the new path (we denote it by $\widetilde{\mathcal{P}}_1$) should be shorter than $\mathcal{P}_d$. We can see an example of such modification on Figure 32. On this figure we show the parties of the paths which are different for $\mathcal{P}_d$ and $\widetilde{\mathcal{P}}_1$. The piece $ADBCLHGK$ belongs to $\mathcal{P}_d$ and the piece $ADEFGK$ belongs to $\widetilde{\mathcal{P}}_1$.

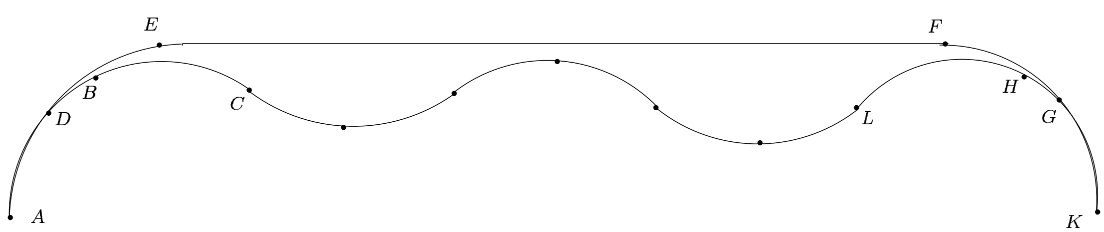


Figure 32

10.2 Some remarks on the aspect of the path $\mathcal{P}_d$.

Definition $(*)$ *We say that there is a "lace" between two consecutive points of zero curvature if there exists at least one intersection point of the path $\mathcal{P}_d$ on this interval.*

Remark that in Section 9 we have introduced other definition of "lace" (more precisely, we say that there is a "lace" on some piece of $\mathcal{P}$ if the tangent angle makes a turn of at least 2π on this piece). But when we use the word "lace" in Section 10, we consider Definition $(*)$.

Lemma 10.2 *There exists a lace on any interval $[t_i, t_{i+1}] \subset [O, T]$ whose length is at least $\sqrt{2.926\pi}$ (i.e. $|\alpha_{i+1} - \alpha_i| \geq 1.4626\pi$).*

See the proof of Lemma 10.2 in Appendix F.1.

Using the results obtained in case II and Lemma 10.2, we obtain the following lemma:

Lemma 10.3 *If the distance between the initial and final points of the path $\mathcal{P}$ is greater than $90.5\sqrt{\pi}$ and if there exists some interval $[t_i, t_{i+1}] \subset [O, T]$ such that $|t_{i+1} - t_i| \geq \sqrt{2.926\pi}$ (i.e. $|\alpha_{i+1} - \alpha_i| \geq 1.4626\pi$), then we can shorten the given path $\mathcal{P}$.*

See the proof of Lemma 10.2 in Appendix F.2.

Remark 10.4 *From now on in Section 10 we consider only paths $\mathcal{P}_d$ consisting of intervals whose lengths are smaller than $\sqrt{2.926\pi}$ (i.e. the difference between the tangent angles at the end and at the beginning of the interval is smaller than 1.4626π).*

10.3 Construction of some path $\widetilde{\mathcal{P}}_1$ and general description of two possible cases depending on the values of the parameters σ and θ

Remind that in Section 10 we call *interval of $\mathcal{P}_d$* the piece of the axis Ot corresponding to the piece of $\mathcal{P}_d$ between two consecutive points with zero curvature (and not the piece between two consecutive switching points). We denote by $d_{\mathcal{P}_d}$ the distance between the initial and final points of the path $\mathcal{P}_d$. Consider an interval $[t_s, t_{s+1}]$ such that $t_s \leq d_{\mathcal{P}_d}/2$ and $t_{s+1} > d_{\mathcal{P}_d}/2$. We denote by A (by C) the point of the path $\mathcal{P}_d$ corresponding to $t = t_s$ (to $t = t_{s+1}$), see Figure 33.

We consider an interval $[t_q, t_{q+1}]$ such that

1) $t_s < t_{s+1} < t_q < t_{q+1}$,

2) the distance between the points corresponding to $t = t_s$ and $t = t_{q+1}$ is at least $13.4R = 6.7\sqrt{\pi}$ ($R = \sqrt{\pi}/2$, see formula (8)).

The requirement that the distance between the points corresponding to $t = t_s$ and $t = t_{q+1}$ should be at least $13.4R = 6.7\sqrt{\pi}$ will be explained in Remark 10.5.

We denote by L (by K) the point of the path $\mathcal{P}_d$ corresponding to $t = t_q$ (to $t = t_{q+1}$), see Figure 34.

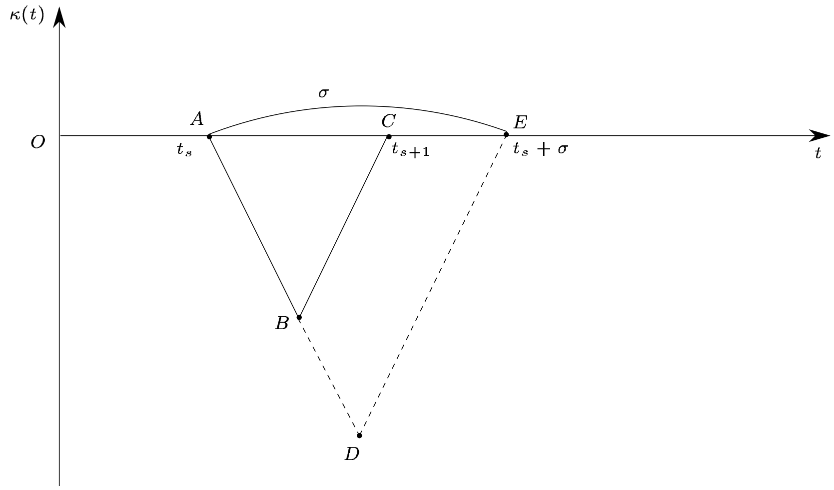


Figure 33

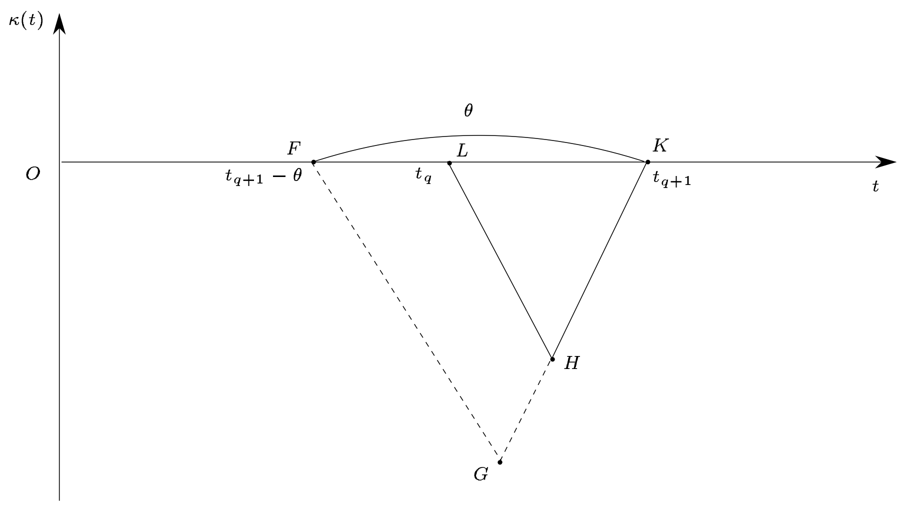


Figure 34

Modify the graph of the curvature $\kappa(t)$ on the interval $[t_s, t_s+\sigma]$ (see Figure 33).

On this figure the pieces AB and BC belong to the graph of $\mathcal{P}_d$ and the pieces AD and DE belong to the graph of the new path $\widetilde{\mathcal{P}}_1$.

The parameter σ varies in the interval $[0, d']$, where d' is some positive constant which is defined below. We denote by α_A, α_E the values of the tangent angle α at the points A and E respectively.

We can increase the absolute value of the angle α_E by increasing σ from 0 to d' because the curvature doesn't change sign on the interval $[t_s, t_s+d']$ and the angle $\alpha_E - \alpha_A$ is the integral of the curvature on this interval:

$$\alpha_E - \alpha_A = \int_0^\sigma \kappa(t)dt \ .$$

The absolute value of $\alpha_E - \alpha_A$ is equal to the area of the triangle ADE, i.e. to $\sigma^2/2$. If we want that $|\alpha_E - \alpha_A| \leq 2\pi$, then, $\sigma^2 \leq 4\pi$ and $\sigma \leq 2\sqrt{\pi}$. Thus,

$$d' = 2\sqrt{\pi} \ . \tag{56}$$

Hence, when σ varies in $[0, 2\sqrt{\pi}]$, then the tangent angle α_E assumes continuously all values from $[\alpha_A, \alpha_A + 2\pi]$ or $[\alpha_A - 2\pi, \alpha_A]$ (the choice of the interval depends on the sign of the curvature on the interval (t_s, t_{s+1})).

Modify the graph $\kappa(t)$ of the curvature of the path $\mathcal{P}_d$ on the interval $[t_{q+1} - \theta, t_{q+1}]$ by means of the parameter θ (see Figure 34). The parameter θ varies in the interval $[0, 2\sqrt{\pi}]$.

We denote by $\mathcal{C}_{A,3R}$ (by $\mathcal{C}_{K,3R}$) the circle with center at the point A (at the point K) and with radius $3R = 3\sqrt{\pi}/2$ (remind that R is defined by (8), see Section 2).

Remark 10.5 *If the distance between the points A and K is at least $13.4R = 6.7\sqrt{\pi}$, then*

1) $t_s + \sigma < t_{q+1} - \theta$,
2) $\mathcal{C}_{A,3R} \cap \mathcal{C}_{K,3R} = \oslash$,
3) the point $E \in \mathcal{C}_{A,3R}$ and the point $F \in \mathcal{C}_{K,3R}$ for every value of σ , θ from the interval $[0, 2\sqrt{\pi}]$.

This is important for the construction of a path $\widetilde{\mathcal{P}}_1$.

Remark that in order this three conditions to hold, it is sufficient that the distance between the points A and K should be at least $8R = 4\sqrt{\pi}$. We consider some greater distance because this simplifies the proof (see Subsections 10.5, 10.6 and 10.7).

Evidently, if the distance between the points A and K is at least $13.4R = 6.7\sqrt{\pi}$, then $t_s + \sigma < t_{q+1} - \theta$ (because $\sigma \in [0, 2\sqrt{\pi}]$, $\theta \in [0, 2\sqrt{\pi}]$) and $\mathcal{C}_{A,3R} \cap \mathcal{C}_{K,3R} = \oslash$.

The distances between the points A and D, D and E, K and G, G and F are smaller than $3R/2 = 3\sqrt{\pi}/4$ (see Proposition 5.3 from [11]). Hence, the distances between the points A and E, K and F are smaller than $3R = 3\sqrt{\pi}/2$. But as the distance between the points A and K is at least $13.4R = 6.7\sqrt{\pi}$, then $E \in \mathcal{C}_{A,3R}$ and $F \in \mathcal{C}_{K,3R}$ for every value of σ , θ from the interval $[0, 2\sqrt{\pi}]$.

Construction of some path $\widetilde{\mathcal{P}}_1$.

We denote by V_A, V_E, V_K and V_F the tangent vectors at the points A, E, K and F. Vary σ so that the vectors V_E and V_K should have the same directions. Hence, the obtained value of the parameter σ depends on the point K. Now we modify the path $\mathcal{P}_d$ on the interval $[t_{q+1} - \theta, t_{q+1}]$ by means of the parameter θ. Simultaneously for every value of the parameter θ we find the corresponding value of the parameter σ so that the vectors V_E and V_F should have the same directions. Thus, σ is a function of θ. For some values of σ and θ the tangent lines at the points E and F coincide (we denote this line by l). So, we obtain the path $\widetilde{\mathcal{P}}_1$.

Now we must prove that varying σ and θ on the interval $[0, 2\sqrt{\pi}]$ in the above-mentioned way, we can always construct some path $\widetilde{\mathcal{P}}_1$.

Consider some value of parameter σ (denote it by σ_1) such that the tangent line at the point of the path corresponding to the point E of the graph $\kappa(t)$ (see Figure 33) should be vertical for $\sigma = \sigma_1$ (see Figure 35). Denote this point of the path by E_1. Denote by V_{E_1} the tangent vector at the point E_1.

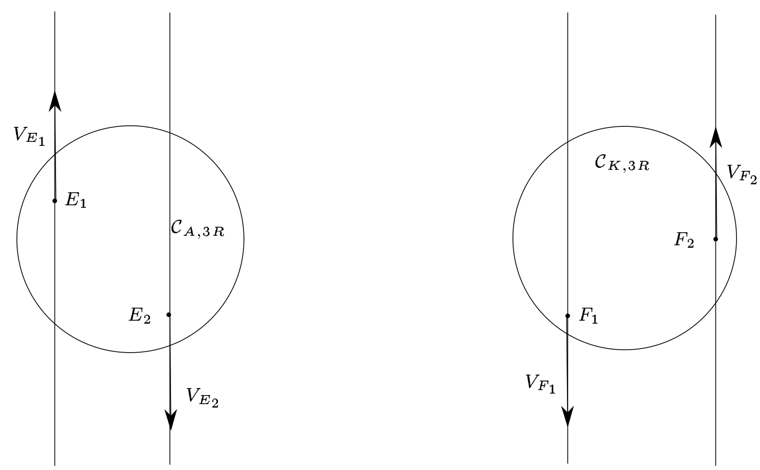


Figure 35

Choose now another value of the parameter σ (denote it by σ_2) such that the tangent vectors V_{E_1} and V_{E_2} should have opposite directions (here we denote by E_2 the point of the path corresponding to the point E of the graph for $\sigma = \sigma_2$ and we denote by V_{E_2} the tangent vector at the point E_2).

Also we choose two values of the parameter θ (denote them by θ_1 and θ_2) such that the vectors V_{E_1} and V_{F_1} (V_{E_2} and V_{F_2}) should have opposite directions (here we denote by V_{F_1}

and V_{F_2} the tangent vectors at the points corresponding to the point F of the graph (see Figure 34) for $\theta = \theta_1$ and $\theta = \theta_2$ respectively).

So, the oriented distances between the tangent lines at the points of the path corresponding to the points E and F of the graph for these two pairs of values σ and θ (i.e. for $(\sigma = \sigma_1, \theta = \theta_1)$ and for $(\sigma = \sigma_2, \theta = \theta_2)$) have opposite signs. These distances don't equal zero because $\mathcal{C}_{A,3R} \cap \mathcal{C}_{K,3R} = \oslash$ (see Remark 10.5). The oriented distance from the point F to the tangent line at the point E of the path is some continuous function of the parameter σ (see Proposition 10.6). It follows from Bolzano's Theorem that there exists at least one value of $\sigma \in [\sigma_1, \sigma_2]$ for which the oriented distance equals zero (it gives the path $\widetilde{\mathcal{P}}_1$).

So, varying σ and θ in the interval $[0, 2\sqrt{\pi}]$ in the above-mentioned way, we can always construct some path $\widetilde{\mathcal{P}}_1$.

Proposition 10.6 *The oriented distance from the point F of the path to the tangent line at the point E is some continuous function of the parameter σ.*

Proof

Denote the tangent line at the point E of the path by p_E. The straight line p_E admits some parametrisation of the class C^1 in σ, the coordinates of the point F of the path are some functions of the class C^2 in θ. As we require that the vectors V_E and V_F have the same direction, so the parameter θ is some function of the parameter σ. Really, we can express the tangent angles at the points E and F by the following formulas:

$$\alpha_E = \alpha_A + \int_0^{\sigma} \kappa(t)dt = \alpha_A \pm \sigma^2/2 \ ,$$

$$\alpha_F = \alpha_K - \int_0^{\theta} \kappa(t)dt = \alpha_K \pm \theta^2/2$$

(the choice of sign depends on the sign of the curvature $\kappa(t)$ on the intervals $(t_s, t_s + \sigma)$, $(t_{q+1} - \theta, t_{q+1})$).

Thus, the condition

$$\Phi(\sigma, \theta) = \alpha_E(\sigma) - \alpha_F(\theta) - \pi = 0$$

defines some function $\theta(\sigma)$ of class C, because

1) $\Phi(\sigma, \theta) \in C^1(\sigma, \theta)$;

2) $\Phi(\sigma, \theta) = \pm\sigma^2/2 \pm \theta^2/2 + \alpha_A - \alpha_K - \pi$ is some strictly increasing or strictly decreasing function of θ, hence, there exists one and only one value of $\theta = \theta(\sigma)$ for which $\Phi(\sigma, \theta) = \alpha_E(\sigma) - \alpha_F(\theta) - \pi = 0$.

It follows from the implicit function theorem that the function $\theta(\sigma)$ belongs to the class C.

The proposition is proved. □

Now we must prove that the thus obtained path $\widetilde{\mathcal{P}}_1$ is shorter than the path $\mathcal{P}_d$ or that $\widetilde{\mathcal{P}}_1$ can be modified so that the new path should be shorter than $\mathcal{P}_d$.

Lemma 10.7 *In case III if the path $\mathcal{P}$ consists of intervals whose lengths are smaller than $\sqrt{2.926\pi}$ and if the distance between the initial and final points of $\mathcal{P}$ is greater than $135.5\sqrt{\pi}$, then we can modify $\mathcal{P}$ so that the obtained path should be shorter than $\mathcal{P}$ and that it should satisfy the initial and final conditions.*

Lemma 10.7 follows directly from Lemmas 10.8 and 10.9.

In the proof of Lemma 10.7 there are two cases to consider.

Remind, at first, that it follows from Lemma 10.2 that there exists one lace on any interval $[t_i, t_{i+1}] \subset [O, T]$ whose length is at least $\sqrt{2.926\pi}$ (i.e. $|\alpha_{i+1} - \alpha_i| \geq 1.4626\pi$). Hence, if $\sigma \geq \sqrt{2.926\pi}$ (or $\theta \geq \sqrt{2.926\pi}$), then the constructed path $\widetilde{\mathcal{P}}_1$ has at least one lace (remind that when we use the word "lace" in Section 10, we consider Definition $(*)$ (see the beginning of Subsection 10.2)).

Set $\sqrt{2.926\pi} = \Gamma$. So, we must consider the two following cases.

1. The case when $\sigma < \Gamma$ and $\theta < \Gamma$ (i.e. the modified path $\mathcal{P}_d$ and the constructed path $\widetilde{\mathcal{P}}_1$ have no lace).
2. The case when at least one of the parameters σ, θ is in $[\Gamma, 2\sqrt{\pi}]$ (i.e. the case when the modified path $\mathcal{P}_d$ has no lace, but the constructed path $\widetilde{\mathcal{P}}_1$ has at least one lace).

The first case is studied in Subsections 10.4, 10.9, the second case is studied in Subsections 10.5–10.8.

10.4 General description of the case when $\sigma < \Gamma$ and $\theta < \Gamma$ (i.e. when the paths $\mathcal{P}$ and $\widetilde{\mathcal{P}}_1$ have no lace)

Lemma 10.8 *In case III if the distance between the initial and final points of the path $\mathcal{P}$ is greater than $135.5\sqrt{\pi}$ and if $\sigma < \Gamma$ and $\theta < \Gamma$ (i.e. when the paths $\mathcal{P}$ and $\widetilde{\mathcal{P}}_1$ have no lace), then we can modify $\mathcal{P}$ so that the obtained path should be shorter than $\mathcal{P}$ and that it should satisfy the initial and final conditions.*

See the proof of Lemma 10.8 in Subsection 10.9.

There exist four cases depending on the values of σ and θ:

1) $t_{s+1} - t_s < \sigma < \Gamma$,
2) $t_{q+1} - t_q < \theta < \Gamma$,
3) $\sigma \leq t_{s+1} - t_s < \Gamma$,
4) $\theta \leq t_{q+1} - t_q < \Gamma$,

but in all cases we use the same method of proof.

This method is described in detail in Subsection 10.9. We give here only some general ideas of the proof.

1⁰. If $|\widetilde{\mathcal{P}}_1| < |\mathcal{P}_d|$, then, Lemma 10.8 is proved. If $|\widetilde{\mathcal{P}}_1| \geq |\mathcal{P}_d|$, then instead of the interval $[t_q, t_{q+1}]$ we consider the next interval $[t_{q+1}, t_{q+2}]$, instead of the interval $[t_s, t_{s+1}]$ we consider the previous interval $[t_{s-1}, t_s]$, and we construct some new path $\widetilde{\mathcal{P}}_2$ by modifying the graph $\kappa(t)$ of the initial path $\mathcal{P}_d$ on two intervals $[t_{s-1}, t_{s-1}+\sigma]$, $[t_{q+2}-\theta, t_{q+2}]$.

2⁰. Now there are two possibilities: either $|\widetilde{\mathcal{P}}_2| < |\mathcal{P}_d|$ or $|\widetilde{\mathcal{P}}_2| \geq |\mathcal{P}_d|$. In the last case we construct some new path $\widetilde{\mathcal{P}}_3$ by modifying the graph $\kappa(t)$ of $\mathcal{P}_d$ on two intervals $[t_{s-2}, t_{s-2}+\sigma]$, $[t_{q+3}-\theta, t_{q+3}]$.

3⁰. We obtain some sequence of paths $\widetilde{\mathcal{P}}_1, \widetilde{\mathcal{P}}_2, \ldots, \widetilde{\mathcal{P}}_j, \ldots, \widetilde{\mathcal{P}}_K$ such that we construct every new path $\widetilde{\mathcal{P}}_j$ by modifying the graph $\kappa(t)$ of the initial path $\mathcal{P}_d$ on two intervals $[t_{s-(j-1)}, t_{s-(j-1)}+\sigma]$, $[t_{q+j}-\theta, t_{q+j}]$. If we suppose that for every path $\widetilde{\mathcal{P}}_j$ we have $|\widetilde{\mathcal{P}}_j| \geq |\mathcal{P}_d|$, then the sequence of $\Delta_{-j}+\Delta_j$ is bounded from below by some increasing geometric progression, see Proposition 10.22. Here by Δ_j we denote the difference between the length of the interval $[t_{q+j-1}, t_{q+j}]$ and the sum of the lengths of two chords connecting the points of $\mathcal{P}_d$ corresponding to $t = t_{q+j-1}$, $t = t_{q+j}$ with the switching point of $\mathcal{P}_d$ belonging to $[t_{q+j-1}, t_{q+j}]$; we denote by Δ_{-j} the difference between the length of the interval $[t_{s-(j-1)}, t_{s-(j-2)}]$ and the sum of the lengths of two chords connecting the points of $\mathcal{P}_d$ corresponding to $t = t_{s-(j-1)}$, $t = t_{s-(j-2)}$ with the switching point of $\mathcal{P}_d$ belonging to $[t_{s-(j-1)}, t_{s-(j-2)}]$.

It follows from Remark 10.4 that in Section 10 we consider only paths consisting of intervals whose lengths are smaller than $\sqrt{2.926\pi}$. We set $\sqrt{2.926\pi} = \Gamma$. Hence, for some value $j = K$ the sum of the length of the piece of $\mathcal{P}_d$ corresponding to Δ_K and the length of the piece of $\mathcal{P}_d$ corresponding to Δ_{-K} becomes greater than the admissible constant 2Γ (because the sequence of $\Delta_{-j}+\Delta_j$ is bounded from below by some increasing geometric progression).

4⁰. Then we prove that the sequence $\Delta_{-K}+\Delta_K, \Delta_{-(K-1)}+\Delta_{K-1}, \ldots, \Delta_{-1}+\Delta_1$ and the sequence of the lengths of the corresponding arcs $l_{-K}+l_K, l_{-(K-1)}+l_{K-1}, \ldots, l_{-1}+l_1$ are two decreasing sequences bounded from above by some geometric progressions (see Proposition 10.25 and Lemma 10.26).

As a corollary of Lemma 10.26 we obtain the desired result (see Lemma 10.27): if the distance between the initial and final points is greater than some constant C, then $|\widetilde{\mathcal{P}}_K| < |\mathcal{P}_d|$ (the estimation of the constant C is given in Lemma 10.28: $C = 135.5\sqrt{\pi}$).

So, Lemma 10.8 follows from Lemmas 10.27 and 10.28.

10.5 General description of the case when at least one of the parameters σ, θ belongs to $[\Gamma, 2\sqrt{\pi}]$ (i.e. when the path $\mathcal{P}$ has no lace, but the path $\widetilde{\mathcal{P}}_1$ has at least one lace).

Lemma 10.9 *In case III if the distance between the initial and the final points of the path $\mathcal{P}$ is greater than $135.5\sqrt{\pi}$ and if at least one of the parameters σ, θ belongs to $[\Gamma, 2\sqrt{\pi}]$ (i.e. when the path $\mathcal{P}$ has no lace, but the path $\widetilde{\mathcal{P}}_1$ has at least one lace), then we can modify $\mathcal{P}$ so that the obtained path should be shorter than $\mathcal{P}$ and that it should satisfy the initial and final conditions.*

See the proof of Lemma 10.9 in Subsection 10.8.

Without loss of generality we consider only the case when $\Gamma \leq \sigma \leq 2\sqrt{\pi}$ (for the two other cases we use the same reasoning).

In this subsection we consider the case when, constructing some path $\widetilde{\mathcal{P}}_1$, we obtain $\sqrt{2.926\pi} \leq \sigma \leq 2\sqrt{\pi}$ (i.e. $1.4626\pi \leq |\alpha_E - \alpha_A| \leq 2\pi$). In this case we obtain one lace between the points A and E, hence, we cannot apply the method described in Subsection 10.4 and we must introduce some new method.

Denote by V_A the tangent vector at the point A. Denote by a the tangent line at the point A and denote by $a_\perp$ the straight line passing through the point A and which is perpendicular to the straight line a.

The straight line $a_\perp$ divides also the plane in two half-planes ('half-plane I' and 'half-plane II').

Definition We call 'half-plane I' a half-plane such that it doesn't contain the vector V_A. We call 'half-plane II' a half-plane such that it contains the vector V_A.

Now we can divide all paths $\mathcal{P}_d$ in two classes ('class I' and 'class II').

Definition We say that a path $\mathcal{P}_d$ belongs to the 'class I' (to the 'class II') if the first intersection point of the straight line a and of the path $\mathcal{P}_d$ on the interval $(t_s, T]$ belongs to the half-plane I (to the half-plane II).

The general idea is to modify the path $\mathcal{P}_d$ on the interval $[t_s, t_{s+1}]$ changing the sign of the curvature (see an example of some modification of this type on Figure 36).

Construct some path $\widetilde{\mathcal{P}}_1$ as in Subsection 10.3, varying the parameters σ and θ (but now the curvature $\kappa(t)$ is positive on the interval $(t_s, t_s + \sigma)$ and $\kappa(t) < 0$ on the interval $(t_{q+1} - \theta, t_{q+1})$ for the constructed path, see Figures 36 and 34; we denote this path by $\widetilde{\mathcal{P}}_{1new}$).

Denote by E^*, F^*, l^* the points E, F and the line segment belonging to the path $\widetilde{\mathcal{P}}_1$ constructed in Subsection 10.3 and denote by E^{**}, F^{**}, l^{**} the points E, F and the line segment belonging to $\widetilde{\mathcal{P}}_{1new}$. Denote by V_{E^*} (by $V_{E^{**}}$) the tangent vector at the point E^* (at the point E^{**} respectively).

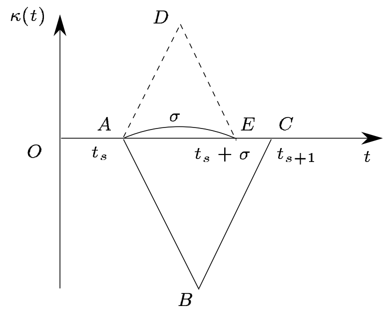


Figure 36

10.5.1 Plan of the study of the case when at least one of the parameters σ, θ belongs to $[\Gamma, 2\sqrt{\pi}]$ (namely, when $\Gamma \leq \sigma \leq 2\sqrt{\pi}$)

1) At first we consider the subcase when the path $\mathcal{P}_d$ belongs to the class I and we obtain that this modification shortens the path $\mathcal{P}_d$ at the left (see Lemma 10.11, Subsection 10.6).

2) Then we consider the subcase when the path $\mathcal{P}_d$ belongs to the class II and we obtain that we can modify $\mathcal{P}_d$ on the interval $[t_{s-1}, t_{q+1}]$ so that either the new path $\widetilde{\mathcal{P}}_{2new}$ has no lace (so, we obtain the case already studied in Subsection 10.4), or we modify the path $\mathcal{P}_d$ on the interval $[t_{s-2}, t_{q+1}]$ so that this modification shortens the path $\mathcal{P}_d$ at the left (see Lemma 10.18, Subsection 10.7).

In the two cases (i.e. when $\mathcal{P}_d$ belongs to the class I or to the class II) we use some property concerning the relative position of the tangent lines at the points A and E^{**}.

It follows from the construction of the path $\widetilde{\mathcal{P}}_{1new}$ that on the interval $(t_s, t_{s+1}]$ the paths $\mathcal{P}_d$ and $\widetilde{\mathcal{P}}_{1new}$ are in different half-planes (with respect to the straight line a).

Denote by ψ the angle between the vector $V_{E^{**}}$ and the vector $-V_A$. So, $\psi = |\alpha_{E^{**}} - \alpha_A| - \pi$ (see Figure 37).

Proposition 10.10 *The angle ψ belongs to $(\pi/6, \pi]$.*

See the proof of Proposition 10.10 in Appendix G.

Denote by $P \in \mathcal{P}_d$ a point such that $|\widehat{ABP}| = |\widehat{ADE}^{**}|$ and we denote by P_{pr} the projection of the point P on the straight line l^{**}.

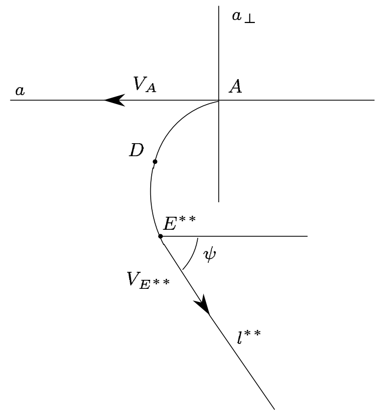


Figure 37

10.6 The case when $\Gamma \leq \sigma \leq 2\sqrt{\pi}$ – the subcase when the path $\mathcal{P}_d$ belongs to the class I

Lemma 10.11 *In the case when $\Gamma \leq \sigma \leq 2\sqrt{\pi}$ and the path $\mathcal{P}_d$ belongs to the class I this modification shortens the path $\mathcal{P}_d$ at the left (because the points P_{pr}, E^{**} and F^{**} are in this order on the straight line l^{**}).*

Plan of the proof of Lemma 10.11

1) At first we prove that in the case when $\psi \in (\pi/6, \pi/2)$ this modification shortens the path $\mathcal{P}_d$ at the left (because the points P_{pr}, E^{**} and F^{**} are in this order on the straight line l^{**}, see Lemma 10.12).

2) Then we prove that in the case when $\psi \in [\pi/2, 5\pi/6]$ this modification shortens the path $\mathcal{P}_d$ at the left (because the points P_{pr}, E^{**} and F^{**} are in this order on the straight line l^{**}, see Lemma 10.13).

3) Finally we prove that in the case when $\psi \in (5\pi/6, \pi]$ this modification shortens the path $\mathcal{P}_d$ at the left (because the points P_{pr}, E^{**} and F^{**} are in this order on the straight line l^{**}, see Lemma 10.16).

4) Thus, as the angle $\psi \in (\pi/6, \pi]$ (see Proposition 10.10), so , Lemma 10.11 follows from Lemmas 10.12, 10.13 and 10.16.

10.6.1 The case when $\psi \in (\pi/6, \pi/2)$

Lemma 10.12 *In the case when $\psi \in (\pi/6, \pi/2)$ this modification shortens the path $\mathcal{P}_d$ at the left (because the points P_{pr}, E^{**} and F^{**} are in this order on the straight line l^{**}).*

To prove Lemma 10.12 we need some auxiliary paths. Consider the part of the axis Ot between the points t_s and $t_s + \sqrt{3\pi}$. Consider the part of the path $\mathcal{P}_d$ and the part of the path $\widetilde{\mathcal{P}}_{1new}$ on $[t_s, t_s + \sqrt{3\pi}]$. The path $\widetilde{\mathcal{P}}_{1new}$ consists on $[t_s, t_s + \sqrt{3\pi}]$ of two arcs of half-clothoid and of a line segment (see its graph of the curvature on Figure 38: it consists of three segments – AD, DE and EH). The path $\mathcal{P}_d$ consists on $[t_s, t_{s+1}]$ of two arcs of half-clothoid (on Figure 38 they are the segments AB and BC); we know only that the path $\mathcal{P}_d$ on $[t_{s+1}, t_s + \sqrt{3\pi}]$ consists of at least one arc of half-clothoid, but its exact aspect isn't important for the proof. Consider some path C_0 on $[t_s, t_s + \sqrt{3\pi}]$: it consists of two arcs of half-clothoid (see its graph of the curvature on Figure 38: it consists of two segments – AZ and ZH). Consider some path C_1 on $[t_s, t_s + \sqrt{3\pi}]$: it consists of two arcs of half-clothoid and of a line segment, the tangent angle is continuous, but the curvature has a point of discontinuity (see its graph of the curvature on Figure 38: it consists of three segments – AZ, ZR_d and RH; q is a constant belonging to $(0, \sqrt{3\pi}]$, we choose a value of q such that the tangent vector at the point R of the path C_1 should be parallel to the tangent vector at the point E^{**} of the path $\widetilde{\mathcal{P}}_{1new}$ (see Proposition H.2)).

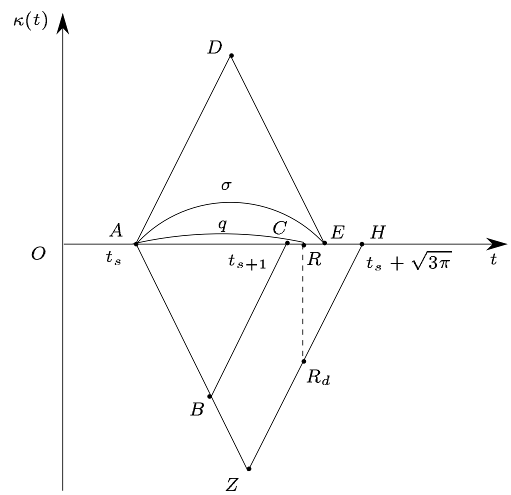


Figure 38

Denote by $P_1 \in C_1$ a point such that $|\widehat{ABP_1}| = |\widehat{ADE}^{**}|$ and denote by P_{1pr} the projection of the point P_1 on the straight line l^{**}.

Proof of Lemma 10.12

We prove Lemma 10.12 in three stages.

1⁰ At first we compare the positions of the points P_{1pr}, E^{**} and F^{**} on the straight line l^{**} – we obtain the following statement: the points P_{1pr}, E^{**} and F^{**} are in this order on the straight line l^{**} (see Proposition H.1, Appendix H).

2⁰ Then we compare the positions of the points P_{pr}, P_{1pr}, E^{**} and F^{**} on the straight line l^{**} – we obtain the following statement: if the points P_{1pr}, E^{**} and F^{**} are in this order on the straight line l^{**}, then the points P_{pr}, P_{1pr}, E^{**} and F^{**} are in this order on the straight line l^{**} (see Proposition H.6, Appendix H).

3⁰ Finaly, using the results obtained in Propositions H.1 and H.6, we obtain the desired statement: in the case when $\psi \in (\pi/6, \pi/2)$, the points P_{pr}, E^{**} and F^{**} are in this order on the straight line l^{**} (i.e. this modification shortens the path $\mathcal{P}_d$ at the left); so Lemma 10.12 is proved.

10.6.2 The case when $\psi \in [\pi/2, 5\pi/6]$

Lemma 10.13 *In the case when $\psi \in [\pi/2, 5\pi/6]$ this modification shortens the path $\mathcal{P}_d$ at the left (because the points P_{pr}, E^{**} and F^{**} are in this order on the straight line l^{**}).*

To prove Lemma 10.13 we need two auxiliary propositions (namely Proposition 10.14 and Proposition 10.15).

Introduce now some notation. Denote by S_1 (by S_2) the first (the second) point belonging to the arc $\widehat{ABC}$ such that its tangent line is orthogonal to the straight line l^{**} (i.e. $|\alpha_{S_2} - \alpha_{S_1}| = \pi$), see Figure 39.

Proposition 10.14 *In the case when $\psi \in [\pi/2, 5\pi/6]$ the point S_2 belongs to the arc $\widehat{BC}$.*

Proposition 10.15 *In the case when $\psi \in [\pi/2, 5\pi/6]$ the point P belongs to the arc $\widehat{S_1S_2}$.*

See the proof of Proposition 10.14 (of Proposition 10.15) in Appendix I.1 (in Appendix I.2).

Proof of Lemma 10.13

Denote by S_3 a point belonging to the arc $\widehat{ADE}^{**}$ such that $|\widehat{AS}_3| = |\widehat{AS}_1|$, and denote by S_{3pr} the projection of the point S_3 on the straight line l^{**} (see Figure 39).

As $\kappa(t) \geq 0$ on the arc $\widehat{AS}_3$ and $\kappa(t) \leq 0$ on the arc $\widehat{AS}_1$, then the points S_{1pr}, S_{3pr}, E^{**} and F^{**} are in this order on the straight line l^{**}.

As $P \in \widehat{S_1S_2}$ (it follows from Proposition 10.15), then the whole projection of the arc $\widehat{S_1P}$ on the straight line l^{**} is to the left with respect of the point S_{1pr}.

The whole projection of the arc $\widehat{S_3E}^{**}$ on the straight line l^{**} is to the right with respect to the point S_{3pr}.

Hence, the points P_{pr}, E^{**} and F^{**} are in this order on the straight line l^{**} (i.e. this modification shortens the path $\mathcal{P}_d$ to the left).

The lemma is proved. □

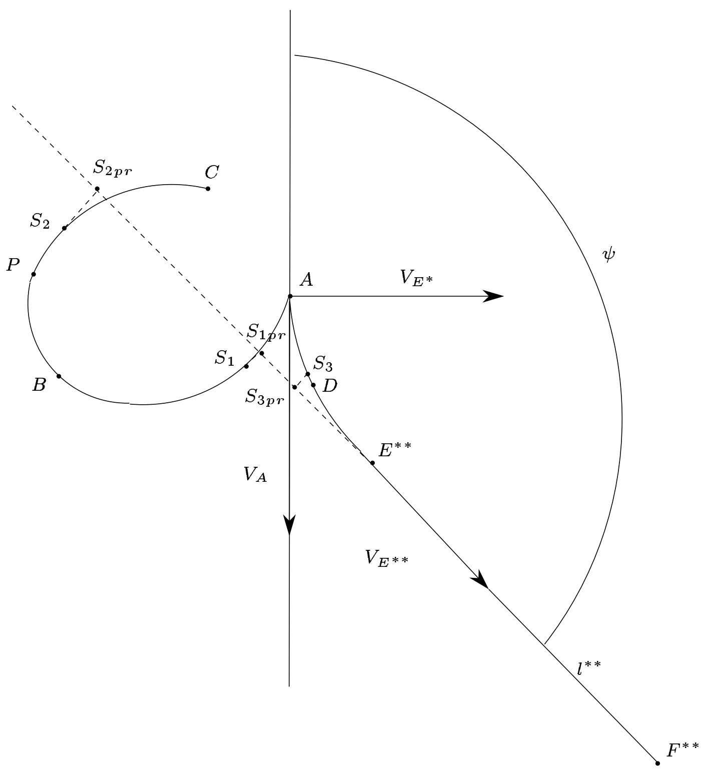


Figure 39

10.6.3 The case when $\psi \in (5\pi/6, \pi]$

Lemma 10.16 *In the case when $\psi \in (5\pi/6, \pi]$ this modification shortens the path $\mathcal{P}_d$ to the left (because the points P_{pr}, E^{**} and F^{**} are in this order on the straight line l^{**}).*

In order to prove Lemma 10.16 we need some auxiliary proposition (namely Proposition 10.17).

As in the previous subsubsection denote by S_1 the first point belonging to the arc $\widehat{ABC}$ such that its tangent line is orthogonal to the straight line l^{**}.

Proposition 10.17 *In the case when $\psi \in (5\pi/6, \pi]$ the point $P \in \widehat{AS}_1$.*

See the proof of Proposition 10.17 in Appendix J.

Proof of Lemma 10.16

The point P belongs to the arc $\widehat{AS}_1$ (see Proposition 10.17). The curvature $\kappa(t)$ is non-negative on the arc $\widehat{ADE}^{**}$ and non-positive on the arc $\widehat{AP}$. Hence, the points P_{pr}, E^{**} and F^{**} are in this order on the straight line l^{**}.

The lemma is proved. □

10.7 The case when $\Gamma \leq \sigma \leq 2\sqrt{\pi}$ – the subcase when the path $\mathcal{P}_d$ belongs to the class II

Lemma 10.18 *In the case when $\Gamma \leq \sigma \leq 2\sqrt{\pi}$ and the path $\mathcal{P}_d$ belongs to the class II, we modify $\mathcal{P}_d$ on the interval $[t_{s-1}, t_{q+1}]$ so that either the new path $\widetilde{\mathcal{P}}_{2new}$ has no lace (so, we obtain the case already studied in Subsection 10.4), or we modify the path $\mathcal{P}_d$ on the interval $[t_{s-2}, t_{q+1}]$ so that this modification shortens the path $\mathcal{P}_d$ to the left.*

See the proof of Lemma 10.18 at the end of Subsection 10.7.

Remind that it follows from Proposition 10.10 that the angle ψ belongs to $(\pi/6, \pi]$ for all values $|\alpha_{E^*} - \alpha_A| \in [1.4626\pi, 2\pi)$.

In the case when $\Gamma \leq \sigma \leq 2\sqrt{\pi}$ and the path $\mathcal{P}_d$ belongs to the class II, we modify $\mathcal{P}_d$ on the interval $[t_{s-1}, t_{q+1}]$ (instead of the interval $[t_s, t_{q+1}]$) by means of two parameters σ and θ (as in Subsection 10.3). We obtain some path $\widetilde{\mathcal{P}}_{2new}$ (see an example of graphs of the curvature of the path $\mathcal{P}_d$ and of the path $\widetilde{\mathcal{P}}_{2new}$ on the interval $[t_{s-1}, t_{s-1}+\sigma]$ on Figure 40: the segments JI, IA, AB and BC belong to the graph of $\mathcal{P}_d$ and the segments JR and RS belong to the graph of $\widetilde{\mathcal{P}}_{2new}$). Remark that σ can be greater than $t_s - t_{s-1}$ or not greater than $t_s - t_{s-1}$ (même si on Figure 40 we have $\sigma > t_s - t_{s-1}$).

If for this path $\widetilde{\mathcal{P}}_{2new}$ we obtain $\sigma < \Gamma$ (i.e. there is no lace on the interval $[t_{s-1}, t_{s-1}+\sigma]$), then we have the case already studied in Subsection 10.4. If not, the path $\widetilde{\mathcal{P}}_{2new}$ has one lace on the interval $[t_{s-1}, t_{s-1}+\sigma]$. We must study this possibility.

As we have obtained one lace on the interval $[t_{s-1}, t_{s-1}+\sigma]$, then we apply the method introduced in Subsection 10.5, i.e. we construct some path $\widetilde{\mathcal{P}}_{3new}$ modifying the initial path

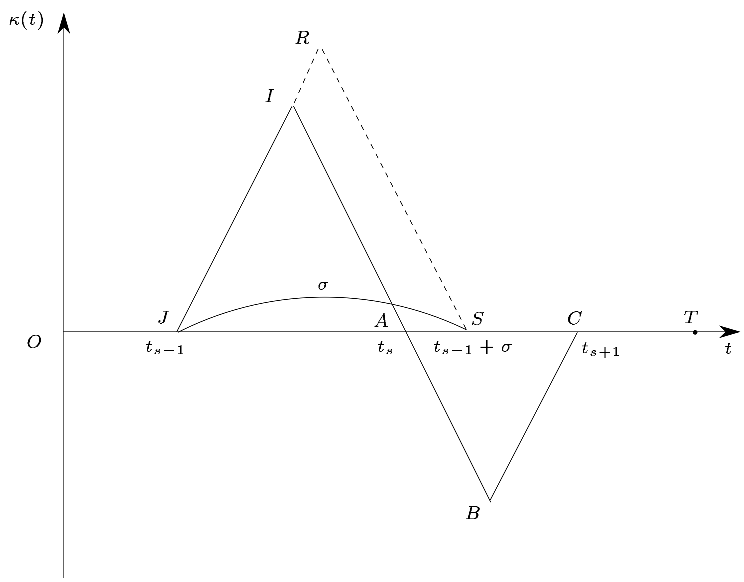


Figure 40

$\mathcal{P}_d$ on the interval $[t_{s-1}, t_{q+1}]$ by means of two parameters σ and θ, but now $\kappa(t) < 0$ on the interval $(t_{s-1}, t_{s-1} + \sigma)$ and $\kappa(t) < 0$ on the interval $(t_{q+1} - \theta, t_{q+1})$ for the path $\widetilde{\mathcal{P}}_{3new}$ (see Figures 41 and 34).

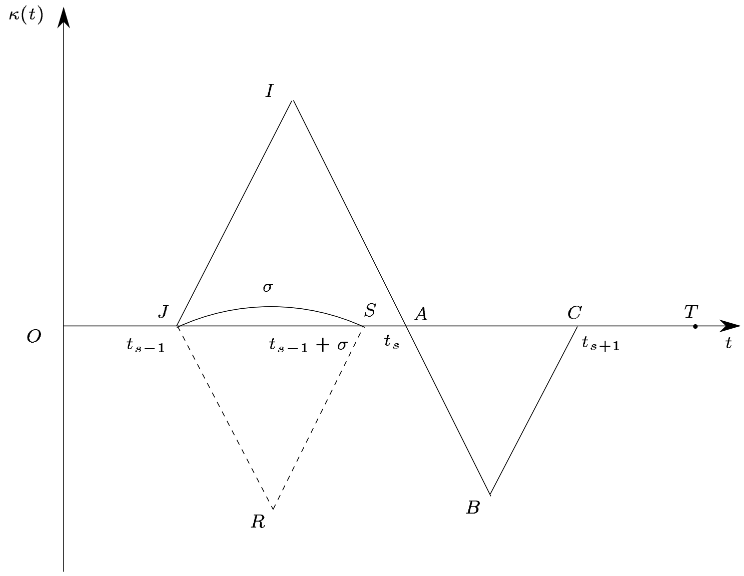


Figure 41

Remark that we don't know whether σ is greater than $t_s - t_{s-1}$ or not, but it isn't important for the proof.

Denote by $\widetilde{R}$ (by $\widetilde{S}$) the point belonging to the path $\widetilde{\mathcal{P}}_{3new}$ and corresponding to the point R (to the point S) of the graph $\kappa(t)$ (see Figure 41). Denote by $\widetilde{F}$ the point belonging to the path $\widetilde{\mathcal{P}}_{3new}$ and corresponding to the point F of the graph $\kappa(t)$ (see Figure 34). Denote by $\tilde{l}$ the line segment belonging to the path $\widetilde{\mathcal{P}}_{3new}$. Denote by $\widetilde{P}$ a point belonging to the path $\mathcal{P}_d$ on the interval $[t_{s-1}, T]$ such that $|\widehat{J\widetilde{P}}| = |\widehat{J\widetilde{R}\widetilde{S}}|$ and denote by $\widetilde{P}_{pr}$ the projection of the point $\widetilde{P}$ on the straight line $\tilde{l}$.

Plan of the proof of Lemma 10.18

We prove Lemma 10.18 in four stages.

> $\mathbf{1^0}$ At first we construct some auxiliary path $\mathcal{C}_0$ and we study the connection between the paths $\mathcal{C}_0$ and $\widetilde{\mathcal{P}}_{2new}$ – we obtain the following statement: considering all cases when the path $\mathcal{C}_0$ has one lace on the interval $[t_{s-1}, t_{s-1}+\sigma]$, we consider all cases when the path $\widetilde{\mathcal{P}}_{2new}$ has one lace on this interval (see Remark 10.19).

2⁰ Then we calculate the minimal length of the arc $\widehat{JIA}$ such that the path $\mathcal{C}_0$ has one lace on the interval $[t_{s-1}, t_{s-1}+\sigma]$ – we obtain the following inequality:

$$|\widehat{JIA}| \geq \sqrt{(4/3 - 2 \times 0.0374)\pi}$$

(see Proposition K.1, Appendix K).

3⁰ Then we calculate the maximal length of the arc $\widehat{J\widetilde{R}\widetilde{S}}$ – we obtain the following inequality:

$$|\widehat{J\widetilde{R}\widetilde{S}}| \leq \sqrt{2\pi}$$

(see Proposition K.2, Appendix K).

4⁰ And at the end, using the results obtained in Propositions K.1 and K.2, we obtain the desired statement: in the case when $\Gamma \leq \sigma \leq 2\sqrt{\pi}$ and the path $\mathcal{P}_d$ belongs to the class II, we modify the path $\mathcal{P}_d$ on the interval $[t_{s-2}, t_{q+1}]$ so that the points $\widetilde{P}$, $\widetilde{S}$ and $\widetilde{F}$ are in this order on the straight line $\tilde{l}$ (i.e. this modification shortens the path $\mathcal{P}_d$ to the left, see the proof of Lemma 10.18).

Construct some auxiliary path $\mathcal{C}_0$ in the following way: modify the path $\mathcal{P}_d$ on the interval $[t_{s-1}, t_s]$ by means of the parameter σ – choose a value of the parameter σ such that the tangent vector at the point S (denote it by V_S) should be collinear to the vector $V_{E^{**}}$ of the path $\widetilde{\mathcal{P}}_{1new}$ (see an example of the graph of the curvature of such modification on Figure 40). Remark that σ can be greater than $t_s - t_{s-1}$ or not greater than $t_s - t_{s-1}$.

The path $\mathcal{C}_0$ consists of two arcs of half-clothoid (on $[t_{s-1}, t_{s-1}+\sigma]$) and of a line segment (on $[t_{s-1}+\sigma, T]$). This line segment (denote it by $l_{\mathcal{C}_0}$) is parallel to the line segment of the straight line l^{**}. See the graph of the curvature of the path $\mathcal{C}_0$ on Figure 40: it consists of three segments – JR, RS (which correspond to the arcs of half-clothoid) and ST (which corresponds to the line segment).

See an example of some path $\mathcal{C}_0$ (namely the arcs $\widehat{JR}$, $\widehat{RS}$ and the line segment ST) on Figure 42.

Remark 10.19 *It follows from the construction of the paths $\mathcal{C}_0$ and $\widetilde{\mathcal{P}}_{2new}$ that if $\widetilde{\mathcal{P}}_{2new}$ has a lace on the interval $[t_{s-1}, t_{s-1}+\sigma]$, then, $\mathcal{C}_0$ has also a lace on this interval, and if $\mathcal{C}_0$ has a lace on the interval $[t_{s-1}, t_{s-1}+\sigma]$, then, $\widetilde{\mathcal{P}}_{2new}$ can have or not have a lace on this interval. Hence, considering all cases when the path $\mathcal{C}_0$ has a lace on the interval $[t_{s-1}, t_{s-1}+\sigma]$, we consider, evidently, all cases when the path $\widetilde{\mathcal{P}}_{2new}$ has a lace on this interval.*

Proof of Lemma 10.18

If modifying the path $\mathcal{P}_d$ on the interval $[t_{s-1}, t_{q+1}]$, we obtain that the new path $\widetilde{\mathcal{P}}_{2new}$ has no lace, then this is the case already studied in Subsection 10.4. If not, we modify the path $\mathcal{P}_d$ on the interval $[t_{s-2}, t_{q+1}]$ and we obtain some path $\widetilde{\mathcal{P}}_{3new}$.

Consider now the paths $\mathcal{P}_d$ and $\widetilde{\mathcal{P}}_{3new}$ on the interval $[t_{s-2}, t_{q+1}]$.

Introduce some notations (as in Subsection 10.5).

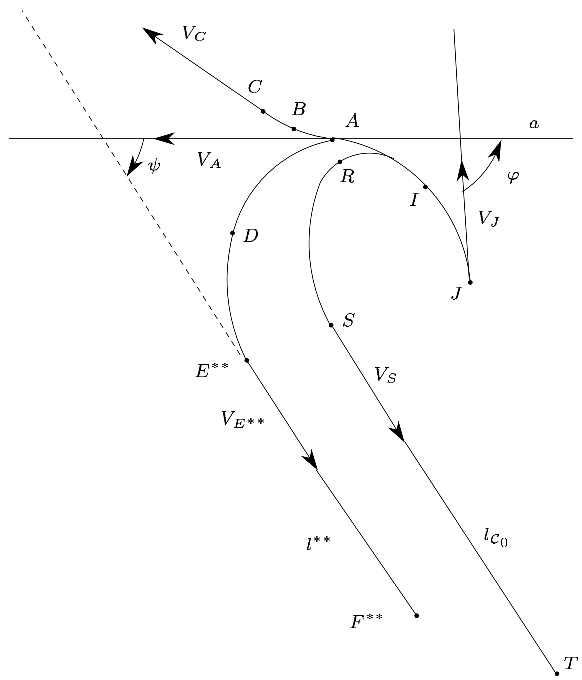


Figure 42

The straight line $j_\perp$ divides the plane in two half-planes. We call them 'half-plane I_J' and 'half-plane II_J'.

Definition We call 'half-plane I_J' a half-plane such that it doesn't contain the vector V_J. We call 'half-plane II_J' a half-plane which contains the vector V_J.

Now we can divide all paths $\mathcal{P}_d$ in two classes ('class I_J' and 'class II_J').

Definition We say that the path $\mathcal{P}_d$ under consideration belongs to the 'class I_J' (to the 'class II_J') if the first intersection point of the straight line j and of the path $\mathcal{P}_d$ on the interval $(t_{s-1}, T]$ belongs to the half-plane I_J (to the half-plane II_J).

We have proved in Subsection 10.6 that if the path $\mathcal{P}_d$ belongs to the class I, then the points P_{1pr}, E^{**}, F^{**} are in this order on the straight line l^{**} (i.e. this modification shortens the path $\mathcal{P}_d$ to the left). By analogy we can prove that if the path $\mathcal{P}_d$ belongs to the class I_J, then the points $\widetilde{P}_{pr}$, $\widetilde{S}$, $\widetilde{F}$ are in this order on the straight line $\tilde{l}$ (i.e. this modification shortens the path $\mathcal{P}_d$ to the left).

Hence, we must study only the case when the path $\mathcal{P}_d$ belongs to the class II_J. There are two subcases: the case when $\widetilde{P} \in \widehat{JIA}$ and the case when $\widetilde{P} \notin \widehat{JIA}$, i.e. $|\widehat{J\widetilde{P}}| > |\widehat{JIA}|$.

If the point $\widetilde{P}$ belongs to the arc $\widehat{JIA}$, then, the points $\widetilde{P}_{pr}$, $\widetilde{S}$, $\widetilde{F}$ are in this order on the straight line $\tilde{l}$ (i.e. this modification shortens the path $\mathcal{P}_d$ to the left – we prove this statement by analogy to the proof in Subsection 10.6).

So we consider only the case when $\widetilde{P} \notin \widehat{JIA}$. As the path $\mathcal{P}_d$ belongs to the class II_J, the first intersection point of the straight line j and of the path $\mathcal{P}_d$ must belong to the half-plane II_J. Hence, a point $W \in \widehat{AT}$ such that the tangent line at the point W should be perpendicular to the straight line a must exist (see Figure 43).

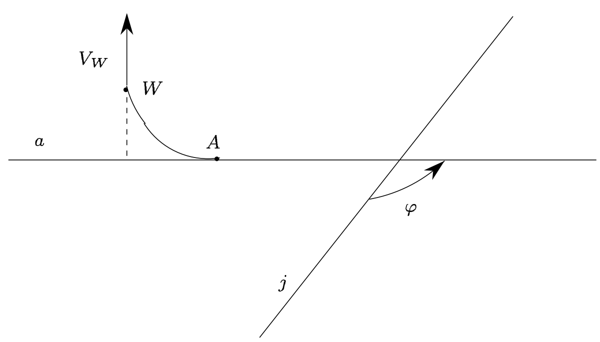


Figure 43

The point $\widetilde{P}$ belongs to the arc $\widehat{JAW}$. Really, we obtain the minimal value of the length of the arc $\widehat{AW}$ if the curvature $\kappa(t)$ on the interval $[t_s, t_W]$ is defined by the following equation:

$$\kappa(t) = -2(t - t_s) \ , \quad \text{for } t \in [t_s, t_W] \ .$$

The following equalities hold:

$$|\alpha_W - \alpha_A| = \pi/2 \ , \quad |\alpha_W - \alpha_A| = |\widehat{AW}|^2 \ ,$$

hence,

$$|\widehat{AW}| = \sqrt{\pi/2} \ .$$

To prove that the point $\widetilde{P}$ belongs to the arc $\widehat{JAW}$, we must prove the following inequality:

$$|\widehat{JIA}| + |\widehat{AW}| > |\widehat{J\widetilde{P}}| = |\widehat{J\widetilde{R}\widetilde{S}}| \ ,$$

i.e. (using the results of Propositions K.1 and K.2, see Appendix K) we must prove that

$$\sqrt{(4/3 - 2 \times 0.0374)\pi} + \sqrt{\pi/2} > \sqrt{2\pi} \ ,$$

i.e.

$$4/3 - 2 \times 0.0374 > 2 + 1/2 - 2\sqrt{2}\sqrt{1/2} = 1/2$$

(it is correct). Hence, the point P belongs to the arc $\widehat{JAW}$, so, in the case when the point $\widetilde{P} \notin \widehat{JIA}$, the points $\widetilde{P}_{pr}$, $\widetilde{S}$, $\widetilde{F}$ are in this order on the straight line $\tilde{l}$ (we prove this statement by analogy to the proof in Subsection 10.6).

The lemma is proved. □

10.8 Proof of Lemma 10.9

In case III if we obtain for the constructed path $\mathcal{P}_1$ that at least one parameter σ, θ belongs to $[\Gamma, 2\sqrt{\pi}]$, then, either we can apply Lemma 10.11 (if the path $\mathcal{P}_1$ belongs to class I), or we can apply Lemma 10.18 (if the path $\mathcal{P}_1$ belongs to class II). We apply Lemma 10.8 on all intervals where for the new path we obtain $\sigma \in [0, \Gamma)$, $\theta \in [0, \Gamma)$.

The lemma is proved. □

10.9 Detailed description of the case when $\sigma < \Gamma$ and $\theta < \Gamma$ (i.e. when the paths $\mathcal{P}$ and $\widetilde{\mathcal{P}}_1$ have no lace)

Denote by X_t (by $\widetilde{X}_t$) a point corresponding to t and belonging to the path $\mathcal{P}_d$ (to the new path $\widetilde{\mathcal{P}}$ respectively). Denote by X_t^{pr} the projection of the point X_t on the straight line l.

Consider now the graph of the curvature of the path $\widetilde{\mathcal{P}}_1$ (see Figure 44). On this figure we see an example of the graph of the curvature of $\widetilde{\mathcal{P}}_1$ where $\sigma \in [0, t_{s+1} - t_s)$ and $\theta \in [0, t_{q+1} - t_q)$, but it isn't important because the following reasoning is correct for all values of σ and θ de $[0, \Gamma)$.

Denote by $\widetilde{X}_{t_{s+1}}$ (by $\widetilde{X}_{t_q}$) the point of $\widetilde{\mathcal{P}}_1$ corresponding to $t = t_{s+1}$ (to $t = t_q$), denote by $X_{t_{s+1}}$ (by X_{t_q}) the point of $\mathcal{P}_d$ corresponding to $t = t_{s+1}$ (to $t = t_q$) and denote by $X_{t_{s+1}}^{pr}$ (by $X_{t_q}^{pr}$) the projection of the point $X_{t_{s+1}}$ (of the point X_{t_q}) on l.

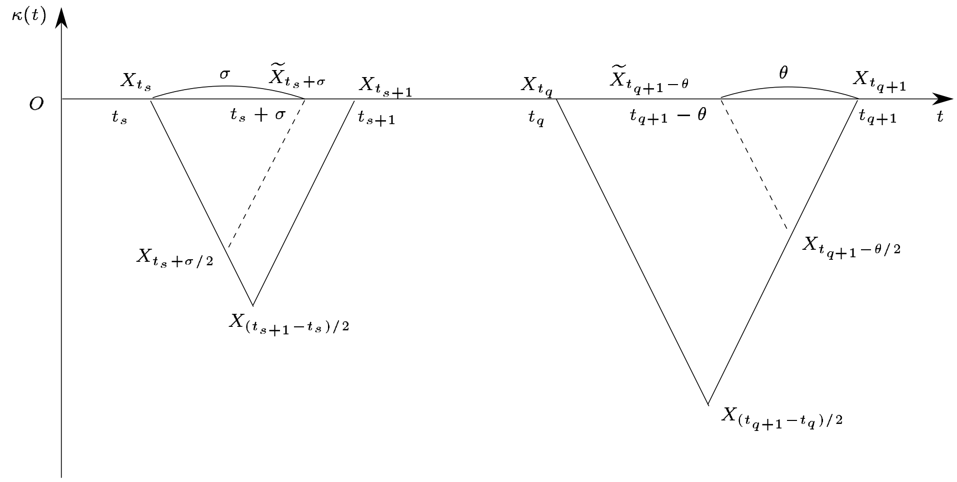


Figure 44

Set $\pm|\widetilde{X}_{t_{s+1}}X^{pr}_{t_{s+1}}| = \alpha_{-1}$, $\pm|\widetilde{X}_{t_q}X^{pr}_{t_q}| = \alpha_1$. Here we write $''+''$ if the point $X^{pr}_{t_{s+1}}$ (the point $X^{pr}_{t_q}$) is to the right (to the left) with respect to the straight line perpendicular to the straight line l and passing through the point $\widetilde{X}_{t_{s+1}}$ (through the point $\widetilde{X}_{t_q}$). If not, we write $''-''$.

At first we explain the sense of the expression "shorten the path $\mathcal{P}_d$ to the left (to the right)".

Definition *We say that some modification shortens the path $\mathcal{P}_d$ to the left (to the right) if for the constructed path $\widetilde{\mathcal{P}}_1$ we obtain $\alpha_{-1} < 0$ ($\alpha_1 < 0$).*

Remark 10.20 *If for some path $\widetilde{\mathcal{P}}_1$ we obtain $\alpha_1 < 0$ and $\alpha_{-1} < 0$, then, $|\widetilde{\mathcal{P}}_1| < |\mathcal{P}_d|$. If we obtain that for $\widetilde{\mathcal{P}}_1$ at least one of the following inequalities holds: $\alpha_1 \geq 0$, $\alpha_{-1} \geq 0$, then, we don't know whether $|\widetilde{\mathcal{P}}_1| < |\mathcal{P}_d|$ or whether $|\widetilde{\mathcal{P}}_1| \geq |\mathcal{P}_d|$; so this case needs to be studied separately.*

See an example where $\alpha_{-1} > 0$, $\alpha_1 > 0$ on Figure 45.

Plan of the proof of Lemma 10.8.

$\mathbf{1^0}$. If $|\widetilde{\mathcal{P}}_1| < |\mathcal{P}_d|$, then Lemma 10.8 is proved. If $|\widetilde{\mathcal{P}}_1| \geq |\mathcal{P}_d|$, then instead of the interval $[t_q, t_{q+1}]$ we consider the next interval $[t_{q+1}, t_{q+2}]$, instead of the interval $[t_s, t_{s+1}]$ we consider the previous interval $[t_{s-1}, t_s]$, and we construct some new path $\widetilde{\mathcal{P}}_2$ by modifying the graph $\kappa(t)$ of the initial path $\mathcal{P}_d$ on two intervals $[t_{s-1}, t_{s-1}+\sigma]$, $[t_{q+2}-\theta, t_{q+2}]$.

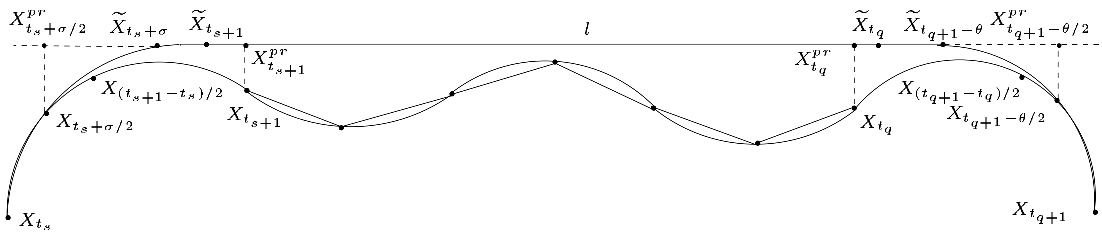


Figure 45

2⁰. Now there are two possibilities: either $|\widetilde{\mathcal{P}}_2| < |\mathcal{P}_d|$ or $|\widetilde{\mathcal{P}}_2| \geq |\mathcal{P}_d|$. In the last case we construct some new path $\widetilde{\mathcal{P}}_3$ by modifying the graph $\kappa(t)$ of $\mathcal{P}_d$ on two intervals $[t_{s-2}, t_{s-2}+\sigma]$, $[t_{q+3}-\theta, t_{q+3}]$.

3⁰. We obtain some sequence of paths $\widetilde{\mathcal{P}}_1$, $\widetilde{\mathcal{P}}_2, \dots, \widetilde{\mathcal{P}}_j$, $\dots, \widetilde{\mathcal{P}}_K$ such that we construct every new path $\widetilde{\mathcal{P}}_j$ by modifying the graph $\kappa(t)$ of the initial path $\mathcal{P}_d$ on two intervals $[t_{s-(j-1)}, t_{s-(j-1)}+\sigma]$, $[t_{q+j}-\theta, t_{q+j}]$. If we suppose that for every path $\widetilde{\mathcal{P}}_j$ we have $|\widetilde{\mathcal{P}}_j| \geq |\mathcal{P}_d|$, then the sequence of $\Delta_{-j}+\Delta_j$ is bounded from below by some increasing geometric progression, see Proposition 10.22. Here by Δ_j we denote the difference between the length of the interval $[t_{q+j-1}, t_{q+j}]$ and the sum of the lengths of two chords connecting the points of $\mathcal{P}_d$ corresponding to $t = t_{q+j-1}$, $t = t_{q+j}$ with the switching point of $\mathcal{P}_d$ belonging to $[t_{q+j-1}, t_{q+j}]$; we denote by Δ_{-j} the difference between the length of the interval $[t_{s-(j-1)}, t_{s-(j-2)}]$ and the sum of the lengths of two chords connecting the points of $\mathcal{P}_d$ corresponding to $t = t_{s-(j-1)}$, $t = t_{s-(j-2)}$ with the switching point of $\mathcal{P}_d$ belonging to $[t_{s-(j-1)}, t_{s-(j-2)}]$.

It follows from Remark 10.4 that in Section 10 we consider only paths consisting of intervals whose lengths are smaller than $\sqrt{2.926\pi}$. We set $\sqrt{2.926\pi} = \Gamma$. Hence, for some value $j = K$ the sum of the length of the piece of $\mathcal{P}_d$ corresponding to Δ_K and the length of the piece of $\mathcal{P}_d$ corresponding to Δ_{-K} becomes greater than the admissible constant 2Γ (because the sequence of $\Delta_{-j}+\Delta_j$ is bounded from below by some increasing geometric progression).

4⁰. Then we prove that the sequence $\Delta_{-K}+\Delta_K, \Delta_{-(K-1)}+\Delta_{K-1}, \dots, \Delta_{-1}+\Delta_1$ and the sequence of the lengths of the corresponding arcs $l_{-K}+l_K, l_{-(K-1)}+l_{K-1}, \dots, l_{-1}+l_1$ are two decreasing sequences bounded from above by some geometric progressions (see Proposition 10.25 and Lemma 10.26).

As a corollary of Lemma 10.26 we obtain Lemma 10.27 which contains the desired result: if the distance between the initial and final points of $\mathcal{P}$ is greater than some constant C, then $|\widetilde{\mathcal{P}}_K| < |\mathcal{P}_d|$; we give the estimation of the constant C in Lemma 10.28: $C = 135.5\sqrt{\pi}$).

So, Lemma 10.8 folows from Lemmas 10.27 and 10.28.

At first, we introduce some notation.

We construct some path $\widetilde{\mathcal{P}}_j$ by modifying the graph $\kappa(t)$ on two intervals $[t_{s-(j-1)}, t_{s-(j-1)}+\sigma]$, $[t_{q+j}-\theta, t_{q+j}]$ $(j \geq 1)$.

Definition *Denote by Δ_0 the difference between the length of the arc $X_{t_{s+1}}\widehat{\ }X_{t_q} \in \mathcal{P}_d$ and the sum of the lengths of the chords between the points of zero curvature and the nearest switching points of the path $X_{t_{s+1}}\widehat{\ }X_{t_q} \in \mathcal{P}_d$.*

Denote by α_j (by α_{-j}) the distance between the point of $\widetilde{\mathcal{P}}_j$ corresponding to $t = t_{q+j-1}$ (to $t = t_{s-(j-2)}$) and the projection of the point of $\mathcal{P}_d$ corresponding to the same value of t on l_j, i.e. $\alpha_j = \pm|\widetilde{X}_{t_{q+j-1}} X^{pr}_{t_{q+j-1}}|$ and $\alpha_{-j} = \pm|\widetilde{X}_{t_{s-(j-2)}} X^{pr}_{t_{s-(j-2)}}|$ – we choose "+" or "−" in the same way as for α_1, α_{-1}).

Denote by Δ_j (by Δ_{-j}) the difference between the length of the interval $[t_{q+j-1}, t_{q+j}]$ (of the interval $[t_{s-(j-1)}, t_{s-(j-2)}]$) and the sum of the lengths of the corresponding chords.

Definition *We say that some modification shortens the path $\mathcal{P}_d$ to the left (to the right respectively) if we have $\alpha_{-j} < 0$ ($\alpha_j < 0$) for the constructed path $\widetilde{\mathcal{P}}_j$.*

Set

$$\Sigma_{-j} = \Delta_{-1} + \Delta_{-2} + \ldots + \Delta_{-j} \ , \quad \Sigma_j = \Delta_1 + \Delta_2 + \ldots + \Delta_j \ .$$

Proposition 10.21 *If for some path $\widetilde{\mathcal{P}}_j$ we have $|\widetilde{\mathcal{P}}_j| \geq |\mathcal{P}_d|$, then the following inequality holds:*

$$\alpha_{-j} + \alpha_j > \Sigma_{-(j-1)} + \Sigma_{j-1} \ .$$

In this subsection we use some auxiliary propositions from Appendix L.

Proposition 10.21 follows from Proposition L.3 (see Appendix L).

Proposition 10.22 *Construct a path $\widetilde{\mathcal{P}}_j$ by modifying the graph $\kappa(t)$ on two intervals $[t_{s-(j-1)}, t_{s-(j-1)}+\sigma]$, $[t_{q+j}-\theta, t_{q+j}]$ $(j \geq 2)$. If $|\widetilde{\mathcal{P}}_j| \geq |\mathcal{P}_d|$, then we have the following inequality:*

$$\Delta_{-j} + \Delta_j > \frac{2}{M}(\Delta_{-1} + \Delta_1)(1 + 2/M)^{j-2} \qquad \text{for} \quad j \geq 2 \ . \tag{57}$$

See the proof of Proposition 10.22 in Appendix L.

Proposition 10.23 *If for any path $\widetilde{\mathcal{P}}_j$ from the sequence of the paths $\widetilde{\mathcal{P}}_j$ $(j \geq 2)$ we have $|\widetilde{\mathcal{P}}_j| \geq |\mathcal{P}_d|$, then the sequence of $\Delta_{-j} + \Delta_j$ is bounded from below by some increasing geometric progression and for some value $j \geq 2$ the sum of the length of the piece of $\mathcal{P}_d$ corresponding to Δ_{-j} and Δ_j becomes greater than the admissible constant 2Γ.*

Proof

Set

$$\hat{d} = 2(\Delta_{-1} + \Delta_1)/M > 0 \ , \qquad q = 1 + 2/M > 1 \ .$$

Then we can rewrite inequality (57) as follows:

$$\Delta_{-j} + \Delta_j > \hat{d}q^{j-2} \qquad \text{for } j \geq 2 \ , \tag{58}$$

where $\hat{d}q^{j-2}$ is an increasing geometric progression (so, the first statement of the proposition is proved).

Consider a function $h(t) = |\widehat{OW}| - |OW|$ for any point W belonging to the half-clothoid (see Figure 82). From Proposition 8.9 of [11] we know that the function $h(t)$ is monotonously increasing. By definition, $\Delta_j = 2h\left((t_{q+j} - t_{q+j-1})/2\right)$, $\Delta_{-j} = 2h\left((t_{s-(j-2)} - t_{s-(j-1)})/2\right)$.

Remind that in Section 10 we consider only paths $\mathcal{P}_d$ consisting of intervals whose lengths are smaller than $\sqrt{2.926\pi}$ (Remark 10.4). We have set $\sqrt{2.926\pi} = \Gamma$.

Hence, we obtain from (58) that for some $j \geq 2$ the sum of the lengths of the pieces of $\mathcal{P}_d$ corresponding to Δ_j and Δ_{-j} becomes greater than the admissible constant 2Γ.

The proposition is proved. □

Definition *Denote by Δ_K the maximal admisible value of Δ_j (i.e. $l_K \leq \Gamma$ and $l_{K+1} > \Gamma$: here we denote by l_K (l_{K+1}) the length of the arc of clothoid corresponding to Δ_K (Δ_{K+1})). Respectively we denote by Δ_{-K} the maximal admisible value of Δ_{-j}.*

Estimate the value of the sum $\Sigma_{-K} + \Sigma_K$.

Proposition 10.24 *For Σ_K and Σ_{-K} we have the following estimation:*

$$\Sigma_{-K} < (6 + 8\sqrt{2})R \quad \text{and} \quad \Sigma_K < (6 + 8\sqrt{2})R \ ,$$

hence,

$$\Sigma_{-K} + \Sigma_K < (12 + 16\sqrt{2})R \ ,$$

(where $R = \sqrt{\pi}/2$).

See the proof of Proposition 10.24 in Appendix L.

Proposition 10.25 *$\Delta_{-K}+\Delta_K, \Delta_{-(K-1)}+\Delta_{K-1}, \ldots, \Delta_{-1}+\Delta_1$ form a decreasing sequence bounded from above by a decreasing geometric progression.*

Proof

It follows from Proposition 10.21 that if for some path $\widetilde{\mathcal{P}}_j$ ($j \geq 2$) we have $|\widetilde{\mathcal{P}}_j| \geq |\mathcal{P}_d|$, then the following inequality holds:

$$\alpha_{-j} + \alpha_j > \Sigma_{-(j-1)} + \Sigma_{j-1} \ .$$

It follows from Proposition L.2 that

$$\alpha_{-j} < \frac{M}{2}\Delta_{-j} \quad \text{and} \quad \alpha_j < \frac{M}{2}\Delta_j \ .$$

Hence,

$$\Delta_{-j} + \Delta_j > \frac{2}{M}(\Sigma_{-(j-1)} + \Sigma_{j-1}) \ .$$

Using this inequality we obtain

$$\Sigma_{-j} + \Sigma_j = (\Sigma_{-(j-1)} + \Delta_{-j}) + (\Sigma_{j-1} + \Delta_j) > (\Sigma_{-(j-1)} + \frac{2}{M}\Sigma_{-(j-1)}) + (\Sigma_{j-1} + \frac{2}{M}\Sigma_{j-1}) =$$

$$= (1 + 2/M)(\Sigma_{-(j-1)} + \Sigma_{j-1}) = \frac{M+2}{M}(\Sigma_{-(j-1)} + \Sigma_{j-1}) \ ,$$

i.e.

$$\Sigma_{-(j-1)} + \Sigma_{j-1} < \frac{M}{M+2}(\Sigma_{-j} + \Sigma_j) \qquad \text{for any } j \geq 2 \ . \tag{59}$$

Consider inequality (59) for $2 \leq j \leq K$:

$$\Sigma_{-(K-1)} + \Sigma_{K-1} < \frac{M}{M+2}(\Sigma_{-K} + \Sigma_K) \ ,$$

$$\Sigma_{-(K-2)} + \Sigma_{K-2} < \frac{M}{M+2}(\Sigma_{-(K-1)} + \Sigma_{K-1}) < \left(\frac{M}{M+2}\right)^2 (\Sigma_{-K} + \Sigma_K) \qquad \text{etc.}$$

Hence,

$$\Sigma_{-(K-s)} + \Sigma_{K-s} < \left(\frac{M}{M+2}\right)^s (\Sigma_{-K} + \Sigma_K) \qquad \text{for } s = 1, \ldots, K-1 \ ,$$

and, using the inequality $\Delta_{-(K-s)} + \Delta_{K-s} < \Sigma_{-(K-s)} + \Sigma_{K-s}$, we obtain

$$\Delta_{-(K-s)} + \Delta_{K-s} < \left(\frac{M}{M+2}\right)^s (\Sigma_{-K} + \Sigma_K) \qquad \text{for } s = 1, \ldots, K-1 \ . \tag{60}$$

From Proposition 10.24 we know that $\Sigma_{-K} + \Sigma_K < (12 + 16\sqrt{2})R$. Hence, all $\Delta_{-(K-s)} + \Delta_{K-s}$ $(s = 1, \ldots, K-1)$ form a decreasing sequence bounded by a decreasing geometric progression.

The proposition is proved. □

Lemma 10.26 *The sequence of the sums of the lengths of the arcs $l_{-K} + l_K, l_{-(K-1)} + l_{K-1}, \ldots, l_{-1} + l_1$ is decreasing as a geometric progression and, hence, the sum of these lengths is finite.*

Proof

Consider the function $h(t) = |\widehat{OW}| - |OW|$ for any point W belonging to the half-clothoid (see Figure 82). From Proposition 8.9 of [11] we know that the function $h(t)$ is monotonously increasing positive-valued. We have $\Delta_j = 2h\left((t_{q+j} - t_{q+j-1})/2\right)$, $\Delta_{-j} = 2h\left((t_{s-(j-2)} - t_{s-(j-1)})/2\right)$.

From Proposition 10.25 we know that $\Delta_{-K} + \Delta_K, \Delta_{-(K-1)} + \Delta_{K-1}, \ldots, \Delta_{-1} + \Delta_1$ form a decreasing sequence bounded from above by a decreasing geometric progression. Hence, if we consider the values of Δ_j, Δ_{-j} outside a small half-neighbourhood of zero, we obtain that the sums of the lengths of the correspondings arcs $l_{-K} + l_K, l_{-(K-1)} + l_{K-1}, \ldots, l_{-1} + l_1$ form a desreasing sequence bounded from above by a decreasing geometric progression. Consider now the function $h(t)$ in a small half-neighbourhood of zero. We obtain for the small t

$$h(t) = |\widehat{OW}| - |OW| = t - \sqrt{\left(\int_0^t \cos\tau^2 d\tau\right)^2 + \left(\int_0^t \sin\tau^2 d\tau\right)^2} =$$

$$= t - \left(t - 2t^5/45 + O(t^9)\right) = 2t^5/45 + O(t^9)\ .$$

Hence, in a small positive half-neighbourhood of zero $\Delta_j = 2t^5/45 + O(t^9)$ $(\Delta_{-j} = 2t^5/45 + O(t^9))$ and, hence, $l_j \approx (45\Delta_j/2)^{\frac{1}{5}}$ $(l_{-j} \approx (45\Delta_j/2)^{\frac{1}{5}})$.

The lemma is proved. □

As a corollary of Lemma 10.26 we obtain Lemma 10.27 (the main result of this subsection).

Lemma 10.27 *If we require that the distance d between the initial and final points be greater than some positive constant C, then*

$$|\widetilde{\mathcal{P}}_K| < |\mathcal{P}_d|\ .$$

Proof

Suppose that for maximal admisible Δ_K we have $|\widetilde{\mathcal{P}}_K| \geq |\mathcal{P}_d|$. The sum of the lengths of corresponding arcs $l_{-K}, \ldots, l_{-1}$, $l_K, \ldots, l_1$ is some finite number, see Lemma 10.26 (denote this number by L). Hence, if the sum of the distances between the points corresponding to t_q and t_{q+J} and to t_{s+1} and $t_{s-(K-1)}$ is greater than L, then the part of the path $\mathcal{P}_d$ between the points corresponding to $t_{s-(K-1)}$ and t_{q+J} can't exist. Hence, if the distance between the initial and final points is greater than some positive constant C, then $|\widetilde{\mathcal{P}}_K| < |\mathcal{P}_d|$.

The lemma is proved. □

The estimation of C is given in Lemma 10.28.

Lemma 10.28 *If the distance d between the initial and the final points of $\mathcal{P}$ is greater than $135.5\sqrt{\pi}$, then*

$$|\widetilde{\mathcal{P}}_K| < |\mathcal{P}_d| \ ,$$

i.e. we can construct some path $\widetilde{\mathcal{P}}$ such that it should be shorter than $\mathcal{P}$ and that it should satisfy the initial and final conditions.

See the proof of Lemma 10.28 in Appendix M.

11 Proof of Theorem 2.6

To prove Theorem 2.6 we summarize the results obtained in Sections 3–10.

Remind that we have given the scheme of all cases on Figure 4.

In all cases (except case II described in Section 9 and case III described in Section 10) we can apply the method studied in Section 8 for case I and obtain using suitable modifications of types A, B and C some new path with the given initial and the final conditions, which is shorter than the initial one and which is a finite concatenation of arcs of clothoid (see Section 8, Lemma 8.1 and Appendix B). In all these cases we consider the initial and final points which are situated not far from each other.

Case II (i.e. the case when $y_p > 0$ for any even p, $y_p < 0$ for any odd p (with the possible exception of the initial and the final points) and there exists at least one even index p (one odd index p) such that $\kappa_p \leq 0$ ($\kappa_p \geq 0$)) is studied in Section 9. In this case we prove that if the distance between the initial and final points is greater than $320\sqrt{\pi}$, then we can modify the path $\mathcal{P}$ so that the new path should be shorter than the given one and should satisfy the initial and final conditions (see Lemma 9.1). As a result we obtain some new path which belongs to other class of paths, i.e. which consists of a line segment and a finite number of arcs of clothoid.

Case III (i.e. the case when $y_p > 0$, $\kappa_p > 0$ for any even p and $y_p < 0$, $\kappa_p < 0$ for any odd p (with the possible exception of the initial and the final points) is studied in Section 10. In this case we prove that if the distance between the initial and final points is greater than $135.5\sqrt{\pi}$, then we can modify the path $\mathcal{P}$ so that the new path should be shorter than the given one and should satisfy the initial and final conditions (see Lemma 10.1). As a result we obtain some new path which belongs to other class of paths, i.e. which consists of a line segment and a finite number of arcs of clothoid.

Thus, summarizing, we have proved that if the distance between the initial and final points is greater than $320\sqrt{\pi}$, then we can shorten the given path $\mathcal{P}$ preserving the initial and final conditions. Hence, the optimal path can't consist of a finite number of concatenated arcs of clothoids.

The theorem is proved. □

A Appendix: Proofs of Propositions 6.4 and 6.5

A.1 Proof of Proposition 6.4

By definition we have the following formulas:

$$x_B(T_B) = \int_0^{T_B} \cos\alpha_B(t)dt \ , \qquad y_B(T_B) = \int_0^{T_B} \sin\alpha_B(t)dt \ ,$$

$$x^T = \int_0^{T} \cos\alpha(t)dt \ , \qquad y^T = \int_0^{T} \sin\alpha(t)dt \ .$$

Hence, we obtain

$$x_B(T_B) - x^T = \int_0^{T_B} \cos\alpha_B(t)dt - \int_0^{T_B} \cos\alpha(t)dt - \int_{T_B}^{T} \cos\alpha(t)dt =$$

$$= \int_0^{T_B} (\cos\alpha_B(t) - \cos\alpha(t))dt - \int_{T_B}^{T} \cos\alpha(t)dt \ . \tag{61}$$

At first we calculate $\int_0^{T_B}(\cos\alpha_B(t) - \cos\alpha(t))dt$:

$$\int_0^{T_B} (\cos\alpha_B(t) - \cos\alpha(t))dt = -2\int_0^{T_B} \sin\frac{\alpha_B(t)-\alpha(t)}{2}\sin\frac{\alpha_B(t)+\alpha(t)}{2}dt \ .$$

For a small δ_B we have $\alpha_B(t) - \alpha(t) = O(\delta_B)$, hence,

$$\sin\frac{\alpha_B(t)-\alpha(t)}{2} = \frac{\alpha_B(t)-\alpha(t)}{2} + O(\delta_B^2) \ .$$

So,

$$\int_0^{T_B} (\cos\alpha_B(t) - \cos\alpha(t))dt = -\int_0^{T_B} (\alpha_B(t) - \alpha(t))\sin\frac{\alpha_B(t)+\alpha(t)}{2}dt + O(\delta_B^2) =$$

$$= -\int_0^{T_B} (\alpha_B(t) - \alpha(t))\sin\alpha(t)dt + O(\delta_B^2) = -\int_0^{T_B} (\alpha_B(t) - \alpha(t))\dot{y}(t)dt + O(\delta_B^2) \ .$$

Now we integrate by parts and we obtain

$$\int_0^{T_B} (\cos\alpha_B(t) - \cos\alpha(t))dt = -[(\alpha_B(t) - \alpha(t))y(t)]|_0^{T_B} + \int_0^{T_B} (\dot{\alpha}_B(t) - \dot{\alpha}(t))y(t)dt + O(\delta_B^2) =$$

$$= -y(T_B)(\alpha_B(T_B) - \alpha(T_B)) + \int_0^{T_B} (\kappa_B(t) - \kappa(t))y(t)dt + O(\delta_B^2) \ .$$

For calculate $-y(T_B)(\alpha_B(T_B)-\alpha(T_B))$ we remark that $(\alpha_B(T_B)-\alpha(T_B))$ is equal to the area of $NGKLF$ (see Figure 7). We denote this area by S_{NGKLF} and we obtain the following formula:

$$S_{NGKLF} = -|\kappa^T|\delta_B K_2 + \delta_B^2 K_2^2$$

(remind that δ_B is always negative).

Really, $\alpha_B(T_B)-\alpha(T_B) = S_{NGKTF}$ if $\kappa^T \geq 0$. If not, $\alpha_B(T_B)-\alpha(T_B) = -|\kappa^T|\delta_B K_2 - \delta_B^2 K_2^2$. But in two these cases

$$-y(T_B)(\alpha_B(T_B)-\alpha(T_B)) = \delta_B y(T_B)|\kappa^T|K_2 + O(\delta_B^2)\ . \tag{62}$$

We calculate now $\int_0^{T_B}(\kappa_B(t)-\kappa(t))y(t)dt$:

$$\int_0^{T_B}(\kappa_B(t)-\kappa(t))y(t)dt = \int_{t_{f-2}}^{t_{f-1}} -4\delta_B K_1 y(t)dt + \int_{t_{f-1}}^{T_B} 2\delta_B K_2 y(t)dt =$$

$$= -4\delta_B K_1 \int_{t_{f-2}}^{t_{f-1}} y(t)dt + 2\delta_B K_2 \int_{t_{f-1}}^{T_B} y(t)dt =$$

$$= -4\delta_B K_1 (t_{f-1}-t_{f-2})y(\mu) + 2\delta_B K_2 (T_B - t_{f-1})y(\nu) =$$

$$= -\delta_B\left[4K_1(t_{f-1}-t_{f-2})y(\mu) - 2K_2(T-t_{f-1})y(\nu)\right] + O(\delta_B^2)$$

(here the point μ (the point ν) is a point of the interval $[t_{f-2}, t_{f-1}]$ ($[t_{f-1}, T]$) such that $y(\mu)$ ($y(\nu)$) is equal to the mean value of the function $y(t)$ in this interval).

Thus, we obtain the following formula:

$$\int_0^{T_B}(\cos\alpha_B(t)-\cos\alpha(t))dt =$$

$$= \delta_B y(T_B)\kappa^T K_2 - \delta_B\left[4K_1(t_{f-1}-t_{f-2})y(\mu) - 2K_2(T-t_{f-1})y(\nu)\right] + O(\delta_B^2) =$$

$$= \delta_B\left[y(T_B)\kappa^T K_2 - 4K_1(t_{f-1}-t_{f-2})y(\mu) + 2K_2(T-t_{f-1})y(\nu)\right] + O(\delta_B^2)\ . \tag{63}$$

We calculate now $\int_{T_B}^{T}\cos\alpha(t)dt$: for the curvature in the intervale of integration $[T_B, T]$ we have the following formula:

$$\kappa(t) = -2t + (\kappa^T + 2T) \quad \text{ for } t\in[T_B, T]\ .$$

So, we have the following equalities for $\alpha(t)$ in the interval of the integration $[T_B, T]$:

$$\alpha(t) = \alpha(T_B) + \int_{T_B}^{t}\kappa(\tau)d\tau = \alpha(T_B) + \int_{T_B}^{t}(-2\tau + (\kappa^T + 2T))d\tau =$$

$$= \alpha(T_B) - \tau^2|_{T_B}^{t} + [(\kappa^T+2T)\tau]|_{T_B}^{t} = \alpha(T_B) - (t^2 - T_B^2) + (\kappa^T + 2T)(t - T_B)\ .$$

We change the variable t to $\tau = t - T_B = t - T - \delta_B K_2$. Hence,

$$\int_{T_B}^{T} \cos\alpha(t)dt = \int_0^{-\delta_B K_2} \cos\alpha(\tau)d\tau = \int_0^{-\delta_B K_2} \cos(\alpha(T_B) + O(\tau))d\tau =$$

$$= \int_0^{-\delta_B K_2} (\cos\alpha(T_B)\cos O(\tau) - \sin\alpha(T_B)\sin O(\tau))d\tau = -\delta_B K_2 \cos\alpha(T_B) + O(\delta_B^2) \ . \quad (64)$$

Summarizing the results obtained in the formulas (61), (63) and (64), we obtain

$$x_B(T_B) = x^T + \delta_B K_x^{(B)} + O(\delta_B^2) \ ,$$

where

$$K_x^{(B)} = K_2(y(T_B)|\kappa^T| + \cos\alpha(T_B)) - d'_B y(\mu) + d''_B y(\nu) = K_2(y(T_B)|\kappa^T| + \cos\alpha(T_B)) + d''_B y(\nu) \ ,$$

$$d'_B = 4K_1(t_{f-1} - t_{f-2}) \ , \qquad d''_B = 2K_2(T - t_{f-1})$$

(remind that $y(\mu) = 0$ for the interval $[t_{f-2}, t_{f-1}]$, see Remark 3.1).

We obtain the formula for $y_B(T)$ by analogy. We have

$$y_B(T_B) - y^T = \int_0^{T_B} (\sin\alpha_B(t) - \sin\alpha(t))dt - \int_{T_B}^{T} \sin\alpha(t)dt \ . \quad (65)$$

At first we calculate $\int_0^{T_B}(\sin\alpha_B(t) - \sin\alpha(t))dt$:

$$\int_0^{T_B} (\sin\alpha_B(t) - \sin\alpha(t))dt = 2\int_0^{T_B} \sin\frac{\alpha_B(t) - \alpha(t)}{2}\cos\frac{\alpha_B(t) + \alpha(t)}{2}dt =$$

$$= \int_0^{T_B} (\alpha_B(t) - \alpha(t))\cos\frac{\alpha_B(t) + \alpha(t)}{2}dt + O(\delta_B^2) =$$

$$= \int_0^{T_B} (\alpha_B(t) - \alpha(t))\cos\alpha(t)dt + O(\delta_B^2) = \int_0^{T_B} (\alpha_B(t) - \alpha(t))\dot{x}(t)dt + O(\delta_B^2) \ .$$

Now we integrate by parts and we obtain (using (62))

$$\int_0^{T_B} (\sin\alpha_B(t) - \sin\alpha(t))dt = [(\alpha_B(t) - \alpha(t))x(t)]|_0^{T_B} - \int_0^{T_B} (\dot{\alpha}_B(t) - \dot{\alpha}(t))x(t)dt + O(\delta_B^2) =$$

$$= x(T_B)(\alpha_B(T_B) - \alpha(T_B)) - \int_0^{T_B} (\kappa_B(t) - \kappa(t))x(t)dt + O(\delta_B^2) =$$

$$= -\delta_B x(T_B)|\kappa^T|K_2 - \delta_B\left[-4K_1(t_{f-1} - t_{f-2})x(\vartheta) + 2\delta_B K_2(T - t_{f-1})x(\chi)\right] + O(\delta_B^2)$$

(here the point ϑ (the point χ) is a point from the interval $[t_{f-2}, t_{f-1}]$ ($[t_{f-1}, T]$) such that $x(\vartheta)$ ($x(\chi)$) is equal to the mean value of the function $x(t)$ on this interval).

Thus we obtain the following formula:

$$\int_0^{T_B} (\sin \alpha_B(t) - \sin \alpha(t))dt =$$

$$= \delta_B \left[-x(T_B)|\kappa^T|K_2 + 4K_1(t_{f-1} - t_{f-2})x(\vartheta) - 2K_2(T - t_{f-1})x(\chi) \right] + O(\delta_B^2) \ . \quad (66)$$

Now we calculate $\int_{T_B}^T \sin \alpha(t)dt$:

$$\int_{T_B}^T \sin \alpha(t)dt = \int_0^{-\delta_B K_2} \sin \alpha(\tau)d\tau = \int_0^{-\delta_B K_2} \sin(\alpha(T_B) + O(\tau))d\tau =$$

$$= \int_0^{-\delta_B K_2} (\sin \alpha(T_B) \cos O(\tau) + \cos \alpha(T_B) \sin O(\tau))d\tau = -\delta_B K_2 \sin \alpha(T_B) + O(\delta_B^2) \ . \quad (67)$$

Summarizing the results obtained in the formulas (65)–(67), we obtain

$$y_B(T_B) = y^T + \delta_B K_y^{(B)} + O(\delta_B^2) \ ,$$

where

$$K_y^{(B)} = K_2(-x(T_B)|\kappa^T| + \sin \alpha(T_B)) + d'_B x(\vartheta) - d''_B x(\chi) \ ,$$

$$d'_B = 4K_1(t_{f-1} - t_{f-2}) \ , \qquad d''_B = 2K_2(T - t_{f-1}) \ .$$

The proposition is proved. □

A.2 Proof of Proposition 6.5

We prove Proposition 6.5 in the same way as Proposition 6.4, but there are some changings in calculations.

1) For calculate $-y(T_B)(\alpha_B(T_B) - \alpha(T_B))$ we remark that $(\alpha_B(T_B) - \alpha(T_B))$ is equal to the area of $NKTF$ (see Figure 8). We denote this area by S_{NKTF} and we obtain the following formula:

$$S_{NKTF} = -|\kappa^T|\delta_B K_2 - \delta_B^2 K_2^2$$

(remind that δ_B is always negative).

Really, $\alpha_B(T_B) - \alpha(T_B) = S_{NKTF}$ if $\kappa^T \geq 0$. If not, $\alpha_B(T_B) - \alpha(T_B) = -|\kappa^T|\delta_B K_2 + \delta_B^2 K_2^2$. But in two these cases

$$-y(T_B)(\alpha_B(T_B) - \alpha(T_B)) = \delta_B y(T_B)|\kappa^T|K_2 + O(\delta_B^2) \ . \quad (68)$$

2) We calculate now $\int_0^{T_B} (\kappa_B(t) - \kappa(t))y(t)dt$:

$$\int_0^{T_B} (\kappa_B(t) - \kappa(t))y(t)dt = \int_{t_{f-2}}^{t_{f-1}} 4\delta_B K_1 y(t)dt - \int_{t_{f-1}}^{T_B} 2\delta_B K_2 y(t)dt =$$

$$= 4\delta_B K_1 \int_{t_{f-2}}^{t_{f-1}} y(t)dt - 2\delta_B K_2 \int_{t_{f-1}}^{T_B} y(t)dt =$$

$$= 4\delta_B K_1 (t_{f-1} - t_{f-2}) y(\mu) - 2\delta_B K_2 (T_B - t_{f-1}) y(\nu) =$$

$$= \delta_B \left[4K_1(t_{f-1} - t_{f-2})y(\mu) - 2K_2(T - t_{f-1})y(\nu)\right] + O(\delta_B^2)$$

So, we obtain the following formula:

$$\int_0^{T_B} (\cos\alpha_B(t) - \cos\alpha(t))dt =$$

$$= \delta_B y(T_B)|\kappa^T| K_2 + \delta_B \left[4K_1(t_{f-1} - t_{f-2})y(\mu) - 2K_2(T - t_{f-1})y(\nu)\right] + O(\delta_B^2) =$$

$$= \delta_B \left[y(T_B)|\kappa^T|K_2 + 4K_1(t_{f-1} - t_{f-2})y(\mu) - 2K_2(T - t_{f-1})y(\nu)\right] + O(\delta_B^2) \ .$$

3) We calculate now $\int_{T_B}^T \cos\alpha(t)dt$: for the curvature in the interval of integration $[T_B, T]$ we have the following formula:

$$\kappa(t) = 2t + (\kappa^T - 2T) \qquad \text{for } t \in [T_B, T] \ .$$

So, for $\alpha(t)$ in the interval of integration $[T_B, T]$ we obtain

$$\alpha(t) = \alpha(T_B) + \int_{T_B}^t \kappa(\tau)d\tau = \alpha(T_B) + (t^2 - T_B^2) + (\kappa^T - 2T)(t - T_B) \ .$$

We change the variable t to $\tau = t - T - \delta_B K_2$. Hence,

$$\int_{T_B}^T \cos\alpha(t)dt = \int_0^{-\delta_B K_2} \cos\alpha(\tau)d\tau = \int_0^{-\delta_B K_2} \cos(\alpha(T_B) + O(\tau))d\tau =$$

$$= \int_0^{-\delta_B K_2} (\cos\alpha(T_B)\cos O(\tau) - \sin\alpha(T_B)\sin O(\tau))d\tau = -\delta_B K_2 \cos\alpha(T_B) + O(\delta_B^2) \ .$$

4) So, we obtain

$$x_B(T_B) = x^T + \delta_B K_x^{(B)} + O(\delta_B^2) \ ,$$

where

$$K_x^{(B)} = K_2(y(T_B)|\kappa^T| + \cos\alpha(T_B)) + d'_B y(\mu) - d''_B y(\nu) = K_2(y(T_B)|\kappa^T| + \cos\alpha(T_B)) - d''_B y(\nu) \ ,$$

$$d'_B = 4K_1(t_{f-1} - t_{f-2}) \ , \qquad d''_B = 2K_2(T - t_{f-1}) \ .$$

5) We obtain the formula for $y_B(T)$ by analogy. We calculate as in the proof of Proposition 6.4 and we obtain (using (68)):

$$\int_0^{T_B} (\sin\alpha_B(t) - \sin\alpha(t))dt =$$

$$= \delta_B \left[-x(T_B)|\kappa^T|K_2 - 4K_1(t_{f-1} - t_{f-2})x(\vartheta) + 2K_2(T - t_{f-1})x(\chi) \right] + O(\delta_B^2) \ .$$

We calculate $\int_{T_B}^T \sin\alpha(t)dt$ as in the proof of Proposition 6.4:

$$\int_{T_B}^T \sin\alpha(t)dt = \int_0^{-\delta_B K_2} \sin\alpha(\tau)d\tau = \int_0^{-\delta_B K_2} \sin(\alpha(T_B) + O(\tau))d\tau =$$

$$= -\delta_B K_2 \sin\alpha(T_B) + O(\delta_B^2) \ .$$

6) Hence, we obtain

$$y_B(T_B) = y^T + \delta_B K_y^{(B)} + O(\delta_B^2) \ ,$$

where

$$K_y^{(B)} = K_2(-x(T_B)|\kappa^T| + \sin\alpha(T_B)) - d'_B x(\vartheta) + d''_B x(\chi) \ ,$$

$$d'_B = 4K_1(t_{f-1} - t_{f-2}) \ , \qquad d''_B = 2K_2(T - t_{f-1}) \ .$$

The proposition is proved. □

B Appendix: Proof of the non-optimality of the path $\mathcal{P}$ – some cases

B.1 The case when $y_p > 0$ for any p

Consider some switching point ($t = t_r$) corresponding to a local minima. Then modifying the graph of $\kappa(t)$ on two intervals $[t_p - \zeta, t_p]$, $[t_l, t_{l+1}]$ (as in Section 8, here by t_p we denote the switching point corresponding to some local maxima on the graph $\kappa(t)$ of the path $\mathcal{P}$) we obtain system (26). In two cases (when either k is an even number and $p > 0$ or k is an odd number and $p < 0$) we express from this system δ_{A1} and δ_{C1} as some functions of δ_B (see the proof of Lemma 8.4 – the cases **1**) and **4**)).

In the two other cases (when either k is an odd number and $p > 0$ or k is an even number and $p < 0$) we modify the graph of $\kappa(t)$ on two intervals $[t_r - \beta, t_r]$, $[t_m, t_{m+1}]$ (see Figure 46). On this figure we denote by dotted line the pieces of the new graph. For this modification we have $\delta_{C2} > 0$.

Recall that $y(t) > 0$ if $t \in [t_r - \beta, t_r]$ and that the mean values of the y-coordinates on every interval belonging to $[0, T]$ except the first and the last one are equal to zero. Hence, for the thus obtained path $\mathcal{P}_{B1C2}$ we have system (27) with $K_x^{(C2)}$ defined by the following formula

$$K_x^{(C2)} = C_{C2}\beta(y(\kappa) - y(\nu)) = C_{C2}\beta y(\kappa) > 0 \ . \tag{69}$$

So, in the two cases (when either k is an odd number and $p > 0$ or k is an even number and $p < 0$) we express from system (27) (with $K_x^{(C2)}$ defined by formula (69)) δ_{A1} and δ_{C2} as some functions of δ_B (see the proof of Lemma 8.4 – the cases **2**) and **3**)).

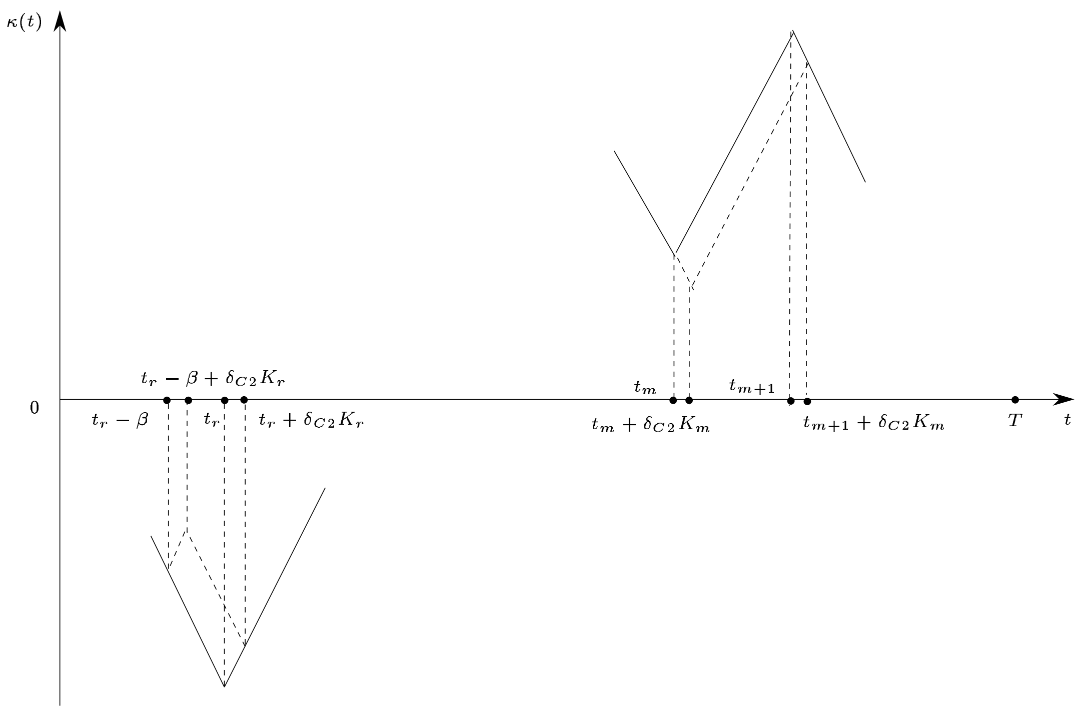

$\kappa(t)$
0
$t_r - \beta + \delta_{C2} K_r$
$t_r - \beta$
t_r
$t_r + \delta_{C2} K_r$
t_m
t_{m+1}
$t_m + \delta_{C2} K_m$
$t_{m+1} + \delta_{C2} K_m$
T
t

Figure 46

B.2 The case when there exists t_s such that $y_s = 0$, $y'_s \neq 0$

In this case for any small neighbourhood $[t_s - \gamma, t_s + \gamma]$ of the point t_s the function $y(t)$ has one sign in the left half-neighbourhood $[t_s - \gamma, t_s]$ and another sign in the right half-neighbourhood $[t_s, t_s + \gamma]$. Hence, in this case we can use the same technique as in Section 8, i.e. modify, at first, the given graph $\kappa(t)$ on two intervals $[t_s - \gamma, t_s]$ and $[t_l, t_{l+1}]$ (as on Figure 9) and then modify the graph $\kappa(t)$ on two intervals $[t_s, t_s + \gamma]$ and $[t_l, t_{l+1}]$. As a result of these two modifications we obtain two systems (26) and (27) and then depending on the sign of p we can consider the suitable system to express δ_{A1} and δ_{C1} (respectively δ_{C2}) as some functions of δ_B.

B.3 The case when there exists t_s such that $y_s = 0$, $y'_s = 0$, $y''_s \neq 0$

Consider now the case when there exists t_s such that $y_s = 0$, $y'_s = 0$, $y''_s \neq 0$ (the point $(x(t_s), y(t_s))$ isn't the inflexion point of a clothoid).

In this case there exists some small neighbourhood of t_s ($[t_s - \gamma, t_s + \gamma]$) such that for all $t \in [t_s - \gamma, t_s + \gamma]$ the function $y(t)$ is either positive or negative (except the point $t = t_s$). Hence, we can use the same technique as in the case when the y-coordinates of the path $\mathcal{P}$ at all switching points are non-zero (see Sections 8–9 and Subsection B.1) because the mean value of y on $[t_s - \gamma, t_s + \gamma]$ is positive or negative.

C Appendix: Subcase A – some auxiliary propositions

To prove Lemma 9.2 we need of some auxiliary propositions (more precisely, Propositions C.1–C.3).

Consider some point belonging to $\mathcal{P}$ whose tangent angle equals zero (modulo 2π), denote it by A. Denote by B the first following point belonging to $\mathcal{P}$ such that $\alpha_B = \alpha_A + \pi/2$ and denote by C the first following point belonging to $\mathcal{P}$ such that $\alpha_C = \alpha_B + \pi/2$. Denote by c the straight line passing throught the point C and perpendicular to the tangent vector at the point C and denote by D the intersection point of the straight line c and of the path $\mathcal{P}$ (see an example on Figure 47).

Proposition C.1 *The following estimation holds:*

$$|\widehat{BC}| \geq |\widehat{AB}|/4 \; . \tag{70}$$

See the proof of the proposition in Appendix C.1.

Proposition C.2 *Consider some path $\mathcal{D}$ of the class $\mathcal{C}^2$ which is a concatenation of arcs of half-clothoids ($\dot\kappa(t) = \pm 2$, $\kappa(0) = \kappa_0$) and of length $\kappa_0/2$. Then the euclidian distance between the initial and final points of $\mathcal{D}$ is less than $\sqrt{\pi}$.*

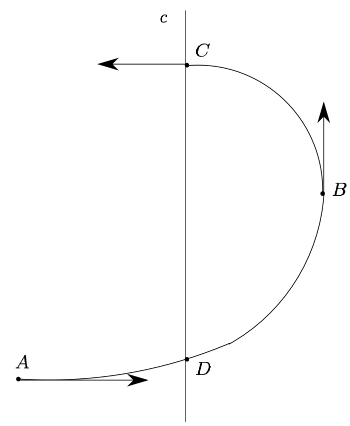


Figure 47

Proof

Consider some path cl on the interval $[0, \kappa_0/2]$ which has the same initial conditions $(x_0, y_0, \alpha_0, \kappa_0)$ that the path $\mathcal{D}$ and which is some piece of half-clothoid with the curvature defined by the equation $\kappa = -2t + \kappa_0$ (see Figure 48).

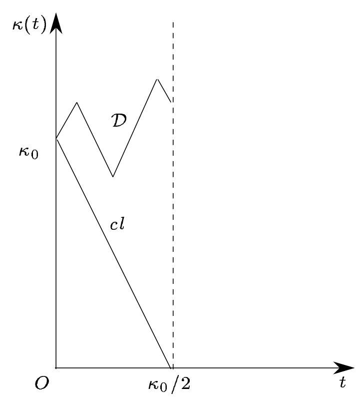


Figure 48

Denote by O_{cl} the centre of cl, by $\vec{\rho}_{cl}(t)$ the radius-vector of a point of cl in the coordinate system with centre at the point O_{cl}. Denote by $\vec{\rho}_{\mathcal{D}}(t)$ the radius-vector of a point of $\mathcal{D}$ in this coordinate system. For $t = 0$ there is $\vec{\rho}_{cl}(0) = \vec{\rho}_{\mathcal{D}}(0)$.

So, it follows from Lemma 7.1 of [11] (one can find the same statement in [13] (Lemma 3.13) and in [10] (Lemma 3.10)) that the following statement holds: for the path $\mathcal{D}$ (defined as behind) and for the path cl we have

$$\rho_{cl}(t) > \rho_{\mathcal{D}}(t), \quad \text{for any } t \in (0, \kappa_0/2] \ .$$

Any half-clothoid is situated in the circle with centre at the centre of the half-clothoid and of radius $R = \sqrt{\pi}/2$. So, the distance between the initial and final points of cl is smaller than $2R = \sqrt{\pi}$. Hence, as $\vec{\rho}_{cl}(0) = \vec{\rho}_{\mathcal{D}}(0)$ and $\rho_{cl}(t) > \rho_{\mathcal{D}}(t)$ for any $t \in (0, \kappa_0/2]$, then, the distance between the initial and final points of $\mathcal{D}$ is smaller than $2R = \sqrt{\pi}$.

The proposition is proved. □

Denote by N the first point belonging to $\mathcal{P}$ whose tangent angle equals zero. Denote by N the first following point belonging to $\mathcal{P}$ whose tangent angle equals zero and such that between N and M there are at least 10 switching points. Denote by N_{pr} (by M_{pr}) the projection of the point N (of the point M) on the axis Ox. Denote by $d_{N_{pr}M_{pr}}$ the distance between the points N_{pr} and M_{pr}.

Proposition C.3 *If the curvature of the path $\mathcal{P}$ is non-negative, then, for the length of the piece of $\mathcal{P}$ between the points N and M (denote it by l_{NM}) we have the following estimation:*

$$l_{NM} > \frac{5}{3} d_{N_{pr}M_{pr}} \ . \tag{71}$$

Proof

As the mean value of the y-coordinate on any interval between two consecutive switching points equals zero and as the curvature is non-negative, then, between any t_q, t_{q+3} switching points the tangent angle makes a turn of at least 2π.

Consider Figure 47. We have proved (see Proposition C.1) that

$$|\widehat{BC}| \geq |\widehat{AB}|/4 = (|\widehat{AC}| - |\widehat{BC}|)/4 \ .$$

Hence,

$$|\widehat{BC}| \geq |\widehat{AC}|/5 \ . \tag{72}$$

When some point moves along the path $\mathcal{P}$ from A to C (see Figure 47), the length $|\widehat{DBC}|$ is "useless" in the following sense: the projections of the points D and C on the axis Ox coincide. Hence, the projection of the point of the path on the axis Ox doesn't advance with respect to the final point from D to C.

Consider the case when the projection of the point D is situated closer to the projection of the final point of $\mathcal{P}$ than the point A. The following inequality holds:

$$|\widehat{DB}| > |\widehat{BC}| \tag{73}$$

(evidently, in the case when the projection of the point D is situated further than the point A, when a point moves along the path $\mathcal{P}$ from A to C it doesn't advance in the sense to shorten the distance between it and the projection of the final point on the axis Ox; so, in this case the length $|\widehat{AC}|$ is "useless").

Thus, using inequalities (72) and (73), we obtain

$$|\widehat{DB}| > |\widehat{BC}| \geq |\widehat{AC}|/5 \ ,$$

hence,

$$|\widehat{DB}| + |\widehat{BC}| \geq \frac{2}{5}|\widehat{AC}|$$

and

$$|\widehat{AD}| = |\widehat{AC}| - (|\widehat{DB}| + |\widehat{BC}|) \leq \frac{3}{5}|\widehat{AC}| \ . \tag{74}$$

So, we obtain that the "usefull length" (i.e. $|\widehat{AD}|$) of the arc $\widehat{AC}$ is at most $3/5$ of the length of all arc $\widehat{AC}$.

We have proved that if the curvature is non-negative on the interval $[t_N, t_M]$, then between any 4 consecutive switching points the tangent angle makes a turn of at least 2π. Hence, the path "loses" $2/5$ of its length on any turn on 2π.

Thus, we have the following inequality:

$$d_{N_{pr}M_{pr}} < \frac{3}{5} l_{NM} \ ,$$

i.e.

$$l_{NM} > \frac{5}{3} d_{N_{pr}M_{pr}} \ .$$

The proposition is proved. □

C.1 Proof of Proposition C.1

Plan of the proof

The main idea is to modify the path ABC so that

1) the tangent angle and the curvature at the point B of the new path $\widetilde{A}B\widetilde{C}$ coincide with the ones of the path ABC,
2) $\alpha_B - \alpha_{\widetilde{A}} = \alpha_B - \alpha_A = \pi/2$ and $\alpha_C - \alpha_B = \alpha_{\widetilde{C}} - \alpha_B = \pi/2$,
3) $|\widetilde{A}B| \geq |AB|$ and $|B\widetilde{C}| \leq |BC|$.

If we prove for the path $\widetilde{A}B\widetilde{C}$ the following inequality holds:

$$|B\widetilde{C}| \geq |\widetilde{A}B|/4 \;, \tag{75}$$

then (as $|\widetilde{A}B| \geq |AB|$ and $|B\widetilde{C}| \leq |BC|$) we obtain inequality (70).

At first we modify the piece BC of the path ABC. Denote by κ_B the value of the curvature at the point B. If we fix κ_B, then in the case when the curvature of the arc $B\widetilde{C}$ is defined by the following formula:

$$\kappa(t) = 2(t - t_B) + \kappa_B$$

(see Figure 49) we obtain $|B\widetilde{C}| < |BC|$ (because the tangent angle is equal to the integral of the curvature on the arc, so, as $\alpha_C - \alpha_B = \alpha_{\widetilde{C}} - \alpha_B$, then we obtain that $|B\widetilde{C}| < |BC|$ for any piece BC).

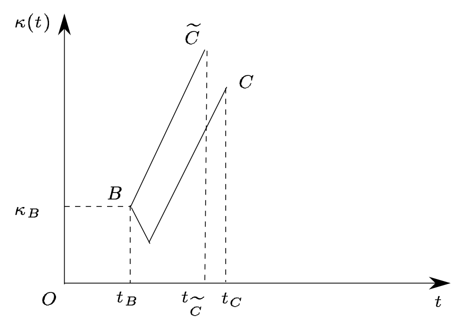


Figure 49

Now modify the piece AB of the path ABC. As the mean value of the y-coordinate on any interval between two consecutive switching points is equal to zero and as $\alpha_B - \alpha_A = \pi/2$, then between the points A and B there exist at most two switching points. Really, any piece of clothoid delimited by two switching points intersects the axis Ox. The condition $\alpha_B - \alpha_A = \pi/2$ imposes that there exists at most one intersection point with the axis Ox on the piece AB, i.e. that at most two switching points belong to the piece AB. Hence, we must consider the three following cases:

1) there is no switching point belonging to the piece AB of the path ABC,
2) there is one switching point belonging to the piece AB of the path ABC,
3) there are two switching points belonging to the piece AB of the path ABC.

I. Consider the case **1**). There are two possibilities:

a) either for the piece AB one has $\kappa(t) = 2(t - t_A) + \kappa_A$ on the interval $[t_A, t_B]$ (see Figure 50 a)),

b) or for the piece AB one has $\kappa(t) = -2(t - t_A) + \kappa_A$ on the interval $[t_A, t_B]$ (see Figure 50 b)).

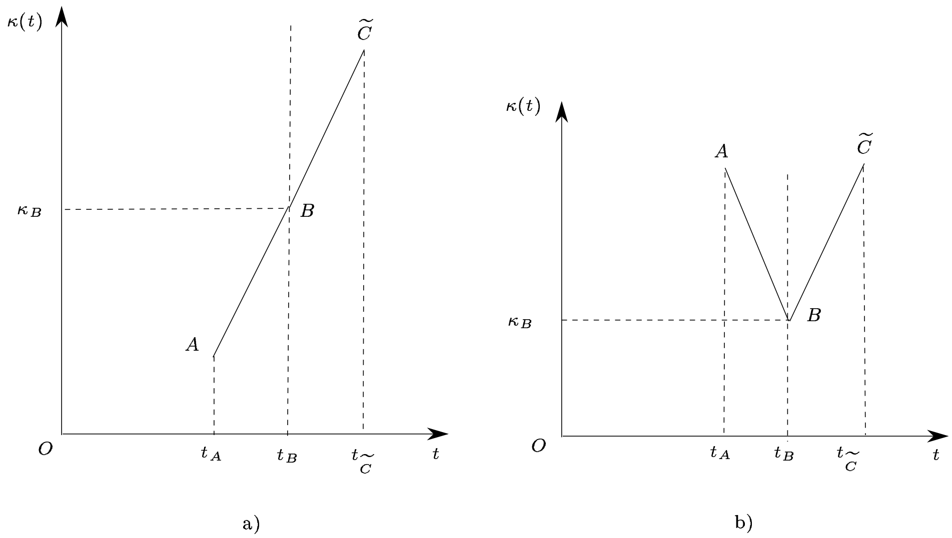


Figure 50

a) Consider the case 1 a). In this case we don't modify the piece AB of the path ABC, i.e. we consider as the path $\widetilde{A}B\widetilde{C}$ the path $AB\widetilde{C}$, and we want to obtain the following estimation:

$$|B\widetilde{C}| > |AB|/4 . \tag{76}$$

Really, if we set $|B\widetilde{C}| = k$, $|AB| = p$ and $\kappa_B = q$, then we obtain the following equalities:

$$\alpha_{\widetilde{C}} - \alpha_B = kq + k^2 = \pi/2 ,$$

$$\alpha_B - \alpha_A = pq - p^2 = \pi/2 .$$

Hence,

$$k^2 + kq + p^2 - pq = 0 ,$$

i.e.

$$k = -q/2 + \sqrt{q^2/4 - p^2 + pq}$$

(because $k > 0$). We must prove that

$$k > p/4 ,$$

i.e. that

$$\sqrt{q^2/4 - p^2 + pq} > q/2 + p/4 ,$$

i.e.

$$q^2/4 - p^2 + pq > q^2/4 + p^2/16 + pq/4 ,$$

i.e.

$$3q/4 > 17p/16 ,$$

i.e.

$$q > 17p/12$$

(it's true because $q \geq 2p$).

So, in the case 1 a) inequality (76) is proved.

b) Consider the case 1 b). There are two possibilities: if κ_B is rather great, then we construct the piece $\widetilde{A}B$ as on Figure 51 i); if not, we construct the piece $\widetilde{A}B$ as on Figure 51 j).

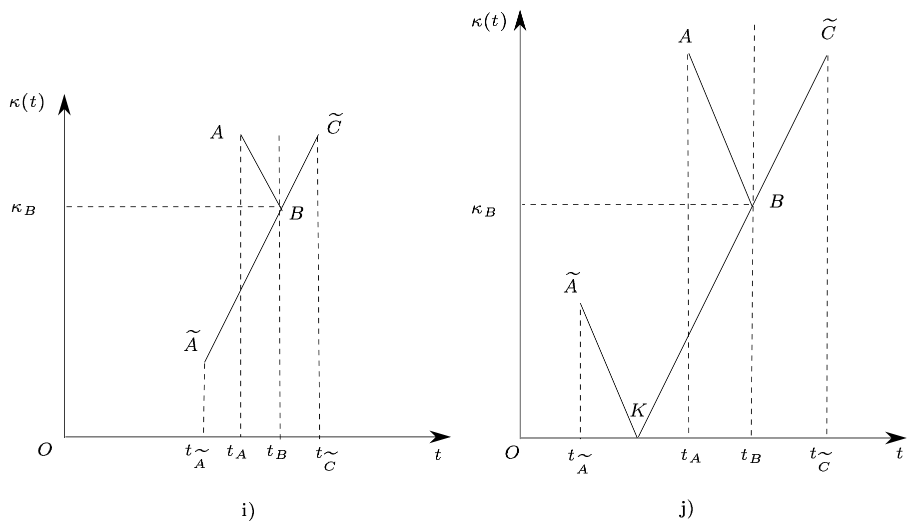


Figure 51

In the first case the piece $\widetilde{A}B$ is as the piece AB in the case 1 a). Hence, for this path $\widetilde{A}B\widetilde{C}$ inequality (75) holds.

In the second case the piece $\widetilde{A}B$ is as the piece $\widetilde{A}B$ in the second subcase of the case 2 a) (see Figure 53 j). So, for this path $\widetilde{A}B\widetilde{C}$ inequality (75) will be proved during the study of the case 2 a) (see below).

II. Consider the case **2**) (i.e. the case when there is one switching point belonging to the piece AB of the path ABC). There are two possibilities:

a) either this switching point is some local minimum of the graph of the curvature on the interval $[t_A, t_B]$ (see Figure 52 a)),

b) or this switching point is some local maximum of the graph of the curvature on the interval $[t_A, t_B]$ (see Figure 52 b)).

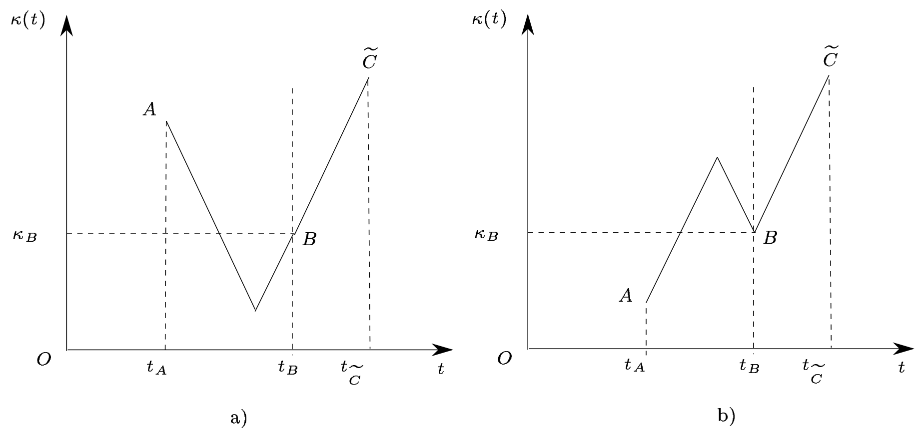


Figure 52

a) Consider the case 2 a). There are two possibilities: if κ_B is rather great, then we construct the piece $\widetilde{A}B$ as on Figure 53 i); if not, we construct the piece $\widetilde{A}B$ as on Figure 53 j).

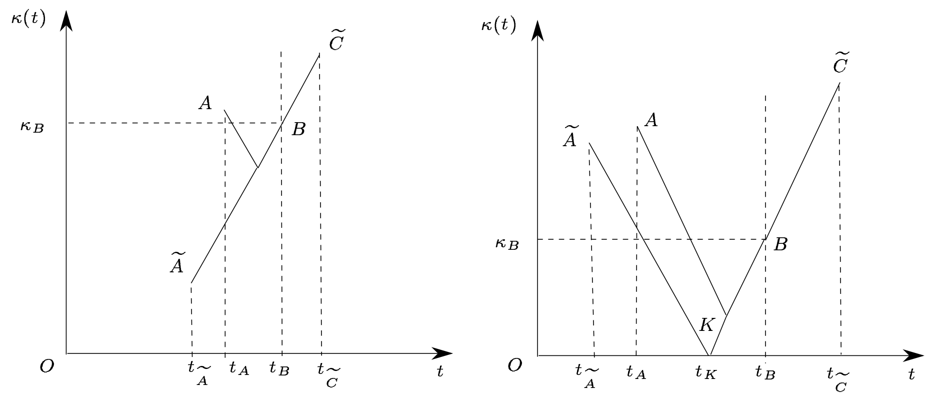
In the first case the piece $\widetilde{A}B$ is as the piece AB in the case 1 a). Hence, for this path $\widetilde{A}B\widetilde{C}$ inequality (75) holds.

In the second case we must prove inequality (75) for the constructed path $\widetilde{A}B\widetilde{C}$ (see Figure 53 j).

Set $|B\widetilde{C}| = k$, $|\widetilde{A}K| = m$ and $KB = n$, so, we obtain the following equalities:

$$\alpha_{\widetilde{C}} - \alpha_B = 2kn + k^2 = \pi/2 \ ,$$

$$\alpha_B - \alpha_{\widetilde{A}} = m^2 + n^2 = \pi/2 \ .$$



i) j)

Figure 53

Hence,

$$k^2 + 2kn - m^2 - n^2 = 0 \ ,$$

i.e.

$$k = -n + \sqrt{2n^2 + m^2}$$

(because $k > 0$). We must prove that

$$k > (m + n)/4 \ ,$$

i.e. that

$$\sqrt{2n^2 + m^2} > n + (n + m)/4 = 5n/4 + m/4 \ ,$$

i.e.

$$2n^2 + m^2 > 25n^2/16 + m^2/16 + 5nm/8 \ ,$$

i.e.

$$7n^2/16 + 15m^2/16 - 5nm/8 > 0 \ ,$$

i.e.

$$n^2 + 15m^2/7 - 10mn/7 > 0 \ ,$$

i.e.

$$(n - 5m/7)^2 + 80m^2/49 > 0$$

(it's true).

So, in the case 2 a) inequality (75) is proved.

b) Consider the case 2 b). There are three possibilities: if κ_B is rather great, then we construct the piece $\widetilde{A}B$ as on Figure 54 i); if not, either we construct the piece $\widetilde{A}B$ as on Figure 54 j), or we construct the piece $\widetilde{A}B$ as on Figure 54 k).

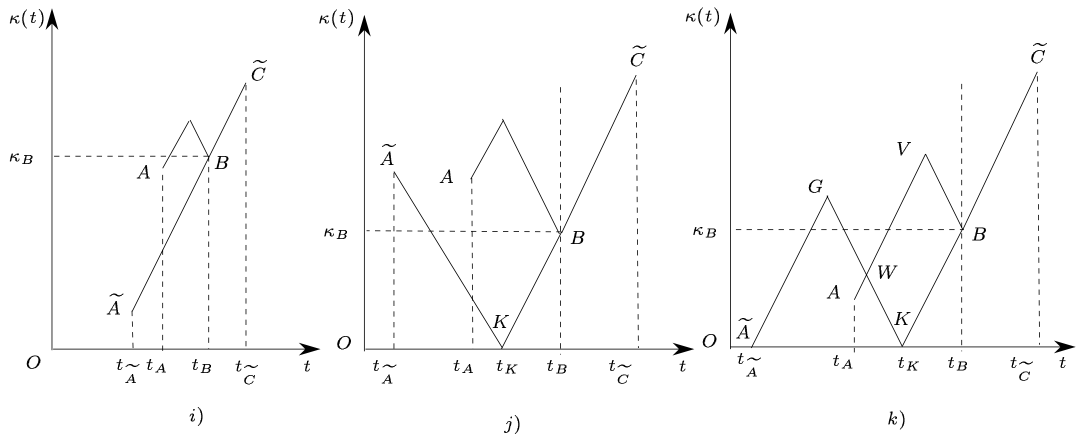


Figure 54

More precisely, we construct the piece $\widetilde{A}B$ as on Figure 54 j) if the straight line passing throught the point K and such that $\dot{\kappa}(t) = -2$ doesn't intersect the piece AB of the graph of the curvature. It's evident, that in this case the constructed piece $\widetilde{A}KB$ is longer than the piece AB. The piece $\widetilde{A}B$ is as the piece $\widetilde{A}B$ in the second subcase of the case 2 a) (see Figure 53 j). Hence, for this path $\widetilde{A}B\widetilde{C}$ inequality (75) holds.

We construct the piece $\widetilde{A}GKB$ as on Figure 54 k) if the straight line passing throught the point K and such that $\dot{\kappa}(t) = -2$ intersects the piece AB of the graph of the curvature. It's evident, that in this case the constructed piece $\widetilde{A}KB$ is longer than the piece AB (because in this case we can always find the points G and $\widetilde{A}$ such that the area of $\widetilde{A}GWAt_A$ should be equal to the area of $WKBV$). The piece $\widetilde{A}B$ is as the piece $\widetilde{A}B$ in the second subcase of the case 3 a) (see Figure 56 k). Hence, for this path $\widetilde{A}B\widetilde{C}$ inequality (75) will be proved during the study of the case 3 a) (see below).

III. Consider the case **3**) (i.e. the case when there are two switching points belonging to the piece AB of the path ABC). There are two possibilities:

a) either the first switching point is some local maximum of the graph of the curvature on the interval $[t_A, t_B]$ (see Figure 55 a)),

b) or the first switching point is some local minimum of the graph of the curvature on the interval $[t_A, t_B]$ (see Figure 55 b)).

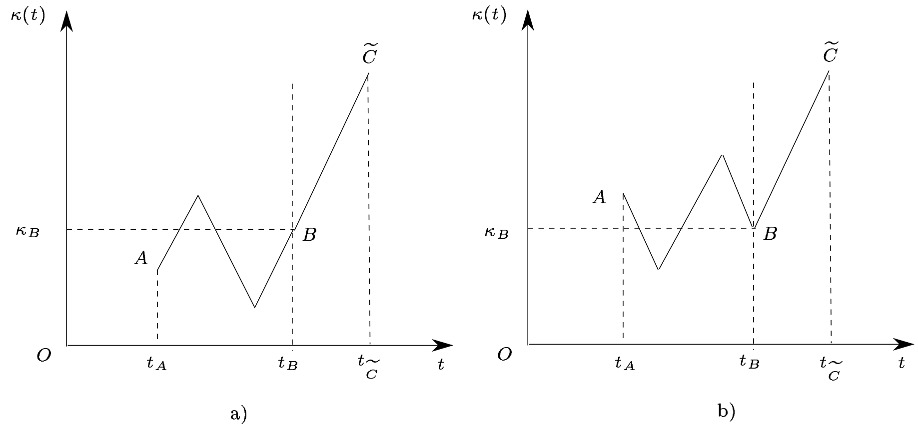


Figure 55

a) Consider the case 3 a). There are three possibilities: if κ_B is rather great, then we construct the piece $\widetilde{A}B$ as on Figure 56 i); if not, either we construct the piece $\widetilde{A}B$ as on Figure 56 j), or we construct the piece $\widetilde{A}B$ as on Figure 56 k).

More precisely, we construct the piece $\widetilde{A}KB$ as on Figure 56 j) if the straight line passing throught the point K and such that $\dot{\kappa}(t) = -2$ doesn't intersect the piece AB of the graph of the curvature. It's evident, that in this case the constructed piece $\widetilde{A}KB$ is longer than the piece AB. The piece $\widetilde{A}B$ is as the piece $\widetilde{A}B$ in the second subcase of the case 2 a) (see Figure 53 j). Hence, for this path $\widetilde{A}B\widetilde{C}$ inequality (75) holds.

We construct the piece $\widetilde{A}GKB$ as on Figure 56 k) if the straight line passing throught the point K and such that $\dot{\kappa}(t) = -2$ intersects the piece AB of the graph of the curvature. It's evident, that in this case the constructed piece $\widetilde{A}GKB$ is longer than the piece AB (because in this case we can always find points G and $\widetilde{A}$ such that the area of $\widetilde{A}GWAt_A$ should be equal to the area of $WVUK$). So, we must prove inequality (75) for the constructed path $\widetilde{A}B\widetilde{C}$.

Set $|B\widetilde{C}| = k$, $|\widetilde{A}G| = |GK| = a$ and $KB = b$. We obtain the following equalities:

$$\alpha_{\widetilde{C}} - \alpha_B = 2kb + k^2 = \pi/2 \ ,$$

$$\alpha_B - \alpha_{\widetilde{A}} = 2a^2 + b^2 = \pi/2 \ .$$

Hence,

$$k^2 + 2kb - 2a^2 - b^2 = 0 \ ,$$

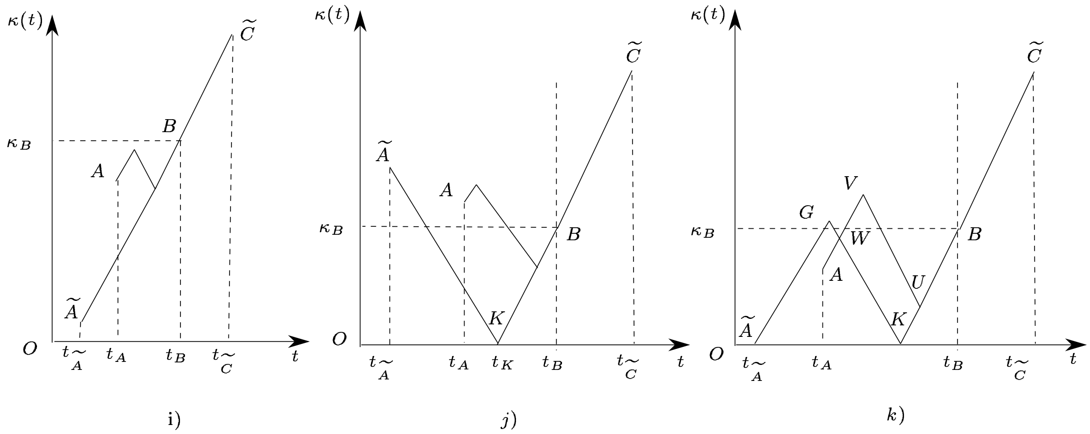


Figure 56

i.e.

$$k = -b + \sqrt{2(a^2 + b^2)}$$

(because $k > 0$). We must prove that

$$k > (2a + b)/4 \ ,$$

i.e. that

$$\sqrt{2(a^2 + b^2)} > b + (2a + b)/4 = (2a + 5b)/4 \ ,$$

i.e.

$$2a^2 + 2b^2 > 25b^2/16 + a^2/4 + 5ab/4 \ ,$$

i.e.

$$7a^2/4 + 7b^2/16 - 5ab/4 > 0 \ ,$$

i.e.

$$a^2 + b^2/4 - 5ab/7 > 0 \ ,$$

i.e.

$$(a - 5b/14)^2 + (1/4 - 25/196)b^2 > 0$$

(it's true).

Thus, in the case 3 a) inequality (75) is proved.

b) Consider the case 3 b). There are three possibilities: if κ_B is rather great, then we construct the piece $\widetilde{A}B$ as on Figure 57 i); if not, either we construct the piece $\widetilde{A}B$ as on

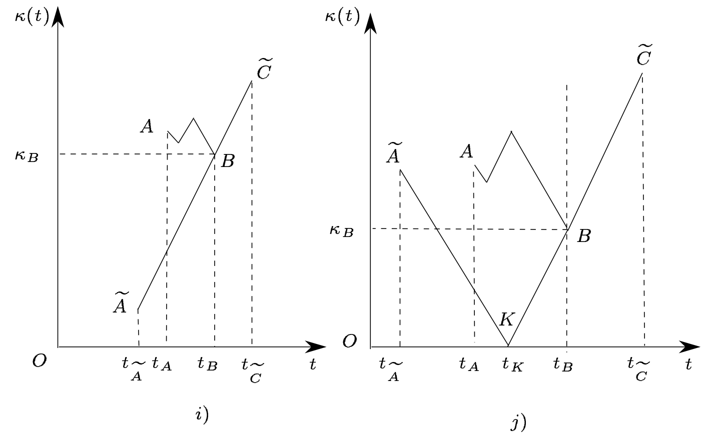


Figure 57

Figure 57 j), or in order to construct the piece $\widetilde{A}B$ we introduce some other method (see its description below).

More precisely, we construct the piece $\widetilde{A}KB$ as on Figure 57 j) if the straight line passing throught the point K and such that $\dot{\kappa}(t) = -2$ doesn't intersect the piece AB of the graph of the curvature. It's evident, that in this case the constructed piece $\widetilde{A}KB$ is longer than the piece AB. The piece $\widetilde{A}B$ is as the piece $\widetilde{A}B$ in the second subcase of the case 2 a) (see Figure 53 j). Hence, for this path $\widetilde{A}B\widetilde{C}$ inequality (75) holds.

So, we must consider only the case when the straight line passing throught the point K and such that $\dot{\kappa}(t) = -2$ intersects the piece AB of the graph of the curvature.

We modify the piece $AUVB$ of the graph (see Figure 58) so that the value of $|t_A - t_B|$ should increase and that the surface under the graph should not change. For this we replace the part AU by the piece $\widetilde{A}U$ or $\widetilde{A}'WU$; the choice of the case depends of the surface under AU (it equals the one under the $\widetilde{A}U$ or $\widetilde{A}'WU$ respectively). The case when AU is replaced by $\widetilde{A}U$ brings us to the case 2 b) already considered.

So, we consider the case when AU is replaced by $\widetilde{A}'WU$. Two situations are possible:

1^0 the curvature at the point $\widetilde{A}'$ is greater or equal to the one at the point B,

2^0 the curvature at the point $\widetilde{A}'$ is smaller than the one at the point B,

In situation 1^0 we replace the graph $\widetilde{A}'WVB$ by the graph which is symmetric to it with respect to the straight line m, the symmetral of $t_{\widetilde{A}'}t_B$ (see Figure 59). We obtain the graph $PQRS$. As the curvature at the point S can be strictly greater than the one at the point B,

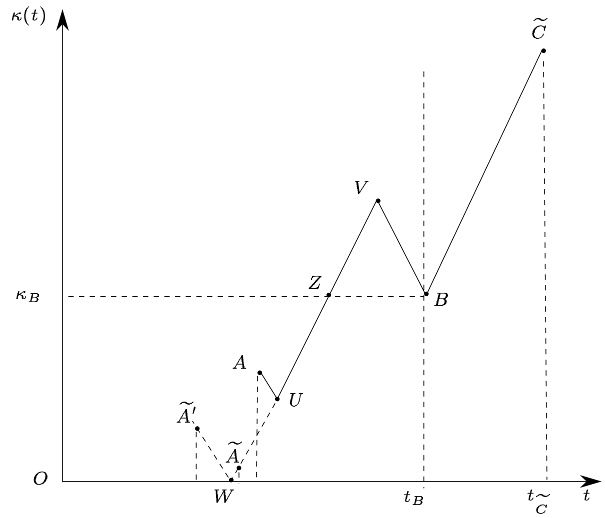


Figure 58

this fact obliges us to change eventually the part $B\widetilde{C}$ also (it is replaced by $S\widetilde{C}'$, the surface under $S\widetilde{C}'$ equals the one under $B\widetilde{C}$).

Thus,

$$|t_S - t_{\widetilde{C}'}| < |t_B - t_{\widetilde{C}}| \ .$$

It follows from the case 3 a) that

$$|t_S - t_{\widetilde{C}'}| \geq \frac{1}{4}|t_P - t_S| = \frac{1}{4}|t_{\widetilde{A}'} - t_B| \ .$$

Hence,

$$|t_B - t_C| \geq |t_B - t_{\widetilde{C}}| \geq \frac{1}{4}|t_{\widetilde{A}'} - t_B| \geq \frac{1}{4}|t_A - t_B| \ ,$$

i.e. inequality (70) is proved in situation 1^0.

In situation 2^0 we replace the graph $\widetilde{A}'WVB$ by the graph $EFKHB$ defined as follows:

– the piece KHB is obtained from the piece $\widetilde{A}'WZ$ by translation,
– the surface under EFK equals the one under ZVB,
– the curvatures at the points E, $\widetilde{A}'$, K are equal (as they are smaller than the ones at the points Z and B, we have $|t_E - t_K| > |t_Z - t_B|$), so, $|t_E - t_B| > |t_{\widetilde{A}'} - t_B|$.

The graph $EFHB$ corresponds to the case 3 a) already considered, hence,

$$|t_B - t_C| \geq |t_B - t_{\widetilde{C}}| \geq \frac{1}{4}|t_E - t_B| > \frac{1}{4}|t_{\widetilde{A}'} - t_B| \geq \frac{1}{4}|t_A - t_B| \ ,$$

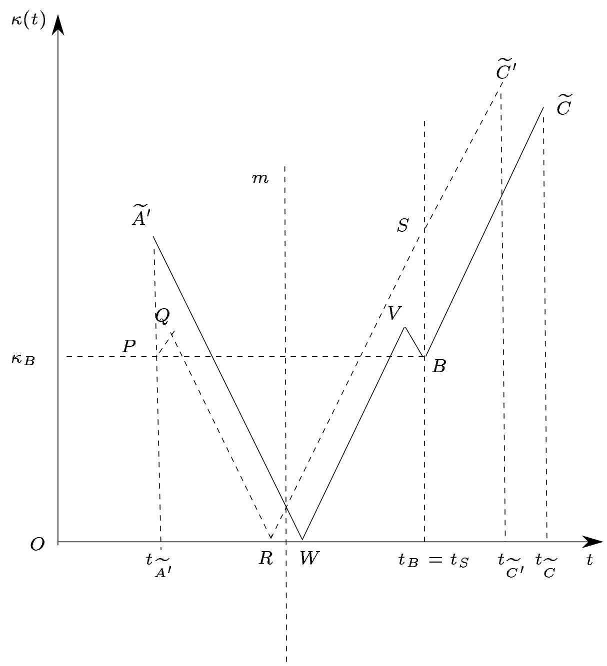

$\kappa(t)$
$\widetilde{C}'$
$\widetilde{C}$
m
$\widetilde{A}'$
S
Q
V
P
κ_B
B
O
$t_{\widetilde{A}'}$
R
W
$t_B = t_S$
$t_{\widetilde{C}'}$
$t_{\widetilde{C}}$
t

Figure 59

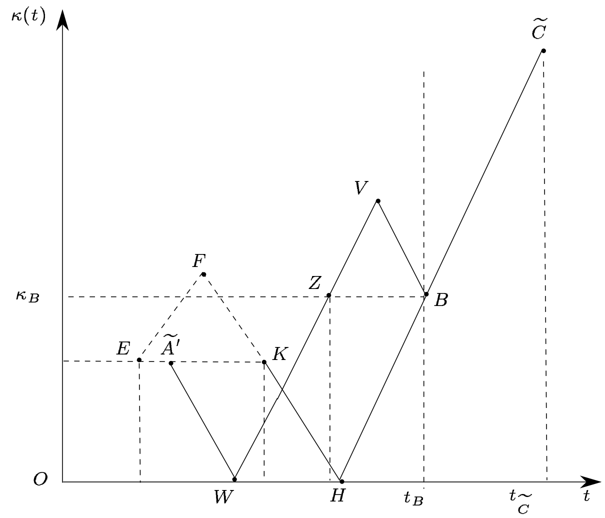


Figure 60

i.e. inequality (70) is proved in situation 2^0. Thus, the case 3 b) is considered.

Hence, we have proved inequality (70) in all cases.

The proposition is proved. □

D Appendix: Subcase B – some auxiliary propositions to prove Proposition 9.10 and the proofs of Propositions 9.10 and 9.11

To prove this proposition we prove two auxiliary propositions (Propositions D.1 and D.2); see the proof of Proposition 9.10 in Appendix D.1.

We denote by t_1 (by t_2) the length of the arc $\widehat{OR}$ (the length of the arc $\widehat{ORI}$). One can see the graphs of the curvature of the arcs $\widehat{ORX}$ and $\widehat{ORI}$ on Figure 61.

Proposition D.1 *If the tangent angle at the point I is equal to the tangent angle at the point X, then,*

$$t_1 = t_2/\sqrt{2}\ . \tag{77}$$

Proof

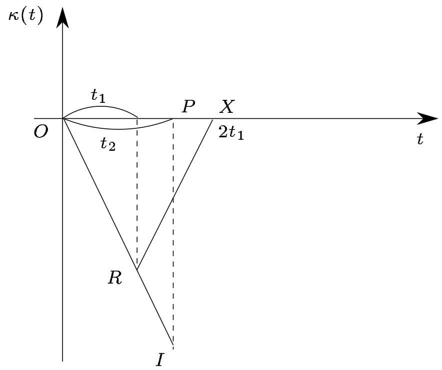


Figure 61

We have the following equalities:

$$|\widehat{OR}| = |\widehat{RX}| = t_1 \ , \quad |\widehat{ORI}| = t_2 \ .$$

We denote by α_I (by α_X) the tangent angle at the point I (at the point X). We denote by S_{OIP} (by S_{ORX}) the area of OIP (of ORX respectively), see Figure 61. So,

$$\alpha_I = \alpha_O - S_{OIP} = \alpha_O - t_2^2 \ ,$$

$$\alpha_X = \alpha_O - S_{ORX} = \alpha_O - 2t_2^1 \ .$$

As $\alpha_I = \alpha_X$, we obtain the following formula:

$$t_1 = t_2/\sqrt{2} \ .$$

The proposition is proved. □

Proposition D.2 *For $|HI|$ we have the following formula:*

$$|HI| = \sqrt{\alpha_O}(\sqrt{2} - 1) \ . \qquad (78)$$

Proof

It is more easy to estimate $\max_{\alpha_O \in [0,\pi]} |HJ|$ if one considers the paths ORX and ORI in another coordinate system (see Figure 62).

So, we have some given angle $\alpha_O \in [0, \pi]$. We consider the coordinate system Oxy such that the angle between the axis Ox and the straight line

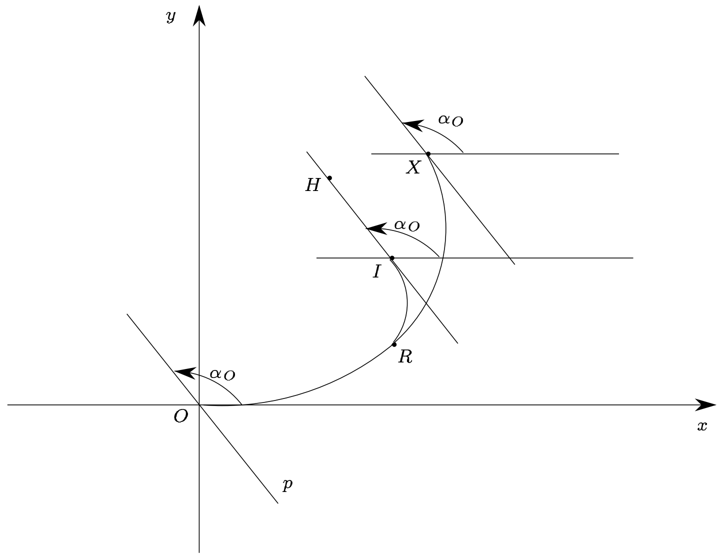


Figure 62

p is equal to α_O (see Figure 62).

The graphs of the curvature of the paths ORX and ORI in this coordinate system are defined as follows:

$$ORX: \quad \kappa(t) = \begin{cases} 2t \ , & \text{for } t \in [0, t_1] \ , \\ -2t + 4t_1 \ , & \text{for } t \in (t_1, 2t_1] \ , \end{cases}$$

$$ORI: \quad \kappa(t) = 2t \ , \quad \text{for } t \in [0, t_2] \ .$$

Remind that the tangent angle of the path ORI (of the path ORX) at the point corresponding to $t = t_2$ (to $t = 2t_1$) is equal to α_O.

There are the following equalities:

$$|HI| = |\widehat{ORX}| - |\widehat{ORI}| = 2t_1 - t_2 \ .$$

Using formula (77), we obtain

$$|HI| = 2t_1 - t_2 = t_2(2/\sqrt{2} - 1) = t_2(\sqrt{2} - 1) = \sqrt{\alpha_O}(\sqrt{2} - 1) \ .$$

The proposition is proved. □

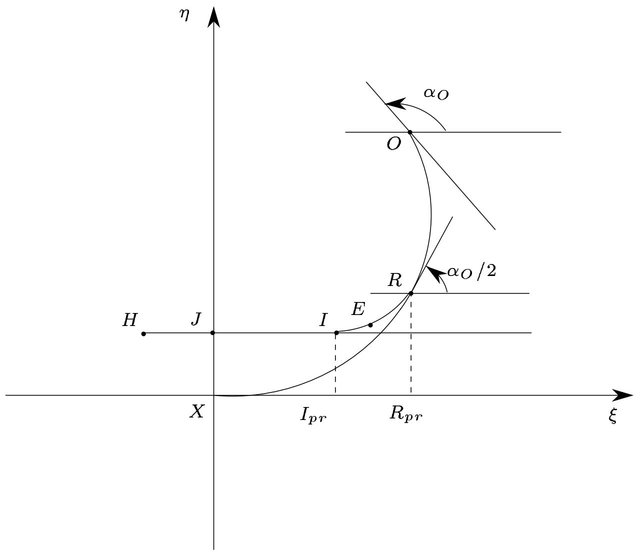


Figure 63

D.1 Proof of Proposition 9.10

We consider the paths ORX and ORI in the coordinate system $X\xi\eta$ (see Figure 63).

We denote by α_R the tangent angle at the point R ($\alpha_R = t_1^2 = \alpha_O/2$) and we denote by R_{pr} (by I_{pr}) the projection of the point R (of the point I) on the axis $X\xi$.

We have the following equalities (see Figure 63):

$$|HJ| = |HI| - |JI| = |HI| - |XI_{pr}| = |HI| - |XR_{pr}| + |I_{pr}R_{pr}| \ .$$

It follows from formula (78) that

$$|HI| = \sqrt{\alpha_O}(\sqrt{2} - 1) \ .$$

For $|XR_{pr}|$ we have the following formula:

$$|XR_{pr}| = \int_0^{\sqrt{\alpha_R}} \cos t^2 dt = \int_0^{\sqrt{\alpha_O/2}} \cos t^2 dt \ .$$

Now we calculate $|I_{pr}R_{pr}|$. For this purpose we consider a point $E \in \widehat{RI}$, E corresponds to $t \in [t_1, t_2]$. We denote by α_E the tangent angle at the point E. For the curvature of the path $\mathcal{H}_2$ on the interval $[t_1, t_2]$ there is the following formula:

$$\kappa(t) = -2t \ , \qquad \text{for } t \in [t_1, t_2] \ .$$

Hence,

$$\alpha_E = \alpha_R + \int_{t_1}^{t} (-2\tau)d\tau = \alpha_R - t^2 + t_1^2 = t_1^2 - t^2 + t_1^2 = \alpha_O - t^2 \ , \quad \text{for } t \in [t_1, t_2] \ .$$

Now we calculate $|I_{pr}R_{pr}|$:

$$|I_{pr}R_{pr}| = \int_{t_1}^{t_2} \cos(\alpha_O - t^2)dt = \int_{\sqrt{\alpha_O/2}}^{\sqrt{\alpha_O}} \cos(\alpha_O - t^2)dt \ .$$

Thus, we obtain the following formula for $|HJ|$:

$$|HJ| = \sqrt{\alpha_O}(\sqrt{2}-1) - \int_0^{\sqrt{\alpha_O/2}} \cos t^2 dt + \int_{\sqrt{\alpha_O/2}}^{\sqrt{\alpha_O}} \cos(\alpha_O - t^2)dt \ .$$

Remind that $\alpha_O \in [0, \pi]$. We calculate the maximal possible value of the length $|HJ|$ for $\alpha_O \in [0, \pi]$. For this purpose we must give an estimation for every term.

Set

$$I_1 = \int_0^{\sqrt{\alpha_O/2}} \cos t^2 dt \ , \quad I_2 = \int_{\sqrt{\alpha_O/2}}^{\sqrt{\alpha_O}} \cos(\alpha_O - t^2)dt \ .$$

Thus we obtain

$$\max_{\alpha_O \in [0,\pi]} |HJ| = \max_{\alpha_O \in [0,\pi]} \sqrt{\alpha_O}(\sqrt{2}-1) - \min_{\alpha_O \in [0,\pi]} I_1 + \max_{\alpha_O \in [0,\pi]} I_2 \ .$$

To estimate I_2 we use the following inequality: $\cos(\alpha_O - t^2) \le 1$ for any $t \in [t_1, t_2]$. Hence,

$$I_2 < \sqrt{\alpha_O} - \sqrt{\alpha_O/2} \ . \tag{79}$$

Now we calculate $\min_{\alpha_O \in [0,\pi]} I_1$. After the change of the variable $\tau = t^2$ the expression I_1 becomes:

$$I_1 = \int_0^{\sqrt{\alpha_O/2}} \cos t^2 dt = \int_0^{\alpha_O/2} \frac{\cos\tau}{2\sqrt{\tau}} d\tau \ .$$

To find $\min_{\alpha_O \in [0,\pi]} I_1$ we consider instead of the function $\cos\tau$ the function $f(\tau) = -2\tau/\pi + 1$ (because $\cos\tau \ge f(\tau)$ for $\tau \in [0, \pi/2]$, see Figure 64).

Hence,

$$I_1 > \int_0^{\alpha_O/2} (-2\tau/\pi + 1)/(2\sqrt{\tau})d\tau = \int_0^{\alpha_O/2} (-\sqrt{\tau}/\pi + 1/(2\sqrt{\tau}))d\tau =$$

$$= -\frac{2}{3}\frac{\tau\sqrt{\tau}}{\pi}\Bigg|_0^{\alpha_O/2} + \sqrt{\tau}\Bigg|_0^{\alpha_O/2} = -\frac{\alpha_O\sqrt{\alpha_O}}{3\pi\sqrt{2}} + \frac{\sqrt{\alpha_O}}{\sqrt{2}} \ . \tag{80}$$

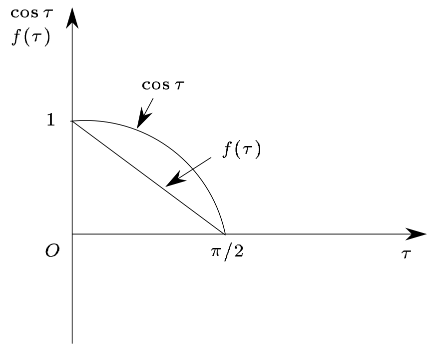


Figure 64

Using formulas (79), (80), we obtain

$$|HJ| < \sqrt{\alpha_O}(\sqrt{2}-1) - \left(-\frac{\alpha_O\sqrt{\alpha_O}}{3\pi\sqrt{2}} + \frac{\sqrt{\alpha_O}}{\sqrt{2}}\right) + \sqrt{\alpha_O} - \sqrt{\alpha_O/2} =$$

$$= \sqrt{\alpha_O}\left(\sqrt{2} - 1 - \frac{1}{\sqrt{2}} + 1 - \frac{1}{\sqrt{2}}\right) + \frac{\alpha_O\sqrt{\alpha_O}}{3\pi\sqrt{2}} = \frac{\alpha_O\sqrt{\alpha_O}}{3\pi\sqrt{2}} \ .$$

Hence,

$$\max_{\alpha_O\in[0,\pi]}|HJ| < \frac{\pi\sqrt{\pi}}{3\pi\sqrt{2}} = \frac{\sqrt{\pi}}{3\sqrt{2}} = 0.4177713791 \ . \tag{81}$$

The proposition is proved. □

D.2 Proof of Proposition 9.11

Remind that it follows from Proposition 5.3 of [11] that the maximal distance between two points of a half-clothoid is smaller than $3R/2 = 3\sqrt{\pi}/4$ (because $R = \sqrt{\pi}/2$). Hence, the maximal distance between the points O and X (and between the points T and Y) is smaller than $3R = \sqrt{\pi}/2$ (because the pieces OX and TY of the path $\widetilde{\mathcal{P}}$ consists of two arcs of a half-clothoid).

We obtain $\max_{\substack{\alpha_O \in [0,\pi] \\ \alpha_T \in [0,\pi]}} (|l| - |l_{min}|)$ if $IW \in p$ and the line segment XY is of the kind of the line segment shown on Figure 65.

There are the following equalities (see Figure 65):

$$|IW| = 40\sqrt{\pi} - 3\sqrt{\pi}/2 - 3\sqrt{\pi}/2 = 37\sqrt{\pi} \ ,$$

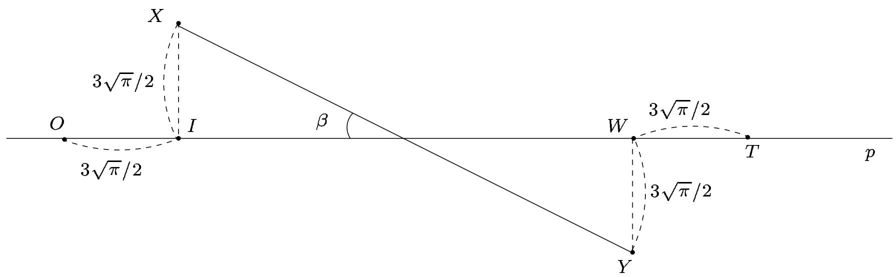


Figure 65

$$|IX| + |WY| = 3\sqrt{\pi} \ ,$$
$$\tan\beta = \frac{|IX| + |WY|}{|IW|} = \frac{3\sqrt{\pi}}{37\sqrt{\pi}} = \frac{3}{37} \ ,$$
$$\beta = \arctan(3/37) \approx 0.0809041 \approx \pi/38.83 \ .$$

Hence,

$$|XY| = |IW|/\cos\beta \ ,$$
$$|XY| - |IW| = |IW|(1/\cos\beta - 1) = 37\sqrt{\pi}(1/\cos(\arctan(3/37)) - 1) \approx$$
$$\approx 37\sqrt{\pi}(1/0.9967290481 - 1) \approx 0.2152155686 \ .$$

Thus,

$$\max_{\substack{\alpha_O \in [0,\pi] \\ \alpha_T \in [0,\pi]}} (|l| - |l_{min}|) < 0.2152155686 \ .$$

The proposition is proved. □

E Appendix: Subcase C – proof of Propositions 9.13, 9.14 and 9.15

E.1 Proof of Proposition 9.13

We must consider two cases:

a) the curvature on the interval following the arc $\widehat{DF}$ is non-positive (see Figure 66, here the following interval is the arc $\widehat{FG}$),

b) the curvature on the interval following the arc $\widehat{DF}$ changes sign (see Figure 67, here $\kappa(t) \leq 0$ on the arc $\widehat{FJ}$ and $\kappa(t) \geq 0$ on the arc $\widehat{JU}$).

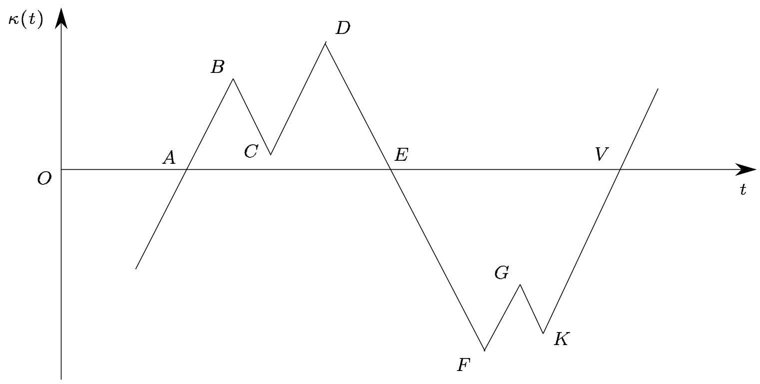


Figure 66

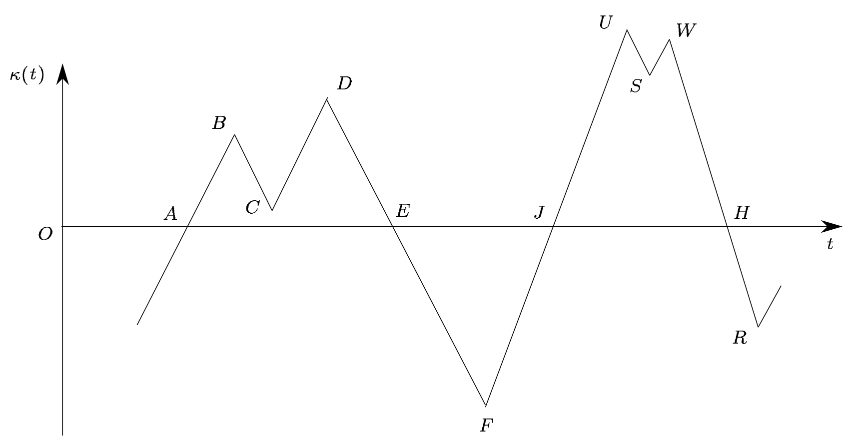


Figure 67

1⁰ Consider the case a). In this case there is one lace between the points F and V because the mean value of the y-coordinate on the intervals between F and G and between G and K equals zero, the curvature doesn't change sign between the points F and V and on the arc $\widehat{LF}$ there is at least one point with vertical tangent line. Here we use the term "lace" in the following sens: yhe tangent angle makes a turn of at least 2π on this piece.

Hence, we use the method introduced in subcase B and we construct some path $\widetilde{\mathcal{P}}$ which satisfies the initial and final conditions and such that $|\widetilde{\mathcal{P}}| < |\mathcal{P}|$. Thus, the path $\mathcal{P}$ isn't optimal.

2⁰ Consider the case b). Denote by Z the first point belonging to the arc $\widehat{LF}$ with vertical tangent line (see Figure 28).

The following equality holds

$$|\alpha_Z - \alpha_L| = \pi/2$$

(see Figure 28).

But $L \in \widehat{EF}$, hence,

$$|\alpha_Z - \alpha_E| > \pi/2 \ .$$

The point Z belongs to the arc $\widehat{EF}$, the arcs $\widehat{EF}$ and $\widehat{FJ}$ are symmetric with respect to the perpendicular to the tangent line at the point F (see Figure 67), hence, as

$$|\alpha_F - \alpha_E| \geq |\alpha_Z - \alpha_E| > \pi/2 \ ,$$

then

$$|\alpha_J - \alpha_F| > \pi/2 \ . \tag{82}$$

If the piece $\widehat{ZJ}$ intersects the tangent line at the point Z (at some point different from the point Z), then between E and this intersection point the tangent angle makes a turn of at least 2π, so, there is a "lace" on $\widehat{EJ}$, hence, we use the method introduced in subcase B and we prove that the path $\mathcal{P}$ isn't optimal.

If the piece $\widehat{ZJ}$ doesn't intersect the tangent line at the point Z (at some point different from the point Z), then all length $|\widehat{ZJ}|$ is "useless" (because the projection of the point of the path on the axis Ox doesn't advance with respect to the final point from Z to J).

As the point Z belongs to the arc $\widehat{LF}$, then it follows from (82) that

$$|\alpha_J - \alpha_Z| > \pi/2 \ . \tag{83}$$

The piece $\widehat{ZJ}$ consists of at most two arcs of half-clothoid (we obtain two arcs when the points F and Z don't coincide, if not, we obtain one arc). For fixed $|\alpha_J - \alpha_Z|$ we obtain the minimal length of $\widehat{ZJ}$ in the case when F coincides with Z.

Really, as $|\alpha_J - \alpha_Z|$ is fixed, then the area of $OZFJ$ equals the area of $O\widetilde{Z}\widetilde{J}$ (see Figure 68, here by $\widetilde{Z}$ (by $\widetilde{J}$) we denote the point Z (the point J) in the case when F coincides with Z). Hence,

$$|O\widetilde{J}| < |OJ| \ .$$

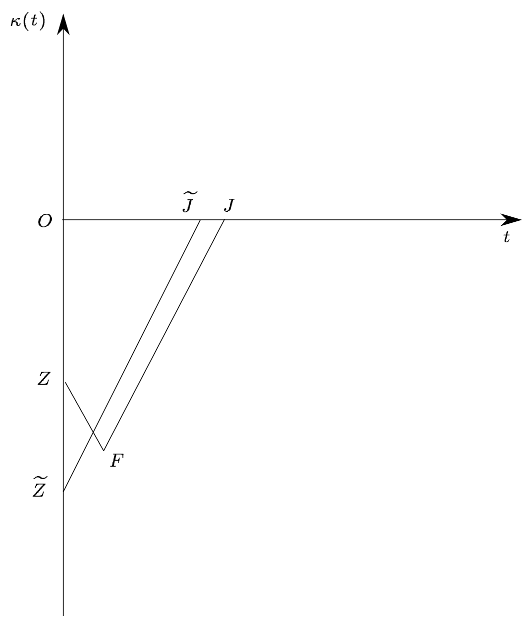


Figure 68

So, as $|\alpha_J - \alpha_Z| > \pi/2$ (see (83)), then we obtain

$$\min |\widehat{ZJ}| > |\widehat{OP}|$$

(see Figure 69).

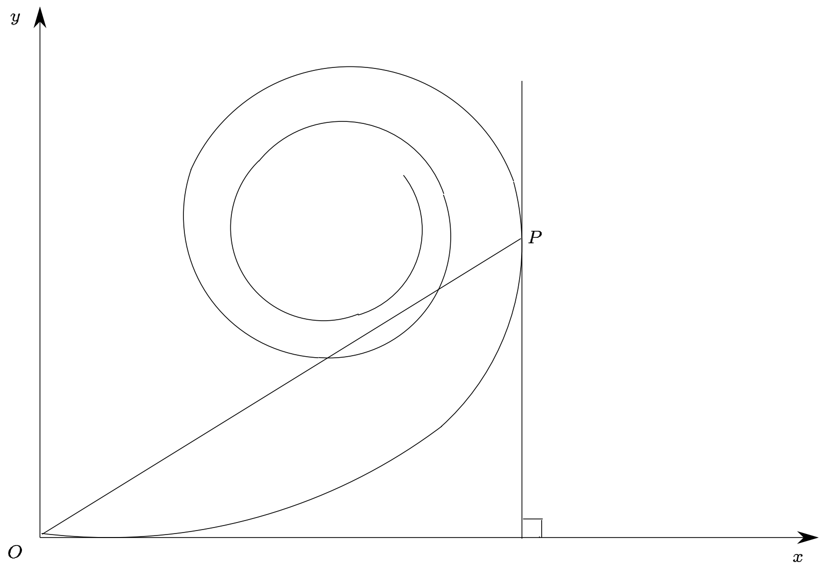


Figure 69

We have

$$|\alpha_P - \alpha_O| = \pi/2 \ ,$$

hence,

$$|\widehat{OP}| = \sqrt{|\alpha_P - \alpha_O|} = \sqrt{\pi/2} \approx 1.253314 \ .$$

So,

$$|\widehat{ZJ}| > |\widehat{OP}| > 1.25 > l_{min} \approx 1.139456743 \ ,$$

where by l_{min} we denote the minimal "useless" length (see Lemma 9.6, this minimal "useless" length is sufficient to apply the method introduced in subcase B).

Hence, in the case when the piece $\widehat{ZJ}$ doesn't intersect the tangent line at the point Z (at some point different from the point Z) the path $\mathcal{P}$ isn't optimal either.

Thus, in all cases we obtain that we can apply the method introduced in subcase B, hence, the path $\mathcal{P}$ isn't optimal.

The proposition is proved. □

E.2 Proof of Proposition 9.14

1⁰ At first we consider the case when the curvature on the interval following the arc $\widehat{DF}$ is non-positive (see Figure 66).

As the curvature doesn't change sign between the points E and V and as on the arc $\widehat{LF}$ there is no point with vertical tangent line, then in order the mean value of y on the intervals between the points F and G and between the points G and K to be equal to zero, the tangent angle between the points E and V must make a turn of at least 2π (remind that the mean value of y on any interval between two consecutive switching points equals zero and, hence, the piece of the path corresponding to any such interval intersects the axis Ox).

So, the path $\mathcal{P}$ has a "lace" between the points F and V (i.e. $\mathcal{P}$ has a "useless" length), hence, we can apply the method introduced in subcase B and, hence, the path $\mathcal{P}$ isn't optimal.

2⁰ Now we consider the case when the curvature changes sign on the interval following the arc $\widehat{DF}$ and the curvature is non-negative on the interval following the arc $\widehat{FU}$ (see Figure 67).

In this case as the position of the point J is defined by the position of the point E (the arcs $\widehat{EF}$ and $\widehat{FJ}$ are symmetric), as the curvature doesn't change sign between the points J and H and as the mean value of y on the intervals between U and S and between S and W equals zero, then, the tangent angle must make a turn of at least 2π between the points J and H, i.e. there is a "lace" between the points J and H (in both cases: the case when on the arc $\widehat{JU}$ there is at least one point with vertical tangent line and the case when on the arc $\widehat{JU}$ there is no point with vertical tangent line). Really, the path intersects the axis Ox at least three times between the points J and H (more precisely, there is at least one intersection point between J and U, between U and S and between S and W). Hence, we can apply the method introduced in subcase B and, hence, the path $\mathcal{P}$ isn't optimal.

3⁰ To study any interval belonging to the path $\mathcal{P}_{dr}$ we apply the same method and we obtain the following result: if the path $\mathcal{P}_{dr}$ has at least one interval such that the curvature doesn't change sign on this interval, then, there exists a "lace" on the path $\mathcal{P}_{dr}$. Hence, we can apply the method introduced in subcase B and, so, the path $\mathcal{P}$ isn't optimal.

The proposition is proved. □

E.3 Proof of Proposition 9.15

Denote by I (by F) the initial point (the final point) of $\mathcal{P}$ and denote by P (by R) the first (the last) point of zero curvature.

If the distance between the initial and final points of the path $\mathcal{P}$ is greater than $320\sqrt{\pi}$ and if the distance between the points P and R is at most $269\sqrt{\pi}$, then the sum of the distances between the points I and P and between the points R and F is greater than $51\sqrt{\pi}$. Hence, at least one among these distances is greater than $25.25\sqrt{\pi}$. So, it follows from Lemma 9.2 that the corresponding piece of $\mathcal{P}$ isn't optimal. Thus, the path $\mathcal{P}$ isn't optimal.

Hence, we consider the case when the distance between the points P and R is greater than $269\sqrt{\pi}$. Denote by $\mathcal{P}_d$ the part of the path $\mathcal{P}$ between the points P and R. Thus, the distance between the initial and final points of the path $\mathcal{P}_d$ is greater than $269\sqrt{\pi}$. It follows from Lemma 7.1 of [11] (one can find the same statement in [13] (Lemma 3.13) and in [10] (Lemma 3.10)) that the piece $ABCDE$ of the path $\mathcal{P}$ is situated inside the circle of radius $R = \sqrt{\pi}/2$. Hence, the maximal possible distance between the points P and R is $\sqrt{\pi}$ and, so, there exists some piece of $\mathcal{P}_d$ such that the curvature changes sign on any interval belonging to this piece and that the distance between the initial and final points of this piece is greater than $134\sqrt{\pi}$ ($(269\sqrt{\pi} - \sqrt{\pi})/2 = 134\sqrt{\pi}$). So, this piece of $\mathcal{P}_d$ corresponds to case III (see Section 10). Hence, we apply the method introduced in case III and we obtain that this piece of $\mathcal{P}_d$ isn't optimal. Thus, the path $\mathcal{P}$ isn't optimal.

The proposition is proved. □

F Appendix: Proof of Lemmas 10.2 et 10.3

F.1 Proof of Lemma 10.2

Consider some interval $[t_I, t_H] \subset [O, T]$ (see Figure 70).

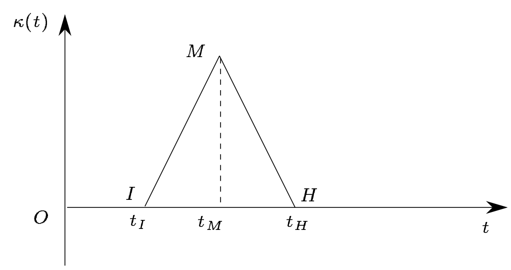


Figure 70

We want to calculate a minimal length of the interval $[t_I, t_H]$ such that there exists a lace on this interval.

Construct two arcs of half-clothoid corresponding to the graph $\kappa(t)$ of Figure 70. We can calculate the minimal interval on which there exists a lace if we consider that the points I and H coincide (see Figure 71). So the vectors $\overrightarrow{IM}$ and V_M must be perpendicular (here we denote by V_M the tangent vector at the point M, $t_M - t_I = t_H - t_M$).

In the coordinate system Ixy the vector $\overrightarrow{IM}$ has the coordinates $(\int_0^t \cos\tau^2 d\tau, \int_0^t \sin\tau^2 d\tau)$; the vector V_M has the coordinates $(\cos t^2, \sin t^2)$. Thus, we must solve the following equation

$$\Phi(t) = \cos t^2 \int_0^t \cos\tau^2 d\tau + \sin t^2 \int_0^t \sin\tau^2 d\tau = 0$$

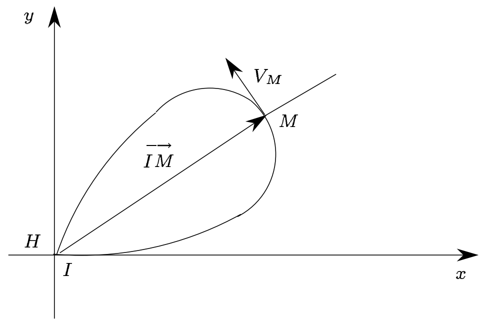


Figure 71

on the interval $[0, 2\pi]$. We solve this equation by means of MAPLE: $\Phi(\sqrt{2.297439573}) = 0$.

Calculate corresponding values of $\alpha_M - \alpha_I$ and $\alpha_H - \alpha_I$:

$$\alpha_M - \alpha_I = 2.297439573 ,$$

$$\alpha_H - \alpha_I = 2 \times 2.297439573 = 4.594879146 = 1.462595458\pi .$$

But

$$\alpha_H - \alpha_I = (t_H - t_I)^2/2 .$$

Hence,

$$t_H - t_I = \sqrt{2(\alpha_H - \alpha_I)} = \sqrt{2.925090916\pi} < \sqrt{2.926\pi} .$$

So, if $|t_H - t_I| \geq \sqrt{2.926\pi}$, then there exists a lace on this interval.

The lemma is proved. □

F.2 Proof of Lemma 10.3

We consider case III when the distance between the initial and final points of the path $\mathcal{P}$ is greater than $90.5\sqrt{\pi}$ and when there exists some interval $[t_i, t_{i+1}] \subset [O, T]$ such that $|t_{i+1} - t_i| \geq \sqrt{2.926\pi}$ (i.e. $|\alpha_{i+1} - \alpha_i| \geq 1.4626\pi$).

Remind the general ideas of the proof in subcase B of case II.

In subcase B of case II we denote by $\mathcal{P}_d$ (by $\widetilde{\mathcal{P}}$) the modified path (the constructed path) and we denote by $\mathcal{P}_{min}$ some path such that it isn't longer than the optimal one and that it satisfies the initial and final conditions (but it may not satisfy the condition of continuity of variables). We prove the inequality $|\widetilde{\mathcal{P}}| < |\mathcal{P}_d|$ in three stages:

a) at first we compare the lengths of the paths $\mathcal{P}_d$ and $\mathcal{P}_{min}$ – we obtain the following inequality:

$$|\mathcal{P}_d| - |\mathcal{P}_{min}| > 1.139456743$$

(see Lemma 9.8),

b) then we compare the lengths of the paths $\widetilde{\mathcal{P}}$ and $\mathcal{P}_{min}$ – we obtain the following inequality:

$$|\widetilde{\mathcal{P}}| - |\mathcal{P}_{min}| > 1.050758327$$

(see Lemma 9.9),

c) then we compare (using the results obtained in Lemmas 9.8, 9.9) the lengths of the paths $\widetilde{\mathcal{P}}$ and $\mathcal{P}_d$ – we obtain the desired inequality:

$$|\widetilde{\mathcal{P}}| < |\mathcal{P}_d|$$

(see Lemma 9.3).

Construct some path $\mathcal{P}_{min}$ in case III in the same way as in subcase B of case II.

In subcase B of case II the crucial point of the proof is the presence of some "useless" length of the path $\mathcal{P}_d$. More precisely, in Lemma 9.8 we use some result of Lemma 9.6 (namely, the statement that this "useless" length is at least 1.139456743) and we prove that $\mathcal{P}_d$ is longer than $\mathcal{P}_{min}$ by at least this "useless" length. As in case III we have proved that if there exists an interval $[t_i, t_{i+1}]$ such that $|t_{i+1} - t_i| \geq \sqrt{2.926\pi}$ (i.e. $|\alpha_{i+1} - \alpha_i| \geq 1.4626\pi$), then there exists a lace on this interval (see Lemma 10.2), i.e. there exists this "useless" length, so we can prove the lemma analogous to Lemma 9.8 in this case.

In case III we construct some path $\widetilde{\mathcal{P}}_1$ (see Subsection 10.3) and we can obtain the inequality

$$|\widetilde{\mathcal{P}}_1| - |\mathcal{P}_{min}| > 1.050758327$$

using the method of the proof of Lemma 9.9.

Then, using these two estimations (analogues to **a**) and **b**) for subcase B of case II), we obtain the inequality

$$|\widetilde{\mathcal{P}}_1| < |\mathcal{P}_d| .$$

Thus, we have proved the following statement: in case III if the distance between the initial and final points of the path $\mathcal{P}$ is greater than $90.5\sqrt{\pi}$ and if there exists some interval $[t_i, t_{i+1}] \subset [O, T]$ such that $|t_{i+1} - t_i| \geq \sqrt{2.926\pi}$ (i.e. $|\alpha_{i+1} - \alpha_i| \geq 1.4626\pi$), then we can shorten the given path $\mathcal{P}$.

The lemma is proved. □

G Appendix: Proof of Proposition 10.10.

To prove Proposition 10.10 we need of the following auxiliary proposition.

Proposition G.1 *As the distance between the point A (corresponding to $t = t_s$) and K (corresponding to $t = t_{q+1}$) is at least $13.4R = 6.7\sqrt{\pi}$ (see the definition of the intervals $[t_s, t_{s+1}]$, $[t_q, t_{q+1}]$ at the beginning of Subsection 10.3), then the angle between the straight lines l^* and l^{**} is smaller than 0.2954π.*

Proof

Really, the distances between the points A and D, D and E, K and G, G and F (see Figures 36 and 34) are smaller than $3R/2 = 3\sqrt{\pi}/4$ (see Proposition 5.3 from [11]). Hence, the distances between the points A and E, K and F are smaller than $3R = 3\sqrt{\pi}/2$. Thus, the points E^*, E^{**} (the points F^*, F^{**}) are in the circle with center at the point A (at the point K respectively) and with radius $3R = 3\sqrt{\pi}/2$ (see Figure 72).

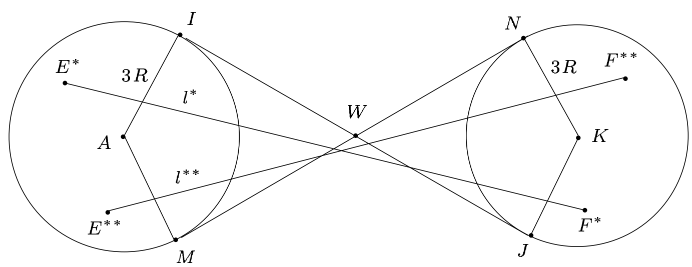


Figure 72

Consider two tangent lines of these circles (denote them by IJ and MN and denote by W their intersection point). We have

$$|AW| = |WK| \geq 6.7R \ , \quad |AI| = |AM| = |KN| = |KJ| = 3R \ , \quad |AI| = |AW| \sin \triangle AWI$$

(here we denote by $\triangle AWI$ the value of the angle AWI).

Hence,

$$\sin \triangle AWI = \frac{|AI|}{|AW|} \leq 3R/6.7R \approx 0.448$$

and

$$\triangle AWI \leq \arcsin 0.448 \approx 0.1477\pi \ , \quad \triangle IWM = 2\triangle AWI \leq 0.2954\pi \ .$$

So, the values of the angles IWM and NWJ are at most 0.2954π. Hence, the values of the angles between the straight lines l^* and l^{**} are smaller than 0.2954π.

The proposition is proved. □

Proof of Proposition 10.10

Really, recall that $|\alpha_{E^*} - \alpha_A| \in [1.4626\pi, 2\pi)$ and that on the interval $(t_s, t_{s+1}]$ the paths $\mathcal{P}_d$ and $\widetilde{\mathcal{P}}_{1new}$ are in different half-planes (with respect to the straight line a).

Hence, using the result of Proposition G.1, we obtain that for all values $|\alpha_{E^*} - \alpha_A| \in [1.4626\pi, 2\pi)$ the angle ψ belongs to $((1.4626 - 1 - 0.2954)\pi, \pi]$, i.e. $\psi \in (\pi/6, \pi]$.

The proposition is proved. □

H Appendix: The case when $\psi \in (\pi/6, \pi/2)$ – proof of some auxiliary propositions (namely, Propositions H.1 – H.6)

1^0. Comparison of the positions of the points P_{1pr}, E^{**} and F^{**} on the straight line l^{**}.

Proposition H.1 *In the case when $\psi \in (\pi/6, \pi/2)$ the points P_{1pr}, E^{**} and F^{**} are in this order on the straight line l^{**}.*

To prove Proposition H.1 we need some auxiliary propositions (namely Propositions H.4 and H.5; see the proof of Proposition H.1 at the end of $\mathbf{1^0}$).

At first we calculate a value of q (see the definition of the path C_1 at the beginning of Subsubsection 10.6.1) such that the tangent vector at the point $R \in C_1$ should be parallel to the tangent vector at the point $E^{**} \in \widetilde{\mathcal{P}}_{1new}$.

Denote the tangent vector at the point $R \in C_1$ by V_R. Recall that the point $E^{**} \in \widetilde{\mathcal{P}}_{1new}$ corresponds to the point E of its graph of the curvature. We don't know whether $q \geq t_{s+1}$, whether $\sigma \geq q$, but it isn't important for the proof.

Proposition H.2 *In order the vector V_R to be parallel to the vector $V_{E^{**}}$ we must choose $q = \sqrt{3\pi} - \sqrt{\pi/2 - \psi}$.*

Proof

At first we denote by S_{AZH} (by S_{AZR_dR}, by S_{RR_dH}) the area of AZH (of AZR_dR, of RR_dH respectively). We have the following formulas:

$$\alpha_H - \alpha_A = -S_{AZH} = -(t_s + \sqrt{3\pi} - t_s)^2/2 = -3\pi/2 \ ,$$

$$\alpha_{R_d} - \alpha_A = -S_{AZR_dR} = -(S_{AZH} - S_{RR_dH}) = -(3\pi/2 - (\sqrt{3\pi} - q)^2) \ ,$$

$$\alpha_{E^{**}} - \alpha_A = -(\pi + \psi) \ .$$

As we want to choose a value of q such that V_R should be parallel to $V_{E^{**}}$, then

$$\alpha_{R_d} - \alpha_A = \alpha_{E^{**}} - \alpha_A \ .$$

Hence,

$$\pi + \psi = 3\pi/2 - (\sqrt{3\pi} - q)^2 \ ,$$

i.e.

$$\pi/2 - \psi = (\sqrt{3\pi} - q)^2 \ ,$$

so,

$$q = \sqrt{3\pi} - \sqrt{\pi/2 - \psi} \ .$$

Thus, if we choose $q = \sqrt{3\pi} - \sqrt{\pi/2 - \psi}$, then the vector V_R is parallel to the vector $V_{E^{**}}$.

The proposition is proved. □

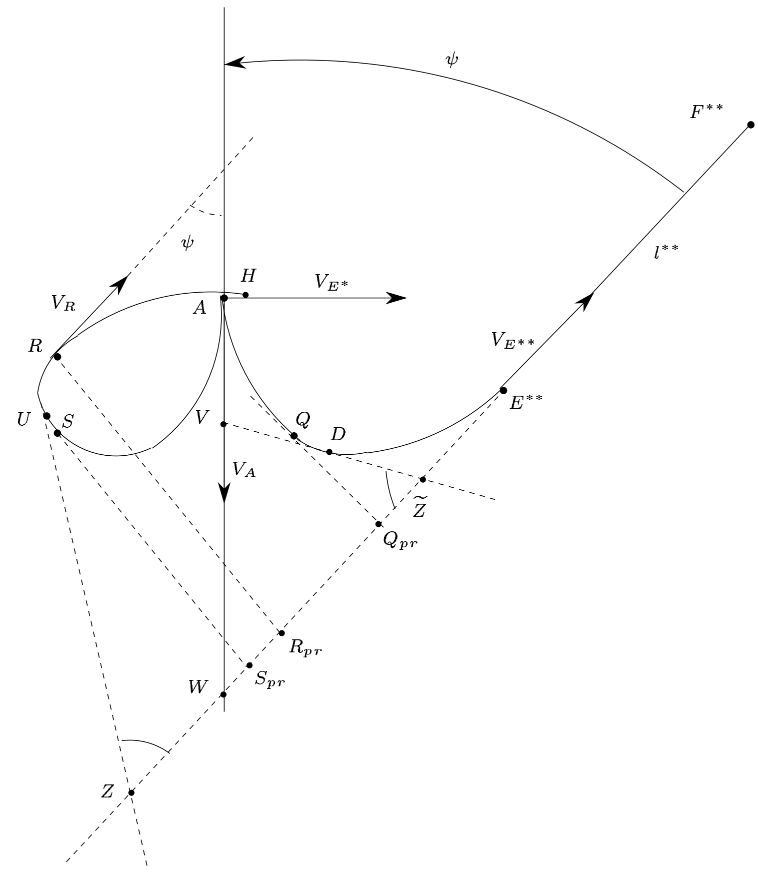


Figure 73

Remark H.3 *From now on we assume that we construct some path C_1 on $[t_s, t_s + \sqrt{3\pi}]$ such that $q = \sqrt{3\pi} - \sqrt{\pi/2 - \psi}$ (see Proposition H.2).*

Introduce now some notations. Denote by Q (by S) a point belonging to $\widetilde{\mathcal{P}}_{1new}$ (to C_1) such that its tangent line is ortogonal to l^{**}. Denote by φ the angle between the tangent vector at a point belonging to the path C_1 (or to the path $\widetilde{\mathcal{P}}_{1new}$) and the vector $V_{E^{**}}$. Denote by U the first point belonging to C_1 such that $\cos\varphi_U = \cos\varphi_D$. Denote by S_{pr} (by R_{pr}, by Q_{pr}) the projection of the point S (of the point R, of the point Q) on the line l^{**}.

Plan of the proof of Proposition H.1

We prove Proposition H.1 in three stages:

a) at first we compare the lengths of the arcs $\widehat{SR}$ and $\widehat{QE}^{**}$ – we obtain the following inequality:

$$|\widehat{SR}| < |\widehat{QE}^{**}|$$

(see Proposition H.4),

b) then we compare the lengths of the arcs $\widehat{ASR}$, $\widehat{AQE}^{**}$ and the segment $S_{pr}Q_{pr}$ – we obtain the following inequality:

$$|\widehat{AQE}^{**}| - |\widehat{ASR}| < |S_{pr}Q_{pr}|$$

(see Proposition H.5),

c) then we compare (using the results obtained in Propositions H.4 and H.5) the positions of the points P_{1pr}, E^{**} and F^{**} – we obtain the desired statement: in the case when $\psi \in (\pi/6, \pi/2)$, the points P_{1pr}, E^{**} and F^{**} are in this order on the straight line l^{**} (see Proposition H.1).

Proposition H.4 *For any* $\psi \in (\pi/6, \pi/2)$ *the following inequality holds:*

$$|\widehat{SR}| < |\widehat{QE}^{**}| \ .$$

See the proof of Proposition H.4 in Subsubsection H.1.

Proposition H.5 *For any* $\psi \in (\pi/6, \pi/2)$ *the following inequality holds:*

$$|\widehat{AQE}^{**}| - |\widehat{ASR}| < |S_{pr}Q_{pr}| \ .$$

See the proof of Proposition H.5 in Subsubsection H.2.

Now, using the results obtained in Propositions H.4 and H.5, we prove Proposition H.1.

Proof of Proposition H.1

It follows from the definition of the point $P_1 \in C_1$ that

$$|\widehat{ABP_1}| = |\widehat{AQE}^{**}| \ .$$

But

$$|\widehat{ABP_1}| = |\widehat{ASR}| + |RP_1| \ .$$

Hence,

$$|RP_1| = |\widehat{AQE}^{**}| - |\widehat{ASR}| \ .$$

It follows from Proposition H.5 that

$$|\widehat{AQE}^{**}| - |\widehat{ASR}| < |S_{pr}Q_{pr}| \ .$$

So,

$$|RP_1| < |S_{pr}Q_{pr}| \ . \tag{84}$$

It follows from Proposition H.4 that

$$|\widehat{SR}| < |\widehat{QE}^{**}| \ . \tag{85}$$

Hence, it follows from inequalities (84) and (85) that the projection of the point P_1 on the straight line l^{**} (remind that we denote it by P_{1pr}) is to the left with respect to the point E^{**}.

The proposition is proved. □

2^0 Comparison of the positions of the points P_{pr}, P_{1pr}, E^{**} and F^{**} on the straight line l^{**}.

Proposition H.6 *If the points P_{1pr}, E^{**} and F^{**} are in this order on the straight line l^{**}, then the points P_{pr}, P_{1pr}, E^{**} and F^{**} are in this order on the straight line l^{**}.*

Proof

The absolute value of the curvature of the path C_1 on $[t_s, t_s + q]$ is not smaller than the absolute value of the curvature of the path $\mathcal{P}_d$; hence, the tangent vector at any point belonging to the path C_1 rotates faster and becomes parallel to $V_{E^{**}}$ sooner than the tangent vector at the point corresponding to the same value of t and belonging to the path $\mathcal{P}_d$; the tangent vector at any point belonging to the path C_1 on $[t_s + q, t_s + \sqrt{3\pi}]$ rests parallel to $V_{E^{**}}$, so, the projection of any point of C_1 on l^{**} moves not less quickly than the projection of the point corresponding to the same value of t and belonging to the path $\mathcal{P}_d$.

The proposition is proved. □

H.1 Proof of Proposition H.4

To prove Proposition H.4 we need two auxiliary propositions (namely Propositions H.7 and H.8).

Proposition H.7 *For any $\psi \in [\pi/6, \pi/2)$ the following inequality holds:*

$$|\widehat{SU}| < |\widehat{QD}| \ .$$

Proposition H.8 *For any $\psi \in [\pi/6, \pi/2)$ the following inequality holds:*

$$|\widehat{UR}| < |\widehat{DE}^{**}| \ .$$

Proposition H.4 follows directly from Propositions H.7 and H.8.

See the proof of Proposition H.7 (Proposition H.8) in Subsubsection H.1.1 (Subsubsection H.1.2).

To prove these two propositions we need the following auxiliary proposition:

Proposition H.9 *For any $\psi \in [\pi/6, \pi/2)$ the following inequalities hold:*

$$|\kappa_Q| < |\kappa_S| \ , \quad |\kappa_D| < |\kappa_U| \ .$$

Proof

1. At first we prove the first inequality.

a) Search for the formula for κ_Q. Using the definition of the points Q and D, we obtain the following formulas:

$$\alpha_Q - \alpha_A = \pi/2 - \psi \ , \qquad \alpha_D - \alpha_A = (\pi - \psi)/2 \ .$$

Hence,

$$\alpha_Q - \alpha_A < \alpha_D - \alpha_A \ \text{ for any } \psi \in [\pi/6, \pi/2) \ .$$

Thus, $Q \in \widehat{AD}$ and Q doesn't coincide with the point D.

As $Q \in \widehat{AD}$, the following formulas hold:

$$\kappa_Q = 2(t_Q - t_A) \ , \qquad \alpha_Q - \alpha_A = (t_Q - t_A)^2 \ .$$

Hence,

$$\kappa_Q = 2\sqrt{\alpha_Q - \alpha_A} = 2\sqrt{\pi/2 - \psi} \ .$$

So, κ_Q is some decreasing function of ψ for $\psi \in [\pi/6, \pi/2)$ and

$$\max_{\psi \in [\pi/6, \pi/2)} \kappa_Q = 2\sqrt{\pi/2 - \pi/6} = 2\sqrt{\pi/3}$$

(i.e. we consider κ_Q for $\psi = \pi/6$).

b) Now we look for the formula for κ_S.

Using the definition of the point S, we obtain the following equality:

$$|\alpha_S - \alpha_A| = \pi/2 + \psi \ . \tag{86}$$

So, $|\alpha_S - \alpha_A|$ is some increasing function of ψ for $\psi \in [\pi/6, \pi/2)$. For $\psi = \pi/6$ we obtain $|\alpha_S - \alpha_A| = 2\pi/3$, for $\psi = \pi/4$ we obtain $|\alpha_S - \alpha_A| = 3\pi/4$ and for $\psi = \pi/2$ we obtain $|\alpha_S - \alpha_A| = \pi$. For the switching point Z the following formula holds (it doesn't depend on ψ):

$$|\alpha_Z - \alpha_A| = 3\pi/4 \ .$$

Hence, for $\psi \in [\pi/6, \pi/4)$ the point S belongs to the arc $\widehat{AZ}$; for $\psi = \pi/4$ the point S coincides with the point Z and for $\psi \in (\pi/4, \pi/2)$ the point S belongs to the arc $\widehat{ZH}$. Thus, $|\kappa_S - \kappa_A| = |\kappa_S|$ is some increasing function of ψ on the interval $[\pi/6, \pi/4]$ and $|\kappa_S|$ is some decreasing function of ψ on the interval $(\pi/4, \pi/2)$. So,

$$\min_{\psi\in[\pi/6,\pi/2)} |\kappa_S| = \min(\min_{\psi\in[\pi/6,\pi/4]} |\kappa_S|, \min_{\psi\in(\pi/4,\pi/2)} |\kappa_S|) \ .$$

As $|\kappa_S|$ is some increasing function of ψ on the interval $[\pi/6, \pi/4]$, we obtain that

$$\min_{\psi\in[\pi/6,\pi/4]} |\kappa_S| = |\kappa_S|\Big|_{\psi=\pi/6} \ .$$

For $\psi \in [\pi/6, \pi/4]$ the point S belongs to the arc $\widehat{AZ}$, hence, for κ_S and $|\alpha_S - \alpha_A|$ the following formulas hold:

$$|\kappa_S| = 2(t_S - t_A) \ , \qquad |\alpha_S - \alpha_A| = (t_S - t_A)^2 \ .$$

Thus, we have

$$|\kappa_S| = 2\sqrt{|\alpha_S - \alpha_A|}$$

and for $\psi = \pi/6$ we obtain

$$|\kappa_S| = 2\sqrt{2\pi/3} \ .$$

So,

$$\min_{\psi\in[\pi/6,\pi/4]} |\kappa_S| = 2\sqrt{2\pi/3} \ .$$

As $|\kappa_S|$ is some decreasing function of ψ on the interval $(\pi/4, \pi/2)$, we have

$$\min_{\psi\in(\pi/4,\pi/2)} |\kappa_S| = |\kappa_S|\Big|_{\psi=\pi/2} \ .$$

For $\psi \in (\pi/4, \pi/2)$ the point S belongs to the arc $\widehat{ZH}$, so, for κ_S and $|\alpha_S - \alpha_A|$ we have the following formulas:

$$\kappa_S = 2(t_S - t_A - \sqrt{3\pi}) \ , \qquad |\alpha_S - \alpha_A| = 3\pi/2 - (t_S - t_A - \sqrt{3\pi})^2 \ .$$

Hence,

$$|t_S - t_A - \sqrt{3\pi}| = \sqrt{3\pi/2 - |\alpha_S - \alpha_A|} = \sqrt{\pi - \psi}$$

(see formula (86)) and

$$|\kappa_S| = 2\sqrt{\pi - \psi} \ .$$

Thus, for $\psi = \pi/2$ we obtain

$$|\kappa_S| = \sqrt{2\pi} \ .$$

So,

$$\min_{\psi \in (\pi/4, \pi/2)} |\kappa_S| = \sqrt{2\pi} \ .$$

Hence,

$$\min_{\psi \in [\pi/6, \pi/2)} |\kappa_S| = \min(\min_{\psi \in [\pi/6, \pi/4]} |\kappa_S|, \min_{\psi \in (\pi/4, \pi/2)} |\kappa_S|) =$$

$$= \min(2\sqrt{2\pi/3}, \sqrt{2\pi}) = \sqrt{2\pi} \ .$$

c) Thus, we have obtained the following equalities:

$$\max_{\psi \in [\pi/6, \pi/2)} \kappa_Q = 2\sqrt{\pi/3} \ ,$$

$$\min_{\psi \in [\pi/6, \pi/2)} |\kappa_S| = \sqrt{2\pi} \ .$$

But

$$2\sqrt{\pi/3} < \sqrt{2\pi} \ .$$

Hence, for any $\psi \in [\pi/6, \pi/2)$ the following inequality holds:

$$|\kappa_Q| < |\kappa_S| \ .$$

Thus, the first inequality of the proposition is proved.

2. Now we prove the second inequality.

a) Look for the formula for κ_D. Using the definition of the point D, we obtain the following formulas:

$$\alpha_D - \alpha_A = (\pi - \psi)/2 \ , \quad \kappa_D - \kappa_A = \kappa_D = 2\sqrt{\alpha_D - \alpha_A} = 2\sqrt{(\pi - \psi)/2} = \sqrt{2(\pi - \psi)} \ .$$

So, κ_D is some decreasing function of ψ for $\psi \in [\pi/6, \pi/2)$.

b) Now we look for the formula for κ_U. Denote by $\triangleleft W\widetilde{Z}V$ (by $\triangleleft \widetilde{Z}WV$, by $\triangleleft WV\widetilde{Z}$, by $\triangleleft WZU$) the absolute value of the angle $W\widetilde{Z}V$ (of the angle $\widetilde{Z}WV$, of the angle $WV\widetilde{Z}$, of the angle WZU respectively). We have the following equalities:

$$\triangleleft W\widetilde{Z}V = \pi - \triangleleft\widetilde{Z}WV - \triangleleft WV\widetilde{Z} \ , \quad \triangleleft \widetilde{Z}WV = \psi \ , \quad \triangleleft WV\widetilde{Z} = (\pi - \psi)/2 \ .$$

Hence,

$$\triangleleft W\widetilde{Z}V = \pi - \psi - (\pi - \psi)/2 = (\pi - \psi)/2 \ .$$

Using the definition of the point U, we obtain the following formula:

$$\lhd WZU = \lhd W\widetilde{Z}V = (\pi - \psi)/2 \ .$$

Hence,

$$|\alpha_U - \alpha_A| = \pi - (\lhd WZU - \psi) = \pi - ((\pi - \psi)/2 - \psi) \ ,$$

so,

$$|\alpha_U - \alpha_A| = \pi/2 + 3\psi/2 \ . \tag{87}$$

Thus, $|\alpha_U - \alpha_A|$ is some increasing function of ψ for $\psi \in [\pi/6, \pi/2)$. For $\psi = \pi/6$ we obtain $|\alpha_U - \alpha_A| = \pi/2 + \pi/4 = 3\pi/4$. For the switching point Z we have the following formula (it doesn't depend on ψ):

$$|\alpha_Z - \alpha_A| = 3\pi/4 \ .$$

Hence, for $\psi = \pi/6$ the point U coincides with the point Z and the point U belongs to the arc $\widehat{ZH}$ for $\psi \in (\pi/6, \pi/2)$.

Represent now $|\kappa_U|$ as function of ψ on the interval $[\pi/6, \pi/2)$.

For $\psi \in [\pi/6, \pi/2)$ the point U belongs to the arc $\widehat{ZH}$, thus for κ_U and $|\alpha_U - \alpha_A|$ we have the following formulas:

$$|\kappa_U| = 2(t_U - t_A - \sqrt{3\pi}) \ , \qquad |\alpha_U - \alpha_A| = 3\pi/2 - (t_U - t_A - \sqrt{3\pi})^2 \ .$$

Hence,

$$|t_U - t_A - \sqrt{3\pi}| = \sqrt{3\pi/2 - |\alpha_U - \alpha_A|} = \sqrt{\pi - 3\psi/2}$$

(see the formula (87)) and

$$|\kappa_U| = 2\sqrt{\pi - 3\psi/2} \ .$$

Thus, we have obtained that $|\kappa_D|$ and $|\kappa_U|$ are two decreasing functions of ψ for $\psi \in [\pi/6, \pi/2)$:

$$|\kappa_D| = \sqrt{2(\pi - \psi)} \ , \tag{88}$$

$$|\kappa_U| = 2\sqrt{\pi - 3\psi/2} \ . \tag{89}$$

For $\psi = \pi/2$ we obtain (it follows from the formulas (88), (89)):

$$|\kappa_D| = \sqrt{\pi} \ , \qquad |\kappa_U| = \sqrt{\pi} \ .$$

Calculate $d|\kappa_D|/d\psi$ and $d|\kappa_U|/d\psi$:

$$\frac{d|\kappa_D|}{d\psi} = -\frac{1}{\sqrt{2(\pi - \psi)}} \ , \quad \frac{d|\kappa_U|}{d\psi} = -\frac{3}{2\sqrt{\pi - 3\psi/2}} \ .$$

So, we obtain

$$\left|\frac{d|\kappa_U|}{d\psi}\right| > \left|\frac{d|\kappa_D|}{d\psi}\right|$$

(because $\sqrt{2(\pi-\psi)} > \frac{2}{3}\sqrt{\pi - 3\psi/2}$ for $\psi \in [\pi/6, \pi/2)$).

Hence,

$$|\kappa_U| > |\kappa_D| \quad \text{for } \psi \in [\pi/6, \pi/2) \ .$$

So, the second inequality of the proposition is proved.

The proposition is proved. □

H.1.1 Proof of Proposition H.7

1. At the beginning of the proof of Proposition H.9 we have obtained the following result: the point Q belongs to the arc $\widehat{AD}$ and doesn't coincide with the point D.

2. Now we study all possible positions of the point S for any $\psi \in [\pi/6, \pi/2)$. Remind the formula (86):

$$|\alpha_S - \alpha_A| = \pi/2 + \psi \ .$$

Thus, the function $|\alpha_S - \alpha_A|$ is some increasing function of ψ. For $\psi = \pi/6$ we obtain $|\alpha_S - \alpha_A| = 2\pi/3$, for $\psi = \pi/4$ we obtain $|\alpha_S - \alpha_A| = 3\pi/4$ and for $\psi = \pi/2$ we obtain $|\alpha_S - \alpha_A| = \pi$. Remind that for the switching point Z we have $|\alpha_Z - \alpha_A| = 3\pi/4$.

So, we can make the following comclusion: for $\psi \in [\pi/6, \pi/4)$ the point S belongs to the arc $\widehat{AZ}$, for $\psi \in (\pi/4, \pi/2)$ the point S belongs to the arc $\widehat{ZH}$, for $\psi = \pi/4$ the point S coincides with the switching point Z.

3. Now we study all possible positions of the point U for any $\psi \in [\pi/6, \pi/2)$. Remind the formula (87):

$$|\alpha_U - \alpha_A| = \pi/2 + 3\psi/2 \ .$$

Thus, the function $|\alpha_U - \alpha_A|$ is some increasing function of ψ. For $\psi = \pi/6$ we obtain $|\alpha_U - \alpha_A| = 3\pi/4$.

So we can make the following conclusion: for $\psi \in (\pi/6, \pi/2)$ the point U belongs to the arc $\widehat{ZH}$ and for $\psi = \pi/6$ the point U coincides with the switching point Z.

4. So, after the studying of all possible positions of the points S and U belonging to the arc $\widehat{AH} \subset C_0$ and after the studying of all possible positions of the points Q and D belonging to the arc $\widehat{AE}^{**} \subset \widetilde{\mathcal{P}}_{1new}$ we can describe all possible relative positions of the points S, U, Z, Q and D on the graph of the absolute value of the curvature $|\kappa(t)|$ on $[t_s, t_s + \sqrt{3\pi}]$ (see Figure 74).

The following equality holds:

$$|\alpha_U - \alpha_S| = |\alpha_D - \alpha_Q| \tag{90}$$

(see the definition of the points U, S, D and Q), i.e. the area of $QQ'D'D$ is equal to the area of $SS'U'U$ (see Figure 74).

It follows from Proposition H.9 that

$$|\kappa_Q| < |\kappa_S| \ , \quad |\kappa_D| < |\kappa_U| \quad \text{for } \psi \in [\pi/6, \pi/2) \ . \tag{91}$$

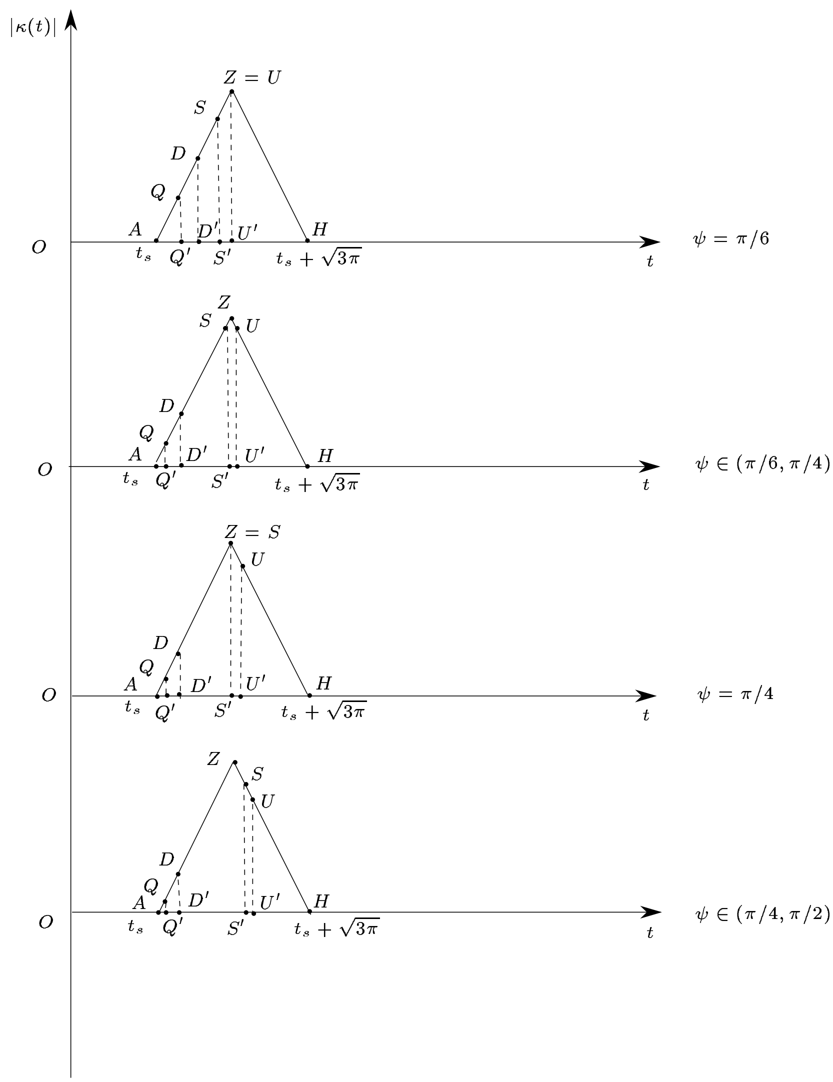

$|\kappa(t)|$
$Z = U$
S
D
Q
A
D'
U'
H
O
t_s
Q'
S'
$t_s + \sqrt{3\pi}$
t
$\psi = \pi/6$
Z
S
U
D
Q
A
D'
U'
H
O
t_s
Q'
S'
$t_s + \sqrt{3\pi}$
t
$\psi \in (\pi/6, \pi/4)$
$Z = S$
U
D
Q
A
D'
U'
H
O
t_s
Q'
S'
$t_s + \sqrt{3\pi}$
t
$\psi = \pi/4$
Z
S
U
D
Q
A
D'
U'
H
O
t_s
Q'
S'
$t_s + \sqrt{3\pi}$
t
$\psi \in (\pi/4, \pi/2)$

Figure 74

Hence, from (90) and (91) we obtain the desired result:

$$|\widehat{SU}| < |\widehat{QD}| \quad \text{for } \psi \in [\pi/6, \pi/2) \ .$$

The proposition is proved. □

H.1.2 Proof of Proposition H.8

1. At first we study all possible positions of the point R for any $\psi \in [\pi/6, \pi/2)$.

Using the definition of the point R, we obtain the following formula:

$$|\alpha_R - \alpha_A| = \pi + \psi \ .$$

So, $|\alpha_R - \alpha_A|$ is some increasing function of ψ for $\psi \in [\pi/6, \pi/2)$. For the switching point Z we have the following formula (it doesn't depend on ψ):

$$|\alpha_Z - \alpha_A| = 3\pi/4 \ .$$

Hence, for $\psi \in [\pi/6, \pi/2)$ the point R belongs to the arc $\widehat{ZH}$ and the point R don't coincide with the point Z.

2. We study all possible positions of the point U for any $\psi \in [\pi/6, \pi/2)$ as in the proof of Proposition H.7 and we obtain the following result: for $\psi \in (\pi/6, \pi/2)$ the point U belongs to the arc $\widehat{ZH}$ and for $\psi = \pi/6$ the point U coincides with the switching point Z.

3. So, after the studying of all possible positions of the points R and U belonging to the arc $\widehat{AH} \subset C_0$ we can describe all possible relative positions of the points U, Z, R, E and D on the graph of the absolute value of the curvature $|\kappa(t)|$ on $[t_s, t_s + \sqrt{3\pi}]$ (see Figure 75).

We have the following equality:

$$|\alpha_R - \alpha_U| = |\alpha_{E^{**}} - \alpha_D| \tag{92}$$

(see the definition of the points R, U, D and E^{**}), i.e. the area of $UU'R'R$ is equal to the area of $DD'E$ (see Figure 75).

It follows from Proposition H.9 that

$$|\kappa_D| < |\kappa_U| \quad \text{for } \psi \in [\pi/6, \pi/2) \ . \tag{93}$$

It follows from the definition of the point R that

$$|\kappa_R| > |\kappa_E| = 0 \quad \text{for } \psi \in [\pi/6, \pi/2) \ . \tag{94}$$

Hence, from (92)–(94) we obtain the desired result:

$$|\widehat{UR}| < |\widehat{DE}^{**}| \quad \text{for } \psi \in [\pi/6, \pi/2) \ .$$

The proposition is proved. □

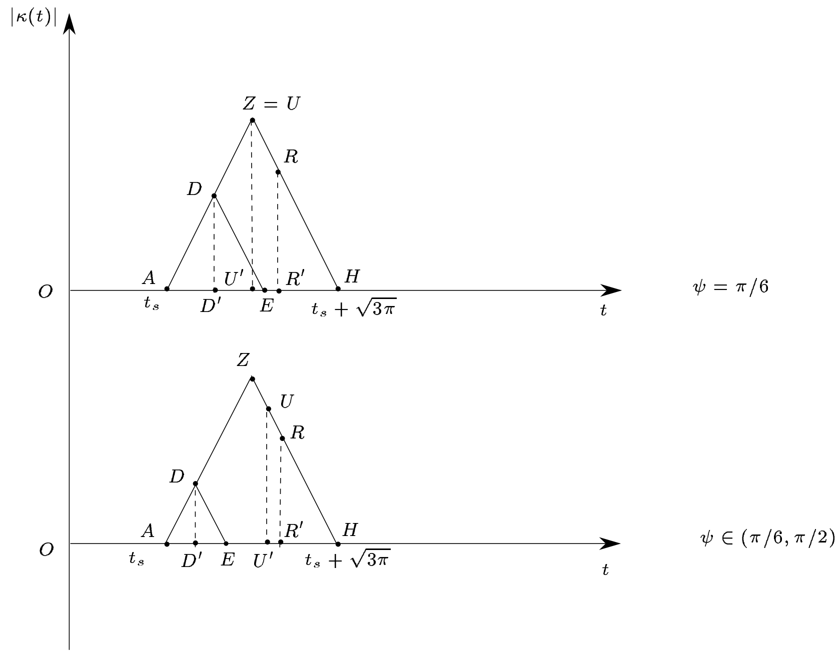

$|\kappa(t)|$
$Z = U$
R
D
O
A
U'
R'
H
t_s
D'
E
$t_s + \sqrt{3\pi}$
t
$\psi = \pi/6$
Z
U
R
D
O
A
R'
H
t_s
D'
E
U'
$t_s + \sqrt{3\pi}$
t
$\psi \in (\pi/6, \pi/2)$

Figure 75

H.2 Proof of Proposition H.5

To prove Proposition H.5 we need some auxiliary propositions (namely Propositions H.10 and H.11).

Proposition H.10 *For any $\psi \in [\pi/6, \pi/2)$ the following equalities hold:*

$$|\widehat{AE}^{**}| = \sqrt{2(\pi - \psi)} \ , \tag{95}$$

$$|\widehat{ASR}| = \sqrt{3\pi} - \sqrt{\pi/2 - \psi} \ . \tag{96}$$

Proposition H.11 *For any $\psi \in [\pi/6, \pi/2)$ the following inequality holds:*

$$|S_{pr}Q_{pr}| > 1/\sqrt{3\pi} \ .$$

See the proof of Proposition H.10 (Proposition H.11) in Subsubsection H.2.1 (in Subsubsection H.2.2).

Proof of Proposition H.5

It follows from Proposition H.10 that the equalities (95), (96) hold:

$$|\widehat{AQE}^{**}| = \sqrt{2(\pi - \psi)} \ , \qquad |\widehat{ASR}| = \sqrt{3\pi} - \sqrt{\pi/2 - \psi} \ .$$

So,

$$|\widehat{AQE}^{**}| - |\widehat{ASR}| = \sqrt{2(\pi - \psi)} - \sqrt{3\pi} + \sqrt{\pi/2 - \psi} \ . \tag{97}$$

It follows from Proposition H.11 that for $|S_{pr}Q_{pr}|$ the following inequality holds:

$$|S_{pr}Q_{pr}| > 1/\sqrt{3\pi} \ . \tag{98}$$

At first we prove inequality (99):

$$\sqrt{2(\pi - \psi)} - \sqrt{3\pi} + \sqrt{\pi/2 - \psi} < 1/\sqrt{3\pi} \ . \tag{99}$$

Really,

$$\sqrt{2(\pi - \psi)} + \sqrt{\pi/2 - \psi}$$

is some decreasing function of ψ for $\psi \in [\pi/6, \pi/2)$. Hence,

$$\max_{\psi \in [\pi/6, \pi/2)} \left(\sqrt{2(\pi - \psi)} + \sqrt{\pi/2 - \psi} \right) = \sqrt{2(\pi - \pi/6)} +$$

$$+ \sqrt{\pi/2 - \pi/6} = \sqrt{5\pi/3} + \sqrt{\pi/3}$$

(i.e. we consider $\psi = \pi/6$).

Thus, we must prove that

$$\sqrt{5\pi/3} - \sqrt{3\pi} + \sqrt{\pi/3} < 1/\sqrt{3\pi} \ ,$$

i.e.

$$\sqrt{5}\pi - 3\pi + \pi < 1 \ ,$$

i.e.

$$\sqrt{5}\pi < 2\pi + 1 \ ,$$

i.e.

$$\pi^2 < 4\pi + 1$$

(this inequality is correct).

Hence, we have proved inequality (99).

So, using (97)–(99), we obtain the desired inequality:

$$|\widehat{AQE}^{**}| - |\widehat{ASR}| < 1/\sqrt{3\pi} < |S_{pr}Q_{pr}| \ .$$

The proposition is proved. □

H.2.1 Proof of Proposition H.10

1. At first we prove equality (95). It follows from the definition of the point E^{**} that we have $|\widehat{AE}^{**}| = \sigma$ (see Figure 38). But

$$\alpha_{E^{**}} - \alpha_A = \sigma^2/2$$

(see Figure 38) and

$$\alpha_{E^{**}} - \alpha_A = \pi - \psi$$

(see Figure 73).

Hence,

$$|\widehat{AE}^{**}| = \sigma = \sqrt{2(\pi - \psi)} \ .$$

2. Now we prove equality (96). The following equality holds (see Figure 73):

$$|\widehat{ASR}| = |\widehat{ASH}| - |\widehat{RH}| \ . \tag{100}$$

For the arc $\widehat{ASH}$ we have the following formula:

$$|\widehat{ASH}| = \sqrt{3\pi} \ . \tag{101}$$

It follows from the definition of the point R that for $|\alpha_H - \alpha_R|$ the following equality holds:

$$|\alpha_H - \alpha_R| = \pi/2 - \psi \ .$$

So, $|\alpha_H - \alpha_R|$ is some decreasing function of ψ for $\psi \in [\pi/6, \pi/2)$. Hence,

$$\max_{\psi \in [\pi/6, \pi/2)} |\alpha_H - \alpha_R| = |\alpha_H - \alpha_R| \Big|_{\psi = \pi/6} .$$

We obtain

$$\max_{\psi \in [\pi/6, \pi/2)} |\alpha_H - \alpha_R| = \pi/2 - \pi/6 = \pi/3 .$$

Hence, the point R belongs to the arc $\widehat{ZH}$ for any $\psi \in [\pi/6, \pi/2)$ (remind that $|\alpha_H - \alpha_Z| = 3\pi/4$).

As $R \in \widehat{ZH}$ we obtain

$$|\alpha_H - \alpha_R| = |t_H - t_R|^2 .$$

So, we obtain the equalities

$$|\widehat{RH}| = |t_H - t_R| = \sqrt{|\alpha_H - \alpha_R|} = \sqrt{\pi/2 - \psi} . \tag{102}$$

Using formulas (100)–(102), we obtain equality (96):

$$|\widehat{ASR}| = |\widehat{ASH}| - |\widehat{RH}| = \sqrt{3\pi} - \sqrt{\pi/2 - \psi} .$$

The proposition is proved. □

H.2.2 Proof of Proposition H.11

Consider the piece of the path C_1 from the point A to the point S and the piece of the path $\widetilde{\mathcal{P}}_{1new}$ from the point A to the point Q (see Figure 76). Denote by Q_1 the point of intersection of the straight line passing through the points Q, Q_{pr} and of the path C_1. Denote by δ the angle between the vector Q_1Q and the tangent vector at the point Q_1.

Denote by Y the point belonging to the arc $\widehat{AS}$ whose tangent line is parallel to the straight line passing through the points W, E^{**} (see Figure 76). Denote by Y_{pr} the projection of the point Y on this line.

Now we prove the following auxiliary proposition:

Proposition H.12 *For the absolute value of the angle δ the following inequality holds:*

$$|\delta| \leq \pi/2 .$$

Proof

Using the definition of the point Q we obtain the following formula:

$$\alpha_Q - \alpha_A = \pi/2 - \psi .$$

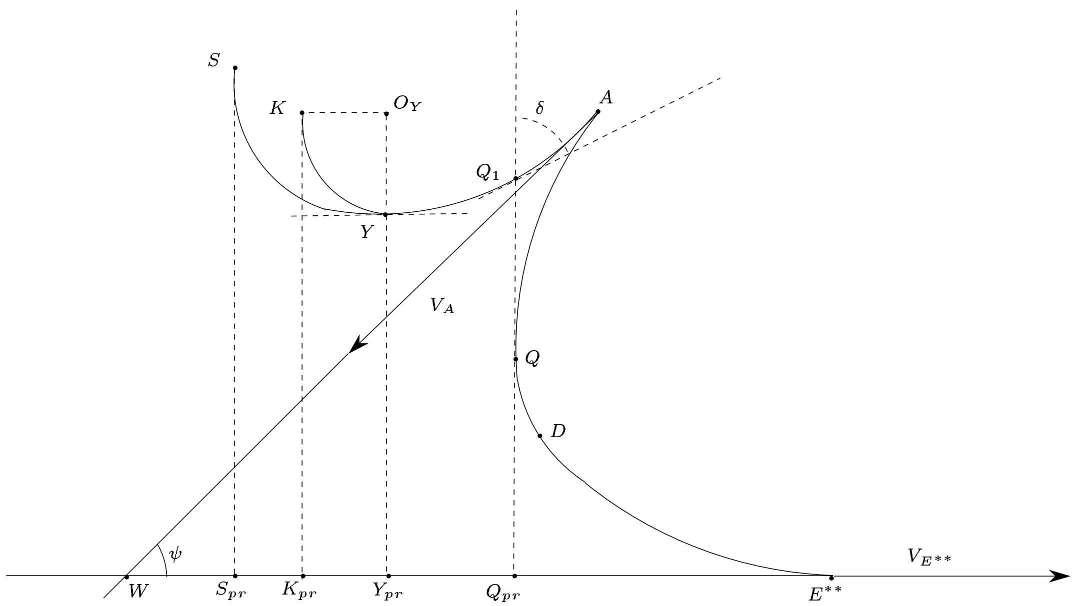


Figure 76

So, $\alpha_Q - \alpha_A$ is some decreasing function of ψ for $\psi \in [\pi/6, \pi/2)$. Hence, $|\widehat{AQ}|$, $|\widehat{AQ_1}|$ and $|\delta|$ are decreasing functions of ψ for $\psi \in [\pi/6, \pi/2)$ and

$$\max_{\psi \in [\pi/6, \pi/2)} |\delta| = |\delta| \Big|_{\psi = \pi/6} .$$

So we consider $\psi = \pi/6$.

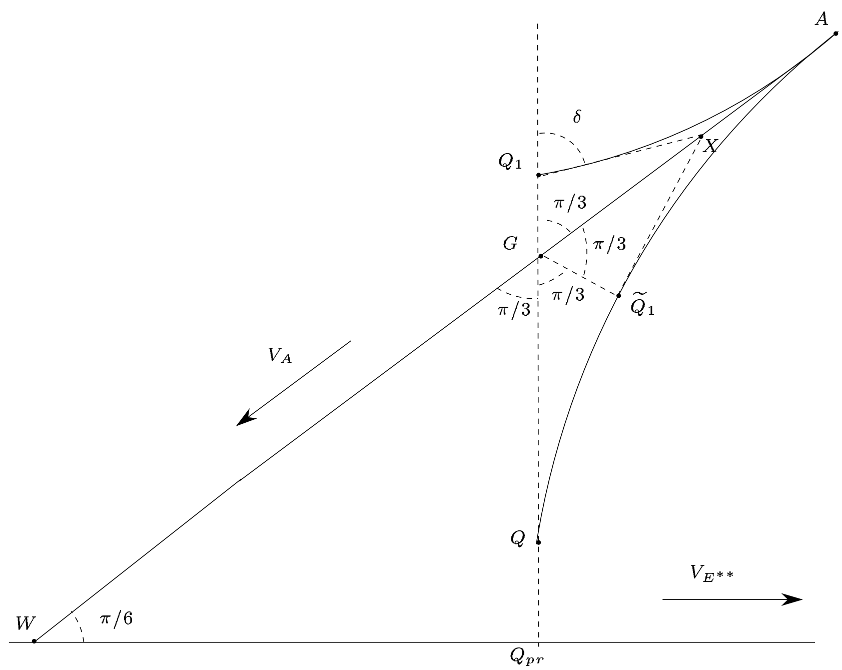


Figure 77

The angle AWQ_{pr} is equal to $\psi = \pi/6$ (see Figure 77). Hence, the angle WGQ (and the angle Q_1GA) equals $\pi/2 - \pi/6 = \pi/3$ (see Figure 77). Denote by $\widetilde{Q}_1$ the point belonging to the arc $\widehat{AQ}$ and which is symmetric to the point Q_1 with respect to the straight line passing through the points A, W. Denote by X the point of intersection of the tangent line at the point Q_1 and of the straight line passing through the points A, W (see Figure 77).

Thus, $|\widehat{AQ_1}| = |\widehat{A\widetilde{Q}_1}|$, the angle $X\widetilde{Q}_1G$ is equal to the angle XQ_1G and it equals $\pi - \delta$.

So, we must prove that the absolute value of the angle $X\widetilde{Q}_1G$ is at least $\pi/2$ (denote this absolute value by $\lhd X\widetilde{Q}_1G$).

Suppose that

$$\lhd X\widetilde{Q}_1G < \pi/2 . \tag{103}$$

Construct the arc symmetric to the arc $A\widetilde{Q}_1$ with respect to the straight line passing through the points G, $\widetilde{Q}_1$ (denote this arc by $\widetilde{Q}_1\widetilde{A}$, see Figure 78). As $\sphericalangle X\widetilde{Q}_1G < \pi/2$ (it follows from the supposition (103)), then the point $\widetilde{A}$ belongs to the straight line passing through the points G, Q_{pr}.

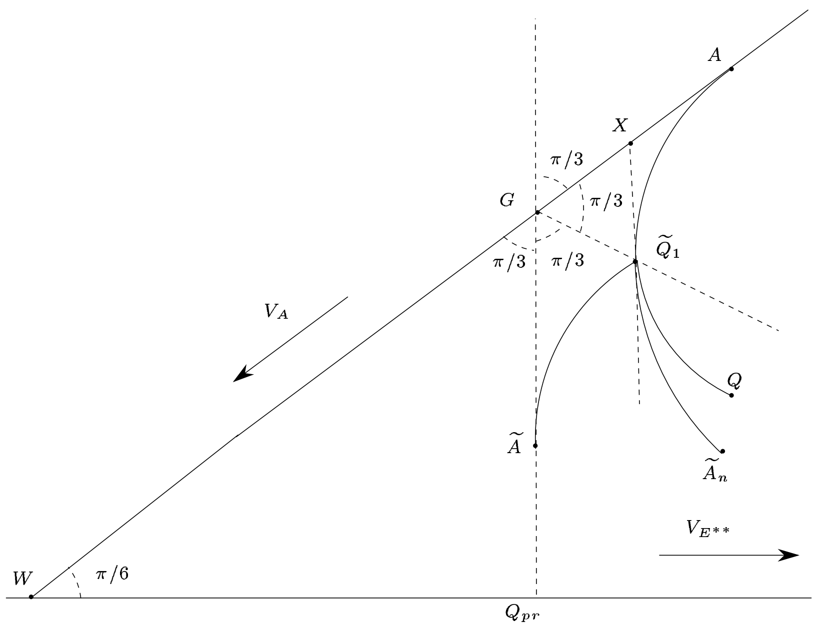


Figure 78

Now rotate the arc $\widetilde{Q}_1\widetilde{A}$ around the point $\widetilde{Q}_1$ anticlockwise till the moment when the tangent line of the arc $\widetilde{Q}_1\widetilde{A}$ at the point $\widetilde{Q}_1$ coincides with the tangent line of the arc $A\widetilde{Q}_1$ at the same point. Denote this arc by $\widetilde{Q}_1\widetilde{A}_n$ (see Figure 78).

On the arc $\widetilde{Q}_1Q$ the curvature $\kappa(t)$ is an increasing function, on the arc $\widetilde{Q}_1\widetilde{A}_n$ it is a decreasing function (see the graphs of the curvature of the arcs $A\widetilde{Q}_1$, $\widetilde{Q}_1\widetilde{A}_n$ and AQ on Figure 79). Hence, the angle $\alpha_Q - \alpha_A > \alpha_{\widetilde{A}_n} - \alpha_A$ and the point Q cannot belong to the straight line passing through the points G, Q_{pr} (see Figure 78) – a contradiction.

So, the supposition (103) isn't correct. Hence,

$$|\delta| = \pi - \sphericalangle X\widetilde{Q}_1G \leq \pi/2 \ .$$

The proposition is proved. □

Now we prove Proposition H.11.

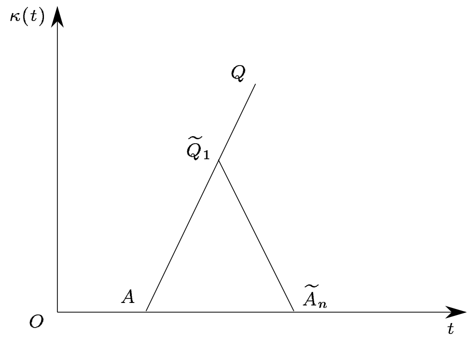


Figure 79

Proof of Proposition H.11

The absolute value of the angle δ is at most $\pi/2$ (see Proposition H.12). Hence, $Y \in \widehat{Q_1S}$ (remind that we denote by Y the point belonging to the arc $\widehat{AS}$ and whose tangent line is parallel to the straight line passing through the points W, E^{**} (see Figure 76)).

The maximal value of the curvature $\kappa(t)$ on the arc $\widehat{ARH}$ (see Figure 73) equals $\sqrt{3\pi}$. Hence, the maximal value of $\kappa(t)$ on the arc $\widehat{Q_1S}$ is at most $\sqrt{3\pi}$.

Consider the tangent circle at the point Y of radius $1/\sqrt{3\pi}$. Denote this circle by $\mathcal{C}_Y$, its centre by O_Y and by K the point belonging to $\mathcal{C}_Y$ the tangent line at which is perpendicular to the straight line passing through the points W, E^{**} ($\kappa(t) < 0$ on the arc $\widehat{YK}$) and denote by K_{pr} the projection of the point K on this straight line.

The tangent vectors at the points K and S are collinear, the curvature at any point of the arc $\widehat{YS}$ is at most $\sqrt{3\pi}$ and the curvature at any point of the arc $\widehat{YK}$ equals $\sqrt{3\pi}$. Hence,

$$|\widehat{YK}| < |\widehat{YS}| \ , \qquad |K_{pr}Y_{pr}| < |S_{pr}Y_{pr}| \ . \tag{104}$$

But $Y \in \widehat{Q_1S}$, so

$$|S_{pr}Y_{pr}| < |S_{pr}Q_{pr}| \ . \tag{105}$$

Using inequalities (104) and (105), we obtain the desired inequality:

$$|S_{pr}Q_{pr}| > |S_{pr}Y_{pr}| > |K_{pr}Y_{pr}| = 1/\sqrt{3\pi} \ .$$

The proposition is proved. □

I Appendix: The case when $\psi \in [\pi/2, 5\pi/6]$ – proof of Propositions 10.14 and 10.15

I.1 Proof of Proposition 10.14

Remind that it follows from Remark 10.4 that in this section we consider only paths $\mathcal{P}_d$ consisting from intervals whose lengths are smaller than $\sqrt{2.926\pi}$.

Hence, $|\widehat{AC}| < \sqrt{3\pi}$ and

$$|\widehat{AB}| < \sqrt{3\pi}/2 = \sqrt{3\pi/4} \ . \qquad (106)$$

Using the definition of the points S_1 and S_2, we obtain

$$|\alpha_{S_2} - \alpha_A| > |\alpha_{S_2} - \alpha_{S_1}| = \pi \ .$$

If the point S_2 belongs to the arc $\widehat{AB}$, then

$$|\widehat{AB}| \geq |\widehat{AS_2}| = \sqrt{|\alpha_{S_2} - \alpha_A|} > \sqrt{\pi} \ ,$$

this is a contradiction with (106). Hence, the point S_2 belongs to the arc $\widehat{BC}$.

The proposition is proved. □

I.2 Proof of Proposition 10.15

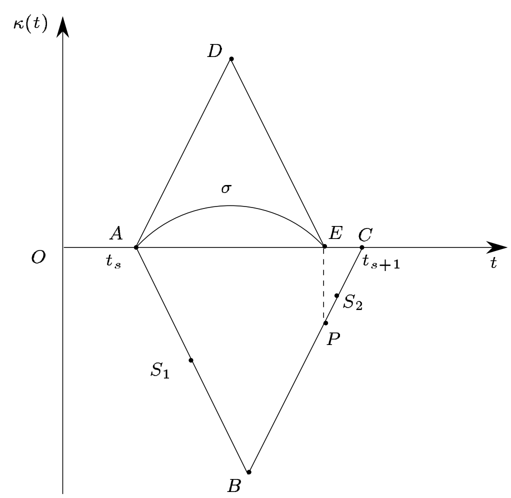


Figure 80

It follows from the definition of the point S_1 that the following equality holds:

$$|\alpha_{S_1} - \alpha_A| = \psi - \pi/2 \ .$$

But we can represent $|\alpha_{S_1} - \alpha_A|$ as a function of $|\widehat{AS}_1|$ (see Figure 80):

$$|\alpha_{S_1} - \alpha_A| = |\widehat{AS}_1|^2 \ .$$

Hence,

$$|\widehat{AS}_1| = \sqrt{|\alpha_{S_1} - \alpha_A|} = \sqrt{\psi - \pi/2} \ . \tag{107}$$

It follows from the definition of the point S_2 that the following equality holds:

$$|\alpha_{S_2} - \alpha_A| = \psi + \pi/2 \ .$$

But, using the result of Proposition 10.14, we can also represent $|\alpha_{S_2} - \alpha_A|$ as a function of $|\widehat{AS}_2|$ (see Figure 80):

$$|\alpha_{S_2} - \alpha_A| = \frac{1}{2}|\widehat{AC}|^2 - (|\widehat{AC}| - |\widehat{AS}_2|)^2 \ .$$

Hence,

$$\frac{1}{2}|\widehat{AC}|^2 - (|\widehat{AC}| - |\widehat{AS}_2|)^2 = \psi + \pi/2 \ ,$$

i.e.

$$|\widehat{AS}_2| = |\widehat{AC}| - \sqrt{\frac{1}{2}|\widehat{AC}|^2 - (\psi + \pi/2)} \ . \tag{108}$$

It follows from the definition of the point P that the following equality holds:

$$|\widehat{AP}| = |\widehat{ADE}^{**}| \ ,$$

but

$$|\alpha_{E^{**}} - \alpha_A| = \frac{1}{2}|\widehat{ADE}^{**}|^2 = \pi - \psi \ ,$$

hence,

$$|\widehat{AP}| = \sqrt{2(\pi - \psi)} \ . \tag{109}$$

It follows from (107) and (108) that $|\widehat{AS}_1|$ and $|\widehat{AS}_2|$ are increasing functions of ψ. It follows from (109) that $|\widehat{AP}|$ is a decreasing function of ψ. Hence, if

$$\min_{\psi\in[\pi/2,5\pi/6]} |\widehat{AS}_2| > \max_{\psi\in[\pi/2,5\pi/6]} |\widehat{AP}| \tag{110}$$

and

$$\max_{\psi\in[\pi/2,5\pi/6]} |\widehat{AS}_1| \leq \min_{\psi\in[\pi/2,5\pi/6]} |\widehat{AP}| \ , \tag{111}$$

then

$$|\widehat{AS}_1| \le |\widehat{AP}| < |\widehat{AS}_2| \quad \text{for } \psi \in [\pi/2, 5\pi/6] \ ,$$

so, the proposition is proved.

As $|\widehat{AS}_2|$ is an increasing function of ψ on the interval $[\pi/2, 5\pi/6]$, then

$$\min_{\psi\in[\pi/2,5\pi/6]} |\widehat{AS}_2| = |\widehat{AS}_2|\Big|_{\psi=\pi/2} \ ,$$

i.e.

$$\min_{\psi\in[\pi/2,5\pi/6]} |\widehat{AS}_2| = |\widehat{AC}| - \sqrt{\frac{1}{2}|\widehat{AC}|^2 - \pi} \ .$$

As $|\widehat{AP}|$ is a decreasing function of ψ on the interval $[\pi/2, 5\pi/6]$, then

$$\max_{\psi\in[\pi/2,5\pi/6]} |\widehat{AP}| = |\widehat{AP}|\Big|_{\psi=\pi/2} \ ,$$

i.e.

$$\max_{\psi\in[\pi/2,5\pi/6]} |\widehat{AP}| = \sqrt{\pi} \ .$$

Thus, to prove inequality (110) we must prove that

$$|\widehat{AC}| - \sqrt{\frac{1}{2}|\widehat{AC}|^2 - \pi} > \sqrt{\pi} \ , \tag{112}$$

i.e.

$$|\widehat{AC}| - \sqrt{\pi} > \sqrt{\frac{1}{2}|\widehat{AC}|^2 - \pi} \ ,$$

i.e.

$$|\widehat{AC}|^2 + \pi - 2|\widehat{AC}|\sqrt{\pi} > \frac{1}{2}|\widehat{AC}|^2 - \pi \ ,$$

i.e.

$$\left(\frac{1}{\sqrt{2}}|\widehat{AC}| - \sqrt{2\pi}\right)^2 > 0 \ .$$

This is correct because $\frac{1}{\sqrt{2}}|\widehat{AC}| - \sqrt{2\pi} = 0$ only for $|\widehat{AC}| = 2\sqrt{\pi}$, but it follows from Remark 10.4 that in this section we consider only paths $\mathcal{P}_d$ consisting of intervals whose lengths are smaller than $\sqrt{2.926\pi} < 2\sqrt{\pi}$.

Thus, inequality (110) is proved. Hence,

$$|\widehat{AP}| < |\widehat{AS}_2| \quad \text{for } \psi \in [\pi/2, 5\pi/6].$$

Now we prove inequality (111). As $|\widehat{AS}_1|$ is an increasing function of ψ on the interval $[\pi/2, 5\pi/6]$, then

$$\max_{\psi\in[\pi/2,5\pi/6]} |\widehat{AS}_1| = |\widehat{AS}_1|\Big|_{\psi=5\pi/6} \ ,$$

i.e.

$$\max_{\psi\in[\pi/2,5\pi/6]} |\widehat{AS}_1| = \sqrt{5\pi/6-\pi/2} = \sqrt{\pi/3}\ .$$

As $|\widehat{AP}|$ is a decreasing function of ψ on the interval $[\pi/2, 5\pi/6]$, then

$$\min_{\psi\in[\pi/2,5\pi/6]} |\widehat{AP}| = |\widehat{AP}|\Big|_{\psi=5\pi/6}\ ,$$

i.e.

$$\min_{\psi\in[\pi/2,5\pi/6]} |\widehat{AP}| = \sqrt{\pi/3}\ .$$

Thus

$$\max_{\psi\in[\pi/2,5\pi/6]} |\widehat{AS}_1| = \min_{\psi\in[\pi/2,5\pi/6]} |\widehat{AP}|\ ,$$

i.e. inequality (111) is proved and, hence,

$$|\widehat{AP}| \geq |\widehat{AS}_1| \quad \text{for} \ \ \psi\in[\pi/2,5\pi/6].$$

So, we obtain that

$$|\widehat{AS}_1| \leq |\widehat{AP}| < |\widehat{AS}_2| \quad \text{for} \ \ \psi\in[\pi/2,5\pi/6].$$

The proposition is proved. □

J Appendix: The case when $\psi \in (5\pi/6, \pi]$ – proof of Proposition 10.17

Formulas (107) and (109) are valid also for $\psi \in (5\pi/6, \pi]$.

It follows from (107) and (109) that $|\widehat{AS}_1|$ ($|\widehat{AP}|$) is an increasing function (a decreasing function) of ψ. Hence if

$$\min_{\psi\in(5\pi/6,\pi]} |\widehat{AS}_1| \geq \max_{\psi\in(5\pi/6,\pi]} |\widehat{AP}|\ ,$$

then

$$|\widehat{AS}_1| \geq |\widehat{AP}| \quad \text{for } \psi\in(5\pi/6,\pi]\ ,$$

so, the proposition is proved.

As $|\widehat{AS}_1|$ is an increasing function of ψ on the interval $(5\pi/6, \pi]$, then

$$\min_{\psi\in(5\pi/6,\pi]} |\widehat{AS}_1| = |\widehat{AS}_1|\Big|_{\psi=5\pi/6}\ ,$$

i.e.

$$\min_{\psi\in(5\pi/6,\pi]} |\widehat{AS}_1| = \sqrt{\pi/3}\ .$$

As $|\widehat{AP}|$ is a decreasing function of ψ on the interval $(5\pi/6, \pi]$, then

$$\max_{\psi\in(5\pi/6,\pi]} |\widehat{AP}| = |\widehat{AP}|\Big|_{\psi=5\pi/6} ,$$

i.e.

$$\max_{\psi\in(5\pi/6,\pi]} |\widehat{AP}| = \sqrt{\pi/3} .$$

Hence,

$$|\widehat{AS_1}| \geq |\widehat{AP}| \quad \text{for } \psi \in (5\pi/6, \pi] .$$

The proposition is proved. □

K Appendix: The case when $\Gamma \leq \sigma \leq 2\sqrt{\pi}$ – the subcase when the path $\mathcal{P}_d$ belongs to the class II (proof of two auxiliary propositions: namely, Propositions K.1 et K.2.)

So, we consider the case when the path $\mathcal{C}_0$ has a lace on the interval $[t_{s-1}, t_{s-1} + \sigma]$.

Denote by j the tangent line at the point J and denote by $j_\perp$ the straight line passing through the point J and which is perpendicular to the straight line j. Denote by $V_{\widetilde{S}}$ the tangent vector at the point $\widetilde{S}$. Denote by $\widetilde{\psi}$ the angle between the vectors $V_{\widetilde{S}}$ and V_J and denote by φ the angle between the vectors V_J and V_A (see Figure 81).

Calculate now the minimal length of the piece $\widehat{JIA} \subset \mathcal{C}_0$ such that for this length the path $\mathcal{C}_0$ has a lace on the interval $[t_{s-1}, t_{s-1} + \sigma]$.

Proposition K.1 *We obtain the minimal value of φ such that the path $\mathcal{C}_0$ has a lace on the interval $(t_{s-1}, t_{s-1} + \sigma)$ (and we obtain the minimal value of the length of the corresponding arc $\widehat{JIA}$) if $\psi = \pi/6$.*

For $\psi = \pi/6$ the following statement holds: the path $\mathcal{C}_0$ has a lace on the interval $(t_{s-1}, t_{s-1} + \sigma)$ if

$$\varphi \geq (2/3 - 0.0374)\pi , \tag{113}$$

i.e. if for the length of the corresponding arc $\widehat{JIA}$ the following inequality holds:

$$|\widehat{JIA}| \geq \sqrt{(4/3 - 2 \times 0.0374)\pi} . \tag{114}$$

Proof

Remind that we denote by ψ the angle between the vector $V_{E^{**}}$ and the vector $-V_A$ (see Figure 81) and that $\psi \in (\pi/6, \pi]$ (see Proposition 10.10).

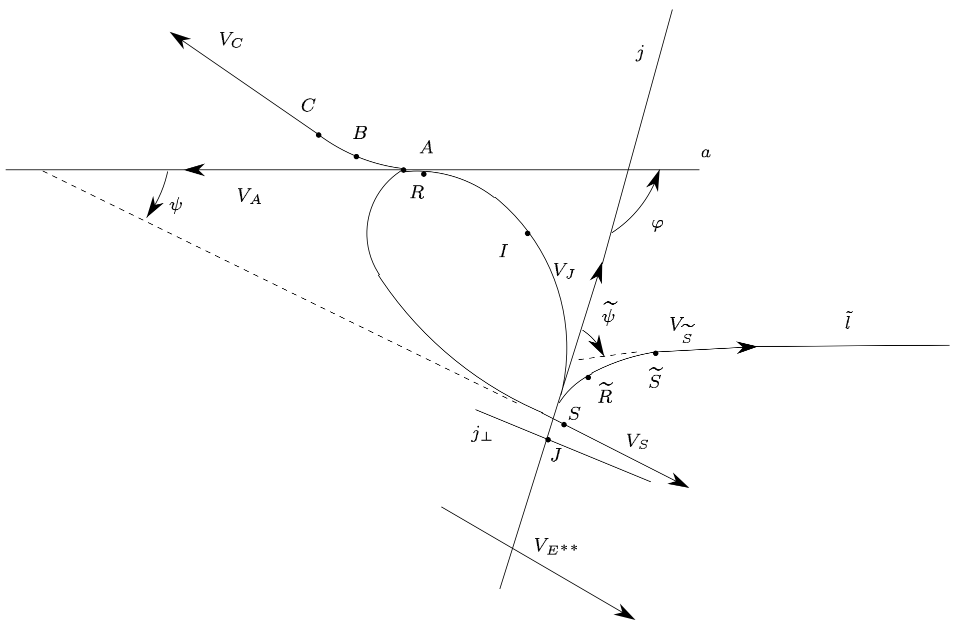

V_C
j
C
B
A
a
V_A
ψ
R
φ
I
V_J
$\widetilde{\psi}$
$V_{\widetilde{S}}$
$\tilde{l}$
$\widetilde{R}$
$\widetilde{S}$
S
$j_\perp$
V_S
J
$V_{E^{**}}$

Figure 81

The following equality holds:

$$|\alpha_S - \alpha_J| = |\alpha_{E^{**}} - \alpha_J| = \pi - \psi + \varphi \ .$$

We obtain a lace if $|\alpha_S - \alpha_J| \geq 1.4626\pi$ (see Lemma 10.2), i.e. if $\pi - \psi + \varphi \geq 1.4626\pi$. Hence, to obtain the minimal value of φ giving a lace, we must consider the maximal value of the expression $\pi - \psi$. As $\psi \in (\pi/6, \pi]$, then we obtain the minimal value of φ giving a lace if $\psi = \pi/6$. So, the first statement of the proposition is proved.

Hence, we obtain

$$\pi - \pi/6 + \varphi \geq 1.4626\pi \ ,$$

i.e.

$$\varphi \geq 1.4626\pi - 5\pi/6 = (2/3 - 0.0374)\pi \ .$$

So, inequality (113) is proved.

Prove now inequality (114). For $|\alpha_A - \alpha_J|$ the following equality holds (see Figure 40):

$$|\alpha_A - \alpha_J| = |\widehat{JIA}|^2/2 \ .$$

Hence,

$$|\widehat{JIA}| = \sqrt{2|\alpha_A - \alpha_J|} = \sqrt{2\varphi} \tag{115}$$

(it follows from the definition of the angle φ).

We have proved that the path $\mathcal{C}_0$ has a lace if $\varphi \geq (2/3 - 0.0374)\pi$. Hence, using (115), we obtain the desired result: the path $\mathcal{C}_0$ has a lace if

$$|\widehat{JIA}| \geq \sqrt{(4/3 - 2 \times 0.0374)\pi} \ .$$

The proposition is proved. □

Proposition K.2 *For the length of the arc* $\widehat{J\widetilde{R}\widetilde{S}}$ *the following inequality holds:*

$$|\widehat{J\widetilde{R}\widetilde{S}}| \leq \sqrt{2\pi} \ . \tag{116}$$

Proof

For $|\alpha_{\widetilde{S}} - \alpha_J|$ the following equality holds (see Figure 41):

$$|\alpha_{\widetilde{S}} - \alpha_J| = |\widehat{J\widetilde{R}\widetilde{S}}|^2/2 \ .$$

Hence,

$$|\widehat{J\widetilde{R}\widetilde{S}}| = \sqrt{2|\alpha_{\widetilde{S}} - \alpha_J|} = \sqrt{2\widetilde{\psi}} \ .$$

So, $|\widehat{J\widetilde{R}\widetilde{S}}|$ is some increasing function of $\widetilde{\psi}$ and the maximal value of $|\widehat{J\widetilde{R}\widetilde{S}}|$ corresponds to the maximal value of $\widetilde{\psi}$, i.e. to π (see Proposition 10.10), hence,

$$|\widehat{J\widetilde{R}\widetilde{S}}| \leq \sqrt{2\pi} \ .$$

The proposition is proved. □

L Appendix: The case when $\sigma < \Gamma$ and $\theta < \Gamma$ – some auxiliary propositions and the proof of Propositions 10.22 and 10.24.

At first we prove some inequalities connecting α_1 and Δ_1, α_{-1} and Δ_{-1}.

Proposition L.1 *We have the following inequalities:*

$$\alpha_1 < M\Delta_1/2 \ , \tag{117}$$

$$\alpha_{-1} < M\Delta_{-1}/2 \ . \tag{118}$$

Proof

As $\Delta_1 > 0$, $\Delta_{-1} > 0$, then for $\alpha_1 \leq 0$ (for $\alpha_{-1} \leq 0$) inequality (117) (inequality (118)) holds. So, we must prove these inequalities respectively for $\alpha_1 > 0$ and $\alpha_{-1} > 0$.

We prove inequality (117). Inequality (118) can be proved by analogy.

Consider a function $h(t) = |\widehat{OW}| - |OW|$ for any point W belonging to the half-clothoid (see Figure 82).

Rappelons qu'on désigne par Δ_1 la différence entre la longueur de l'intervalle $[t_q, t_{q+1}]$ et la somme des longueurs des cordes $X_{t_q}X_{(t_{q+1}-t_q)/2}$, $X_{(t_{q+1}-t_q)/2}X_{t_{q+1}}$, i.e.

$$\Delta_1 = 2(|\widehat{X_{t_q}X}_{(t_{q+1}-t_q)/2}| - |X_{t_q}X_{(t_{q+1}-t_q)/2}|) \ .$$

Remind that from Proposition 8.9 of [11] we know that the function $h(t)$ is monotonously increasing. Hence

$$|\widehat{X_{t_q}X}_{(t_{q+1}-t_q)/2}| - |X_{t_q}X_{(t_{q+1}-t_q)/2}| > |\widehat{\widetilde{X}_{t_{q+1}-\theta}X}_{t_{q+1}-\theta/2}| - |\widetilde{X}_{t_{q+1}-\theta}X_{t_{q+1}-\theta/2}|) \ ,$$

so

$$\Delta_1 = 2(|\widehat{X_{t_q}X}_{(t_{q+1}-t_q)/2}| - |X_{t_q}X_{(t_{q+1}-t_q)/2}|) >$$

$$> 2(|\widehat{\widetilde{X}_{t_{q+1}-\theta}X}_{t_{q+1}-\theta/2}| - |\widetilde{X}_{t_{q+1}-\theta}X_{t_{q+1}-\theta/2}|) \ . \tag{119}$$

From Proposition 7.3 of [11] we have the following inequality:

$$\alpha_1 = |\widetilde{X}_{t_q}X^{pr}_{t_q}| < |\widetilde{X}_{t_{q+1}-\theta}X_{t_{q+1}-\theta/2}| - |\widetilde{X}_{t_{q+1}-\theta}X^{pr}_{t_{q+1}-\theta/2}| \ , \tag{120}$$

where $X^{pr}_{t_{q+1}-\theta/2}$ denotes the projection of the point $X_{t_{q+1}-\theta/2}$ on the line l.

Consider a function $f(t) = \frac{|OW|-|OV|}{|\widehat{OW}|-|OW|}$ for any point W belonging to the half-clothoid (see Figure 82). Set $M = \sup f(t)$.

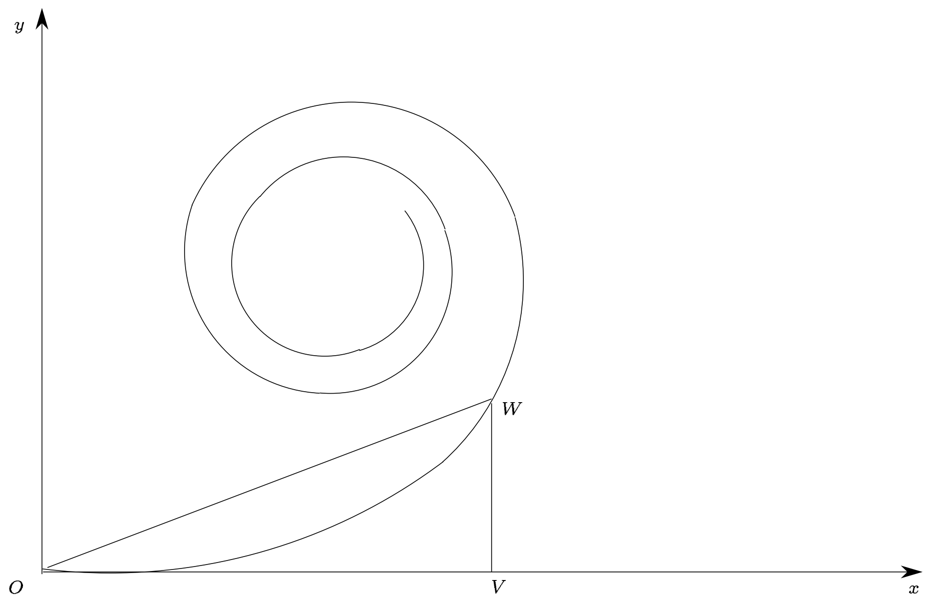


Figure 82

From (119), (120) we obtain

$$\frac{\alpha_1}{\Delta_1} < \frac{|\widetilde{X}_{t_{q+1}-\theta}X_{t_{q+1}-\theta/2}| - |\widetilde{X}_{t_{q+1}-\theta}X^{pr}_{t_{q+1}-\theta/2}|}{2(|\widetilde{X}_{t_{q+1}-\theta}\overset{\frown}{X}_{t_{q+1}-\theta/2}| - |\widetilde{X}_{t_{q+1}-\theta}X_{t_{q+1}-\theta/2}|)} = \frac{1}{2} \times f\left(\frac{\theta}{2}\right) < M/2 \ ,$$

i.e.

$$\alpha_1 < M\Delta_1/2 \ .$$

Thus, we obtain inequality (117).

The proposition is proved. □

Now we generalize the results obtained in Proposition L.1.

Proposition L.2 *Construct a path* $\widetilde{\mathcal{P}}_j$ *by modifying the graph* $\kappa(t)$ *on two intervals* $[t_{s-(j-1)}, t_{s-(j-1)} + \sigma]$, $[t_{q+j} - \theta, t_{q+j}]$. *Then for every* $j \geq 1$ *we have the following inequalities*

$$\alpha_{-j} < M\Delta_{-j}/2 \quad \text{and} \quad \alpha_j < M\Delta_j/2 \ . \tag{121}$$

We can prove Proposition L.2 by analogy with the proof of Proposition L.1.

Proposition L.3 *Construct a path* $\widetilde{\mathcal{P}}_j$ *by modifying the graph* $\kappa(t)$ *on two intervals* $[t_{s-(j-1)}, t_{s-(j-1)} + \sigma]$, $[t_{q+j} - \theta, t_{q+j}]$ $(j \geq 2)$. *If there exists* $J \geq 2$ *such that*

$$\alpha_{-J} + \alpha_J \leq \Sigma_{-(J-1)} + \Sigma_{J-1} \ ,$$

then, for the corresponding path $\widetilde{\mathcal{P}}_J$ *the following inequality holds:*

$$|\widetilde{\mathcal{P}}_J| < |\mathcal{P}_d| \ .$$

Proof

It follows from the definition of Δ_{-j}, Δ_j, Σ_{-j} and Σ_j that

$$\Sigma_{-(J-1)} + \Delta_0 + \Sigma_{J-1} = \Delta_{-(J-1)} + \Delta_{-(J-2)} + \ldots + \Delta_{-1} + \Delta_0 + \Delta_1 + \ldots + \Delta_{J-1} <$$
$$< |X_{t_{s-(J-2)}}\overset{\frown}{X}_{t_q+J}| - l \ . \tag{122}$$

We have

$$\alpha_{-J} + \alpha_J \leq \Sigma_{-(J-1)} + \Sigma_{J-1} \ .$$

Hence,

$$\alpha_{-J} + \alpha_J < \Sigma_{-(J-1)} + \Delta_0 + \Sigma_{J-1} \ ,$$

and, using (122), we obtain the following inequality:

$$\alpha_{-J} + \alpha_J < |X_{t_{s-(J-2)}}\widehat{\ }X_{t_q+J}| - l \ . \tag{123}$$

As

$$|\widetilde{X}_{t_s-(J-2)}\widehat{\ }\widetilde{X}_{t_q+J}| = \alpha_{-J} + l + \alpha_J \ ,$$

then, using (123), we obtain

$$|\widetilde{X}_{t_s-(J-2)}\widehat{\ }\widetilde{X}_{t_q+J}| = \alpha_{-J} + l + \alpha_J < |X_{t_{s-(J-2)}}\widehat{\ }X_{t_q+J}| \ ,$$

i.e.

$$|\widetilde{\mathcal{P}}_J| < |\mathcal{P}_d| \ .$$

The proposition is proved. □

Proof of Proposition 10.22

We prove the proposition by induction on j.

Consider $j = 2$. From Proposition L.2 we have

$$\alpha_{-2} < \frac{M}{2}\Delta_{-2} \quad \text{and} \quad \alpha_2 < \frac{M}{2}\Delta_2 \ .$$

It follows from Proposition 10.21 that if for some path $\widetilde{\mathcal{P}}_2$ we have $|\widetilde{\mathcal{P}}_2| \geq |\mathcal{P}_d|$, then, the following inequality holds:

$$\alpha_{-2} + \alpha_2 > \Sigma_{-1} + \Sigma_1 = \Delta_{-1} + \Delta_1 \ .$$

Thus, we obtain the following chain of inequalities:

$$\Delta_{-1} + \Delta_1 < \alpha_{-2} + \alpha_2 < \frac{M}{2}(\Delta_{-2} + \Delta_2) \ ,$$

and, hence,

$$\Delta_{-2} + \Delta_2 > \frac{2}{M}(\Delta_{-1} + \Delta_1) \ .$$

So, for $j = 2$ inequality (57) holds.

Suppose that inequality (57) holds for any $j \leq I$. Now we must prove that then it should hold for $j = I + 1$, i.e. that the following inequality holds:

$$\Delta_{-(I+1)} + \Delta_{I+1} > \frac{2}{M}(\Delta_{-1} + \Delta_1)(1 + 2/M)^{I-1} \ .$$

Really, it follows from Proposition 10.21 that if for some path $\widetilde{\mathcal{P}}_{I+1}$ we have $|\widetilde{\mathcal{P}}_{I+1}| \geq |\mathcal{P}_d|$, then, the following inequality holds:

$$\alpha_{-(I+1)} + \alpha_{I+1} > \Sigma_{-I} + \Sigma_I \ .$$

It follows from Proposition L.2 we have

$$\alpha_{-(I+1)} < \frac{M}{2}\Delta_{-(I+1)} \quad \text{and} \quad \alpha_{I+1} < \frac{M}{2}\Delta_{I+1} \ .$$

Hence,

$$\Delta_{-(I+1)} + \Delta_{I+1} > \frac{2}{M}(\alpha_{-(I+1)} + \alpha_{I+1}) > \frac{2}{M}(\Sigma_{-I} + \Sigma_I) \ .$$

According to the inductive assumption, inequality (57) holds for any $j \leq I$, hence,

$$\Delta_{-(I+1)} + \Delta_{I+1} > \frac{2}{M}(\Sigma_{-I} + \Sigma_I) > \frac{2}{M}(\Delta_{-1} + \Delta_1)\left(1 + \frac{2}{M} + \frac{2}{M}\left(1 + \frac{2}{M}\right) + \right.$$

$$\left. + \frac{2}{M}\left(1 + \frac{2}{M}\right)^2 + \ldots + \frac{2}{M}\left(1 + \frac{2}{M}\right)^{I-2}\right) = \frac{2}{M}(\Delta_{-1} + \Delta_1)\left(1 + \frac{2}{M} + \right.$$

$$\left. + \left(1 + \frac{2}{M}\right)^{I-1} - \left(1 + \frac{2}{M}\right)\right) = \frac{2}{M}(\Delta_{-1} + \Delta_1)\left(1 + \frac{2}{M}\right)^{I-1} \ .$$

The proposition is proved. □

Proof of Proposition 10.24

We prove the estimation for Σ_K. One can prove the estimation for Σ_{-K} by analogy.

From Proposition 8.2 of [11] for the length of an optimal path (denote it by l_{opt}) we have the following estimation:

$$l_{opt} \leq d + (6 + 8\sqrt{2})R + \left(|\kappa^0| + |\kappa^T|\right)/2 \ .$$

where by d we denote the distance between the initial and final point of the optimal path.

Remind that we assume that the path $\mathcal{P}_d$ is optimal and that $|\kappa^0| = |\kappa^T| = 0$. So, for the length $l_{\mathcal{P}_d}$ of the path $\mathcal{P}_d$ we have the following inequality:

$$l_{\mathcal{P}_d} - d \leq (6 + 8\sqrt{2})R \ . \tag{124}$$

By definition $\Sigma_K = \Delta_1 + \ldots + \Delta_K$ and $\Delta_j = 2h\left((t_{q+j} - t_{q+j-1})/2\right)$. Denote by l_j the length of the arc of clothoid corresponding to Δ_j and denote by d' the distance between the initial point of the arc l_1 and the final point of the arc l_J. Then

$$\Sigma_K = \Delta_1 + \ldots + \Delta_K < 2(l_1 + \ldots + l_K) - d' \ . \tag{125}$$

Now, using (124), we obtain the estimation of $2(l_1 + \ldots + l_K) - d'$. Really, we obtain the greatest possible value of the sum $l_1 + \ldots + l_K$ in the case when all the rest of the path $\mathcal{P}_d$

would be a line segment parallel to the line connecting the initial and final points (because $l_{\mathcal{P}_d}$ is bounded). So from (124) we obtain

$$2(l_1 + \ldots + l_K) - d' \leq (6 + 8\sqrt{2})R \ . \quad (126)$$

Now we obtain from (125), (126) the desired estimation:

$$\Sigma_K < (6 + 8\sqrt{2})R \ .$$

The proposition is proved. □

M Appendix: The case when $\sigma < \Gamma$ and $\theta < \Gamma$ – the estimation of the constant C (i.e. the proof of Lemma 10.28)

Estimate now the constant C. At first we compute the sum of the maximal possible distance between the initial point of the arc l_1 and the final point of the arc l_K and the maximal possible distance between the initial point of the arc l_{-K} and the final point of the arc l_{-1}.

Denote by E the point of the half-clothoid with tangent angle equal to $\pi/512$.

From Proposition 10.25 we know that $\Delta_{-K} + \Delta_K, \ldots, \Delta_{-1} + \Delta_1$ form a decreasing sequence. Consider the first $(Q+1)$-terms of this sequence $\Delta_{-K}+\Delta_K, \ldots, \Delta_{-(K-Q)}+\Delta_{K-Q}$ such that

$$\Delta_{-(K-Q+1)} + \Delta_{K-Q+1} > 2\Delta_E = 4(|\widehat{OE}| - |OE|) \quad \text{and} \quad \Delta_{-(K-Q)} + \Delta_{K-Q} < 2\Delta_E \ .$$

Denote by $\text{dist}_{1,K-Q}$ (by $\text{dist}_{K-Q,K}$, by $\text{dist}_{-K,-(K-Q)}$, by $\text{dist}_{-(K-Q),-1}$, by $\text{dist}_{-K,-1}$, by $\text{dist}_{1,K}$) the maximal possible distance between the initial point of the arc l_1 (of l_{K-Q}, of l_{-K}, of $l_{-(K-Q)}$, of l_{-K}, of l_1) and the final point of the arc l_{K-Q} (of l_K, of $l_{-(K-Q)}$, of l_{-1}, of l_{-1}, of l_K respectively).

To estimate the constant C we use the following method:

1) at first we estimate the sum of the maximal possible distance between the initial point of the arc l_1 and the final point of the arc l_{K-Q} and the maximal possible distance between the initial point of the arc $l_{-(K-Q)}$ and the final point of the arc l_{-1};

2) then we estimate the sum of the maximal possible distance between the initial point of the arc l_{K-Q} and the final point of the arc l_K and the maximal possible distance between the initial point of the arc l_{-K} and the final point of the arc $l_{-(K-Q)}$;

3) and then we add these sums.

The exactness of the estimation depends on the successful choice of the point E. For the point E chosen outside a small half-neighbourhood of the point $\pi/512$ (this half-neighbourhood is defined by the condition $t \in (\pi/512 - \varepsilon, \pi/512)$ for small $\varepsilon > 0$) the estimation is worse than for the point $\pi/512$. We can improve the estimation varying the position of the point E in the small interval $(\pi/512-\varepsilon, \pi/512)$, but we can't essentially improve the estimation.

So, we consider the point E with tangent angle equal to $\pi/512$.

To prove Lemma 10.28 we prove three following lemmas.

Lemma M.1 *The sum of the maximal possible distance between the initial point of the arc* l_{K-Q} *and the final point of the arc* l_K *and the maximal possible distance between the initial point of the arc* l_{-K} *and the final point of the arc* $l_{-(K-Q)}$ *is smaller than* $198R$*, i.e.*

$$\mathrm{dist}_{-K,-(K-Q)} + \mathrm{dist}_{K-Q,K} < 198R\ .$$

Lemma M.2 *The sum of the maximal possible distance between the initial point of the arc* l_1 *and the final point of the arc* l_{K-Q} *and the maximal possible distance between the initial point of the arc* $l_{-(K-Q)}$ *and the final point of the arc* l_{-1} *is smaller than* $80R/\sqrt{2}$*, i.e.*

$$\mathrm{dist}_{-(K-Q),-1} + \mathrm{dist}_{1,K-Q} < 80R/\sqrt{2}\ .$$

The proof of Lemma M.1 (Lemma M.2) is given in Subsection M.1 (Subsection M.2).

Lemma M.3 *For the sum of the maximal possible distance between the initial point of the arc* l_1 *and the final point of the arc* l_K *and the maximal possible distance between the initial point of the arc* l_{-K} *and the final point of the arc* l_{-1} *we have the following estimation:*

$$\mathrm{dist}_{-K,-1} + \mathrm{dist}_{1,K} < 254.6R\ .$$

Proof

Evidently,

$$\mathrm{dist}_{-K,-1} + \mathrm{dist}_{1,K} < \mathrm{dist}_{-K,-(K-Q)} + \mathrm{dist}_{-(K-Q),-1} + \mathrm{dist}_{1,K-Q} + \mathrm{dist}_{K-Q,K}\ .$$

Hence, from Lemma M.1 and Lemma M.2 we obtain:

$$\mathrm{dist}_{-K,-1} + \mathrm{dist}_{1,K} < 198R + 80/\sqrt{2}R \approx (198 + 80/1.414)R \approx$$
$$\approx (198 + 56.576)R = 254.576R < 254.6R\ .$$

The lemma is proved. □

Proof of Lemma 10.28

Recall that the distance between the points corresponding to $t = t_s$ and $t = t_{q+1}$ is no smaller than $13,4R$ (see Subsection 10.3 – the choice of the intervals $[t_s, t_{s+1}]$, $[t_q, t_{q+1}]$).

From Lemma M.3 we know that if the sum of the distance between t_q and T and the distance between 0 and t_{s+1} is no smaller than $254.6R$, then we can construct a suboptimal path $\widetilde{\mathcal{P}}$ such that $|\widetilde{\mathcal{P}}| < |\mathcal{P}_d|$.

Hence, if the distance between the initial and the final points of the path $\mathcal{P}_d$ is no smaller than $(13.4 + 254.6)R = 268R$, then we can construct a suboptimal path $\widetilde{\mathcal{P}}$ such that $|\widetilde{\mathcal{P}}| < |\mathcal{P}_d|$.

In Proposition 5.3 of [13] we obtain the following result: the maximal possible distance between two points of a half-clothoid is smaller than $3R/2$. Recall that

$$0 \leq N - N_d \leq 2$$

(see the beginning of Section 10). Here by N (N_d) we denote the minimal number of switching points of the path $\mathcal{P}$ ($\mathcal{P}_d$) necessary to construct a suboptimal path as in Section 10. Hence,

$$d \geq 268R + 2 \times 3R/2 = 271R = 135.5\sqrt{\pi} \ .$$

The lemma is proved. □

M.1 Proof of Lemma M.1.

From formula (60) we have

$$\Delta_{-(K-s)} + \Delta_{K-s} < \left(\frac{3}{5}\right)^s (\Sigma_{-K} + \Sigma_K) \ , \qquad \text{for } s = 1, \dots K-1 \tag{127}$$

(because $M < 3$, see Corollary 8.8 of [13]).

From Proposition 10.24 we have

$$\Sigma_{-K} + \Sigma_K < (12 + 16\sqrt{2})R \ . \tag{128}$$

Hence, from (127) and (128) we obtain

$$\Delta_{-(K-Q)} + \Delta_{K-Q} < \left(\frac{3}{5}\right)^Q (\Sigma_{-K} + \Sigma_K) < \left(\frac{3}{5}\right)^Q (12 + 16\sqrt{2})R \ .$$

Recall that $\Delta_{K-Q} < \Delta_E$, $\Delta_{K-Q-1} > \Delta_E$, $\Delta_{-(K-Q)} < \Delta_E$ and $\Delta_{-(K-Q-1)} > \Delta_E$. Compute the number Q such that

$$\left(\frac{3}{5}\right)^Q (12 + 16\sqrt{2})R \leq 2\Delta_E \ , \qquad \text{i.e.} \qquad \left(\frac{3}{5}\right)^Q \leq \frac{\Delta_E}{(6 + 8\sqrt{2})R} \ .$$

For this purpose estimate Δ_E from below. From Proposition 8.12 of [13] we have the following equality (see Figure 83):

$$\frac{H(E)}{D(E)} = \frac{H'(M)}{D'(M)} = \frac{H'(\theta t)}{D'(\theta t)} = \cos\alpha(\theta t) \ , \qquad 1/2 < \theta < 1 \ .$$

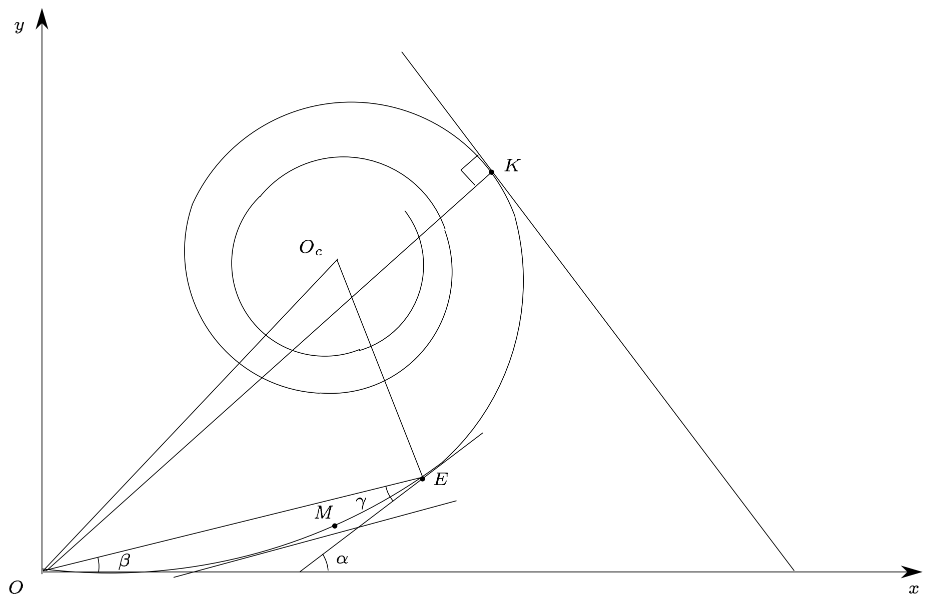


Figure 83

Here by $H(t)$ and $D(t)$ we denote the lengths of the chord and of the arc respectively (by $H'(t)$, $D'(t)$ we denote their derivatives). Then

$$\Delta_E = 2(D(E) - H(E)) = 2D(E)(1 - H(E)/D(E)) = 2D(E)(1 - \cos\alpha(\theta t)) \ . \qquad (129)$$

We have $\alpha_E = \pi/512$, hence, $\alpha(\theta t) > \pi/1024$, hence, $\cos\alpha(\theta t) < \cos(\pi/1024)$ and from equality (129) we obtain the following inequality

$$\Delta_E > 2D(E)\,(1 - \cos(\pi/1024)) \ ,$$

where

$$D(E) = \sqrt{\alpha_E} = 2R\sqrt{\alpha_E/\pi} = 2R/\sqrt{512} = R/(8\sqrt{2}) \ .$$

Hence

$$\Delta_E > R(1 - \cos(\pi/1024))/(4\sqrt{2}) = 0.1767766953 \times (1 - 0.9999952938)R =$$

$$= 0.1767766953 \times 0.47062 \times 10^{-5}R = 0.8319464836 \times 10^{-6}R \ .$$

Now compute Q for which

$$\left(\frac{3}{5}\right)^Q \leq \frac{0.8319464836 \times 10^{-6}R}{(6 + 8\sqrt{2})R} \ .$$

We can rewrite this inequality as follows:

$$\left(\frac{3}{5}\right)^Q \leq \frac{0.8319464836 \times 10^{-6}}{17.31370850} = 0.4805131631 \times 10^{-7} \ .$$

For $Q = 33$ we obtain $(3/5)^{33} = 0.4775196666 \times 10^{-7} < 0.4805131631 \times 10^{-7}$. Hence,

$$\Delta_{-(K-33)} + \Delta_{K-33} < 2\Delta_E \ .$$

Thus, the sum of the distance between the initial point of the arc l_{K-Q} and the final point of the arc l_K and the distance between the initial point of the arc l_{-K} and the final point of the arc $l_{-(K-Q)}$ is smaller than $2 \times 33 \times (2cd_{max})$, where by cd_{max} we denote the maximal length of the chord. From Proposition 5.3 of [13] we have $cd_{max} < 3R/2$.

So, for the sum of the distance between the initial point of the arc l_{K-Q} and the final point of the arc l_K and the distance between the initial point of the arc l_{-K} and the final point of the arc $l_{-(K-Q)}$ we have the following estimation:

$$\text{dist}_{-K,-(K-Q)} + \text{dist}_{K-Q,K} < 2 \times 33 \times 3R = 198R \ .$$

The lemma is proved. □

M.2 Proof of Lemma M.2.

Recall that $l_{-(K-Q)} + l_{K-Q} < 2l_E$, hence,

$$\text{dist}_{-(K-Q),-1} + \text{dist}_{1,K-Q} <$$
$$< [l_{-1} + l_{-2} + \ldots + l_{-(K-Q+1)} + l_E] + [l_1 + l_2 + \ldots + l_{K-Q+1} + l_E] \ . \tag{130}$$

Recall that $\Delta_j = 2h((t_{q+j} - t_{q+j-1})/2)$. From formula (127) we have (because $M < 3$, see Corollary 8.8 of [13]):

$$\Delta_{-(K-s)} + \Delta_{K-s} < \left(\frac{3}{5}\right)^s (\Sigma_{-K} + \Sigma_K) \ , \qquad s = 1, \ldots, K-1 \ ,$$

i.e.

$$h(l_{-(K-s)}/2) + h(l_{K-s}/2) < \left(\frac{3}{5}\right)^s (\Sigma_{-K} + \Sigma_K)/2 \ , \qquad s = 1, \ldots, K-1 \ . \tag{131}$$

Consider two pairs of consecutive intervals ($l_{-(K-s-1)}$, $l_{-(K-s)}$ and l_{K-s}, l_{K-s-1}). From formulas (131) we obtain

$$h(l_{-(K-s)}/2) + h(l_{K-s}/2) < \left(\frac{3}{5}\right)^s (\Sigma_{-K} + \Sigma_K)/2 \ ,$$

$$h(l_{-(K-s-1)}/2) + h(l_{K-s-1}/2) < \left(\frac{3}{5}\right)^{s+1} (\Sigma_{-K} + \Sigma_K)/2 = \left(\frac{3}{5}\right) \times \left(\frac{3}{5}\right)^s (\Sigma_{-K} + \Sigma_K)/2 \ .$$

We want to find an upper bound of the sum

$$[l_1 + l_2 + \ldots + l_{-(K-Q+1)} + l_E] + [l_{-1} + l_{-2} + \ldots + l_{K-Q+1} + l_E] =$$

$$= \left[2\left(h^{-1}(h(l_1/2)) + h^{-1}(h(l_2/2)) + \ldots + h^{-1}(h(l_E/2))\right)\right] +$$

$$+ \left[2\left(h^{-1}(h(l_{-1}/2)) + h^{-1}(h(l_{-2}/2)) + \ldots + h^{-1}(h(l_E/2))\right)\right] \ .$$

Hence, we must consider the sum of the maximal possible values of $h(l_1/2), \ldots, h(l_E/2)$, $h(l_{-1}/2), \ldots, h(l_E/2)$, i.e.

$$\frac{h(l_{K-s-1}/2)}{h(l_{K-s}/2)} = \frac{3}{5} \ , \qquad s = 1, \ldots, K-1 \tag{132}$$

and

$$\frac{h(l_{-(K-s-1)}/2)}{h(l_{-(K-s)}/2)} = \frac{3}{5} \ , \qquad s = 1, \ldots, K-1 \tag{133}$$

Apply now Lemma 8.10 of [13]. Recall that in this lemma we define the connection between the functions h and h^{-1} only for the tangent angle belonging to $(0, \alpha_K)$, where by α_K we denote the tangent angle at the point K (the point with the maximal chord length). So, consider formulas (132) only for the $(K-Q-1)$ first terms of the sequence $l_1, \ldots, l_K$ from l_1 to l_{K-Q}:

$$\frac{h(l_{K-Q-s-1}/2)}{h(l_{K-Q-s}/2)} = \frac{3}{5} \ , \qquad s = 1, \ldots, K-Q-2 \ ,$$

and consider formulas (133) only for the $(K-Q-1)$ first terms of the sequence $l_{-1}, \ldots, l_{-K}$ from l_{-1} to $l_{-(K-Q)}$:

$$\frac{h(l_{-(K-Q-s-1)}/2)}{h(l_{-(K-Q-s)}/2)} = \frac{3}{5} \ , \qquad s = 1, \ldots, K-Q-2 \ .$$

Using Lemma 8.10 of [13] we obtain two following inequalities (in our case $k = 3/5$):

$$\frac{h^{-1}\left(\frac{3}{5}h(l_{K-Q-s-1}/2)\right)}{h^{-1}\left(h(l_{K-Q-s}/2)\right)} < 1 - \frac{1-3/5}{64} = \frac{159}{160} \ , \quad \text{i.e.} \quad l_{K-Q-s-1} < \frac{159}{160} l_{K-Q-s} \ , \qquad (134)$$

and

$$\frac{h^{-1}\left(\frac{3}{5}h(l_{-(K-Q-s-1)}/2)\right)}{h^{-1}\left(h(l_{-(K-Q-s)}/2)\right)} < 1 - \frac{1-3/5}{64} = \frac{159}{160} \ , \quad \text{i.e.} \quad l_{-(K-Q-s-1)} < \frac{159}{160} l_{-(K-Q-s)} \ . \qquad (135)$$

So, we obtain (using inequality (134))

$$l_{K-Q-s} < \left(\frac{159}{160}\right)^s l_{K-Q} \ , \qquad s = 1, \ldots, K-Q-2 \ ,$$

and we obtain (using inequality (135))

$$l_{-(K-Q-s)} < \left(\frac{159}{160}\right)^s l_{-(K-Q)} \ , \qquad s = 1, \ldots, K-Q-2 \ .$$

Recall that $l_{K-Q} + l_{-(K-Q)} < 2l_E$. Hence,

$$l_{K-Q-s} + l_{-(K-Q-s)} < 2\left(\frac{159}{160}\right)^s l_E \ , \qquad s = 1, \ldots, K-Q-2 \ , \qquad (136)$$

Estimate now the sum $[l_1 + l_2 + \ldots + l_{K-Q+1} + l_E] + [l_{-1} + l_{-2} + \ldots + l_{-(K-Q+1)} + l_E]$. Using formulas (136) we obtain

$$[l_1 + l_2 + \ldots + l_{K-Q+1} + l_E] + [l_{-1} + l_{-2} + \ldots + l_{-(K-Q+1)} + l_E] =$$

$$= (l_1 + l_{-1}) + (l_2 + l_{-2}) + \ldots + (l_{K-Q+1} + l_{-(K-Q+1)}) + 2l_E <$$

$$< 2\left[\left(\frac{159}{160}\right)^{N-1} l_E + \left(\frac{159}{160}\right)^{N-2} l_E + \ldots + \left(\frac{159}{160}\right) l_E + l_E\right] =$$

$$= 2l_E\left[1 + \frac{159}{160} + \left(\frac{159}{160}\right)^2 + \ldots + \left(\frac{159}{160}\right)^{N-1}\right] <$$

$$< 2l_E\left(1 + \frac{159}{160} \Big/ \left(1 - \frac{159}{160}\right)\right) = 2l_E \times 160 = 320l_E \ .$$

Recall that $l_E = 2D(E) = 2R/(8\sqrt{2}) = R/(4\sqrt{2})$. Hence

$$[l_1 + l_2 + \ldots + l_{K-Q+1} + l_E] + [l_{-1} + l_{-2} + \ldots + l_{-(K-Q+1)} + l_E] <$$

$$< 320R/(4\sqrt{2}) = 80R/\sqrt{2} \ . \tag{137}$$

Now, using inequalities (130) and (137) we have the following inequality:

$$\mathrm{dist}_{-(K-Q),-1} + \mathrm{dist}_{1,K-Q} < 80R/\sqrt{2} \ .$$

The lemma is proved. □

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