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SUFFICIENT CONDITIONS FOR MIXED BOUNDARY DATA TO HAVE A STEADY-STATE TWO-PHASE STEFAN-SIGNORINI PROBLEM THROUGH VARIATIONAL INEQUALITIES

DES CONDITIONS SUFFISANTES POUR DES DONNEES MIXTES AU BORD POUR OBTENIR UN PROBLEME STATIONNAIRE DE STEFAN-SIGNORINI A DEUX PHASES A TRAVERS DES INEQUATIONS VARIATIONNELLES

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ABSTRACT

We consider a steady-state heat conduction problem in a multidimensional bounded domain Ω which has a regular boundary Γ composed by the union of two parts Γ_1 and Γ_2. We assume, without loss of generality, that the melting temperature is zero degree centigrade. We consider a source term g in the domain. On the boundary Γ_2 we have a positive heat flux q and on the boundary Γ_1 we have a Signorini type condition with a positive external temperature b.

We obtain sufficient conditions on data q, g, b to obtain a change of phase (steady-state, two-phase, Stefan-Signorini problem) in Ω, that is a temperature of non-constant sign in Ω. We use the elliptic variational inequalities theory. We also find that the solution of the corresponding elliptic variational inequality is differentiable with respect to the Neumann datum q on Γ_2. Several properties already obtained for variational equalities can also be generalized for variational inequalities.

Moreover, by using the finite element method, when Ω is a polygonal domain, we also obtain sufficient conditions on data to obtain a steady-state, two-phase, discretized Stefan-Signorini problem in the corresponding discretized domain, that is a discrete temperature of non-constant sign in Ω. Then, error bounds between the continuous and discrete sufficient conditions are obtained in terms of the finite element approximation parameter h.

RÉSUMÉ

On considère un problème stationnaire de conduction de la chaleur dans un domaine multidimensionnel borné Ω qui a une frontière régulière Γ donnée par l'union de deux parties Γ_1 et Γ_2. On suppose, sans perte de généralité, que la température de changement de phase est zéro grade centigrade. On considère une source g dans le domaine. On a un flux de chaleur q positif sur Γ_2 et une condition de type Signorini sur Γ_1 avec une température externe b positive.

On obtient des conditions suffisantes sur les données q, g, b pour obtenir une température de signe non-constante dans Ω. On utilise la théorie des inéquations variationnelles elliptiques. On obtient aussi que la solution de l'inéquation variationnelle elliptique associée est dérivable par rapport à la donnée de Neumann q sur Γ_2 On généralise aux inéquations variationnelles des propriétés déjà obtenues pour des équations variationnelles.

En plus, en utilisant la méthode des éléments finis, quand Ω est un domaine polygonal, on obtient des conditions suffisantes sur les données pour obtenir un problème stationnaire discret de Stefan-Signorini à deux phases dans le domaine discret correspondant, c'est-à-dire une température discrète de signe non-constante dans Ω. Par ailleurs, des estimations de l'erreur entre les conditions suffisantes continues et discrètes, en fonction du paramètre h de l'approximation par éléments finis, son aussi obtenues.

Key words. Steady-state Stefan problem, Stefan-Signorini problem, free boundary problem, finite element method, mixed elliptic problem, numerical analysis, variational inequalities , error bounds.

Mots Clés. Problème de Stefan stationnaire, problème de Stefan Signorini, problème à frontière libre, méthode d'éléments finis, problème elliptique mixte, analyse numérique, inéquations variationnelles, estimation de l'erreur.

1. INTRODUCTION.

We consider a bounded domain $\Omega \in \mathbb{R}^n$ with regular boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ with $|\Gamma_2| = \text{meas}(\Gamma_2) > 0$ and $|\Gamma_1| > 0$. We suppose that $\Gamma_1 = \Gamma_{1_t} \cup \Gamma_{1_s}$ with $|\Gamma_{1_i}| > 0$ for i = t, s.

We consider a steady-state heat conduction problem in Ω. We assume, without loss of generality, that the melting temperature is zero degree centigrade. We consider a source term g in the domain Ω. On the boundary Γ_2 we have a positive heat flux q and on the boundary Γ_{1_t} we impose a positive temperature b. On the boundary Γ_{1_s} we have a Signorini type condition with a positive external temperature b. If θ is the temperature of the material we can consider the new unkown function in Ω defined by [Du,Ta1]

(1) $$u = k_2 \theta^+ - k_1 \theta^-$$

where $k_i > 0$ is the thermal conductivity of the phase i (i = 1: solid phase, i = 2: liquid phase). Let $B = k_2 b > 0$ where $b > 0$ is the temperature imposed on Γ_{1_t}.

We consider the following steady-state, Stefan-Signorini, free boundary problem

(2) $$-\Delta u = g \quad \text{in } \Omega,$$

(3) $$-\frac{\partial u}{\partial n} \Big| _{\Gamma_2} = q \quad \text{on } \Gamma_2,$$

(4) $$u \,|_{\Gamma_{1_t}} = B \quad \text{on } \Gamma_{1t},$$

(5) $$u \geq B, \ \frac{\partial u}{\partial n} \geq 0, \ (u - B) \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_{1_s}.$$

The conditions (5) representing unilateral constraints on Γ_{1_s} are called the Signorini conditions. They describes heat (or a volume) injection through a semi-permeable wall (see [DuLi]).

The mean goal of this paper is to find the continuous and discrete (by finite element method) sufficient conditions on q= Const. > 0 on Γ_2 in order to obtain a temperature u of non-constant sign in Ω, that is a steady-state, two-phase, Stefan-Signorini problem. We follow [Ta3] and [Ta4] where the pioneer ideas for the continuous and discrete analysis were developed respectively without Signorini conditions. When $\Gamma_1 = \Gamma_{1_t}$ (i. e. $\Gamma_{1_s} = \emptyset$) the corresponding free boundary problem without Signorini boundary conditions was studied in [GaTa]. We follow a method similar to the one developed in [BoShTa, GaTa, GoTa, Sa, Ta2, Ta3, Ta4] for some particular cases. The Signorini problem is considered in [DiJi,DuLi]. The Stefan-Signorini problem is also considered for the multidimensional case in [To] and for the unidimensional case in [Ke,Wa].

In Section 2 we obtain the continuous sufficient condition on the heat flux on Γ_2 in order to have a temperature wich takes positive and negative values in Ω. Previously, we obtain that the solution of the corresponding elliptic variational inequality is differentiable with respect to the Neumann datum q on Γ_2 and we give its explicit expression. We also obtain a relationship between the

Stefan-Signorini problem and the corresponding related problem without Signorini condition on Γ_{1_s}.

In Section 3, through the finite element approximation method, we obtain the corresponding discrete results for the related continuous problem carried out in the previous Section.

In Section 4, we obtain the estimates between the continuous and discrete sufficient conditions (obtained in Section 2 and Section 3, respectively) in terms of the finite element approximation parameter h.

2. CONTINUOUS ANALYSIS.

The variational formulation of the problem $(2)-(5)$ is given by

$$(6)\qquad \left| \begin{array}{l} a(u,v-u) \geq L(v-u), \quad \forall\, v \in K_B \\ u \in K_B \end{array} \right.$$

where

$$(7)\qquad \left| \begin{array}{ll} V = H^1(\Omega), & V_o = \left\{ v \in V \,/\, v \mid_{\Gamma_{1_t}} = 0 \right\}, \\ W_o = \left\{ v \in V \,/\, v|_{\Gamma_{1_t}} = v \mid_{\Gamma_{1_s}} = 0 \right\} \subset V_o\,, & \\ K_B = \left\{ v \in V \,/\, v \mid_{\Gamma_{1_t}} = B\,,\, v \mid_{\Gamma_{1_s}} \geq B \right\} = B + K_o\,, & \\ K_o = \left\{ v \in V \,/\, v \mid_{\Gamma_{1_t}} = 0\,,\, v \mid_{\Gamma_{1_s}} \geq 0 \right\} \supset W_0\,, & \end{array} \right.$$

and

$$(8)\qquad a(u,v) = \int_\Omega \nabla u \,.\, \nabla v \, dx\,, \qquad L(v) = L_{qg}(v) = \int_\Omega g\, v\, dx - \int_{\Gamma_2} q\, v\, d\gamma.$$

For $g \in L^2(\Omega)$, $b > 0$ and $q \in L^2(\Gamma_2)$, we have a unique solution $u = u_{qgB}$ (it will be generally denoted by u_q) of the variational inequality (6) [KiSt, LiSt,Ta2].

Lemma 1. The solution u of the variational inequality (6) satisfies the following inequality:

$$(9)\qquad a(u^-,u^-) + \int_\Omega g\, u^-\, dx \leq \int_{\Gamma_2} q\, u^-\, d\gamma\,.$$

where u^- is the negative part of u.

Proof. – If we choose $v = u^+ \in K_B$ in the variational inequality (6) corresponding to the solution $u \in K$ and we take in account that $u = u^+ - u^-$ then we obtain (9). □

Corollary 2. If $g \geq 0$ $(g \not\equiv 0)$ in Ω and $q > 0$ over Γ_2 then we have

$$(10) \qquad u^- \not\equiv 0 \text{ in } \bar{\Omega} \Leftrightarrow u^- \not\equiv 0 \text{ over } \Gamma_2 . \qquad \square$$

Remark 1. In order to obtain a temperature of non-constant sign in Ω, by Corolary 2, it is sufficient to have that the temperature takes negative values on Γ_2. □

We obtain for $u = u_q$ the following properties:

Lemma 3. (i) If u_{q_i} is the solution for data $q_i \in \mathbb{R}$ $(i = 1, 2)$, $B > 0$ and $g \in L^2(\Omega)$, then we have

$$(11) \qquad \alpha \, ||u_{q_2} - u_{q_1}||_V^2 \leq a(u_{q_2} - u_{q_1}, u_{q_2} - u_{q_1}) \leq (q_1 - q_2) \int_{\Gamma_2} (u_{q_2} - u_{q_1}) \, d\gamma,$$

where $\alpha > 0$ is the coercive constant of the bilinear form a.

(ii) The functions $\mathbb{R}^+ \rightarrow V$ and $\mathbb{R}^+ \rightarrow L^2(\Gamma_2)$ given by $q \rightarrow u_q \in V$ and $q \rightarrow u_q \in L^2(\Gamma_2)$ respectively, are Lipschitzian, that is

$$(12) \qquad \begin{cases} \text{(i)} & || u_{q_2} - u_{q_1} ||_V \leq \dfrac{|\Gamma_2|^{\frac{1}{2}} \, || \gamma_0 ||}{\alpha} \, | q_2 - q_1 | \\[2ex] \text{(ii)} & || u_{q_2} - u_{q_1} ||_{L^2(\Gamma_2)} \leq \dfrac{|\Gamma_2|^{\frac{1}{2}} \, || \gamma_0 ||^2}{\alpha} \, | q_2 - q_1 | \end{cases}$$

where γ_0 is the trace operator over V.

(iii) The function $\mathbb{R}^+ \rightarrow \mathbb{R}$,

$$(13) \qquad q \rightarrow \int_{\Gamma_2} u_q \, d\gamma$$

is continuous and strictly decreasing function. Moreover, we have

$$(14) \qquad q_1 \leq q_2 \;\Rightarrow\; \text{(i) } u_{q_1} \leq u_{q_2} \text{ in } \bar{\Omega} \qquad \text{and} \qquad \text{(ii) } \int_{\Gamma_2} u_{q_2} \, d\gamma \leq \int_{\Gamma_2} u_{q_1} \, d\gamma .$$

(iv) There exist $u'_q \in V_0$ such that:

$$(15) \qquad \begin{cases} \text{(i)} & \dfrac{u_{q+\delta} - u_q}{\delta} \rightarrow u'_q \text{ in V-weak, when } \delta \rightarrow 0, \\[2ex] \text{(ii)} & \dfrac{u_{q+\delta} - u_q}{\delta} \rightarrow u'_q \text{ in } L^2(\Gamma_2)\text{-weak, when } \delta \rightarrow 0, \end{cases}$$

and

(16) $$a(u_q, u'_q) = L(u'_q) = \int_\Omega g\, u'_q\, dx - q \int_\Omega u'_q\, d\gamma\,.$$

Proof. – (i) If we choose $v = u_2 \in K_B$ and $v = u_1 \in K_B$ in (6) corresponding to the solution u_1 and u_2 respectively, we add them and we use the coerciveness of the bilinear form a then we obtain the inequalities (11).

(ii) and (iii) From (11) we obtain (14ii) and using the Cauchy-Schwarz inequality and the trace operator γ_0 we deduce (12i) and (12ii).

Let be $q_1 \le q_2$. Let $w = (u_{q_1} - u_{q_2})^-$ be. We shall prove that $w = 0$ in $\bar{\Omega}$, that is $u_{q_2} \le u_{q_1}$ in $\bar{\Omega}$. We have that $w \in K_0 \subset V_0$. We choose $v = u_{q_1} + w \in K_B$ in the variational inequality (6) for u_{q_1} for datum q_1 and $v = u_{q_2} - w \in K_B$ in the variational inequality for u_{q_2} for datum q_2, respectively , we add them and we obtain

(17) $$\alpha \| w \|^2 \le a(w,w) \le -\int_{\Gamma_2} (q_2 - q_1)\, w\, d\gamma \le 0$$

that is $w = 0$ in $\bar{\Omega}$ because $w \in K_0$.

For the other hand, (13) is a continuous function because

(18) $$\left| \int_{\Gamma_2} u_{q+\delta}\, d\gamma - \int_{\Gamma_2} u_q\, d\gamma \right| \le \int_{\Gamma_2} \left| u_q - u_{q+\delta} \right| d\gamma \le \frac{|\Gamma_2| \, \| \gamma_0 \|^2}{\alpha} |\delta| \;\longrightarrow\; 0 \quad \text{when } \delta \to 0\,.$$

(iv) From (12)(i)-(ii) we have that $\frac{u_{q+\delta} - u_q}{\delta}$ is bounded in V and $L^2(\Gamma_2)$, that is there exists $u'_q \in V_0$ such that (15)(i)-(ii) are satisfied.

Moreover if we choose $v = u_{q+\delta} \in K_B$ in the variational inequality (6) for u_q, we divide by δ then we obtain

(19) $$a\Big(u_q, \frac{u_{q+\delta} - u_q}{\delta}\Big) \ge L\Big(\frac{u_{q+\delta} - u_q}{\delta}\Big)$$

and taking the limit $\delta \to 0$ we deduce

(20) $$a(u_q, u'_q) \ge L(u'_q).$$

Similarity, if we choose $v = u_q \in K_B$ in the variational inequality (6) for $u_{q+\delta}$, we divide by δ and take the limit when $\delta \to 0$ we obtain

(21) $$a(u_q, u'_q) \le L(u'_q)$$

that is (16). □

Remark 2. We shall obtain that u'_q can be obtained as the unique solution of a variational inequality (see (36) and (38)) which does not depend on q, we remark that the equality (16) will play an important role in order to obtain this assertion. □

The element u, unique solution of (6), is also characterized by the following minimization problem:

$$(22) \qquad \begin{cases} J(u) \leq J(v), \quad \forall v \in K_B \\ u \in K_B, \end{cases}$$

where

$$(23) \qquad J(v) = J_{qg}(v) = \frac{1}{2} a(v,v) - L_{qg}(v) .$$

We can define the real function $f\colon \mathbb{R}^+ \to \mathbb{R}$ in the following way (see [Ta3])

$$(24) \qquad f(q) = J(u_q) = \frac{1}{2} a(u_q, u_q) - \int_\Omega g\, u_q \, dx + q \int_{\Gamma_2} u_q \, d\gamma$$

where u_q $(= u_{qgB})$ is the unique solution of the variational inequality (6) for each heat flux $q > 0$. Then we obtain the following mean property, wich is the cornerstone of this paper to obtain other results.

Theorem 4. The function f is differentiable. Moreover, f′ is a continuous and strictly decreasing function, and it is given by the following expression

$$(25) \qquad f'(q) = \int_{\Gamma_2} u_q \, d\gamma .$$

Proof. – By using the definitions of function f and some elementary computations we have

$$(26) \qquad \frac{f(q+\delta) - f(q)}{\delta} = \frac{1}{2} a\left(u_{q+\delta} + u_q , \frac{u_{q+\delta} - u_q}{\delta}\right) - \int_\Omega g \frac{u_{q+\delta} - u_q}{\delta} dx + {}$$

$$+ \int_{\Gamma_2} u_{q+\delta} \, d\gamma + q \int_{\Gamma_2} \frac{u_{q+\delta} - u_q}{\delta} \, d\gamma$$

and taking the limit when $\delta \to 0$ we deduce, for all $q > 0$, that

$$(27) \qquad f'(q) = \lim_{\delta \to 0} \frac{f(q+\delta) - f(q)}{\delta} = \int_{\Gamma_2} u_q \, d\gamma + a(u_q, u_q') - L(u_q') = \int_{\Gamma_2} u_q \, d\gamma ,$$

that is (25). □

Remark 3. We have obtained for the derivative of function f, the same expression (25) which we have yet deduced for variational equality [Ta3,GaTa] instead of variational inequality. □

Corollary 5. We have the following properties:

(28) $$\frac{d}{dq}\left(\int_{\Omega} g\, u_q\, dx\right) = \int_{\Omega} g\, u_q'\, dx$$

(29) $$\frac{d}{dq}\left[a(u_q, u_q)\right] = 2\, a(u_q, u_q')$$

(30) $$f''(q) = \int_{\Gamma_2} u_q'\, d\gamma\ .$$

Proof. – We have

(31) $$\frac{d}{dq}\left(\int_{\Omega} g\, u_q\, dx\right) = \lim_{\delta\to 0} \frac{\int_{\Omega} g\, u_{q+\delta}\, dx - \int_{\Omega} g\, u_q\, dx}{\delta} =$$

$$= \lim_{\delta\to 0} \int_{\Omega} g\, \frac{u_{q+\delta} - u_q}{\delta}\, dx = \int_{\Omega} g\, u_q'\, dx$$

(32) $$\frac{d}{dq}\left[\, a(u_q, u_q)\,\right] = \lim_{\delta\to 0} \frac{a(u_{q+\delta}, u_{q+\delta}) - a(u_q, u_q)}{\delta} =$$

$$= \lim_{\delta\to 0} a\left(u_{q+\delta} + u_q, \frac{u_{q+\delta} - u_q}{\delta}\right) = 2\, a(u_q, u_q')$$

By differentiating (24) with respect to q we obtain that

(33) $$f''(q) = \frac{1}{q}\left[\frac{d}{dq}\left(\int_{\Omega} g\, u_q\, dx\right) - \frac{1}{2}\frac{d}{dq}\left[a(u_q, u_q)\right]\right] = \frac{1}{q}\left[\left(\int_{\Omega} g\, u_q'\, dx\right) - a(u_q, u_q')\right] = \int_{\Gamma_2} u_q'\, d\gamma\ .$$

□

Theorem 6. We have the following properties:

(i) The element u_q can be written by

(34) $$u_q = u_{qgB} = B + U_g + q\,\eta$$

where U_g is the unique solution of the following elliptic variational equality

(35) $$\left|\begin{array}{l} a(U_g, v) = \int_{\Omega} g\, v\, dx, \quad \forall\, v \in W_o \\ U_g \in W_o\ . \end{array}\right.$$

and η is the unique solution of the following elliptic variational inequality:

(36) $$\left| \begin{array}{l} a(\eta, v-\eta) \geq - \int_{\Gamma_2} (v-\eta)\, d\gamma\ , \quad \forall\, v \in K_o \\ \eta \in K_o . \end{array} \right.$$

Moreover, $\eta/\Gamma_2 \leq 0$ with

(37) $$- \int_{\Gamma_2} \eta\, d\gamma \geq a(\eta,\eta) \geq \alpha \, \|\eta\|_V^2 \ > 0 \ .$$

(ii) The element u_q' does not depend on q, that is

(38) $$u_q' = \eta \in K_0 \ .$$

(iii) We have

(39) $$f'(q) = (B\,|\Gamma_2| + C_g) \ - \ D\,q \ , \qquad f''(q) = \int_{\Gamma_2} \eta\, d\gamma$$

where

(40) $$C_g = \int_{\Gamma_2} U_g\, d\gamma \ , \qquad D = - \int_{\Gamma_2} \eta\, d\gamma > 0 \ .$$

Proof. – (i) The element $\sigma = B + U_g + q\,\eta \in K_B$ is a solution of (6) by splitting problem (2)-(5) in three parts.

(ii) On the other hand, if we choose $v = -\eta^- \in K_0$ (because $\eta^-/\Gamma_{1_s} = 0$) in (6) we obtain

(41) $$0 \geq -a(\eta^+,\eta^+) = -a(\eta,\eta^+) \geq \int_{\Gamma_2} \eta^+\, d\gamma$$

that is $\eta^+/\Gamma_2 = 0$, then $\eta/\Gamma_2 \leq 0$.

Moreover, by taking $v = 0 \in K_0$ in (6) we obtain

(42) $$a(\eta, -\eta) \geq \int_{\Gamma_2} \eta\, d\gamma$$

that is (37).

(iii) The results (39) are obtained from (24), (25), (34) and (38). □

We can define the real function $R = R(B,g)$ in the following way

(43) $$R(B,g) = \frac{B\,|\Gamma_2| + C_g}{D} \ .$$

Theorem 7. For $B > 0$ and $g \in L^2(\Omega)$, we have:

(44) $$q > R(B,g) \;\Rightarrow\; u \text{ is of non-constant sign in } \Omega,$$

i.e., there exists a steady-state, two-phase, Stefan-Signorini problem.

Proof. – The result (44) is obtained by considering the following equivalence

(45) $$q > R(B,g) \quad \Leftrightarrow \quad f'(q) = \int_{\Gamma_2} u_q \, d\gamma < 0 . \qquad \Box$$

We shall obtain a relationship between problem (2)-(5) with the same problem without Signorini boundary conditions on Γ_{1_s}, that is we consider the following problem [GaTa]

(46) $$\left| \begin{array}{l} -\Delta z = g \quad \text{in } \Omega \\ z/\Gamma_1 = B, \qquad -\frac{\partial z}{\partial n}/\Gamma_2 = q \end{array} \right.$$

where $\Gamma_1 = \Gamma_{1_t} \cup \Gamma_{1_s}$, whose variational inequality is given by

(47) $$\left| \begin{array}{l} a(z, v - z) = L(v - z), \quad \forall\, v \in K \\ z \in K \end{array} \right.$$

where

(48) $$K = \{ v \in V \mid v/\Gamma_1 = B \} \subset K_B .$$

Theorem 8. (i) The unique solution z of the variational equality (47) is given by

(49) $$z = z_{qgB} = B + U_g - q\,\xi$$

where ξ is the unique solution of the variational equality

(50) $$\left| \begin{array}{l} a(\xi, v) = \int_{\Gamma_2} v \, d\gamma, \quad \forall\, v \in W_o \\ \xi \in W_o . \end{array} \right.$$

(ii) We have the following relationship between problems (2)-(5) and (46)

(51) $$\left| \begin{array}{ll} \text{(i)} & u \geq z \quad \text{in } \bar{\Omega}, \\ \text{(ii)} & \eta \geq -\xi \quad \text{in } \bar{\Omega}, \qquad \text{(iii)} \quad \xi > 0 \text{ in } \Omega . \end{array} \right.$$

(iii) We have

$$C_g \equiv \int_{\Gamma_2} U_g \, d\gamma = a(\xi, U_g) = \int_{\Omega} g \xi \, dx . \tag{52}$$

Proof. – (i) The element $B + U_g - q\xi \in K$ is a solution of (47) by splitting problem (46) in three parts. Then, the equality (49) is given by uniqueness of the variational equality (47) [GaTa].

(ii) To prove (51i) we shall consider the equivalence

$$u \geq z \ \text{in } \bar{\Omega} \quad \Leftrightarrow \quad \psi = 0 \ \text{in } \bar{\Omega} \tag{53}$$

where $\psi = (u - z)^{-}$. We have that $\psi/\Gamma_1 = 0$ because

$$\left| \begin{array}{l} (u - z)/\Gamma_{1_t} = u/\Gamma_{1_t} - z/\Gamma_{1_t} = B - B = 0 . \\[2ex] (u - z)/\Gamma_{1_s} = u/\Gamma_{1_s} - z/\Gamma_{1_s} \geq B - B = 0 . \end{array} \right. \tag{54}$$

If we choose $v = z + \psi \in K$ in the variational equality (47) for z, we obtain

$$a(z, \psi) = \int_{\Omega} g \psi \, dx - q \int_{\Gamma_2} \psi \, d\gamma . \tag{55}$$

If we choose $v = u + \psi \in K_B$ in the variational inequality (6) for u, we obtain

$$- a(u, \psi) \leq - \int_{\Omega} g \psi \, dx + q \int_{\Gamma_2} \psi \, d\gamma . \tag{56}$$

If we add (55) and (56) we deduce $a(z - u, \psi) \leq 0$ that is $a(u - z, \psi) \leq 0$. Therefore we get $a(\psi, \psi) \leq 0$, that is $\psi = 0$ in $\bar{\Omega}$ because $\psi/\Gamma_1 = 0$.

From (34), (49) and (51i) we deduce that $\eta \geq -\xi$ in $\bar{\Omega}$.

On the other hand we have $\xi > 0$ in Ω because if we choose $v = \xi^{-} \in W_0$ in the variational equality (47) for ξ we deduce

$$\alpha \| \xi^{-} \|^2 \leq a(\xi^{-}, \xi^{-}) = - a(\xi, \xi^{-}) = - \int_{\Gamma_2} \xi^{-} \, d\gamma \ \leq 0 \tag{57}$$

that is $\xi^{-} = 0$ in $\bar{\Omega}$. □

We define

$$\left| \begin{array}{ll} C = \int_{\Gamma_2} \xi \, d\gamma = a(\xi, \xi) > 0 , & q_0(B) = \dfrac{B|\Gamma_2|}{C} \\[3ex] Q(B, g) = \dfrac{B|\Gamma_2| + C_g}{C} = q_0(B) + \dfrac{C_g}{C} & \end{array} \right. \tag{58}$$

where C is a geometric constant which appears in [Ta3].

Theorem 9. (i) If $g \in L^2(\Omega)$ verifies the condition

(59) $$B|\Gamma_2| + C_g = B|\Gamma_2| + \int_\Omega g\,\xi\,dx > 0$$

then

(60) $$R(B,g) \geq Q(B,g).$$

(ii) Under the hypothesis (59), if there exists a steady-state, two-phase, Stefan-Signorini problem (i. e. u is of non-constant sign in $\bar{\Omega}$) then there exists a steady-state, two-phase, Stefan problem without Signorini boundary condition on Γ_{1_s} (i. e. z is of non-constant sign in $\bar{\Omega}$).
(iii) If $q > Q(B,g)$ then there exists a steady-state, two-phase, Stefan problem without Signorini boundary condition on Γ_{1_s} (i. e. z is of non-constant sign in $\bar{\Omega}$).

Proof. We have

(61) $$0 < \int_{\Gamma_2} \eta\,d\gamma \leq \int_{\Gamma_2} \xi\,d\gamma,$$

then

(62) $$\frac{1}{D} = \frac{1}{-\int_{\Gamma_2} \eta\,d\gamma} \geq \frac{1}{\int_{\Gamma_2} \xi\,d\gamma} = \frac{1}{C}.$$

Therefore, if condition (59) is verified then

(63) $$R(B,g) = \frac{B|\Gamma_2| + C_g}{D} \geq \frac{B|\Gamma_2| + C_g}{C} = Q(B,g).$$

The conclusions (ii) and (iii) follow from (60) and [GaTa], respectively.

Remark 4. – The condition (59) is trivially verified for all $g \in L^2(\Omega)$ such that $g \geq 0$ in $\bar{\Omega}$.

3. NUMERICAL ANALYSIS.

We suppose that $\Omega \subset \mathbb{R}^n$ is a convexe polygonal bounded domain. We consider τ_h, a regular triangulation of the polygonal domain Ω with Lagrange triangles of type 1, constituted by affine-equivalent finite element of class C^0, where $h > 0$ is a parameter which goes to zero. We can take h equal to the longest side of the triangles $T \in \tau_h$ [BrSc, Ci, GlLiTr]. We follow the method developed in [Ta3] in order to obtain the discrete equivalent of the continuous result (44).

The variational formulation of the discretized problem corresponding to the continuous problem (6) is given by

$$(64)\qquad \left| \begin{array}{l} a(u_h, v_h - u_h) \geq L(v_h - u_h), \qquad \forall\, v_h \in K_{B_h} \\ u_h \in K_{B_h} \end{array} \right.$$

where

$$(65)\qquad \left| \begin{array}{l} K_{B_h} = B + K_{0_h} \subset K_B, \quad P_1 = \text{set of polynomials of degree } \leq 1 \\ K_{0_h} = \left\{ v_h \in C^0(\bar{\Omega}) \,/\, v_h/T \in P_1(T),\, \forall\, T \in \tau_h,\, v_h/\Gamma_{1t} = 0,\, v_h/\Gamma_{1_s} \geq 0 \right\} \\ V_{0_h} = \left\{ v_h \in C^0(\bar{\Omega}) \,/\, v_h/T \in P_1(T),\, \forall\, T \in \tau_h,\, v_h/\Gamma_{1t} = 0 \right\} \\ W_{0_h} = \left\{ v_h \in C^0(\bar{\Omega}) \,/\, v_h/T \in P_1(T),\, \forall\, T \in \tau_h,\, v_h/\Gamma_{1t} = 0,\, v_h/\Gamma_{1_s} = 0 \right\} \end{array} \right.$$

with

$$(66)\qquad W_{0_h} \subset K_{0_h} \subset V_{0_h}, \qquad\qquad W_{0_h} \subset W_0,\; K_{0_h} \subset K_0,\; V_{0_h} \subset V_0.$$

For $g \in L^2(\Omega)$, $b > 0$ (i.e. $B > 0$) and $q > 0$, the unique solution of the variational inequality (64) [Ci,KiSt] will be denoted by $u_h = u_{h,q}$. The element $u_{h,q}$ is also characterized by the minimization problem:

$$(67)\qquad \left| \begin{array}{l} J(u_{h,q}) \leq J(v_h), \quad \forall\, v_h \in K_{B_h} \\ u_{h,q} \in K_{B_h}. \end{array} \right.$$

where J are defined by (23).

For each $h > 0$, we define the real function $f_h\colon \mathbb{R}^+ \to \mathbb{R}$ in the following way

$$(68)\qquad f_h(q) = J(u_{h,q}) = \frac{1}{2}\, a\left(u_{h,q}, u_{h,q}\right) - L_q\left(u_{h,q}\right).$$

We obtain the following properties for the discrete solution $u_h = u_{h,q}$ of the elliptic variational inequality (64).

Theorem 10. For all $h > 0$, we have the following properties:

(i) We have

(69) $$a(u_h^-, u_h^-) + \int_\Omega g\, u_h^- \, dx \le \int_{\Gamma_2} q\, u_h^- \, d\gamma .$$

(ii) If $g \ge 0$ $(g \not\equiv 0)$ in Ω and $q > 0$ over Γ_2 then we have

(70) $$u_h^- \not\equiv 0 \text{ in } \bar\Omega \Leftrightarrow u_h^- \not\equiv 0 \text{ over } \Gamma_2 .$$ □

(iii) If u_{h, q_i} is the solution for data $q_i \in \mathbb{R}$ $(i = 1, 2)$, $B > 0$ and $g \in L^2(\Omega)$, then we have

(71) $$\alpha \| u_{h,q_2} - u_{h,q_1} \|_V^2 \le a(u_{h,q_2} - u_{h,q_1}, u_{h,q_2} - u_{h,q_1}) \le (q_1 - q_2) \int_{\Gamma_2} (u_{h,q_2} - u_{h,q_1})\, d\gamma .$$

(iv) The functions $\mathbb{R}^+ \to V$ and $\mathbb{R}^+ \to L^2(\Gamma_2)$ given by $q \to u_{h,q} \in V$ and $q \to u_{h,q} \in L^2(\Gamma_2)$ respectively, are Lipschitzian, that is

(72) $$\begin{cases} \text{(i)} & \| u_{h,q_2} - u_{h,q_1} \|_V \le \dfrac{|\Gamma_2|^{\frac{1}{2}} \| \gamma_0 \|}{\alpha} | q_2 - q_1 | , \\ \text{(ii)} & \| u_{h,q_2} - u_{h,q_1} \|_{L^2(\Gamma_2)} \le \dfrac{|\Gamma_2|^{\frac{1}{2}} \| \gamma_0 \|^2}{\alpha} | q_2 - q_1 | . \end{cases}$$

(v) The function $\mathbb{R}^+ \to \mathbb{R}$,

(73) $$q \to \int_{\Gamma_2} u_{h,q}\, d\gamma$$

is continuous and strictly decreasing function. Moreover, we have

(74) $$q_1 \le q_2 \;\Rightarrow\; \text{(i) } u_{h,q_1} \le u_{h,q_2} \text{ in } \bar\Omega \text{ and } \quad \text{(ii) } \int_{\Gamma_2} u_{h,q_2}\, d\gamma \le \int_{\Gamma_2} u_{h,q_1}\, d\gamma .$$

(vi) For all $q > 0$, there exists $u'_{h,q} \in V_{0_h}$ such that

(75) $$\frac{u_{h,q+\delta} - u_{h,q}}{\delta} \to u'_{h,q} \text{ in } V\text{-weak, when } \delta \to 0$$

(76) $$\frac{u_{h,q+\delta} - u_{h,q}}{\delta} \to u'_{h,q} \text{ in } L^2(\Gamma_2)\text{-weak, when } \delta \to 0$$

(77) $$a(u_{h,q}, u'_{h,q}) = \int_\Omega g\, u'_{h,q}\, dx - q \int_{\Gamma_2} u'_{h,q}\, d\gamma .$$

(vii) The function f_h is differentiable. Moreover, we have the following expressions:

(78) $$\text{(i) } f'_h(q) = \int_{\Gamma_2} u_{h,q}\, d\gamma , \qquad \text{(ii) } f'_h{}'(q) = \int_{\Gamma_2} u'_{h,q} d\gamma .$$

(viii) The element $u_{h,q}$ can be written as

(79) (i) $u_{h,q} = B + U_{h,g} + q\eta_h$, (ii) $\eta_h = u'_{h,q} \in K_{0_h}$

where $U_{h,g}$ and η_h are respectively the unique solutions of the variational equality (40) and inequality (41), that is:

(80) $$\left| \begin{array}{l} a(U_{h,g}, v_h) = \int_\Omega g v_h \, dx , \quad \forall v_h \in W_{0_h} \\ U_{h,g} \in W_{0_h} \end{array} \right.$$

(81) $$\left| \begin{array}{l} a(\eta_h, v_h - \eta_h) = - \int_{\Gamma_2} (v_h - \eta_h) \, d\gamma , \quad \forall v_h \in K_{0_h} \\ \eta_h \in K_{0_h} . \end{array} \right.$$

(ix) We also have that $\eta_h/\Gamma_2 \leq 0$ and

(82) $$- \int_{\Gamma_2} \eta_h \, d\gamma = a(\eta_h, \eta_h) \geq \alpha \, ||\eta_h||^2_V > 0 .$$

(x) We also have

(83) (i) $f_h(q) = (B|\Gamma_2| + C_{h_g}) - D_h q$, (ii) $f_h'(q) = -D_h < 0$

where

(84) $$C_{h,g} = \int_{\Gamma_2} U_{h,g} \, d\gamma , \qquad D_h = - \int_{\Gamma_2} \eta_h \, d\gamma > 0 .$$

(xi) If, for each $h > 0$, we define the real function

(85) $$R_h(B, g) = \frac{B|\Gamma_2| + C_{h,g}}{D_h}$$

then we obtain that

(86) $q > R_h(B,g) \Rightarrow u_h$ is of non-constant sign in Ω,

i.e. there exist a discrete steady-state, two-phase, Stefan-Signorini problem.

Proof. – We use a method similar to the one developed for the continuous case (see Lemmas 1, 3 Theorems 4, 6 and 7).

(i) If we choose $v = u^+ \in K_{B_h}$ in the variational inequality (64) corresponding to the solution $u_h \in K$ and we take in account that $u_h = u_h^+ - u_h^-$ then we obtain (69). (ii) is an elementary corollary of (69).

(iii) If we choose $v = u_2 \in K_{B_h}$ and $v = u_1 \in K_{B_h}$ in (64) corresponding to the solution u_1 and u_2 respectively, we add them and we use the coerciveness of the bilinear form a then we obtain the inequalities (71).

(iv) – (v) From (71) we obtain (74)(ii) and using the Cauchy-Schwarz inequality and the trace operator γ_0 we deduce (72)(i) and (72)(ii).

Let be $q_1 \leq q_2$. Let $w = (u_{h,q_1} - u_{h,q_2})^-$ be. We shall prove that $w = 0$ in $\bar{\Omega}$, that is $u_{h,q_2} \leq u_{h,q_1}$ in $\bar{\Omega}$. We have that $w \in K_{0_h} \subset V_{0_h}$. We choose $v = u_{h,q_1} + w \in K_{B_h}$ in the variational inequality (64) for u_{h,q_1} for datum q_1 and $v = u_{h,q_2} - w \in K_B$ in the variational inequality for u_{h,q_2} for datum q_2, respectively, we add them and we obtain $w = 0$ in $\bar{\Omega}$.

For the other hand, (73) is a continuous function because

(87) $$\left| \int_{\Gamma_2} u_{h,q+\delta}\, d\gamma - \int_{\Gamma_2} u_{h,q}\, d\gamma \right| \leq \int_{\Gamma_2} |u_{h,q} - u_{h,q+\delta}| d\gamma \leq \frac{|\Gamma_2| \|\gamma_0\|^2}{\alpha} |\delta| \to 0 \text{ when } \delta \to 0 .$$

(vi) From (72)(i)-(ii) we have that $\frac{u_{h,q+\delta} - u_{h,q}}{\delta}$ is bounded in V and $L^2(\Gamma_2)$, that is there exists $u'_{h,q} \in V_{0_h}$ such that (75) and (76) are satisfied.

Moreover if we choose $v = u_{h,q+\delta} \in K_{B_h}$ in the variational inequality (64) for $u_{h,q}$, we divide by δ and we take the limit $\delta \to 0$ we deduce

(88) $$a(u_{h,q}, u'_{h,q}) \geq L(u'_{h,q}).$$

Similarity, if we choose $v = u_{h,q} \in K_{B_h}$ in the variational inequality (64) for $u_{h,q+\delta}$, we divide by δ and take the limit when $\delta \to 0$ we obtain

(89) $$a(u_{h,q}, u'_{h,q}) \leq L(u'_{h,q})$$

that is (77).

(vii) By using the definition of the function f and some elementary computations we obtain

(90) $$\frac{f(q+\delta) - f(q)}{\delta} = \frac{1}{2} a\left(u_{h,q+\delta} + u_{h,q}, \frac{u_{h,q+\delta} - u_{h,q}}{\delta}\right) - \int_{\Omega} g \frac{u_{h,q+\delta} - u_{h,q}}{\delta} dx +$$

$$+ \int_{\Gamma_2} u_{h,q+\delta}\, d\gamma + q \int_{\Gamma_2} \frac{u_{h,q+\delta} - u_{h,q}}{\delta} d\gamma$$

and taking the limit when $\delta \to 0$ we deduce, for all $q > 0$, that

(91) $$f'(q) = \lim_{\delta \to 0} \frac{f(q+\delta) - f(q)}{\delta} = \int_{\Gamma_2} u_{h,q}\, d\gamma + a(u_{h,q}, u'_{h,q}) - L(u'_{h,q}) = \int_{\Gamma_2} u_{h,q}\, d\gamma ,$$

that is (78)(i). By differentiating (78)(i) with respect to q we obtain (78)(ii).

(viii) The element $\xi = B + U_{h,g} + q\,\eta_h \in K_{B_h}$ is a solution of (64) by splitting it in three parts.

(ix) On the other hand, if we choose $v = -\eta_h^- \in K_{0_h}^-$ (because $\eta_h^-/\Gamma_{1_s} = 0$) in (81) we obtain

$$(92) \qquad 0 \geq -a(\eta_h^+, \eta_h^+) = -a(\eta_h, \eta_h^+) \geq \int_{\Gamma_2} \eta_h^+ \, d\gamma$$

that is $\eta_h^+/\Gamma_2 = 0$, then $\eta_h/\Gamma_2 \leq 0$.

Moreover, by taking $v = 0 \in K_{0_h}$ in (64) we obtain

$$(93) \qquad a(\eta_h, -\eta_h) \geq \int_{\Gamma_2} \eta_h \, d\gamma$$

that is (82).

(x) The results (83) are obtained from (68), (78)(i)-(ii), (79)(i)-(ii).

(xi) The result (86) is obtained by considering the following equivalence

$$(94) \qquad q > R_h(B,g) \qquad \Leftrightarrow \qquad f_h'(q) = \int_{\Gamma_2} u_{h,q} \, d\gamma < 0 .$$

□

Now, we consider the discretization of the continuous problem (47), that is

$$(95) \qquad \left| \begin{array}{l} a(z_h, v_h) = L(v_h), \quad \forall\, v_h \in K_h \\ z_h \in K_h \end{array} \right.$$

where $K_h = B + W_{o_h} \subset K_{B_h}$. Therefore, we have

Theorem 11. (i) The unique solution z_h of the variational equality (95) is given by

$$(96) \qquad z_h = z_{h,\,qgB} = B + U_{h,g} - q\,\xi_h$$

where ξ_h is the unique solution of the variational equality

$$(97) \qquad \left| \begin{array}{l} a(\xi_h, v_h) = \int_{\Gamma_2} v_h \, d\gamma, \quad \forall\, v_h \in W_{o_h} \\ \xi_h \in W_{o_h} . \end{array} \right.$$

(ii) We have the following relationship between problems (64) and (95)

$$(98) \qquad \left| \begin{array}{ll} \text{(i)} \quad u_h \geq z_h \ \text{ in } \bar{\Omega}, & \\ \text{(ii)} \quad \eta_h \geq -\xi_h \ \text{ in } \bar{\Omega}, & \text{(iii)} \quad \xi_h > 0 \text{ in } \Omega . \end{array} \right.$$

(iii) We have

(99) $$C_{h,g} \equiv \int_{\Gamma_2} U_{h,g}\, d\gamma = a(\xi_h, U_{h,g}) = \int_{\Omega} g\, \xi_h\, dx\,.$$

(iv) If $g \in L^2(\Omega)$ verifies the condition

(100) $$B|\Gamma_2| + C_{h,g} = B|\Gamma_2| + \int_{\Omega} g\, \xi_h\, dx > 0$$

then

(101) $$R_h(B,g) \geq Q_h(B,g)\,.$$

where

(102) $$\left| \begin{array}{l} Q_h(B,g) = q_{o_h}(B) + \dfrac{C_{h,g}}{C_h} \\[2ex] C_h = a(\xi_h, \xi_h) = \displaystyle\int_{\Gamma_2} \xi_h\, d\gamma > 0\,. \end{array} \right.$$

(v) Under the hypothesis (100), if there exists a discretized steady-state, two-phase, Stefan-Signorini problem (i.e. u_h is of non-constant sign in $\bar{\Omega}$) then there exists a discretized steady-state, two-phase, Stefan problem without Signorini boundary condition on Γ_{1_s} (i. e. z_h is of non-constant sign in $\bar{\Omega}$).

(vi) If $q > Q_h(B,g)$ then there exists a discretized, steady-state, two-phase, Stefan problem without Signorini boundary condition on Γ_{1_s} (i.e. z_h is of non-constant sign in $\bar{\Omega}$).

Proof. – (i)-(ii) We follow a similar method used in Theorem 8.
(iii) It follows from (80), (84) and (97).
(iv) It follows from (101) and

(103) $$\frac{1}{D_h} = \frac{1}{-\int_{\Gamma_2} \eta_h\, d\gamma} \geq \frac{1}{\int_{\Gamma_2} \xi_h\, d\gamma} = \frac{1}{C_h}\,.$$

(v)-(vi) They follow from (101) and [Sa]. □

Remark 5. The constant C_h has played an important role in [Ta4]. □

Now we shall obtain estimates for the continuous critical flux $q_c = q_c(B,g)$ (for $B > 0$ and $g \in L^2(\Omega)$) such that [BoShTa, GaTa]

(104) $$\left| \begin{array}{ll} \text{for } q < q_c, & u > 0 \text{ in } \Omega \text{ (no phase change), and} \\[2ex] \text{for } q > q_c, & u \text{ takes negative and positive values in } \Omega \text{ (two phases are present).} \end{array} \right.$$

Similarly, we define the discrete critical flux $q_{c_h} = q_{c_h}(B, g)$ (for $B > 0$ and $g \in L^2(\Omega)$) such that

(104)
$$\left| \begin{array}{l} \text{for } q < q_{c_h}, \quad u_h > 0 \text{ in } \Omega \text{ (no discrete phase change), and} \\ \\ \text{for } q > q_{c_h}, \quad u_h \text{ takes negative and positive values in } \Omega \text{ (two discrete phases are present)} \end{array} \right.$$

Then, we have

Theorem 12. (i) For $B > 0$, the continuous critical flux $q_c(B, g)$ is a non-decreasing function of $g \in L^2(\Omega)$ in the following sense:

(106) $$g_1 \le g_2 \text{ in } \bar{\Omega} \qquad \Rightarrow \qquad q_c(B, g_1) \le q_c(B, g_2) \text{ en } \mathbb{R}.$$

(ii) We have

(107) $$q_c(B, g) \le Q(B, g) .$$

(iii) For $B > 0$, the discretized critical flux $q_{c_h}(B, g)$ is a non-decreasing function of $g \in L^2(\Omega)$ in the following sense:

(108) $$g_1 \le g_2 \text{ in } \bar{\Omega} \qquad \Rightarrow \qquad q_{c_h}(B, g_1) \le q_{c_h}(B, g_2) \text{ en } \mathbb{R}.$$

(iv) We have

(109) $$q_{c_h}(B, g) \le Q_h(B, g) .$$

Proof. – (i) It follows from the maximun principle, that is

(110) $$g_1 \le g_2 \text{ in } \bar{\Omega} \qquad \Rightarrow \qquad u_{qBg_1} \le u_{qBg_2} \text{ in } \bar{\Omega} .$$

(ii) It follows from (104) and (106).
(iii)-(iv) They follow from (105) and the following discrete maximun principle

(111) $$g_1 \le g_2 \text{ in } \Omega \qquad \Rightarrow \qquad u_{h, qBg_1} \le u_{h, qBg_2} \text{ in } \bar{\Omega} .$$ □

4. ERROR BOUNDS.

Let Π_h be the corresponding linear interpolation operator for the finite element approximation.

There exists a constant $C_0 > 0$ (independent of h) such that [BrSc, Ci]

$$(112) \qquad \| v - \Pi_h v \|_V \le C_0 h^{r-1} \|v\|_{r,\Omega}, \qquad \forall\, v \in H^r(\Omega), \qquad r > 1.$$

where $\| . \|_{r,\Omega}$ is the norm in the Sobolev space $H^r(\Omega)$.

If we suppose the regularity properties:

$$(113) \qquad \xi \in H^r(\Omega), \qquad \eta \in H^r(\Omega), \qquad r > 1,$$

we obtain the following error estimates in terms of the finite element approximation parameter h.

Lemma 13. (i) We have the following inequalities and estimations

$$(114) \qquad 0 < C - C_h \le C_1 h^{2(r-1)},$$

$$(115) \qquad 0 < q_{o_h}(B) - q_o(B) \le \frac{C_0^2 \; h^{2(r-1)}}{C} \; | \xi |^2_{r,\Omega} \; q_{o_h}(B).$$

where $C_1 > 0$ is a constant independent of h.

(ii) If we let h, $B > 0$, and $0 < \epsilon_o < 1$ (ϵ_o is a parameter to be chosen arbitrarily), then we have the estimations

$$(116) \qquad q_o(B) < q_{o_h}(B) \le \frac{q_o(B)}{\epsilon_o} \quad \text{and} \quad C_h \ge C\, \epsilon_o, \qquad \forall\, h \le h_r(\epsilon_o),$$

$$(117) \qquad 0 < q_{o_h}(B) - q_o(B) \le C_2 h^{2(r-1)}, \qquad \forall\, h \le h_r(\epsilon_o),$$

where

$$(118) \qquad h_r(\epsilon_o) = \left(\frac{C\,(1-\epsilon_o)}{C_o^2 \, | \xi |^2_{r,\Omega}} \right)^{\frac{1}{2(r-1)}}$$

and $C_2 > 0$ is a constant independent of h.

(iii) We have the following estimation

$$(119) \qquad \| \xi - \xi_h \| \le C_3 h^{r-1}$$

where $C_3 > 0$ is a constant independent of h.

(iv) For all $B > 0$, we have

(120) $$\lim_{h\to 0^+} q_{o_h}(B) = q_o(B) .$$

Proof. – (i)-(ii) We can take (see [Ta4])

(121) $$C_1 = C_o^2 \mid \xi \mid_{r,\Omega}^2 > 0 , \qquad C_2 = \frac{C_o^2 \mid \xi \mid_{r,\Omega}^2}{C \; \epsilon_o} \, q_o(B) > 0 .$$

(iii) We have

(122) $$\alpha \| \xi - \xi_h \|_V^2 \le a(\xi - \xi_h , \xi - \xi_h) = a(\xi , \xi) - a(\xi_h , \xi_h) = C - C_h \le C_1 h^{2(r-1)}$$

that is (119) with

(123) $$C_3 = \sqrt{\frac{C_1}{\alpha}} = \frac{C_o \mid \xi \mid_{r,\Omega}}{\sqrt{\alpha}} .$$

(iv) It follows from (117). □

Lemma 14. (i) We have the following estimates and inequalities

(124) $$\left| C_{h,g} - C_g \right| \le C_4 h^r$$

(125) $$C_h \ge \frac{C}{2} \quad \Leftrightarrow \quad h \le \left(\frac{C}{2C_1}\right)^{\frac{1}{2(r-1)}}$$

(126) $$\left| \frac{C_{h,g}}{C_h} - \frac{C_g}{C} \right| \le \begin{cases} C_5 h^r & \text{if} \quad r \ge 2 \\ C_5 h^{2(r-1)} & \text{if} \quad 1 < r < 2 \end{cases} \qquad \forall h \le \min\left(1 , \left(\frac{C}{2C_1}\right)^{\frac{1}{2(r-1)}} \right)$$

where C_4 and C_5 are two positive constants independent of the paramenter h.
(ii) We also have the estimations

(127) $$\left| Q_h(B , g) - Q(B , g) \right| \le \begin{cases} C_6 h^r & \text{if} \quad r \ge 2 \\ C_6 h^{2(r-1)} & \text{if} \quad 1 < r < 2 \end{cases}$$

$$\forall h \le \min\left(1 , \left(\frac{C}{2C_1}\right)^{\frac{1}{2(r-1)}} \right).$$

(iii) We have

(128) $$\lim_{h\to 0^+} Q_h(B , g) = Q(B , g) .$$

Proof. – (i) Taking into account the inverse inequalities [Ci] and (119), we have

(129) $$\left|C_{h,g} - C_g\right| = \left| \int_\Omega g\,(\xi_h - \xi)\; dx \right| \le \|g\|_{L^2(\Omega)}\, \|\xi_h - \xi\|_{L^2(\Omega)} \le C_4\, h^r$$

where C_4 is a positive constant independent of parameter h, that is (124) holds.

The equivalence (125) follows from (114) and the following equivalence

(130) $$C - C_1\, h^{2(r-1)} \ge \frac{C}{2} \qquad \Leftrightarrow \qquad \forall\, h \le \left(\frac{C}{2\,C_1}\right)^{\frac{1}{2(r-1)}} .$$

Therefore, we obtain

(131) $$\left|\frac{C_{h,g}}{C_h} - \frac{C_g}{C}\right| = \frac{\left|C_{h,g}\, C - C_g\, C_h\right|}{C_h\, C} \le \frac{C\left|C_{h,g} - C_g\right| + \left|C_g\right|(C_h - C)}{C\, C_h} \le$$

$$\le \frac{2}{C^2}\left[\, C\, C_4\, h^r + \|g\|_{L^2(\Omega)}\, \|\xi\|_{L^2(\Omega)}\, C_1\, h^{2(r-1)}\right] \le$$

$$\le \begin{cases} C_5\, h^r & \text{if} \quad r \ge 2 \\ C_5\, h^{2(r-1)} & \text{if} \quad 1 < r < 2 \end{cases} \qquad \forall\, h \le \min\left(1, \left(\frac{C}{2\,C_1}\right)^{\frac{1}{2(r-1)}}\right)$$

where C_5 is a positive constant independent of h which is given by

(132) $$C_5 = \frac{2}{C^2}\left[\, C\, C_4 + \|g\|_{L^2(\Omega)}\, \|\xi\|_{L^2(\Omega)}\right] .$$

(ii) Taking into account (117), (126) and the fact

(133) $$\left|Q_h(B,g) - Q(B,g)\right| \le \left(q_{o_h}(B) - q_o(B)\right) + \left|\frac{C_{h,g}}{C_h} - \frac{C_g}{C}\right|$$

we get (127).

(iii) It follows from (127). □

Theorem 15. (i) We have that η_h is a bounded sequence in V, that is

(134) $$\|\eta_h\|_V \le C_7 \,, \qquad \forall\, h > 0$$

where C_7 is a positive constant independent of h.

(ii) There exists a positive constant C_8, independent of h, such that

(135) $$\|\eta - \eta_h\|_V^2 \le C_8 \inf_{v_h \in K_{o_h}} \|v_h - \eta\|_V .$$

(iii) We have the following estimations

(136) $$\|\eta-\eta_h\|_V^2 \leq C_9\, h^{r-1}$$

where C_9 is a positive constant independent of h.

Proof. – (i) By using (82) we obtain

(137) $$\alpha\|\eta_h\|_V^2 \leq -\int_{\Gamma_2} \eta_h\, d\gamma \leq |\Gamma_2|^{\frac{1}{2}}\|\gamma_o\|\|\eta_h\|_V \quad , \qquad \forall\, h>0$$

that is (134) with $C_7 = \dfrac{|\Gamma_2|^{\frac{1}{2}}\|\gamma_o\|}{\alpha} > 0$.

(ii) Following [Ci, Fa], for all $v \in K_o$ and $v_h \in K_{o_h}$, we obtain that

(138) $$\alpha\|\eta-\eta_h\|^2 \leq a(\eta-\eta_h,\eta-\eta_h) \leq$$

$$\leq a(\eta, v-\eta_h) + a(\eta_h, \eta-v_h) + \int_{\Gamma_2}(v-\eta_h)\, d\gamma + \int_{\Gamma_2}(v_h-\eta)\, d\gamma \leq$$

$$\leq \|\eta\|_V\|v-\eta_h\|_V + \|\eta_h\|_V\|\eta-v_h\|_V + |\Gamma_2|^{\frac{1}{2}}\|v-\eta_h\|_{L^2(\Gamma_2)} + |\Gamma_2|^{\frac{1}{2}}\|v_h-\eta\|_{L^2(\Gamma_2)} =$$

$$= \left[\|\eta\|_V + |\Gamma_2|^{\frac{1}{2}}\|\gamma_o\|\right]\|v-\eta_h\|_{L^2(\Gamma_2)} + |\Gamma_2|^{\frac{1}{2}}\|\gamma_o\|\left(1+\tfrac{1}{\alpha}\right)\|v_h-\eta\|_V \,.$$

Therefore, taking into account that $K_{o_h} \subset K_o$ we deduce (135) with

(139) $$C_8 = \frac{\alpha+1}{\alpha^2}|\Gamma_2|^{\frac{1}{2}}\|\gamma_o\| > 0\,.$$

(iii) If we take $v_h = \Pi_h\,\eta \in K_{o_h}$ in (112) we deduce

(140) $$\|\eta-\eta_h\|_V^2 \leq C_8\|\Pi_h\,\eta-\eta\|_V \leq C_8\, C_o\,|\eta|_{r,\Omega}\, h^{r-1} = C_9\, h^{r-1}$$

where we have choosen $C_9 = C_8\, C_o\,|\eta|_{r,\Omega} > 0$. □

Theorem 16. (i) We have the following estimations and inequalities

(141) $$|D-D_h| \leq C_{10}\, h^{\frac{r-1}{2}}$$

(142) $$D_h \geq \frac{D}{2} \qquad \forall\, h \leq \left(\frac{D}{2\,C_{10}}\right)^{\frac{2}{r-1}}$$

where C_{10} is a positive constant independent of h.

(ii) We also have the estimations

$$(143) \qquad \left|R(B,g)-R_h(B,g)\right| \le C_{11}\, h^{\frac{r-1}{2}}\ , \qquad \forall\, h \le \min\left(1, \left(\frac{D}{2\,C_{10}}\right)^{\frac{2}{r-1}}\right)$$

where C_{11} is a positive constant independent of h.

(iii) For all $B>0$ and $g \in L^2(\Omega)$ we have

$$(144) \qquad \lim_{h\to 0^+} R_h(B,g) = R(B,g)\ .$$

Proof. – (i) We have

$$(145) \qquad \left|D-D_h\right| = \left|\int_{\Gamma_2} (\eta-\eta_h)\, d\gamma\right| \le |\Gamma_2|^{\frac{1}{2}} \|\eta-\eta_h\|_{L^2(\Gamma_2)} \le \|\gamma_o\| |\Gamma_2|^{\frac{1}{2}} \|\eta-\eta_h\|_V \le C_{10}\, h^{\frac{r-1}{2}}$$

where we have choosen $C_{10} = \|\gamma_o\| |\Gamma_2|^{\frac{1}{2}} \sqrt{C_9} > 0$.

From (141) we have

$$(146) \qquad D - C_{10}\, h^{\frac{r-1}{2}} \le D_h \le D + C_{10}\, h^{\frac{r-1}{2}}$$

then we get (142).

(ii) By (124) and (141), for all $h \le \min\left(1, \left(\frac{D}{2\,C_{10}}\right)^{\frac{2}{r-1}}\right)$, we get

$$(147) \qquad \left|R(B,g)-R_h(B,g)\right| = \left|\frac{B|\Gamma_2|+C_g}{D} - \frac{B|\Gamma_2|+C_{h,g}}{D_h}\right| =$$

$$= \left|\frac{\left(B|\Gamma_2|+C_g\right)(D_h-D) - D\left(C_g - C_{gh}\right)}{D\, D_h}\right| \le$$

$$\le \frac{2}{D^2}\left[\left(B|\Gamma_2| + \|g\|_{L^2(\Omega)} \|\xi\|_{L^2(\Omega)}\right) C_{10}\, h^{\frac{r-1}{2}} + C_4\, h^r\right] \le C_{11}\, h^{\frac{r-1}{2}}$$

where we have choosen

$$(148) \qquad C_{11} = \frac{2}{D^2}\left[C_4 + C_{10}\left(B|\Gamma_2| + \|g\|_{L^2(\Omega)} \|\xi\|_{L^2(\Omega)}\right)\right] > 0.$$

(iii) It follows from (143). □

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