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Abstract: The solution of Maxwell's equations in a non-convex polyhedral domain is less regular than in a smooth or convex polyhedral domain. In this paper we show that this solution can be decomposed into the orthogonal sum of a singular part and a regular part, and we give a characterization of the singular part. We also notice that the decomposition is linked to the one associated to the scalar Laplacian.

Key-words: Maxwell's equations, polyhedral domains, corner singularities.

(Résumé : tsvp)

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Caractérisation de la partie singulière de la solution des équations de Maxwell dans un ouvert polyédrique

Résumé : La solution des équations de Maxwell dans un ouvert polyédrique non convexe est moins régulière que dans le cas d'un domaine régulier ou convexe. Dans ce papier nous montrons que la solution peut être décomposée en la somme orthogonale d'une partie singulière et d'une partie régulière, et nous donnons une caractérisation de la partie singulière. Nous prouvons également que cette décomposition est liée à celle du laplacien scalaire.

Mots-clé : Equations de Maxwell, domaines polyédriques, singularités de coin.

1 Introduction

When solving Maxwell's equations with regular source terms in a non convex polygonal or polyhedral domain (with Lipschitz-continuous boundary) the solutions, instead of being in $H^1(\Omega)^3$ as in the case of a convex domain, are only in $H(\mathbf{curl}\,,\Omega) \cap H(\mathrm{div}\,,\Omega)$. In the same way, when solving a problem involving the scalar Laplace operator with data in $L^2(\Omega)$, the solution instead of being in $H^2(\Omega)$ as in the regular case (convex polygonal or polyhedral, or with a smooth boundary) is only in $H^{1+s}(\Omega)$, with $0 < s < 1$. Grisvard [10] showed that a solution of the scalar Laplace operator in the general case could be decomposed into the sum of a regular part and a singular part. This decomposition is based on a decomposition of $L^2(\Omega)$ into the sum of the image space of the regular parts and its orthogonal. In the case of a polygonal domain of $\mathbb{R}^2$, Grisvard [11] completely characterized these two spaces; starting from this result, we introduced an orthogonal decomposition of the solution of Maxwell's equations and proposed a method for its numerical computation [2], [3]. This method can be generalized to a polyhedral domain with Lipschitz continuous boundary provided an orthogonal decomposition of $L^2(\Omega)$ can be obtained and each of its terms can be fully described.

In this article we would like to generalize the results obtained in [2], [3] to the three dimensional case, proving in particular that the solution of Maxwell's equations can also be decomposed into the orthogonal sum of a regular term and a singular term. We shall show that this decomposition is still linked to the decomposition of $L^2(\Omega)$ associated to the scalar Laplace operator.

The article is organized as follows. First, we shall recall the characterization of the orthogonal decomposition of $L^2(\Omega)$ in the case of a non convex polyhedral domain Ω with Lipschitz continuous boundary obtained by Assous and Ciarlet [1].Then for a model problem associated to the steady-state Maxwell equations, we shall introduce a decomposition of the space of solutions, which will enable us to characterize the singular solutions as well as the regular solutions.

In the last part, we obtain results that are complementary to those obtained by Costabel and Dauge [6], who worked on the explicit study of the singular part of the solution. In this spirit, Bonnet-Ben Dhia *et al* [4] have recently worked on the solution of the frequential Maxwell equations by a regularizing method that can be related to the theory developed by Costabel and Dauge. The originality of our approach lies on the one hand on the theoretical analysis of the decomposition of $L^2(\Omega)$ and, on the other hand, in the introduction of orthogonal decompositions, which enable us among other things to describe the space of singular solutions more precisely.

2 A characterization of the orthogonal of $\Delta(H^2(\Omega) \cap H^1_0(\Omega))$ in $L^2(\Omega)$

Let Ω be a connected and simply connected polyhedral open set of $\mathbb{R}^3$ with a connected and Lipschitz-continuous boundary Γ. We denote by $(\Gamma_i)_{1 \le i \le N_F}$ the faces of Γ.

As we mentionned in the introduction, the space $L^2(\Omega)$ can be decomposed in the following way.

Theorem 2.1 *The image by the Laplace operator of the space $H^2(\Omega) \cap H^1_0(\Omega)$, denoted by $\Delta(H^2(\Omega) \cap H^1_0(\Omega))$, is a closed subspace of $L^2(\Omega)$, and we have the following orthogonal decomposition:*

$$L^2(\Omega) = \Delta(H^2(\Omega) \cap H^1_0(\Omega)) \overset{\perp}{\oplus} N. \tag{1}$$

Proof: This result was proved by Grisvard [10], and by Dauge [7]. ∎

One of the goals of this paper is to characterize the elements p of N. To that aim, we denote by $D(\Delta, \Omega)$ the space $\{q \in L^2(\Omega);\ \Delta q \in L^2(\Omega)\}$.

By definition, each face Γ_j is a polygon, hence its boundary $\partial\Gamma_j$ is Lipschitz-continuous. For any point $\mathbf{x} \in \Gamma_j$, we denote by $\rho_j(\mathbf{x})$ the distance of $\mathbf{x}$ to $\partial\Gamma_j$. We then have the following definition (cf [11])

Definition 2.1 *$\widetilde{H}^{1/2}(\Gamma_j)$ is the set of functions f of $H^{1/2}(\Gamma_j)$ such that $f/\sqrt{\rho_j}$ also belongs to $L^2(\Gamma_j)$. We denote by $||f||_{\sim,1/2,\Gamma_j} = (||f||^2_{0,\Gamma_j} + ||f/\sqrt{\rho_j}||^2_{0,\Gamma_j})^{1/2}$ the associated norm.*

Finaly, we denote by $\widetilde{H}^{-1/2}(\Gamma_j)$ the dual space of $\widetilde{H}^{1/2}(\Gamma_j)$.

Now, let us recall the theorem which is proved in Assous and Ciarlet [1].

Theorem 2.2 *$p \in N$ if and only if*

$$p \in D(\Delta, \Omega), \quad \Delta p = 0, \quad p_{|\Gamma_i} = 0 \text{ in } \widetilde{H}^{-1/2}(\Gamma_i), \text{ for } 1 \le i \le N_F.$$

The proof is based on several technical results, which we recall as we shall also need them in the present paper, and on the classical theory developped by Gagliardo [8] and Necas [12]. Let Γ_j be a fixed face.

Proposition 2.1 *(i) The normal trace on Γ_j mapping, $\mathbf{g} \mapsto \mathbf{g} \cdot \mathbf{n}_{|\Gamma_j}$, is linear and continuous from $\{\mathbf{g} \in H^1(\Omega)^3,\ \mathbf{g} \times \mathbf{n}_{|\Gamma} = 0\}$ into $\widetilde{H}^{1/2}(\Gamma_j)$.*

(ii) The trace of the normal derivative on Γ_j mapping, $u \mapsto (\partial u/\partial n)_{|\Gamma_j}$, is linear and continuous from $H^2(\Omega) \cap H^1_0(\Omega)$ into $\widetilde{H}^{1/2}(\Gamma_j)$.

Next, we define the space

$$H_j(\Omega) = \{v \in H^2(\Omega) \cap H^1_0(\Omega),\ \ (\partial u/\partial n)_{|\Gamma_k} = 0, \text{for } k \neq j\}.$$

Proposition 2.2 *Let μ be an element of $\widetilde{H}^{1/2}(\Gamma_j)$. Then there exists a lifting u belonging to $H_j(\Omega)$ such that*

$$\frac{\partial u}{\partial n}_{|\Gamma_j} = \mu.$$

Proposition 2.3 *For a given face Γ_j, there exists a constant $C(\Gamma_j)$ such that*

$$\forall \mu \in \widetilde{H}^{1/2}(\Gamma_j), \quad \exists u \in H_j(\Omega),\ \frac{\partial u}{\partial n}_{|\Gamma_j} = \mu \text{ and } ||u||_2 \le C\,||\mu||_{\sim,1/2,\Gamma_j}.$$

Proposition 2.4 *The space $H^2(\Omega)$ is dense in $D(\Delta,\Omega)$ endowed with the graph norm $||q||_D = \{||q||_0^2 + ||\Delta q||_0^2\}^{1/2}$.*

Proposition 2.5 *When p belongs to $D(\Delta, \Omega)$, we have $p_{|\Gamma_i} \in \widetilde{H}^{-1/2}(\Gamma_i)$ for all $1 \le i \le N_F$. In addition, for each face Γ_i, there exists $C_i > 0$ such that*

$$\forall p \in D(\Delta, \Omega), ||p||_{\sim,-1/2,\Gamma_i} \le C_i\,||p||_D. \tag{2}$$

Moreover, the following integration by parts formula holds

$$\forall (p, v) \in D(\Delta, \Omega) \times (H^2(\Omega) \cap H^1_0(\Omega)),$$
$$\int_\Omega p \Delta v \, d\mathbf{x} - \int_\Omega v \Delta p \, d\mathbf{x} = \sum_i \langle p_{|\Gamma_i}, (\frac{\partial v}{\partial n})_{|\Gamma_i} \rangle_{\widetilde{H}^{-1/2}(\Gamma_i), \widetilde{H}^{1/2}(\Gamma_i)}. \tag{3}$$

3 Application to the Maxwell equations

3.1 The model problem

Consider

$$X = \{\mathbf{u} \in H(\mathbf{curl}\,,\Omega) \cap H(\mathrm{div}\,,\Omega),\ \mathbf{u} \times \mathbf{n} = 0 \text{ on } \Gamma\}$$

the Hilbert space endowed with the canonical inner product of $H(\mathbf{curl}\,,\Omega) \cap H(\mathrm{div}\,,\Omega)$ and

$$V = \{\mathbf{u} \in H(\mathbf{curl}\,,\Omega) \cap H(\mathrm{div}\,,\Omega),\ \mathrm{div}\,\mathbf{u} = 0,\ \mathbf{u} \times \mathbf{n} = 0 \text{ on } \Gamma\}$$

the Hilbert space endowed with the canonical inner product of $H(\mathbf{curl}\,,\Omega) \cap H(\mathrm{div}\,,\Omega)$.

Given a function $g \in L^2(\Omega)$, and a function $\mathbf{f} \in L^2(\Omega)^3$ verifying $\mathrm{div}\,\mathbf{f} = 0$ and $\mathbf{f} \cdot \mathbf{n} = 0$ on Γ, we consider the following problem:
Find $\mathbf{u} \in H(\mathbf{curl}\,,\Omega) \cap H(\mathrm{div}\,,\Omega)$ *such that:*

$$\mathbf{curl}\,\mathbf{u} = \mathbf{f} \text{ in } \Omega \tag{4}$$

$$\mathrm{div}\,\mathbf{u} = g \text{ in } \Omega \tag{5}$$

$$\mathbf{u} \times \mathbf{n} = 0 \text{ on } \Gamma. \tag{6}$$

Theorem 3.1 *Let* $\mathbf{f} \in L^2(\Omega)^3$ *with* $\mathrm{div}\,\mathbf{f} = 0$ *and* $\mathbf{f} \cdot \mathbf{n} = 0$ *on* Γ *and* $g \in L^2(\Omega)$. *Then problem (4)-(6) admits a unique solution* $\mathbf{u} \in H(\mathbf{curl}\,,\Omega) \cap H(\mathrm{div}\,,\Omega)$.

Proof: The proof can be based on the theory of mixed problems by introducing a Lagrange multiplier for the divergence condition (5) (see Girault-Raviart [9]).

First, the solution of (4)-(6), if it exists, is unique, because from Weber [13], we know that $||\mathbf{v}||_X = \{||\mathrm{div}\,\mathbf{v}||_0^2 + ||\mathbf{curl}\,\mathbf{v}||_0^2\}^{1/2}$ is a norm equivalent to the canonical norm on X. Then, it is also solution of the variational problem (if $p = 0$):
Find $(\mathbf{u}, p) \in X \times L^2(\Omega)$ *such that*

$$\int_\Omega \mathbf{curl}\,\mathbf{u} \cdot \mathbf{curl}\,\mathbf{v}\,d\mathbf{x} + \int_\Omega p\,\mathrm{div}\,\mathbf{v}\,d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{curl}\,\mathbf{v}\,d\mathbf{x} \quad \forall \mathbf{v} \in X, \tag{7}$$

$$\int_\Omega \mathrm{div}\,\mathbf{u}\,q\,d\mathbf{x} = \int_\Omega g\,q\,d\mathbf{x} \quad \forall q \in L^2(\Omega). \tag{8}$$

So there remains to prove that the variational problem (7)-(8) has a unique solution. We shall do this with the help of the inf-sup theory (see for example [9]). From Weber [13], it is clear that the bilinear form $(\mathbf{u}, \mathbf{v}) \mapsto \int_\Omega \mathbf{curl}\,\mathbf{u} \cdot \mathbf{curl}\,\mathbf{v}\,d\mathbf{x}$ is coercive on the kernel of the second bilinear form (8), that is V. Moreover the inf-sup condition is satisfied. Indeed, Let $q \in L^2(\Omega)$. Then taking $\mathbf{v} = \nabla\xi$ with $\xi \in H_0^1(\Omega)$ such that $\Delta\xi = q$: we have $\mathbf{v} \in X$ and

$$\int_\Omega q\,\mathrm{div}\,\mathbf{v}\,d\mathbf{x} = \int_\Omega q\Delta\xi\,d\mathbf{x} = \int_\Omega q^2\,d\mathbf{x}.$$

It follows that

$$\inf_{q \in L^2(\Omega)} \sup_{\mathbf{v} \in X} \frac{\int_\Omega q\,\mathrm{div}\,\mathbf{v}\,d\mathbf{x}}{||\mathbf{v}||_X ||q||_0} \geq 1.$$

Hence the problem (7)-(8) has a unique solution. Finaly, let $\mathbf{v} = \nabla\xi$ with $\xi \in H_0^1(\Omega)$ such that $\Delta\xi = p$: we have $\mathbf{v} \in X$. Then (7) yields

$$\int_\Omega p^2\,d\mathbf{x} = 0,$$

which enables us to conclude. Indeed, (7) becomes

$$\int_\Omega \mathbf{curl}\,\mathbf{u} \cdot \mathbf{curl}\,\mathbf{v}\,d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{curl}\,\mathbf{v}\,d\mathbf{x} \quad \forall \mathbf{v} \in X.$$

Whence $\mathbf{curl}\,(\mathbf{curl}\,\mathbf{u}-\mathbf{f})=0$ and consequently, there exists $\varphi\in H^1(\Omega)$ such that $\mathbf{curl}\,\mathbf{u}-\mathbf{f}=\nabla\varphi$. In particular,

$$||\nabla\varphi||_0^2=\int_\Omega \mathbf{curl}\,\mathbf{u}\cdot\nabla\varphi\,d\mathbf{x}-\int_\Omega \mathbf{f}\cdot\nabla\varphi\,d\mathbf{x}.$$

As $\mathbf{u}\in X$, the first term cancels by integration by parts. The same is true for the second term, thanks to the hypotheses on $\mathbf{f}$. The conclusion follows. ■

Remark 3.1 The equation (5) can be brought back to the case $g=0$ by letting $\mathbf{v}=\mathbf{u}-\nabla\psi$, ψ being the unique element of $H_0^1(\Omega)$ verifying $\Delta\psi=g$. The function ψ verifies a Laplace problem (which has been studied by Grisvard [10]) that can be solved with a classical variational formulation. In order to simplify our presentation we shall suppose in the sequel that $g=0$. ■

3.2 Decomposition of the space of solutions

Let us introduce the space of regular solutions

$$X_R=\{\mathbf{v}\in H^1(\Omega)^3,\ \mathbf{v}\times\mathbf{n}=0 \text{ on } \Gamma\},\ X_{R,j}=\{\mathbf{v}\in X_R,\ \mathbf{v}\cdot\mathbf{n}_{|\Gamma_k}=0 \text{ for } k\neq j\}$$

and

$$V_R=\{\mathbf{z}\in H^1(\Omega)^3,\ \mathrm{div}\,\mathbf{z}=0,\ \mathbf{z}\times\mathbf{n}=0 \text{ on } \Gamma\}.$$

Proposition 3.1 *The spaces X_R and V_R are closed respectively in X and V.*

Proof: Costabel showed in [5] that on the space X_R we have the equality of the bilinear forms

$$\int_\Omega \nabla\mathbf{v}:\nabla\mathbf{w}\,d\mathbf{x}=\int_\Omega \mathbf{curl}\,\mathbf{v}\cdot\mathbf{curl}\,\mathbf{w}\,d\mathbf{x}+\int_\Omega \mathrm{div}\,\mathbf{v}\mathrm{div}\,\mathbf{w}\,d\mathbf{x}. \tag{9}$$

This equality remains obviously verified on V_R. The claimed results are a straightforward consequence. ■

We have the following results, for a given face Γ_j:

Proposition 3.2 *Let μ belong to $\widetilde{H}^{1/2}(\Gamma_j)$. Then there exists a lifting $\mathbf{v}\in X_{R,j}$ such that*

$$\mathbf{v}\cdot\mathbf{n}_{|\Gamma_j}=\mu.$$

Proof: A face Γ_j being fixed, we consider $\mu\in\widetilde{H}^{1/2}(\Gamma_j)$. We assume that the face Γ_j is imbedded in the plane of equation $x_3=0$. The reasoning is done in two stages:

(a) Case of a scalar function.

If we denote by $\tilde\mu$ the extension of μ by 0 on Γ, we have $\tilde\mu\in H^{1/2}(\Gamma)$. After Necas [12]:

$$\exists z_3\in H^1(\Omega),\ z_{3|\Gamma}=\tilde\mu.$$

(b) Case of a vector function.

If we take $z_1=z_2=0$ and denote by $\mathbf{z}=(z_1,z_2,z_3)^T$, we have $\mathbf{z}\in H^1(\Omega)^3$. By construction,

$$\mathbf{z}_{|\Gamma_k}=0 \text{ for } k\neq j,\ \mathbf{z}\times\mathbf{n}_{|\Gamma_j}=0 \text{ and } \mathbf{z}\cdot\mathbf{n}_{|\Gamma_j}=\mu.$$

In other terms, $\mathbf{z}\in X_{R,j}$ is a lifting of μ. ■

Proposition 3.3 *For a given face Γ_j, there exists a constant $C(\Gamma_j)$ such that*

$$\forall \mu \in \widetilde{H}^{1/2}(\Gamma_j), \quad \exists \mathbf{v} \in X_{R,j}, \quad \mathbf{v}\cdot\mathbf{n}_{|\Gamma_j} = \mu \text{ and } ||\mathbf{v}||_1 \leq C\,||\mu||_{\sim,1/2,\Gamma_j}.$$

Proof: We now consider the mapping:

$$\begin{aligned} X_{R,j} &\longrightarrow \widetilde{H}^{1/2}(\Gamma_j) \\ \mathbf{v} &\longmapsto \mathbf{v}\cdot\mathbf{n}_{|\Gamma_j}. \end{aligned}$$

Due to propositions 2.1 and 3.2 respectively, it is linear and continuous on the one hand, and onto on the other hand. Moreover its kernel is $\{\mathbf{v} \in X_{R,j},\ \mathbf{v}\cdot\mathbf{n}_{|\Gamma_j} = 0\} = H_0^1(\Omega)^3$.

The mapping $\mathbf{v} \longmapsto \mathbf{v}\cdot\mathbf{n}_{|\Gamma_j}$ is hence bijective, linear and continuous from $X_{R,j}/H_0^1(\Omega)^3$ into $\widetilde{H}^{1/2}(\Gamma_j)$.

The inverse mapping is thus also continuous thanks to the Banach-Steinhaus theorem. The conclusion follows. ∎

Proposition 3.4 *We have the following integration by parts formula:*

$$\begin{aligned} \forall (p,\mathbf{v}) \ &\in \ D(\Delta,\Omega)\times X_R, \\ &< \nabla p, \mathbf{v} >_{X_R',X_R} + \int_\Omega p \operatorname{div}\mathbf{v}\, d\mathbf{x} = \sum_i \langle p_{|\Gamma_i}, \mathbf{v}\cdot\mathbf{n}_{|\Gamma_i}\rangle_{\widetilde{H}^{-1/2}(\Gamma_i),\widetilde{H}^{1/2}(\Gamma_i)}. \end{aligned} \tag{10}$$

Proof: Let $p \in D(\Delta,\Omega)$. Due to proposition 2.4 we can choose a sequence $(p_k)_k$ of elements of $H^2(\Omega)$ such that $p_k \to p$ in $D(\Delta,\Omega)$. For $\mathbf{v} \in X_R$, we have the relation

$$\int_\Omega \nabla p_k\,\mathbf{v}\,d\mathbf{x} + \int_\Omega p_k \operatorname{div}\mathbf{v}\,d\mathbf{x} = \sum_i \int_{\Gamma_i} p_{k|\Gamma_i}\,\mathbf{v}\cdot\mathbf{n}_{|\Gamma_i}\,d\sigma_i.$$

As $\mathbf{v}\cdot\mathbf{n}_{|\Gamma_j} \in \widetilde{H}^{1/2}(\Gamma_j)$ thanks to proposition 2.1, and as we know from (2) that the trace mapping is continuous from $D(\Delta,\Omega)$ onto $\widetilde{H}^{-1/2}(\Gamma_j)$, we get

$$\int_{\Gamma_j} p_{k|\Gamma_j}\,\mathbf{v}\cdot\mathbf{n}_{|\Gamma_j}\,d\sigma_j \to < p_{|\Gamma_j}, \mathbf{v}\cdot\mathbf{n}_{|\Gamma_j}\rangle_{\widetilde{H}^{-1/2}(\Gamma_j),\widetilde{H}^{1/2}(\Gamma_j)}.$$

On the other hand, as $\int_\Omega p_k \operatorname{div}\mathbf{v}\,d\mathbf{x} \to \int_\Omega p \operatorname{div}\mathbf{v}\,d\mathbf{x}$, the term $\int_\Omega \nabla p_k\,\mathbf{v}\,d\mathbf{x}$ admits a limit when $k \to +\infty$. Moreover, for $k \neq l$,

$$\begin{aligned} \left|\int_\Omega (\nabla p_k - \nabla p_l)\,\mathbf{v}\,d\mathbf{x}\right| \ &\leq \ ||p_k - p_l||_0\,||\operatorname{div}\mathbf{v}||_0 + \sum_i ||p_k - p_l||_{\sim,-1/2,\Gamma_i}\,||\mathbf{v}\cdot\mathbf{n}||_{\sim,1/2,\Gamma_i} \\ &\leq \ C\,||p_k - p_l||_D\,||\mathbf{v}||_X, \end{aligned} \tag{11}$$

due to proposition 2.5 for $(p_k - p_l)$ and proposition 2.1 for $\mathbf{v}$. Thus, $(\nabla p_k)_k$ is a Cauchy sequence in the dual space of X_R. Hence it has a limit in this space. On the other hand,

$$\nabla p_k \to \nabla p \text{ in } H^{-1}(\Omega)^3.$$

As moreover $H_0^1(\Omega)^3 \subset X_R$, we have $X_R' \subset H^{-1}(\Omega)^3$ and consequently $(\nabla p_k)_k$ converges in X_R' to ∇p. The conclusion follows.

∎

In the case where the boundary Γ is smooth or when the domain Ω is convex we have the equality $V_R = V$. But in our case V_R is strictly included in V. Let us denote by V_S the orthogonal of V_R in V for the norm $\mathbf{v} \mapsto ||\mathbf{curl}\,\mathbf{v}||_0$ (which is indeed a norm equivalent to the canonical norm after [13]). We then have the decomposition into a direct orthogonal sum

$$V = V_R \overset{\perp}{\oplus} V_S. \tag{12}$$

We have the following characterization of the space V_S:

Theorem 3.2 *Let* $\mathbf{u}$ *be an element of* V. *Then* $\mathbf{u}$ *belongs to* V_S *if and only if there exists a* $p \in N$, *unique, such that* $\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} = \nabla p$ *in* $H_0(\mathbf{curl}\,, \Omega)'$.

Theorem 3.2 can be proved in the following way, which takes two steps. First we have the theorem

Theorem 3.3 *Let* $\mathbf{u} \in V$. *Then* $\mathbf{u}$ *belongs to* V_S *if and only if there exists a function* $p \in L^2(\Omega)$, *unique, such that*

$$\int_\Omega \mathbf{curl}\,\mathbf{u} \cdot \mathbf{curl}\,\mathbf{v}\, d\mathbf{x} + \int_\Omega p \,\mathrm{div}\,\mathbf{v}\, d\mathbf{x} = 0 \quad \forall \mathbf{v} \in X_R\,.$$

Proof: Let $\mathbf{u} \in V$, we have $\mathbf{u} \in V_S$ if and only if

$$\int_\Omega \mathbf{curl}\,\mathbf{u} \cdot \mathbf{curl}\,\mathbf{z}\, d\mathbf{x} = 0 \quad \forall \mathbf{z} \in V_R\,.$$

Consider then the linear form l defined by

$$l: \quad \mathbf{v} \mapsto < l, \mathbf{v} > = \int_\Omega \mathbf{curl}\,\mathbf{u} \cdot \mathbf{curl}\,\mathbf{v}\, d\mathbf{x}$$

defined and continuous on X_R which cancels on V_R. In particular, it is a continuous linear form on $H_0^1(\Omega)^3$ which cancels on

$$V = \{\mathbf{v} \in H_0^1(\Omega)^3; \mathrm{div}\,\mathbf{v} = 0\}\,.$$

Due to the de Rham theorem, there exists $p \in L^2(\Omega)$ (defined for the moment up to a constant) such that

$$< l, \mathbf{v} > = -\int_\Omega p \,\mathrm{div}\,\mathbf{v}\, d\mathbf{x} \quad \forall \mathbf{v} \in H_0^1(\Omega)^3$$

that is

$$\int_\Omega \mathbf{curl}\,\mathbf{u} \cdot \mathbf{curl}\,\mathbf{v}\, d\mathbf{x} + \int_\Omega p \,\mathrm{div}\,\mathbf{v}\, d\mathbf{x} = 0 \quad \forall \mathbf{v} \in H_0^1(\Omega)^3\,.$$

Now let $\mathbf{v} \in X_R$; first we assume that

$$\int_\Omega \mathrm{div}\,\mathbf{v}\, d\mathbf{x} = 0\,.$$

After [9], there exists a function $\mathbf{w} \in H_0^1(\Omega)^3$ such that $\mathrm{div}\,\mathbf{w} = -\mathrm{div}\,\mathbf{v}$.

Then the function $\mathbf{v} + \mathbf{w}$ of X_R verifies $\mathrm{div}\,(\mathbf{v} + \mathbf{w}) = 0$, so that $\mathbf{v} + \mathbf{w} \in V_R$. This implies

$$\int_\Omega \mathbf{curl}\,\mathbf{u} \cdot \mathbf{curl}\,(\mathbf{v} + \mathbf{w})\, d\mathbf{x} = 0,$$

that is

$$\int_\Omega \mathbf{curl}\,\mathbf{u} \cdot \mathbf{curl}\,\mathbf{v}\, d\mathbf{x} = -\int_\Omega \mathbf{curl}\,\mathbf{u} \cdot \mathbf{curl}\,\mathbf{w}\, d\mathbf{x} = \int_\Omega p \,\mathrm{div}\,\mathbf{w}\, d\mathbf{x} = -\int_\Omega p \,\mathrm{div}\,\mathbf{v}\, d\mathbf{x},$$

or

$$\int_\Omega \mathbf{curl}\,\mathbf{u} \cdot \mathbf{curl}\,\mathbf{v}\, d\mathbf{x} + \int_\Omega p \,\mathrm{div}\,\mathbf{v}\, d\mathbf{x} = 0.$$

Let us now consider any function $\mathbf{v} \in X_R$. We introduce a function $\phi_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ such that $\int_\Omega \Delta\phi_0 \, d\mathbf{x} = 1$ and set $\mathbf{v}_0 = \mathbf{grad}\, \phi_0$. Then,

$$\mathbf{v}_0 \in X_R, \quad \int_\Omega \operatorname{div} \mathbf{v}_0 \, d\mathbf{x} = 1.$$

We then set

$$\tilde{\mathbf{v}} = \mathbf{v} - (\int_\Omega \operatorname{div} \mathbf{v} \, d\mathbf{x})\mathbf{v}_0 \,,$$

so that $\tilde{\mathbf{v}} \in X_R$ verifies

$$\int_\Omega \operatorname{div} \tilde{\mathbf{v}} \, d\mathbf{x} = 0.$$

Due to the previous considerations, we have

$$\int_\Omega \mathbf{curl}\, \mathbf{u} \cdot \mathbf{curl}\, \tilde{\mathbf{v}} \, d\mathbf{x} + \int_\Omega p \operatorname{div} \tilde{\mathbf{v}} \, d\mathbf{x} = 0 \,.$$

But $\mathbf{curl}\, \tilde{\mathbf{v}} = \mathbf{curl}\, \mathbf{v}$ and so

$$\int_\Omega p \operatorname{div} \tilde{\mathbf{v}} \, d\mathbf{x} = \int_\Omega p \operatorname{div} \mathbf{v} \, d\mathbf{x} - (\int_\Omega p \operatorname{div} \mathbf{v}_0 \, d\mathbf{x}) \int_\Omega \operatorname{div} \mathbf{v} \, d\mathbf{x} = \int_\Omega (p - \lambda(p)) \operatorname{div} \mathbf{v} \, d\mathbf{x} \,,$$

with

$$\lambda(p) = \int_\Omega p \operatorname{div} \mathbf{v}_0 \, d\mathbf{x}.$$

(We notice that $p - \lambda(p)$ is determined uniquely). Replacing $p - \lambda(p)$ by p, we finally obtain that

$$\int_\Omega (\mathbf{curl}\, \mathbf{u} \cdot \mathbf{curl}\, \mathbf{v} + p \operatorname{div} \mathbf{v}) \, d\mathbf{x} = 0 \quad \forall \mathbf{v} \in X_R \,.$$

Conversely if $\mathbf{u} \in V$ satisfies this last relation, we have straightforwardly

$$\int_\Omega \mathbf{curl}\, \mathbf{u} \cdot \mathbf{curl}\, \mathbf{z} \, d\mathbf{x} = 0 \qquad \forall \mathbf{z} \in V_R,$$

so that $\mathbf{u} \in V_S$. Of course, the uniqueness of p comes immediately from the fact that $\mathbf{u} \in V_S$ and the first part of the proof. ■

The second step of the proof consists in a more explicit characterization of p.

Lemma 3.1 *Let* $\mathbf{u}$ *be an element of* $H_0(\mathbf{curl}\,, \Omega)$*. Then* $p \in L^2(\Omega)$ *verifies*

$$\int_\Omega (\mathbf{curl}\, \mathbf{u} \cdot \mathbf{curl}\, \mathbf{v} + p \operatorname{div} \mathbf{v}) \, d\mathbf{x} = 0 \quad \forall \mathbf{v} \in X_R$$

if and only if

$$\mathbf{curl}\, \mathbf{curl}\, \mathbf{u} = \nabla p \text{ in } H_0(\mathbf{curl}\,, \Omega)', \quad \text{and } p \in N.$$

Proof: Taking $\mathbf{v} \in \mathcal{D}(\Omega)^3$, we immediately have $\mathbf{curl}\, \mathbf{curl}\, \mathbf{u} = \nabla p$, in $H_0(\mathbf{curl}\,, \Omega)'$, as $\mathbf{curl}\, (\mathbf{curl}\, \mathbf{u}) \in \mathbf{curl}\, L^2(\Omega)^3 \subset H_0(\mathbf{curl}\,, \Omega)'$. Choosing $\mathbf{v} = \nabla\phi$, with a function $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$, we find

$$\int_\Omega p \, \Delta\phi \, d\mathbf{x} = 0,$$

so that $p \in \Delta(H^2(\Omega) \cap H_0^1(\Omega))^\perp = N$.

Conversely, if $p \in N$ verifies $\mathbf{curl}\, \mathbf{curl}\, \mathbf{u} = \nabla p$ in $H_0(\mathbf{curl}\,, \Omega)'$, we have on the one hand

$$< \mathbf{curl}\, \mathbf{curl}\, \mathbf{u}, \mathbf{v} >_{H_0(\mathbf{curl}\,)', H_0(\mathbf{curl}\,)} = \int_\Omega \mathbf{curl}\, \mathbf{u} \cdot \mathbf{curl}\, \mathbf{v} \, d\mathbf{x} \quad \forall \mathbf{v} \in X_R. \tag{13}$$

On the other hand (cf (10) proposition 3.4)

$$
\begin{aligned}
< \nabla p, \mathbf{v} >_{X_R', X_R} &= -\int_\Omega p \operatorname{div} \mathbf{v} \, d\mathbf{x} + \sum_i \langle p_{|\Gamma_i}, \mathbf{v}\cdot\mathbf{n}_{|\Gamma_i} \rangle_{\widetilde{H}^{-1/2}(\Gamma_i), \widetilde{H}^{1/2}(\Gamma_i)} \\
&= -\int_\Omega p \operatorname{div} \mathbf{v} \, d\mathbf{x} \quad \forall \mathbf{v} \in X_R.
\end{aligned} \tag{14}
$$

But, as X_R is dense in $H_0(\mathbf{curl}\,, \Omega)$, we have

$$
\forall \mathbf{v} \in X_R \quad < \nabla p, \mathbf{v} >_{X_R', X_R} = < \mathbf{curl}\,\mathbf{curl}\,\mathbf{u}, \mathbf{v} >_{H_0(\mathbf{curl}\,)', H_0(\mathbf{curl}\,)} .
$$

The result follows. ∎

Theorem 3.2 is then a straightforward consequence of theorem 3.3 and of lemma 3.1.

Remark 3.2 We also notice that in our case X_R is a genuine subspace of X. Let us denote by X_S the orthogonal of X_R in X for the inner product

$$
(\mathbf{v}, \mathbf{w}) \mapsto \int_\Omega \mathbf{curl}\,\mathbf{v} \cdot \mathbf{curl}\,\mathbf{w} \, d\mathbf{x} + \int_\Omega \operatorname{div} \mathbf{v} \operatorname{div} \mathbf{w} \, d\mathbf{x}.
$$

We then have the following decomposition into a direct orthogonal sum

$$
X = X_R \overset{\perp}{\oplus} X_S. \tag{15}
$$

The solution of (4)-(6) can be written $\mathbf{u} = \mathbf{u}_R' + \mathbf{u}_S'$, with $(\mathbf{u}_R', \mathbf{u}_S') \in X_R \times X_S$. In particular, for any $q \in H^2(\Omega) \cap H_0^1(\Omega)$, ∇q is orthogonal to $\mathbf{u}_s'$ for the above inner product. This reads

$$
\int_\Omega \Delta q \operatorname{div} \mathbf{u}_s' \, d\mathbf{x} = 0.
$$

Hence the definition of N yields that

$$
\operatorname{div} \mathbf{u}_S' \in N.
$$

Thus, if $\operatorname{div} \mathbf{u} = 0$, it follows that the divergence of the regular part is singular, i. e. $\operatorname{div} \mathbf{u}_R' \in N$. ∎

3.3 Saddle-point formulation

We now come to a saddle-point formulation in $H^1(\Omega)^3$ of the problem:
Find $\mathbf{u} \in V$ *solution of*

$$
\mathbf{curl}\,\mathbf{u} = \mathbf{f}\,.
$$

We have already seen, cf (7)-(8), that $\mathbf{u}$ is a solution of

$$
\int_\Omega \mathbf{curl}\,\mathbf{u} \cdot \mathbf{curl}\,\mathbf{v} \, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{curl}\,\mathbf{v} \, d\mathbf{x} \quad \forall \mathbf{v} \in X \tag{16}
$$

$$
\int_\Omega \operatorname{div} \mathbf{u} \, q \, d\mathbf{x} = 0 \quad \forall q \in L^2(\Omega). \tag{17}
$$

We use the decomposition

$$
\mathbf{u} = \mathbf{u}_R + \mathbf{u}_S \qquad \mathbf{u}_R \in V_R, \; \mathbf{u}_S \in V_S \,.
$$

To $\mathbf{u}_S$ we associate the unique function $p \in N$ such that

$$
\mathbf{curl}\,\mathbf{curl}\,\mathbf{u}_S = \nabla p \text{ in } H_0(\mathbf{curl}\,, \Omega)'.
$$

We can then characterize the pair $(\mathbf{u}_R, p)$ which consists of the regular part $\mathbf{u}_R$ and the function p associated to the singular part $\mathbf{u}_S$ of the solution $\mathbf{u}$ as the solution of a saddle-point problem in the space $X_R \times L^2(\Omega)$.

Theorem 3.4 *The pair* $(\mathbf{u}_R, p)$ *is the unique solution of the problem:*
Find $(\mathbf{u}_R, p) \in X_R \times L^2(\Omega)$ solution of

$$\int_\Omega (\mathbf{curl}\,\mathbf{u}_R \cdot \mathbf{curl}\,\mathbf{v} - p\,\mathrm{div}\,\mathbf{v})\,d\mathbf{x} = \int_\Omega \mathbf{f}\cdot\mathbf{curl}\,\mathbf{v}\,d\mathbf{x} \quad \forall \mathbf{v} \in X_R, \tag{18}$$

$$\int_\Omega \mathrm{div}\,\mathbf{u}_R\, q\,d\mathbf{x} = 0 \quad \forall q \in L^2(\Omega). \tag{19}$$

Proof: Let us verify that $(\mathbf{u}_R, p)$ indeed is a solution of the previous problem. Due to (16), we have

$$\int_\Omega (\mathbf{curl}\,\mathbf{u}_R \cdot \mathbf{curl}\,\mathbf{v} + \mathbf{curl}\,\mathbf{u}_S \cdot \mathbf{curl}\,\mathbf{v})\,d\mathbf{x} = \int_\Omega \mathbf{f}\cdot\mathbf{curl}\,\mathbf{v}\,d\mathbf{x} \quad \forall \mathbf{v} \in X\,.$$

Taking $\mathbf{v} \in X_R$ and regarding the definition of p, we have

$$\int_\Omega (\mathbf{curl}\,\mathbf{u}_R \cdot \mathbf{curl}\,\mathbf{v} - p\,\mathrm{div}\,\mathbf{v})\,d\mathbf{x} = \int_\Omega \mathbf{f}\cdot\mathbf{curl}\,\mathbf{v}\,d\mathbf{x} \quad \forall \mathbf{v} \in X_R,$$

which proves our claim.

It remains to verify the uniqueness of the solution. If $(\mathbf{u}_R, p) \in X_R \times L^2(\Omega)$ verifies (18)-(19) with $\mathbf{f} = 0$, taking $\mathbf{v} = \mathbf{u}_R \in V_R$, we obtain $\mathbf{curl}\,\mathbf{u}_R = 0$ and so, as $\mathbf{u} \mapsto ||\mathbf{curl}\,\mathbf{u}||_0$ is a norm on V, $\mathbf{u}_R = 0$. Thus

$$\int_\Omega p\,\mathrm{div}\,\mathbf{v}\,d\mathbf{x} = 0 \quad \forall \mathbf{v} \in X_R.$$

In particular, $\nabla p = 0$, hence p is constant. Whence

$$p\int_\Omega \mathrm{div}\,\mathbf{v}\,d\mathbf{x} = 0 \quad \forall \mathbf{v} \in X_R\,,$$

so $p = 0$. ∎

This is enough to prove theorem 3.4. However, let us now also check the existence and uniqueness of a solution to the saddle-point problem (18)-(19) using the inf-sup theory. First, we have already seen that the bilinear form $\int_\Omega \mathbf{curl}\,\mathbf{u}_R \cdot \mathbf{curl}\,\mathbf{v}\,d\mathbf{x}$ is coercive on V_R. Thus, all we need to check is the inf-sup condition:

$$\sup_{\mathbf{v}\in X_R} \frac{\int_\Omega p\,\mathrm{div}\,\mathbf{v}\,d\mathbf{x}}{||\mathbf{v}||_1} \geq \beta ||p||_0 \quad \forall p \in L^2(\Omega). \tag{20}$$

This will follow straightforwardly from

Proposition 3.5 *The divergence mapping from* $X_R \to L^2(\Omega)$ *is surjective, i.e.*

$$\mathrm{div}\,X_R = L^2(\Omega).$$

Proof: We have the inclusions

$$H_0^1(\Omega)^3 \subset X_R \text{ and } \nabla(H^2(\Omega)\cap H_0^1(\Omega)) \subset X_R.$$

Hence, as $L_0^2(\Omega) = \mathrm{div}\,H_0^1(\Omega)^3$ (see [9]),

$$L_0^2(\Omega) \subset \mathrm{div}\,X_R \text{ and } \Delta(H^2(\Omega)\cap H_0^1(\Omega)) \subset \mathrm{div}\,X_R.$$

We also have $\mathrm{div}\,X_R \subset L^2(\Omega)$, so

$$L_0^2(\Omega) + \Delta(H^2(\Omega)\cap H_0^1(\Omega)) \subset \mathrm{div}\,X_R \subset L^2(\Omega).$$

Moreover, obviously,

$$L^2(\Omega) = L_0^2(\Omega) \oplus I\!R, \tag{21}$$

the sum being orthogonal in $L^2(\Omega)$.

Then, assuming for the moment that there exists $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ such that $\Delta\phi$ does not belong to $L_0^2(\Omega)$, there exists a non vanishing real constant c such that $\Delta\phi = f + c$ with $f \in L_0^2(\Omega) = \operatorname{div} H_0^1(\Omega)^3$. Hence there exists $\mathbf{g} \in H_0^1(\Omega)^3$ such that $f = \operatorname{div} \mathbf{g}$. Then $\mathbf{y} = \nabla\phi - \mathbf{g}$, which belongs to X_R, is such that $\operatorname{div} \mathbf{y} = c$. Now as c is different from 0, any constant is the divergence of an element of X_R ($\mathbf{y}$ multiplied by the appropriate scalar).

Now as we already know that the elements of $L_0^2(\Omega)$ are the divergence of an element of X_R, the result follows thanks to the decomposition (21).

To end the proof we now need to prove that there exists indeed $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ such that $\Delta\phi$ does not belong to $L_0^2(\Omega)$. For this let us assume that $\Delta(H^2(\Omega) \cap H_0^1(\Omega)) \subset L_0^2(\Omega)$ and come to a contradiction:

Take $c \in I\!R \backslash \{0\}$. Then $c \in L^2(\Omega) = \Delta(H^2(\Omega) \cap H_0^1(\Omega)) \oplus N$. Hence

$$c = \Delta\phi + p, \quad \phi \in H^2(\Omega) \cap H_0^1(\Omega),\ p \in N,$$

and $\|c\|_0^2 = \|\Delta\phi\|_0^2 + \|p\|_0^2$. We have, because of the hypothesis, $\int_\Omega \Delta\phi \, d\mathbf{x} = 0$ and, due to the orthogonality of the decomposition, $\int_\Omega p\Delta\phi \, d\mathbf{x} = 0$. Then

$$\|c\|_0^2 = \int_\Omega c(\Delta\phi + p)\, d\mathbf{x} = \int_\Omega cp \, d\mathbf{x} = \int_\Omega (\Delta\phi + p)p \, d\mathbf{x} = \int_\Omega p^2 \, d\mathbf{x} = \|p\|_0^2.$$

Hence $\|\Delta\phi\|_0^2 = 0$, which implies that $c \in N$. Then, from Theorem 2.2, $c|_{\Gamma_j} = 0$ which implies $c = 0$. This contradicts our previous assumption, thus the proposition is proved. ■

To conclude, we note that the divergence mapping is continuous from X_R to $L^2(\Omega)$, and that its kernel is V_R. Thus, due to the Banach-Steinhaus theorem, the inverse mapping is also continuous from $L^2(\Omega)$ to X_R/V_R, which yields (20).

Once the pair $(\mathbf{u}_R, p)$ solution of the saddle-point problem (18)-(19) is obtained, there remains to determine the singular part $\mathbf{u}_S$ from p, that means solving

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{u}_S = \nabla p \text{ in } H_0(\mathbf{curl}\,, \Omega)', \text{ with } \mathbf{u}_S \in V.$$

Of course, this problem admits a unique solution due to theorem 3.2.

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