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François Laubie

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INTRODUCTION TO THE THEORY OF NIM-LINEAR CODES

INTRODUCTION A LA THÉORIE DES CODES NIM-LINÉAIRES

François LAUBIE

Université de Limoges - 123 Avenue Albert Thomas - 87060 Limoges Cedex

Chercheur extérieur - Projet CODES

Abstract - Formalizing the theory of coin turning games of H. W. Lenstra, we describe some vector spaces provided with circuitless directed graphs structure, whose Grundy functions are linear for the Nim addition and the Nim multiplication of natural integers. The kernels of such graphs are linear error-correcting codes which generalize the lexicographic codes of Conway and Sloane. We introduce the notion of lixigraphy which generalizes that of Grundy function and which simplies and enriches their theory.

Résumé - En formalisant la théorie des jeux de pièces tournantes (ou de pile ou face) de H. W. Lenstra, nous décrivons certains espaces vectoriels munis de structure de graphes orientés sans circuit dont les fonctions de Grundy sont linéaires pour la Nim-addition et la Nim-multiplication des entiers naturels. Les noyaux de tels graphes sont des codes correcteurs d'erreurs linéaires qui généralisent les codes lexicographiques de Con way et Sloane. On introduit la notion de lexigraphie qui généralise celle de fonction de Grundy et qui en simplifie et en enrichit la théorie.

Introduction

The linear character of some graphs was discovered in the context of game theory by C.L. Bouton in 1902 [2] ; it was developed by R.P. Sprague in 1935 [10], P.M. Grundy in 1939 [6] and later by other authors such as M.P. Schützenberger, C. Berge... . The ideas of the theory of Nim-linear graphs and Nim-linear codes come from a lecture given by H.W. Lenstra in 1978 at the "Séminaire de théorie des nombres de Bordeaux" [8] when the author announced the discovery of lexicographic codes (or lexicodes) by J.H. Conway. Eight years later, J.H. Conway and N.J.A. Sloane published their first paper on lexicodes [5]. Then R.A. Brualdi and V.S. Pless defined the notion of greedy codes which generalizes lexicodes [3]. Later still, it appeared that the real original paper was published in 1960 by V.I. Levenstein (in Russian) [9] : this author had already defined and investigated greedy codes and lexicodes under the name of "systematic codes".

The Nim-linear graphs are the circuitless directed 1-graphs the vertices of which are the elements of a $\mathbb{F}_2$-vector spaces and their Grundy functions are linear for the Nim-addition of natural integers. Their kernels are (Nim)-linear codes which generalize the "systematic" codes of V.I. Levenstein. In this paper we study some Nim-linear graphs which are a formalization of the so-called "coin turning games" of H.W. Lenstra [8] (see also [1], ch. 14) ; they are called coin turning games graphs or, simpler, Lenstra graphs like in [7].

Although this theory can be considered as a new formulation of the "coin turning games", most of the results are new generalizations of known theorems. Lexicographic and greedy (or systematic) codes are constructed by means of the greedy algorithm applied to $\mathbb{F}_2^n$ provided with a total order [3] ; in the case when $\mathbb{F}_2^n$ is not provided with a total order but only with a Lenstra graph structure, the same idea gives rise to the notion of Nim-linear codes and, as in the previous case, it appears that these codes are linear. Moreover some new proofs of well-known theorems make this paper self-contained. But it constitutes only an introduction ; for example, there are some natural operations in the set of Lenstra graphs : addition, multiplication and extension of scalars and these operations can enable us to construct some new classes of linear binary codes which are not yet classified. Finally let us note that we keep the notations $\oplus$ and $\otimes$ of Conway and Sloane for Nim-addition and Nim-multiplication and, also, for direct sum and tensor product.

1 Lenstra graphs

In the following, the word graph means a circuitless finite directed 1-graph. The Grundy function of such a graph is the only function for which the following property holds : *for every vertex x, $g(x)$ is the least non-negative integer distinct from every $g(x')$ where (x', x) is an arc.*

Let $G = (X, \Gamma)$ be a graph and let $x \in X$; we denote by $\Gamma^-(x)$ (resp. $\Gamma^+(x)$) the set of the vertices $y \in X$ such that (y, x) (resp. (x, y)) is an arc of Γ and we let $\Gamma(x)$ be $\Gamma^-(x) \cup \Gamma^+(x)$.

For every positive integer m, let us consider the graph $\underline{m}$ of which the vertices are the non-negative integers $0, 1, \cdots, m-1$, and the arcs are the pairs (a, b) of such integers with $a < b$; in the following, $\underline{m}$ means the set $\{0, 1, \cdots, m-1\}$ as well.

In the case of $m = 2^n$, the addition in binary without carrying (called Nim-addition and denoted $\oplus$) makes $\underline{2^n}$ an n-dimensional vector space over the field with two elements denoted $\mathbb{F}_2$ or $\underline{2}$.

Let $\mathcal{B} = (e_1, \cdots, e_n)$ be a basis of the vector space $\mathbb{F}_2^n$ (partially or totally) ordered ; its order is simply denoted $\leq$. Let us recall that the support with respect to $\mathcal{B}$ of a vector $u = \sum_{i=1}^{n} u_i e_i \in \mathbb{F}_2^n$ is the set :

$$supp_{\mathcal{B}}(u) = supp(u) = \{e_i \in \mathcal{B}; u_i \neq 0\};$$

such a vector u is said to be $\mathcal{B}$-admissible or simply admissible if $supp(u)$ contains a unique maximal element denoted $e_{\mu(u)}$. Let $\mathcal{R}$ be a set of admissible vectors. Let us consider the graph of which the vertices are the vectors of $\mathbb{F}_2^n$ and the arcs are the pairs (u, v) with $u = \sum_{i=1}^{n} u_i e_i$, $v = \sum_{i=1}^{n} v_i e_i$ such that $t = u + v \in \mathcal{R}$ and $u_{\mu(t)} = 0$ (or equivalently $v_{\mu(t)} = 1$) ; this means that $supp(v)$ contains at least one element greater than every element belonging to $supp(u)$ and not to $supp(v)$; this implies that the graph admits no circuit. Under these hypotheses, this graph is called an n-dimensional Lenstra graph of board $\mathcal{B}$ and rule $\mathcal{R}$; $\underline{2^n}$ is such a graph.

Remark 1 *This terminology is chosen by analogy with that of "coin turning games" [8].*

Remark 2 *Let $G = (\mathbb{F}_2^n, \Gamma)$ be a n-dimensional Lenstra graph of board $\mathcal{B}$ and rule $\mathcal{R}$; for every $t \in \mathbb{F}_2^n$, let $\Gamma^-[t]$ (resp. $\Gamma^+[t]$) be the set of vertices x such that $x + t \in \Gamma^+(x)$ (resp.$x + t \in \Gamma^-(x)$) ; then $\mathcal{R} = \{t \in \mathbb{F}_2^n; \Gamma^-[t] \neq \emptyset\}$ and for every $t \in \mathcal{R}$, $\Gamma^-[t]$ is an hyperplane of $\mathbb{F}_2^n$ such that $\Gamma^+[t] = t + \Gamma^-[t]$.*

Example 1 The so-called lexigraph of length n and distance d with $2 \leq d \leq n+1$ is the graph of which the vertices are the vectors $u = (u_1, u_2, \cdots, u_n) \in \mathbb{F}_2^n$ and the arcs are the pairs (u, v) of distinct vectors such that $\delta(u, v) \leq d - 1$ where δ is the Hamming distance and such that $u_m = 0$ for $m = \max\{k \in \mathbb{N}; u_k \neq v_k\}$. It is clearly the n-dimensional Lenstra graph of board the canonical ordered basis of $\mathbb{F}_2^n$ and of rule the set of non-zero vectors with at least $n - d + 1$ coordinates equal to zero. The kernels of the lexigraphs of length n and distance d are the Conway-Sloane's lexicodes of length n and distance d [5].

The lexigraph of length n and distance $n+1$ is $\underline{2^n}$.

Example 2. Let $\mathcal{B}$ be a totally ordered basis of $\mathbb{F}_2^n$ and let φ be the linear automorphism of $\mathbb{F}_2^n$ which maps the basis $\mathcal{B}$ onto the canonical ordered basis of $\mathbb{F}_2^n$, preserving the orders. Let d be an integer such that $2 \leq d \leq n+1$ and let $\mathcal{A}$ be the set of all the pairs (u, v) such that $(\varphi(u), \varphi(v))$ is an arc of the lexigraph of length n and distance d. The so-called $\mathcal{B}$-greedy graph of length n and designed distance d is the graph of which the set of vertices is $\mathbb{F}_2^n$ and the set of arcs is $\mathcal{A}$. It is clearly the n-dimensional Lenstra graph of board $\mathcal{B}$ and of rule the set of all non zero vectors with at least $n-d+1$ coordinates in the basis $\mathcal{B}$, equal to zero. The kernels of the $\mathcal{B}$-greedy graphs of length n and designed distance d are the Brualdi- Pless, $\mathcal{B}$-greedy codes of length n and designed distance d [3].

2 Lexigraphies

Definition Let $G_1 = (X_1, \Gamma_1)$ and $G_2 = (X_2, \Gamma_2)$ be graphs ; a lexigraphy from G_1 to G_2 is a map $f : X_1 \to X_2$ such that : $\forall x \in X_1, f(\Gamma_1(x)) \subset \Gamma_2(f(x))$ and $f(\Gamma_1^-(x)) \supset \Gamma_2^-(f(x))$. In the case of G_1 and G_2 two Lenstra graphs, a lexigraphy which is a linear map is called a linear lexigraphy.

Lemma 1. If f and g are two lexigraphies then $f \circ g$ is a lexigraphy ; moreover a bijective lexigraphy is a directed graphs isomorphism.

Proof. The first part of this lemma is obvious. Let $f : G_1 \to G_2$ be a bijective lexi graphy, let $x \in X_1$ and $y \in \Gamma_1^-(x)$; since $f(y) \in \Gamma_2(f(x))$ and $x \notin \Gamma_1^-(y)$ then $f(x) \notin \Gamma_2^-(f(y))$, $f(y) \in \Gamma_2^-(f(x))$ and therefore $f(\Gamma_1^-(x)) = \Gamma_2^-(f(x))$.

Lemma 2 There exists one and only one lexigraphy from any graph G to $\underline{2^n}$ (for n large enough), it is the Grundy function of G.

Proof : Let f be a lexigraphy from G to $\underline{2^n}$; since $\{0, 1, \cdots, f(x)-1\} \subset f(\Gamma^-(x)) \subset \mathbb{N}\backslash\{f(x)\}$, the equalities $f(x) = \min(\mathbb{N}\backslash f(\Gamma^-(x)))$ hold for every vertex x.

Lemma 3. Let $G_1 = (X_1, \Gamma_1)$ (resp. $G_2 = (X_2, \Gamma_2)$) be a Lenstra graph of rule $\mathcal{R}_1$ (resp. $\mathcal{R}_2$) and let f be a linear lexigraphy from G_1 to G_2. Then $f(\mathcal{R}_1) = \mathcal{R}_2 \cap f(X_1)$ and $\forall x \in X_1, f(\Gamma_1(x)) = \Gamma_2(f(x)) \cap f(X_1)$.

Proof : Let $t_1 \in \mathcal{R}_1$ and $x \in X_1$; then

$$f(t_1) = f(x) + f(x + t_1) \in f(x) + f(\Gamma_1(x)) \subset f(x) + \Gamma_2(f(x)) = \mathcal{R}_2;$$

let $t_2 \in \mathcal{R}_2 \cap f(X_1)$ and let $\tau_1 \in X_1$ such that $t_2 = f(\tau_1)$; if $f(x) + t_2 \in \Gamma_2^-(f(x))$ there exists $y \in \Gamma_1^-(x)$ such that $f(y) = f(x) + t_2$, therefore $t_2 = f(x+y) \in f(\mathcal{R}_1)$; now, if $f(x) + t_1 \in \Gamma_2^+(f(x))$ then $f(x) \in \Gamma_2^-(f(x) + t_2) = \Gamma_2^-(f(x + \tau_1)) \subset f(\Gamma_1^-(x + \tau_1))$; let $z \in \Gamma_1^-(x + \tau_1)$ such that $f(x) = f(z)$; we obtain $t_2 = f(\tau_1) = f(z) + f(x) + f(\tau_1) = f(z + x + \tau_1) \in f(\mathcal{R}_1)$. Thus : $f(\mathcal{R}_1) = \mathcal{R}_2 \cap f(X_1)$.
Now, let $y_2 \in \Gamma_2(f(x)) \cap f(X_1)$, let $y_1 \in X_1$ such that $y_2 = f(y_1)$ and let $t_2 = y_2 + f(x) \in \mathcal{R}_2$; since $t_2 = f(y_1 + x) \in \mathcal{R}_2 \cap f(X_1)$, there exists $t_1 \in \mathcal{R}_1$ such that $t_2 = f(t_1)$ and then : $y_2 = f(x) + t_2 = f(x + t_1) \in f(\Gamma_1(x))$.

Example. Let $\mathcal{B} = (e_i)_{1 \leq i \leq n}$ (resp. $\mathcal{B}' = (e'_j)_{1 \leq j \leq m}$) be a basis of $\mathbb{F}_2^n$ (resp. of $\mathbb{F}_2^m$).

For every mapping φ from $\mathcal{B}$ to $\mathcal{B}'$ we define a linear map $\overset{\circ}{\varphi}$ from $\mathbb{F}_2^m$ to $\mathbb{F}_2^n$ by $\overset{\circ}{\varphi}(e'_j) = \sum_{\varphi(e_i)=e'_j} e_i$ for every $e'_j \in \mathcal{B}'$. Let us assume that the sets $\mathcal{B}$ and $\mathcal{B}'$ are provided with some orders preserved by φ and such that, for every $e'_j \in \mathcal{B}'$, the subset $\varphi^{-1}(\{e'_j\})$ of $\mathcal{B}$ possesses an unique maximal element. If G' is an m-dimensional Lenstra graph of board $\mathcal{B}'$ and of rule $\mathcal{R}'$ then $\overset{\circ}{\varphi}(\mathcal{R}')$ is the rule of an n-dimensional Lenstra graph G of board $\mathcal{B}$ and $\overset{\circ}{\varphi}$ is a lexigraphy from G' to G.

Let us consider the following particular case : $\mathcal{B} = \mathcal{B}' = \{0, 1, \cdots, 2^n - 1\}$ provided with the usual orders, $\varphi = g =$ the Grundy function of $\mathcal{B}$; then the map $\overset{\circ}{g}$ from $\underline{2^{2^n}}$ to $\underline{2^{\mathcal{B}}} = \underline{2^{2^n}}$ is defined by :

$$g\left(\sum_{k=0}^{2^n-1} m_k\, 2^k\right) = \sum_{k=0}^{2^n-1} m_k \prod_{i=0}^{n-1} (2^{2^i})^{k_i}$$

with $m_k = 0$ or 1, $k_i = 0$ or 1 and $k = \sum_{i=0}^{n-1} k_i 2^i$.

Lexicographic fields. Let us consider an algebraic closure Ω of $\mathbb{F}_2$, a root $x_0 \in \Omega$ of the polynomial $X^2 + X + 1$ and the subfield $K_1 = \mathbb{F}_2(x_0)$ of Ω. Let us construct by recurrence a sequence $K_2 = K_1(x_1), \cdots, K_j = K_{j-1}(x_{j-1}), \cdots$ of subfields of Ω where, for every $n = 1, 2, \cdots, j-1, \cdots, x_n$ means a root of the polynomial $X^2 + X + x_0 x_1 \cdots x_{n-1}$. Then every element of K_n can be identified in a unique way as a polynomial of degree one in every indeterminate of $\mathbb{F}_2[x_0, x_1, \cdots, x_{n-1}]$. Let us order these polynomials in the following way : $0 < 1 < x_0 < x_0 + 1 < x_1 < x_1 + 1 < x_1 + x_0 < x_1 + x_0 + 1 < x_1 x_0 < x_1 x_0 + 1 < x_1 x_0 + x_0 < x_1 x_0 + x_0 + 1 < x_1 x_0 + x_1 < x_1 x_0 + x_1 + 1 < x_1 x_0 + x_1 + x_0 < x_1 x_0 + x_1 + x_0 + 1 < x_2 \cdots$. For any pair (φ, ψ) of such polynomials, let us say that (φ, ψ) is an arc if $\varphi < \psi$; thus K_n becomes a 2^n-dimensional Lenstra graph of which the Grundy function g is defined by :

$$g\left(\sum_{k=0}^{2^n-1} m_k \prod_{i=0}^{n-1} x_i^{k_i}\right) = \sum_{k=0}^{2^n-1} m_k 2^k$$

with $m_k = 0$ or 1, $k_i = 0$ or 1 and $k = \sum_{i=0}^{n-1} k_i 2^i$.

3 Sum of Lenstra graphs

Let $G_1 = (X_1, \Gamma_1)$ (resp. $G_2 = (X_2, \Gamma_2)$) be a Lenstra graph of board $\mathcal{B}_1$ (resp.$\mathcal{B}_2$) and of rule $\mathcal{R}_1(resp.\mathcal{R}_2)$. Let $\mathcal{B} = \mathcal{B}_1 \coprod \mathcal{B}_2$ the disjonctive union of $\mathcal{B}_1$ and $\mathcal{B}_2$ ordered by :

$$x \leq y \Leftrightarrow (x \in \mathcal{B}_1, y \in \mathcal{B}_1, x \leq y) \text{ or } (x \in \mathcal{B}_2, y \in \mathcal{B}_2, x \leq y).$$

Let X be the direct sum of vector spaces $X_1 \oplus X_2$ (so that $\mathcal{B}$ is a basis of X) and let $\mathcal{R}$ be the subset $(\mathcal{R}_1 \oplus 0) \cup (0 \oplus \mathcal{R}_2)$ of X. The Lenstra graph of board $\mathcal{B}$ and rule $\mathcal{R}$ is denoted $G_1 \oplus G_2 = (X, \Gamma)$ with $\Gamma = \Gamma_1 \oplus \Gamma_2$; it is called the sum of G_1 and G_2 ; as a graph, it is the cartesian sum of Γ_1 and Γ_2 because, for every $u_1 \oplus u_2 \in X_1 \oplus X_2$, the following equalities hold :

$$\Gamma^-(u_1 \oplus u_2) = (\Gamma_1^-(u_1) \oplus u_2) \cup (u_1 \oplus \Gamma_2^-(u_2)),$$

$$\Gamma(u_1 \oplus u_2) = (\Gamma_1(u_1) \oplus u_2) \cup (u \oplus \Gamma_2(u_2)).$$

Proposition 1 Let $G_1 = (X_1, \Gamma_1), G_2 = (X_2, \Gamma_2), G_1' = (X_1', \Gamma_1')$ and $G_2' = (X_2', \Gamma_2')$ be Lenstra graphs and let $f_1 : X_1 \to X_1'$ and $f_2 : X_2 \to X_2'$ be lexigraphies. Then the map $f_1 \oplus f_2$ from $X_1 \oplus X_2$ to $X_1' \oplus X_2'$ defined by $f_1 \oplus f_2(u_1 \oplus u_2) = f_1(u_1) \oplus f_2(u_2)$ is a lexigraphy from $G_1 \oplus G_2$ to $G_1' \oplus G_2'$.

Proof It is a straightforward check ; let $\Gamma = \Gamma_1 \oplus \Gamma_2$, $\Gamma' = \Gamma_1' \oplus \Gamma_2', u_1 \in X_1, u_2 \in X_2$, then :

$$\begin{aligned} f_1 \oplus f_2(\Gamma(u_1 \oplus u_2)) &= f_1 \oplus f_2((\Gamma_1(u_1) \oplus u_2) \cup (u_1 \oplus \Gamma_2(u_2))) \\ &= (f_1(\Gamma_1(u_1)) \oplus f_2(u_2)) \cup (f_1(u_1) \oplus f_2(\Gamma_2(u_2))) \\ &\subset (\Gamma_1'(f_1(u_1)) \oplus f_2(u_2)) \cup (f_1(u_1) \oplus \Gamma_2'(f_2(u_2))) \\ &= \Gamma'(f'(u_1) \oplus f_2(u_2)) = \Gamma'(f_1 \oplus f_2(u_1 \oplus u_2)); \\ \Gamma'^-(f_1 \oplus f_2)(u_1 \oplus u_2)) &= (\Gamma_1'^-(f_1(u_1)) \oplus f_2(u_2)) \cup (f_1(u_1) \oplus \Gamma_2'^-(f_2(u_2))) \\ &\subset (f_1(\Gamma_1^-(u_1)) \oplus f_2(u_2)) \cup (f_1(u_1) \oplus f_2(\Gamma_2^-(u_2))) \\ &= f_1 \oplus f_2((\Gamma_1^-(u_1) \oplus u_2) \cup (u_1 \oplus \Gamma_2^-(u_2))) \\ &= f_1 \oplus f_2(\Gamma^-(u_1 \oplus u_2) \end{aligned}$$

Theorem 1 Let $G = (X, \Gamma)$ be a Lenstra graph. The sum mapping

$$\begin{array}{lll} X \oplus X & \to & X \\ u \oplus v & \mapsto & u + v \text{ is a lexigraphy from } G \oplus G \text{ to } G. \end{array}$$

Proof First it is clear that for every $u, v \in X, (u + \Gamma(v)) \cup (\Gamma(u) + v) = \Gamma(u + v)$ because :

$$u + v + t \in \Gamma(u + v) \Longleftrightarrow u + t \in \Gamma(u) \text{ or } v + t \in \Gamma(v).$$

Now let us show that for every $u, v \in X$ we have $\Gamma^-(u+v) \subset (u+\Gamma^-(v)) \cup (\Gamma^-(u)+v)$. Let $w \in \Gamma^-(u+v)$ then $t = u + v + w$ lies in the rule of G. Let $\mathcal{B} = (e_i)_{1 \leq i \leq n}$ be the board of G ; let us write u, v, w and t in the basis $\mathcal{B}$: $u = \sum_{i=1}^{n} u_i e_i, v = \sum_{i=1}^{n} v_i e_i, w = \sum_{i=1}^{n} w_i e_i$ $t = \sum_{i=1}^{n} t_i e_i$ and let $i_0 = \mu(t)$; then $w_{i_0} = 0$ and $u_{i_0} \neq v_{i_0}$; so one of the two following options holds :

$$u_{i_0} + w_{i_0} = 0, v_{i_0} = 1 \Rightarrow w \in u + \Gamma^-(v)$$

or

$$v_{i_0} + w_{i_0} = 0,\ u_{i_0} = 1 \Rightarrow w \in \Gamma^-(u) + v.$$

In order to finish the proof, let us remark that $(u + \Gamma^-(v)) \cup (\Gamma^-(u) + v)$ ($resp(u + \Gamma(v)) \cup (\Gamma(u) + v))$) is the image of $\Gamma^-(u \oplus v)$ ($resp.$ $\Gamma(u \oplus v)$) under the sum.

Corollary 1 Let $G_1 = (X_1, \Gamma_1), G_2 = (X_2, \Gamma_2)$ and $G' = (X', \Gamma')$ be Lenstra graphs and let $f_1 : G_1 \to G'$ and $f_2 : G_2 \to G'$ be two lexigraphies ; then the map from $X_1 \oplus X_2$ to X', still denoted by $f_1 \oplus f_2$, defined by $f_1 \oplus f_2(u_1 \oplus u_2) = f_1(u_1) + f_2(u_2)$ is a lexigraphy.

Proof $f_1 \oplus f_2$ is composed of lexigraphies so let us apply lemma 1.

Corollary 2 The Grundy function of a Lenstra graph is a linear map.

Proof Let $G = (X, \Gamma)$ be an n-dimensional Lenstra graph and let $g : X \to \underline{2^n}$ be its Grundy function ; the mappings : $\begin{array}{ccc} X \oplus X & \to & \underline{2^n} \\ u \oplus v & \mapsto & g(u) \oplus g(v) \end{array}$, $\begin{array}{ccc} X \oplus X & \to & \underline{2^n} \\ u \oplus v & \mapsto & g(u+v) \end{array}$ are lexigraphies from $G \oplus G$ to $\underline{2^n}$; then, from lemma 2, they are the same. (Let us recall that in $\underline{2^n}$, $\oplus$ means Nim-addition).

Corollary 3 . The Grundy function of the Lenstra graph $G_1 \oplus G_2$ is $g_1 \oplus g_2$ where g_1 (*resp.* g_2) is the Grundy function of G_1 (*resp.* G_2).

Proof As above, $g_1 \oplus g_2$ is a lexicography from $G_1 \oplus G_2$ to $\underline{2^n}$ (for n large enough), so let us apply lemma 2.

Corollary 4. The Nim-addition is characterized by
$m \oplus n = \min(\mathbb{N} \setminus \{m' \oplus n, m \oplus n'; m' < m, n' < n\})$ for every $m, n \in \mathbb{N}$.

Proof. Let $k \in \mathbb{N}$ such that $m < 2^k$ and let $\underline{2^k} = (X, \Gamma)$; then $m \oplus n \in X$ can be identified to the image of $m \oplus n \in X \oplus X$ under the sum mapping from $X \oplus X$ to X ; therefore : $\Gamma^-(m \oplus n) = (\Gamma^-(m) \oplus n) \cup (m \oplus \Gamma^-(n))$.

Lemma 4 Let $G = (X, \Gamma)$ be a Lenstra graph of board $\mathcal{B}$ and let $u, v \in X$. If $supp_{\mathcal{B}}(v) \cap supp_{\mathcal{B}}(u) = \emptyset$ then $\Gamma^-(u+v) = (u + \Gamma^-(v)) \cup (\Gamma^-(u) + v)$.

Proof. The inclusion $\Gamma^-(u+v) \subset (u + \Gamma^-(v)) \cup (\Gamma^-(u) + v)$ is already known ; let $w \in u + \Gamma^-(v)$; then $w = u + v + t$ where t is lying in the rule of G ; let us write the vectors u, v, w in the basis $\mathcal{B} = (e_i)_{1 \leq i \leq n}$: $u = \sum_{i=1}^{n} u_i e_i, v = \sum_{i=1}^{n} v_i e_i$, $w = \sum_{i=1}^{n} w_i e_i$; then $v_{\mu(t)} = 1, u_{\mu(t)} = 0$ and $w_{\mu(t)} = 0$, so $w \in \Gamma^-(u+v)$.

4 Product of Lenstra graphs

Let $G_1 = (X_1, \Gamma_1)$ (*resp.* $G_2 = (X_2, \Gamma_2)$) be an n_1-dimensional Lenstra graph (resp. *an* n_2-dimensional Lenstra graph) of board $\mathcal{B}_1 = (e_i^1)_{1 \leq i \leq n_1}$ (*resp.* $\mathcal{B}_2 = (e_i^2)_{1 \leq i \leq n_2}$) of rule $\mathcal{R}_1$ (*resp.* $\mathcal{R}_2$). Let us consider the basis $\mathcal{B} = \mathcal{B}_1 \otimes \mathcal{B}_2 = (e_i^1 \otimes e_j^2)_{\substack{1 \leq i \leq n_1 \\ 1 \leq j \leq n_2}}$ of the tensor product $X = X_1 \otimes X_2$ of the $\mathbb{F}_2$-vector spaces X_1 and X_2. Let us provide $\mathcal{B}$ with the following order :

$$e_{i_1}^1 \otimes e_{j_1}^2 < e_{i_2}^1 \otimes e_{j_2}^2 \iff e_{i_1}^1 < e_{i_2}^1 \text{ and } e_{j_1}^2 < e_{j_2}^2.$$

The $n_1 n_2$-dimensional Lenstra graph of board $\mathcal{B}$ and of rule $\mathcal{R} = \mathcal{R}_1 \otimes \mathcal{R}_2 = \{t_1 \otimes t_2; t_1 \in \mathcal{R}_1, t_2 \in \mathcal{R}_2)$ is called the product of the Lenstra graphs G_1 and G_2 ; it is denoted $G_1 \otimes G_2 = (X, \Gamma)$ with $\Gamma = \Gamma_1 \otimes \Gamma_2$.

Remark For every $u \in \Gamma_1$ and $v \in \Gamma_2$ the following equalities hold :

$$\Gamma(u \otimes v) = \{u^* \otimes v^* + u^* \otimes v + u \otimes v^*; u^* \in \Gamma_1(u), v^* \in \Gamma_2(v)\}$$

$$\Gamma^-(u \otimes v) = \{u' \otimes v' + u' \otimes v + u \otimes v'; u' \in \Gamma_1^-(u), v' \in \Gamma_2^-(v)\}$$

because :

$$u \otimes v + s \otimes t = (u+s) \otimes (v+t) + (u+s) \otimes v + u \otimes (v+t).$$

Proposition 2 The tensor product of two linear lexigraphies is a linear lexigraphy ; that is to say : let $G_1 = (X, \Gamma_1), G_2 = (X_2, \Gamma_2), G_1', G_2'$ be Lenstra graphs and let $f_1 : G_1 \to G_1'$ and $f_2 : G_2 \to G_2'$ be linear lexigraphies ; then the linear map $f_1 \otimes f_2 : X_1 \otimes X_2 \to X_1' \otimes X_2'$ characterized by $f_1 \otimes f_2(x_1 \otimes x_2) = f_1(x_1) \otimes f_2(x_2)$ for every $x_1 \in X_1, x_2 \in X_2$ is a lexigraphy.

Proof Let $\Gamma = \Gamma_1 \otimes \Gamma_2$ and $\Gamma' = \Gamma_1' \otimes \Gamma_2'$. Since every vertex of $G_1 \otimes G_2$ is a sum of elements of the kind $x_1 \otimes x_2$ every pair of them having disjoint support, it is sufficient to prove first that for every $u_1 \in G_1, u_2 \in G_2, f_1 \otimes f_2(\Gamma(u_1 \otimes u_2)) \subset \Gamma'(f_1 \otimes f_2(u_1 \otimes u_2))$ and $\Gamma^-(f_1 \otimes f_2(u_1 \otimes u_2)) \subset f_1 \otimes f_2(\Gamma(u_1 \otimes u_2))$, secondly that, for every $x \in X_1 \otimes X_2$ such that $f_1 \otimes f_2(\Gamma(x)) \subset \Gamma'(f_1 \otimes f_2(x))$ and $\Gamma'^-(f_1 \otimes f_2(x)) \subset f_1 \otimes f_2(\Gamma^-(x))$ and for every $u_1 \in X_1, u_2 \in X_2$ so that $supp(x) \cap supp(u_1 \otimes u_2) = \emptyset$, we have $f_1 \otimes f_2(\Gamma(x + u_1 \otimes u_2)) \subset \Gamma'(f_1 \otimes f_2(x + u_1 \otimes u_2))$ and $\Gamma'^-(f_1 \otimes f_2(x + u_1 \otimes u_2)) \subset f_1 \otimes f_2(\Gamma^-(x + u_1 \otimes u_2))$.
Now we only write this second series of computations because it implies the first one.
Let $x \in X$ such that $f_1 \otimes f_2(\Gamma(x)) \subset \Gamma'(f_1 \otimes f_2(x))$ and $\Gamma'^-(f_1 \otimes f_2(x)) \subset f_1 \otimes f_2(\Gamma^-(x))$ and let $u_1 \in X_1$ and $u_2 \in X_2$; since :

$$\Gamma(x + u_1 \otimes u_2) = (\Gamma(x) + u_1 \oplus u_2) \cup (x + \Gamma(u_1 \oplus \Gamma_2))$$

we have :

$$f_1 \otimes f_2(\Gamma(x + u_1 \otimes u_2)) \subset (\Gamma'(f_1 \otimes f_2(x)) + f_1 \otimes f_2(u_1 \otimes u_2)) \cup (f_1 \otimes f_2(x) + f_1 \otimes f_2(\Gamma(u_1 \otimes u_2)))$$

moreover :

$$f_1 \otimes f_2(\Gamma(u_1 \otimes u_2)) = \{f_1(u_1^*) \otimes f_2(u_2^*) + f_1(u_1) \otimes f_2(u_2^*) + f_1(u_1^*) \otimes f_2(u_2); u_1^* \in \Gamma_1(u_1), u_2^* \in \Gamma_2(u_2)\}$$

$$\subset \{f_1(u_1)^* \otimes f_2(u_2)^* + f_1(u_1) \oplus f_2(u_2)^* + f_1(u_1)^* \oplus f_2(u_2) + f_1(u_1) \otimes f_2(u_2); f_1(u_1)^* \in \Gamma'(f_1(u_1)),$$
$$f_2(u_2)^* \in \Gamma'(f_2(u_2))\}$$

$= \Gamma'(f_1 \otimes f_2(u_1 \otimes u_2))$;
therefore :

$$f_1 \otimes f_2(\Gamma(x + u_1 \otimes u_2)) \subset \Gamma'(f_1 \otimes f_2(x) + f_1 \otimes f_2(u_1 \otimes u_2)).$$

Let us consider now :

$$\Gamma'^-(f_1 \otimes f_2(x + u_1 \otimes u_2)) = \Gamma'^-(f_1 \otimes f_2)(x) + f_1(u_1) \otimes f_2(u_2)))$$

$$\subset (f_1 \otimes f_2(x) + \Gamma'^-(f_1(u_1) \otimes f_2(u_2)) \cup (\Gamma'^-(f_1 \otimes f_2(x) + f_1(u_1) \otimes f_2(u_2))).$$

we have :

$$\begin{aligned} \Gamma'^-(f_1(u_1) \otimes f_2(u_2)) &= \{f_1(u_1)' \otimes f_2(u_2)' + f_1(u_1)' \otimes f_2(u_2) + f_1(u_1) \otimes f_2(u_2)'; \\ &\quad f_1(u_1)' \in \Gamma_1'(f_1(u_1)), f_2(u_2)' \in \Gamma_2'(f_2(u_2))\} \\ &\subset \{f_1 \otimes f_2(u_1' \otimes u_2' + u_1 \otimes u_2); u_1' \in \Gamma_1^-(u_1), u_2' \in \Gamma_2^-(u_2)\} \\ &= f_1 \otimes f_2(\Gamma^-(u_1 \otimes u_2)); \end{aligned}$$

hence

$$\Gamma'^-(f_1 \otimes f_2(x+u_1 \otimes u_2)) \subset (f_1 \otimes f_2(x) + f_1 \otimes f_2(\Gamma^-(u_1 \otimes u_2))) \cup (f_1 \otimes f_2(\Gamma^-(x)) + f_1 \otimes f_2(u_1 \otimes u_2)));$$

finally, if $supp(x) \cap supp(u_1 \otimes u_2) = \emptyset$, applying lemma 4, we get :

$$\Gamma'^-(f_1 \otimes f_2(x + u_1 \otimes u_2)) \subset f_1 \otimes f_2(\Gamma^-(x + u_1 \otimes u_2)).$$

Definition. A Lenstra graphs isomorphism is a vector spaces isomorphism which is also a directed graphs isomorphism.

Theorem 2. Let $\Gamma, \Gamma_1, \Gamma_2, \Gamma_3$ be Lenstra graphs ; we have the following canonical Lenstra graphs isomorphism :
$\Gamma_1 \oplus \Gamma_2 \dot{\rightarrow} \Gamma_2 \oplus \Gamma_1$,
$\Gamma_1 \oplus (\Gamma_2 \oplus \Gamma_3) \dot{\rightarrow} (\Gamma_1 \oplus \Gamma_2) \oplus \Gamma_3$,
$\underline{1} \oplus \Gamma \dot{\rightarrow} \Gamma$, (with $\underline{1} = \underline{2}^o$, of course),
$\Gamma_1 \oplus \Gamma_2 \dot{\rightarrow} \Gamma_2 \otimes \Gamma_1$,
$\Gamma_1 \otimes (\Gamma_2 \otimes \Gamma_3) \dot{\rightarrow} (\Gamma_1 \otimes \Gamma_2) \otimes \Gamma_3$,
$\underline{2} \otimes \Gamma \dot{\rightarrow} \Gamma$,
$\Gamma_1 \otimes (\Gamma_2 \oplus \Gamma_3) \dot{\rightarrow} (\Gamma_1 \otimes \Gamma_2) \oplus (\Gamma_1 \otimes \Gamma_3)$.

Proof These natural verifications are left to the reader.

5 Nim-multiplication

Let a and b be two non-negative integers and let n be such that $a < 2^{2^n}$ and $b < 2^{2^n}$. We define the Nim-product of a and b and we denote by $a \otimes b$ the vertex $g(a \otimes b)$ of $\underline{2}^{2^n}$ where g means the Grundy function of the $\underline{2}$-linear graph $\underline{2}^{2^n} \otimes \underline{2}^{2^n} = (X, \Gamma)$. It is clearly an associative and commutative law on $\underline{2}^{2^n}$; it admits 0 as annihilator because the equality $a \otimes 0 = 0$ holds in X and it admits 1 as unity because $\Gamma^-(a \otimes 1) = \{a' \otimes 1; a' < a\}$; it is distributive with respect to the Nim-addition because the Grundy function g is linear ; finally its recursive definition is

$$a \otimes b = \min(\mathbb{N} \backslash \{a' \otimes b' \oplus a \otimes b' \oplus a' \otimes b; a' < a, b' < b\}).$$

Some tables can be found in [5].

Proposition 3 (by Lenstra [8]). Let $G_1 = (X_1, \Gamma_1)$ and $G_2 = (X_2, \Gamma_2)$ two Lenstra graphs. The Grundy function of $G_1 \otimes G_2$ is the Nim-product of the Grundy functions of G_1 and G_2.

Proof. Let n be an integer large enough for the Grundy functions g_1 of G_1, g_2 of G_2 and g of $G_1 \otimes G_2$ to take their values in $\underline{2}^{2^n}$. Let γ be the Grundy function of $\underline{2}^{2^n} \otimes \underline{2}^{2^n}$. Since $g_1 \otimes g_2 : G_1 \otimes G_2 \to \underline{2}^{2^n} \otimes \underline{2}^{2^n}$ is a lexigraphy (prop. 2), we have $g = \gamma \circ (g_1 \otimes g_2)$ and therefore, for every $u_1 \in X_1$, and $u_2 \in X_2$ the equality $g(u_1 \otimes u_2) = g_1(u_1) \otimes g_2(u_2)$ holds in $\underline{2}^{2^n}$.

Notation. From now, we will use the notation $g = g_1 \otimes g_2$ whenever no confusion with the tensor product $g_1 \otimes g_2$ can arise.

Proposition 4. Let $G = (X, \Gamma)$ be a Lenstra graph of board $\mathcal{B}$ and of rule $\mathcal{R}$. For every $u, v \in X$, let us call uv the componentwise product of u and v with respect to the basis $\mathcal{B}$. Then the linear map from $X \otimes X$ to X characterized by $u \otimes v \mapsto uv$ is a lexigraphy if and only if for every $s, t \in \mathcal{R}$, $st \in \mathcal{R}$.

Proof. Let us assume that the componentwise multiplication is a lexigraphy from $G \otimes G$ to G. Let $s, t \in \mathcal{R}$; since $s \otimes t$ is lying in the rule of $G \otimes G$, we have $st \in \Gamma(0) = \mathcal{R}$.
Now let us assume that $\mathcal{R}$ is invariant by componentwise multiplication and let us write $xy = M(x \otimes y)$ for every $x, y \in X$. For every $u, v \in G$ and $s, t \in \mathcal{R}$, we have

$$(u+s)(v+t) + u(v+t) + (u+s)v = uv + st$$

with $st \in \mathcal{R}$ so $M(\Gamma \otimes \Gamma(u \otimes v)) \subset \Gamma(uv)$.
Moreover if $uv + t \in \Gamma^-(uv)$ for some $t \in \mathcal{R}$ then $uv + t = (u+t)(v+t) + u(v+t) + (u+t)v$ with $u + t \in \Gamma^-(u)$ and $v + t \in \Gamma^-(v)$; therefore $\Gamma^-(uv) \subset M((\Gamma \otimes \Gamma)^-(u \otimes v))$.
The end of this proof can be copied from that of proposition 2.

Corollary 1. Let $G = (X, \Gamma)$ be a Lenstra graph the rule of which being invariant by componentwise multiplication and let g be the Grundy function of G. Then for every $u, v \in X$, $g(uv) = g(u) \otimes g(v)$.

Proof. Let n be an integer large enough for g and $g \otimes g$ to take their values in $\underline{2}^{2^n}$; then $g \otimes g$ and $g \circ M$ are the same lexigraphy from $G \otimes G$ to $\underline{2}^{2^n}$.

Corollary 2. With the same hypothesis as above, the error-correcting code $Ker g$ is a prime ideal of the $\mathbb{F}_2$-algebra X which contains the anihilators all vectors which are not code-words.

Proof : obvious.

6 Extended Lenstra graphs

Let $D = (\Delta, \Gamma_\Delta)$ be a ν-dimensional Lenstra graph of board $H = (\eta_i)_{i \leq \nu}$ and of rule P. Let $\mathcal{B} = \{e_1, e_2, \cdots, e_n\}$ be a (totally or partially) ordered set. Let X be the Abelian group constituted by the formal sums $a = \sum_{i=1}^{n} \alpha_i e_i$ with coefficients α_i in Δ ; we call support relatively to $\mathcal{B}$ of such a formal sum the set :

$$supp_{\mathcal{B}}(a) = supp(a) = \{e_i \in \mathcal{B}; \alpha_i \neq 0\};$$

a formal sum $r = \sum_{i=1}^{n} \rho_i e_i \in X$ is said to be admissible if there exists $\rho \in P$ such that for every $i = 1, 2, \cdots, n, \rho_i = \epsilon_i \rho$ with $\epsilon_i = 0$ or 1 and if $supp(r)$ admits a unique maximal element $e_{\mu(r)}$. Let $\mathcal{R}$ be a set of admissible formal sums $\sum \epsilon_i \rho e_i$ with $\epsilon_i = 0$ or 1, $\rho \in P$ such that $\forall \rho' \in P$, $\sum \epsilon_i \rho' e_i \in \mathcal{R}$; the group X can be provided with a graph structure $G = (X, \Gamma)$ the arcs of which are the pairs (a, b) with

$a = \sum_{i=1}^{n} \alpha_i e_i, \quad b = \sum_{i=1}^{n} \beta_i e_i, \quad r = a + b \in \mathcal{R}$ and $(\alpha_{\mu(r)}, \beta_{\mu(r)})$ being an arc of D.

Under these conditions, G is said to be an n-dimensional Lenstra graph extended to D of board $\mathcal{B}$ and rule $\mathcal{R}$.

Clearly every Lenstra graph is a Lenstra graph extended to $\underline{2}$. Moreover, if $G = (X, \Gamma)$ is such a Lenstra graph extended to D of board $\mathcal{B}$ and rule $\mathcal{R}$ then we can canonically associate to it a Lenstra graph $G_0 = (X_0, \Gamma_0)$ of board $\mathcal{B}$ by choosing as rule $\mathcal{R}_0$ the set of all vectors $\sum_{i=1}^{n} t_i e_i \in X_0 = \mathbb{F}_2^n$ such that $\sum_{i=1}^{n} t_i \rho e_i \in \mathcal{R}$ for some $\rho \in P$. Then, the map

$$\begin{array}{ccc} X_0 \otimes \Delta & \rightarrow & X \\ \sum_{i,j} \lambda_{ij} e_i \otimes \eta_j & \mapsto & \sum_{i=1}^{n} \left(\sum_{i=1}^{\nu} \lambda_{ij} \eta_j\right) e_i \end{array}$$

is a group isomorphism and also a directed graph isomorphism so that G and $G_0 \otimes D$ can be identified.

Let G and G' be two Lenstra graphs extended to D. It is easy to define the sum $G \oplus G'$ as a Lenstra graph extended to D by copying the definition of the sum of two Lenstra graphs. Then, by theorem 2, we get the canonical isomorphisms :

$$G \oplus G' \simeq (G_0 \otimes D) \oplus (G'_0 \otimes D) \simeq (G_0 \oplus G'_0) \otimes D$$

so that the Lenstra graph $(G \oplus G')_0$ associated to $G \oplus G'$ is canonically isomorphic to $G_0 \oplus G'_0$.
Moreover, if we denote by g (resp. g', g_Δ, g_0, g'_0) the Grundy function of G (resp. G', D, G_0, G'_O) the Grundy function of $G \oplus G'$ is :

$$(g_0 \oplus g'_0) \otimes g_\Delta = (g_0 \otimes g_\Delta \oplus (g'_0 \otimes g_\Delta) = g \oplus g'.$$

Finally, by the commutativity of the diagram

$$\begin{array}{ccc} G \oplus G & \xrightarrow{s} & G \\ \downarrow & & \downarrow \\ (G_0 \oplus G_0) \otimes D & \xrightarrow{+\oplus id_\Delta} & G_0 \otimes D \end{array}$$

the sum mapping $\begin{array}{ccc} G \oplus G & \rightarrow & G \\ u \oplus v & \mapsto & u + v \end{array}$ is a lexigraphy.

In the case of $D = \underline{2}^{2^m}$, a Lenstra graph G extended to D is canonically a $\underline{2}^{2^m}$-linear space : let $\mathcal{B} = (e_i)_{1 \leq i \leq n}$ the board of G ; for every $\lambda \in \underline{2}^{2^m}$ and $u = \sum_{i=1}^{n} u_i e_i$ with $u_i \in \underline{2}^{2^m}$, we define

$$\lambda \cdot u = \sum_{i=1}^{n} (\lambda \otimes u_i) e_i$$

where $\lambda \otimes u_i$ is the Nim-product in $\underline{2}^{2^m}$. Let $\mathcal{R}$ be the rule of $G = (X, \Gamma)$ and let $G_0 = (X_0, \Gamma_0)$ be the Lenstra graph associated with G, that is to say : the Lenstra graph of board $\mathcal{B}$ and of rule $\mathcal{R}_0 = \{\sum_{i=1}^{n} t_i e_i \in \mathcal{R}; \forall i = 1, 2, \cdots, n, t_i \in \underline{2}\}$.
The embedding of directed graphs $1 \otimes G_0 \hookrightarrow G$ identifies G_0 to a sub-graph of G and then $\mathcal{R}_0 = \mathcal{R} \cap G_0$. Let us recall that the "scalars extension" map :

$$\begin{array}{lcl} \underline{2}^{2^m} \otimes X_0 & \rightarrow & X \\ \sum_{i,j} \lambda_{ij}\, 2^j \otimes e_i & \mapsto & \sum_{i=1}^{n} \left(\sum_{j=0}^{m-1} \lambda_{ij} 2^j\right) e_i \end{array}$$

(with $\lambda_{ij} = 0$ or 1 so $\lambda_{ij} 2^j = \lambda_{ij} \otimes 2^j$) is an isomorphism of directed graphs. Therefore the Grundy function g of G can be identified to the Grundy function of $\underline{2}^{2^m} \otimes G_0$ that is to say $id \otimes g_0$.

Let $u = \sum_{i=1}^{n} u_i\, e_i$ with $u_i \in \underline{2}^{2^m}$ be a vertex of G and let $\lambda \in \underline{2}^{2^m}$. We can write :

$$g(\lambda u) = g\Big(\sum_{i=1}^{n} (\lambda \otimes u_i) e_i\Big) = (id \otimes g_0)\Big(\sum_{i=1}^{n} (\lambda \otimes u_i) e_i$$

$$= \sum_{i=1}^{n} (\lambda \otimes u_i) \otimes g_0(e_i) = \lambda \otimes \sum_{i=1}^{n} u_i \otimes g_0(e_i)$$

$$= \lambda \otimes g\left(\sum_{i=1}^{n} u_i\, e_i\right) = \lambda \otimes g(u).$$

Thus :
Theorem : The Grundy function of a $\underline{2}^{2^m}$-linear graph is $\underline{2}^{2^m}$-linear.

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