Long time asymptotics for some dynamical noise free non-linear filtering problems, new cases

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Abstract: We are interested here in the long time behaviour of the conditional law for a special case of filtering problem: there is no noise on the state equation and the prior law of the state process concentrate fast in some neighborhood of a limit cycle with strictly negative characteristic exponents. Then assuming a deterministic observability property on the cycle we show the concentration of the conditional law on an arbitrary neighborhood of the current (unknown) state as the time goes to infinity. This work can be considered as illustrating how the tools of dynamical systems theory can be use to study the long time behavior of the filtering process.

Key-words: Non-linear filtering, long time asymptotics, measure concentration

(Résumé : tsvp)

Asymptotiques en temps long pour des problèmes de filtrage non-linéaires sans bruit de dynamique, nouveaux cas.

Résumé : Nous nous intéressons ici au comportement en temps long de la loi conditionnelle pour un problème particulier de filtrage : il n'y a pas de bruit sur l'équation d'état et la loi a priori du processus d'état se concentre rapidement dans un certain voisinage d'un cycle limite dont les exposants caractéristiques sont strictement négatifs. Alors en supposant une propriété d'observabilité déterministe sur le cycle limite on montre la concentration de la loi conditionnelle sur un voisinage arbitraire de l'état courant (inconnu) lorsque le temps croît vers l'infini. Ce travail peut être considéré comme une illustration de l'utilisation de la théorie des systèmes dynamiques pour l'étude du comportement en temps long du processus de filtrage.

Mots-clé : Filtrage non-linéaire, asymptotique en temps long, concentration de mesure

Contents

1 Introduction

This paper is essentially a sequel of our previous work [1] concerning the long time behaviour of some non linear filtering problems (see for example [8] for a review on non linear filtering). There is not much literature on this topic: some results have been proved in [5] and [6] by Kunita when the state process is ergodic. Let us also mention that the results presented in [5] have been extended to the non compact case in [4] by Ji. Concerning the sensitivity of the filter to the prior distribution, some results can be found in [7].

Here we will try to show how the tools developped within the theory of dynamical systens can be used for studying the long time behaviour of the optimal filter. We consider the following problem:

$$\left\{ \begin{array}{rcl} dX_t &=& b(X_t)\,dt \\ dY_t &=& h(X_t)\,dt + dB_t \end{array} \right. \tag{1}$$

where X_t takes values in a compact domain $D \subset \mathbb{R}^d$ and Y_t in $\mathbb{R}^m$, X_0 is a random vector of given law P_{X_0}, with density p_0, B_t is a m-dimensional Wiener process independent of X_0. b is assumed to be C^1. In the sequel, $\Phi_t(x)$ will denote the (deterministic) flow of the state equation; it is the state reached by the system at time t, starting from x at $t = 0$. We will assume that it is well defined for every time t and every initial condition x_0. We will also assume that the dynamical system defined by the vector field b has the following behaviour: most of the trajectories are attracted (in a sense to be precised below) by any neighbourhood $\mathcal{C}^{(\delta)} = \{x \in D, d(x, \mathcal{C}) < \delta\}$ of a periodic orbit $\mathcal{C}$ with $d-1$ characteristic values having negative real parts (see [2]). Then every attracted orbit has an asymptotic phase on the limit cycle $\mathcal{C}$ and its distance from $\mathcal{C}$ decays exponentially fast as t tends to ∞. All these notions will be precised in the next section. Note that in all the sequel by "period" we mean the least period of the considered function. With some more assumptions on h and p_0, it is then sufficient to have some observability property on $\mathcal{C}$ to show the concentration of the conditional law on any small ball centered on the true position $X_t = \Phi_t(X_0)$ of the state equation. Recall from [1] that μ_t being the conditional density, for all $a > 0$:

$$\int_{\{\|x-X_t\| \le a\}} \mu_t(x)\,dx = \frac{\int_{\Phi_t^{-1}(\{\|x-X_t\| \le a\})} f_t(x)\,dx}{\int_{\mathbb{R}^d} f_t(x)\,dx}.$$

where

$$f_t(x) = \exp\left[-\frac{1}{2}\int_0^t \|h(\Phi_s(x)) - h(\Phi_s(X_0))\|^2\,ds + \int_0^t (h(\Phi_s(x)) - h(\Phi_s(X_0)))\,dB_s\right] p_0(x). \tag{2}$$

The main idea is that on an attracted orbit, the observations are very close to the ones given by its asymptotic phase.

2 Assumptions and properties of the system

First we consider a deterministic system associated with (1):

$$\begin{cases} \dfrac{dx_t}{dt} = b(x_t) \\ \\ dy_t = h(X_t)dt \end{cases} \tag{3}$$

with initial condition $x_0 \in D$. For all x_1, $x_2 \in D$ and all $t \geq 0$ we denote

$$I_t(x_1, x_2) = \int_0^t |h(\Phi_s(x_1)) - h(\Phi_s(x_2))|^2 \, ds.$$

Denote by $T > 0$ the period of the periodic orbit $\mathcal{C}$. In all the sequel we make the following assumption on the observation function h:

(A1)

(i) There is some $K_h > 0$ such that or all x_1 and x_2 in D

$$\|h(x_1) - h(x_2)\| \leq K_h \|x_1 - x_2\| \tag{4}$$

(ii) We have the observability condition on $\mathcal{C}$:

$$\forall \eta > 0,\ \exists \varepsilon > 0 \text{ such that } \forall x_1,\ x_2 \in \mathcal{C},\ \|x_1 - x_2\| \geq \eta \ \Rightarrow\ I_T(x_1, x_2) > \varepsilon. \tag{5}$$

Then we have the following lemma, which gives a characterisation of the observability condition:

Lemma 2.1 *Assume h and b are continous (then so is the flow Φ associated with b), then the observability condition (5) is equivalent to any of the following assertions:*

(∗) $h \circ \Phi_t(x)$ is periodic of period T for all $x \in \mathcal{C}$.

(∗∗) $\forall x_1,\ x_2 \in \mathcal{C},\ x_1 \neq x_2,\ \exists\, t > 0$ such that $h(\Phi_t(x_1)) \neq h(\Phi_t(x_2))$ (this is the standard deterministic observability property on $\mathcal{C}$).

Proof (5) ⇒ (∗):

Suppose $h \circ \Phi_t$ has period T', $0 < T' < T$. Then let x_1 and x_2 such that $x_2 = \Phi_{T'}(x_1)$, $x_1 \neq x_2$. Then $\forall t > 0$, $I_t(x_1, x_2) = 0$.

(∗) ⇒ (5):

Suppose: $\exists \eta > 0$ such that $\forall \varepsilon > 0\ \exists x_1,\ x_2 \in \mathcal{C}$, $\|x_1 - x_2\| \geq \eta$ and $I_T(x_1, x_2) < \varepsilon$. Thus

$$\inf_{\substack{(x_1, x_2) \in \mathcal{C}^2 \\ \|x_1 - x_2\| \geq \eta}} I_T(x_1, x_2) = 0.$$

The domain of the infimum is compact and $I_T(.,.)$ is continuous, so there exist x_1, $x_2 \in \mathcal{C}$, $x_1 \neq x_2$ such that $I_T(x_1, x_2) = 0$. Then by continuity in t, $h(\Phi_t(x_1)) = h(\Phi_t(x_2))$, $\forall t > 0$. Let $0 < \tau < T$ such that $\Phi_\tau(x_1) = x_2$ (interchange the role played by x_1 and x_2 if necessary). We have $\forall t > 0$, $h(\Phi_t(x_1)) = h(\phi_{t+\tau}(x_1))$, that is $h \circ \Phi_t(x)$ has period $\tau < T$.

(5) $\Rightarrow$ (**) is obvious.

(**) $\Rightarrow$ (5):

Again by a continuity argument, we choose ε such that

$$0 < \varepsilon < \inf_{\substack{(x_1, x_2) \in \mathcal{C}^2 \\ \|x_1 - x_2\| \geq \eta}} I_T(x_1, x_2).$$

□

Let us now outline some classical notions related to dynamical systems (see for instance [2] for details). From now on, we will consider that b is C^1. First we define the Poincaré map: let $\xi_0 \in \mathcal{C}$ and π an hyperplane tranverse to $\mathcal{C}$ at ξ_0 (i.e. satisfying the equation $\xi.b(\xi_0) = 0$, where the dot denotes the scalar product in $\mathbb{R}^d$). As $b(\xi_0) \neq 0$ (we are on a periodic orbit), an immediate consequence of the implicit function theorem gives that there exists a unique real valued C^1 function $\tau(\xi)$ defined on a neighbourhood of ξ_0 in π, such that $\Phi_{\tau(\xi)}(\xi) \in \pi$ and $\tau(\xi_0) = T$. Then the map $F : \xi \to \Phi_{\tau(\xi)}(\xi)$ defined on the same neighbourhood of ξ_0 is also C^1 and is called the Poincaré map. Consider now the matrix $H(t, \xi) = \frac{\partial \Phi_t(\xi)}{\partial x}$. H is solution of

$$\frac{dH(t,\xi)}{dt} = \frac{\partial b(\Phi_t(\xi))}{\partial x} H(t, \xi),$$

$H(0, \xi) = I$. For $\xi = \xi_0$, the matrix $\frac{\partial b(\Phi_t(\xi_0))}{\partial x}$ is periodic, of period T. Then by the Floquet theory, $H(t, \xi_0)$ has a representation of the form:

$$H(t, \xi_0) = C(t)\, e^{t\Lambda}$$

where $C(t)$ is periodic matrix of period T and Λ is a constant matrix. The eigenvalues $\lambda_1, \ldots, \lambda_d$ of $H(T, \xi_0) = H(\tau(\xi_0), \xi_0) = e^{T\Lambda}$ are the *characteristic roots* of the periodic orbit lying on $\mathcal{C}$ and $T^{-1} \log \lambda_1, \ldots, T^{-1} \log \lambda_d$ are the *characteristic exponents*. Note that only the real parts of the characteristic exponents are uniquely defined (they are defined modulo $2i\pi$ and Λ is not unique). One of the characteristic values, say the last one in a suitable system of coordinates such that $b(\xi_0) = (0, \ldots, 0, 1)$, is 1, and the submatrix obtained from $e^{T\Lambda}$ by deleting the last row and column is the Jacobian of F at ξ_0.

From now on, we will assume that the $d-1$ first characteristic exponents of $\mathcal{C}$ have stictly negative real parts. Using these tools, one can show ([2] theorem IX 11.1) that there exist $\delta > 0$, $L > 0$ and $\sigma > 0$ such that for all $x \in \mathcal{C}^{(\delta)} = \{\xi \in D,\ d(\xi, \mathcal{C}) < \delta\}$ there is some $\tilde{x} \in \mathcal{C}$ verifying

$$\|\Phi_t(x) - \Phi_t(\tilde{x})\| < L\, e^{-\sigma t}. \tag{6}$$

$\tilde{x}$ is called the *asymptotic phase* of x. In order to use these results, we will assume that the following condition is fulfilled:

(A2) The vector field b is C^1 and the $d-1$ characteristic exponents of the periodic orbit lying on $\mathcal{C}$ have stricly negative real parts.

The proof of the following lemma was given to the author by A. M. Davie. As long as we know it cannot be found in the standard literature, so we reproduce Davie's complete proof here.

Lemma 2.2 (Davie) *Assume that (A2) is fulfilled. Then there is a neighbourhood $\mathcal{C}^{(\delta)}$ of $\mathcal{C}$ such that there exists a constant $C > 0$ verifying: for all x_1, $x_2 \in \mathcal{C}^{(\delta)}$ and all $t \geq 0$ we have*

$$\|\Phi_t(x_1) - \Phi_t(x_2)\| \leq C\,\|x_1 - x_2\|. \tag{7}$$

Proof Let π be a tranverse hyperplan to $\mathcal{C}$, and F the associated Poincaré map. Corresponding to the periodic orbit is an attracting fixed point $\gamma \in \pi$ of F ($\{\gamma\} = \mathcal{C} \cap \pi$). Then, for a suitable norm, we have $\|DF(\alpha)\| \leq k < 1$ for α near γ, $\alpha \in \pi$. Let x_1, x_2 be close to the periodic orbit $\mathcal{C}$ (but not close to π). We find s and t so that

$$\Phi_t(x_1) = \alpha \in \pi, \quad \Phi_s(x_2) = \beta \in \pi$$

and

$$\|\alpha - \beta\| \leq C_1 \|x_1 - x_2\|, \quad |s - t| \leq C_1 \|x_1 - x_2\|.$$

This is possible because by an implicit function argument similar to the proof of lemma IX.10.1 in [2] the map $x_1 \mapsto t$ is C^1 in some neighbourhood of x_1. But we have to be careful to avoid singularities near π: if we consider only positive times, we get something close to 0 on one side of π, and close to the period T on the other one. To deal with this problem we may allow negative times and consider different maps. First notice that as t and s (either positive or negative) are bounded, it is sufficient to consider x_1 and x_2 close to each other. Then consider $\varepsilon > 0$ small and define:

$$D_1 = \left(\bigcup_{|u| \leq 2\varepsilon} \Phi_u(\pi) \right) \cap \mathcal{C}^{(\delta)},$$

$$D_2 = \left(\bigcup_{|u| < \varepsilon} \Phi_u(\pi) \right)^c \cap \mathcal{C}^{(\delta)}.$$

If x_1 and x_2 are close enough to each other, then they are both in at least one of the two domains above. On D_1, we allow both positive and negative times, and on D_2 we allow only positive ones. Then these two maps are C^1 on their respective domain.

For $\eta \in \pi$ we let $\tau(\eta)$ be the time t such that $\Phi_t(\eta) = F(\eta)$, then

$$|\tau(\eta_1) - \tau(\eta_2)| \leq C_2 \|\eta_1 - \eta_2\|$$

for all η_1, $\eta_2 \in \pi$ close to γ.

Let $\alpha_n = F^n(\alpha)$ and $\beta_n = F^n(\beta)$. We have $\|DF(\eta))\| \leq k$ for η near γ so $\|DF^n(\eta))\| \leq k^n$ so

$$\|\alpha_n - \beta_n\| \leq k^n\,\|\alpha - \beta\| \leq C_1\,k^n\,\|x_1 - x_2\|$$

and

$$|\tau(\alpha_n) - \tau(\beta_n)| \leq C_1\, C_2\, k^n\, \|x_1 - x_2\|.$$

We have

$$\Phi_{t_n}(x_1) = \alpha_n \quad \text{where} \quad t_n = t + \tau(\alpha_1) + \cdots + \tau(\alpha_{n-1}),$$

$$\Phi_{s_n}(x_2) = \beta_n \quad \text{where} \quad s_n = s + \tau(\beta_1) + \cdots + \tau(\beta_{n-1}).$$

Then we obtain

$$|s_n - t_n| \leq C_3\, \|x_1 - x_2\|,$$

so

$$\|\Phi_{t_n}(x_1) - \Phi_{s_n}(x_2)\| \leq C_4\, \|x_1 - x_2\|,$$

also

$$\|\Phi_{t_n}(x_2) - \Phi_{s_n}(x_2)\| \leq C_4\, \|x_1 - x_2\|,$$

so

$$\|\Phi_{t_n}(x_1) - \Phi_{t_n}(x_2)\| \leq (C_4 + C_5)\, \|x_1 - x_2\|.$$

Since $t_{n+1} - t_n$ is bounded, we then obtain

$$\|\Phi_t(x_1) - \Phi_t(x_2)\| \leq C_6\, \|x_1 - x_2\|, \text{ for } t_n \leq t \leq t_{n+1},$$

for every n, i.e.

$$\|\Phi_t(x_1) - \Phi_t(x_2)\| \leq C\, \|x_1 - x_2\|,$$

where C does not depend on t, x_1, x_2. □

Now we come back to the system (1). We make some assumptions concerning the flow outside $\mathcal{C}^{(\delta)}$ and the prior law P_0.

(A3)

(i) There exists a function $\Delta : \mathbb{R}_+ \longrightarrow \overline{\mathbb{R}}_+$ such that $\lim_{t\to+\infty} \Delta(t) = +\infty$, and for some $\delta > 0$ small enough so that both (7) and (6) are satisfied there is some $K > 0$ such that for all $t > 0$:
$$P_0\left(\Phi_t^{-1}\left(\mathcal{C}^{(\delta),c}\right)\right) \leq K\, e^{-\Delta(t)}. \tag{8}$$

(ii) There exists a function $\alpha : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that for all x_1, $x_2 \in D$, for all $t > 0$:
$$\|\Phi_t(x_1) - \Phi_t(x_2)\| \leq \alpha(t)\, \|x_1 - x_2\|. \tag{9}$$

(iii) There exists a function $t_0 : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ verifying:
$$\frac{t_0(t)}{t} \longrightarrow 0 \text{ as } t \to +\infty, \tag{10}$$

$$\frac{\alpha(t_0(t))}{\sqrt{t}} + \frac{A(t_0(t))}{t} \longrightarrow 0 \text{ as } t \to +\infty \tag{11}$$

and

$$\frac{A(t)+t}{\Delta(t_0(t))} \longrightarrow 0 \text{ as } t \to +\infty \tag{12}$$

where $A(t) = \|\alpha\|_{L^2(0,t)}$.

Remark 2.3 The conditions above should be viewed as conditions on P_0 rather than on the flow. In particular they are trivialy fulfilled in the case where $P_0(\Phi_{t_1}^{-1}(\mathcal{C}^{(\delta)})) = 1$, for some $t_1 \geq 0$, for which we have $\forall t \geq t_1$, $\Delta(t) = +\infty$. Consider now another case: there is one fixed point ξ, but all the other points in a neighbourhood of ξ are attracted by $\mathcal{C}$. To simplify, assume that we are in $\mathbb{R}^2$, $\xi = 0$ and that 0 and $\text{supp}(p_0)$ are in the interior of the curve $\mathcal{C}$. For all $a > 0$, let $B(0,a) = \{x \in \mathbb{D}, \|x\| \leq a\}$. We assume that there is $a_0 > 0$ such that the flow is linear in $B(0,a_0)$, i.e. $\forall x \in B(0,a_0)$, $\Phi_t^{-1}(x) = e^{-t}\, x$ (we remove the constants for simplicity). Assume also that

$$t_1 = \sup_{x \in B(0,a_0)^c \cap \mathcal{C}^{(\delta),c}} \inf \left\{ s > 0,\ \Phi_s(x) \in \mathcal{C}^{(\delta)} \right\} < +\infty.$$

Thus $\alpha(t) = e^t$ To satisfy both (10) and (11) we can choose:

$$t_0(t) = \frac{1}{2+\nu_1} \log t, \text{ with } \nu_1 > 0.$$

Then to satisfy (8) and (12) p_0 must decay very fast around the fixed point 0: consider that we have $p_0(x) \leq K \exp\left(-\exp(\|x\|^{-2-\nu_2})\right)$ for some $\nu_2 > \nu_1$. We get for $t > t_1$:

$$\begin{aligned} P_0\left(\Phi_t^{-1}(\mathcal{C}^{(\delta),c})\right) &\leq P_0\left(\Phi_{t-t_1}^{-1}(B(0,a_0))\right) \\ &= P_0\left(B(0, e^{-t+t_1} a_0)\right) \\ &\leq K\, 2\pi \int_0^{e^{-t+t_1} a_0} r \exp\left[-\exp\left[r^{-2-\nu_2}\right]\right] dr. \end{aligned}$$

Let $a_1 = a_0^{-2-\nu_2}$, then for t large enough:

$$\begin{aligned} P_0\left(\Phi_t^{-1}(\mathcal{C}^{(\delta),c})\right) &\leq K_1\, e^{-2t+2t_1} a_0^2 \exp\left[-\exp\left[e^{(2+\nu_2)(t-t_1)} a_1\right]\right] \\ &\leq K_2 \exp\left[-\exp\left[e^{(2+\nu_2-\frac{1}{2}(\nu_2-\nu_1))t} a_1\right]\right], \end{aligned}$$

for some $K_! > 0$, $K_2 > 0$. So we can choose:

$$\Delta(t) = \exp\left[e^{(2+\nu_2-\frac{1}{2}(\nu_2-\nu_1))t} a_1\right],$$

and then

$$\Delta(t_0(t)) = \exp\left[a_1\, t^{1+\nu_3}\right]$$

for some $\nu_3 > 0$. Finaly we have:

$$\frac{A(t)+t}{\Delta(t_0(t))} = e^{t - a_1\, t^{1+\nu_3}} + t\, e^{-a_1\, t^{1+\nu_3}} \longrightarrow 0 \text{ as } t \to +\infty.$$

□

Lemma 2.4 *Assume that (A2) and (A3) are fulfilled. Then*

$$d(X_t, \mathcal{C}) \xrightarrow{a.s.} 0 \text{ as } t \to +\infty.$$

Proof Let $i(n)$ be an incerasing sequence such that $e^{-\Delta(i(n))} \leq \frac{1}{2^{-n}}$. Then by (8) and the Borel-Cantelli lemma:

$$P_0\left(\bigcap_{n\geq 0}\bigcup_{m\geq n} \Phi_{i(n)}^{-1}\left(\mathcal{C}^{(\delta),c}\right)\right) = 0,$$

i.e.

$$P_0\left(\bigcup_{n\geq 0}\bigcap_{m\geq n} \Phi_{i(n)}^{-1}\left(\mathcal{C}^{(\delta)}\right)\right) = 1.$$

Thus P_0 a.s. there exists some (random) τ such that $X_\tau \in \mathcal{C}^{(\delta)}$ and by (6) $d(\Phi_t(X_\tau), \mathcal{C}) \to 0$ as $t \to +\infty$. □

Finaly we have to estimate the stochastic integral appearing in (2).

Lemma 2.5 *Assume that (A1) and (A3) are fulfilled. Then the following estimate holds:*

$$E\left[\sup_{x\in D}\left|\int_0^t \left(h(\Phi_s(x)) - h(\Phi_s(X_0))\right)\, dB_s\right|\right] \leq K\, A(t)$$

for some $K > 0$.

Proof For all $t > 0$ and x_1, x_2 in D, using (9) and (4):

$$\int_0^t \|h(\Phi_s(x_1) - \Phi_s(x_2)\|^2\, ds \leq K_h^2\, (A(t))^2\, \|x_1 - x_2\|^2.$$

Then the same argument as in [1] proposition 2.1 applies (recall also that the domain D is bounded). □

Then by (12) we have immediatly the

Lemma 2.6 *Assume that (A1) and (A3) are fulfilled. Then the following convergence holds:*

$$\frac{1}{\Delta(t_0(t))}\sup_{x\in D}\left|\int_0^t \left(h(\Phi_s(x)) - h(\Phi_s(X_0))\right)\, dB_s\right| \xrightarrow{P} 0 \text{ as } t \to +\infty. \quad (13)$$

From now on we will denote:

$$B_a^t = \{x \in D,\, \|x - X_t\| \leq a\},$$

Lemma 2.7 *Assume that (A1), (A2) and (A3) are fulfilled. Then the following convergence holds:*

$$\frac{1}{t}\sup_{x\in\Phi_{\tau_1}^{-1}\left(B_a^{\tau_1}\right)}\left|\int_0^t \left(h(\Phi_s(x)) - h(\Phi_s(X_0))\right)\, dB_s\right| \xrightarrow{P} 0 \text{ as } t \to +\infty \quad (14)$$

for all $0 < a < \frac{\delta}{4}$ *and where*

$$\tau_1 = \inf\left\{t > 0,\, B_a^t \subset \mathcal{C}^{(\delta)}\right\}. \quad (15)$$

Proof As X_0 and B are independent, we can consider that $P = P_0 \otimes P_B$ on some probability space $\Omega_0 \times \Omega_B$. We denote by E_0 (resp. E_B) the expectation over Ω_0 (resp. Ω_B). Note that τ_1 is X_0–measurable and is also independent of B. For almost all ω_0, and for all $x_1, x_2 \in \Phi_{\tau_1}^{-1}(B_a^{\tau_1})$ we have:

$$\int_0^t \|h(\Phi_s(x_1)) - h(\Phi_s(x_2))\|^2 \, ds \leq \left(K_1 \, A(\tau_1)^2 + K_2 \, \alpha(\tau_1)^2 \, (t - \tau_1)^+\right) \, \|x_1 - x_2\|^2,$$

for some $K_1 > 0$ and $K_2 > 0$, using (4), (9) for the integral over $[0, \tau_1]$ and (4), (7) for the integral over $[\tau_1, t]$. Then again the same argument as in [1] proposition 2.1 applies to give P_0–p.s.:

$$E_B \left[\sup_{x \in \Phi_{\tau_1}^{-1}(B_a^{\tau_1})} \left| \int_0^t \left(h(\Phi_s(x)) - h(\Phi_s(X_0))\right) \, dB_s \right| \right]$$
$$\leq K_3 \, \left(K_1 \, A(\tau_1)^2 + K_2 \, \alpha(\tau_1)^2 \, (t - \tau_1)\right)^{\frac{1}{2}}$$

for all $t > \tau_1$, for some $K_3 > 0$, which clearly implies the lemma. □

Lemma 2.8 *Assume that (A1), (A2) and (A3) are fulfilled. Then the following convergence holds:*

$$\frac{1}{t} \sup_{x \in \Phi_{t_0(t)}^{-1}\left(\Phi_{t-t_0(t)}^{-1}(B_a^t) \cap \mathcal{C}^{(\delta)}\right)} \left| \int_0^t \left(h(\Phi_s(x)) - h(\Phi_s(X_0))\right) \, dB_s \right| \xrightarrow{P} 0 \text{ as } t \to +\infty. \tag{16}$$

Proof Let Θ_a^t be the domain of the supremum. Then for all $0 < a < \frac{\delta}{4}$, t such that $t_0(t) > \tau_1$ (τ_1 defined in (15)), and $x_1, x_2 \in \Theta_a^t$:

$$\int_0^t \|h(\Phi_s(x_1)) - h(\Phi_s(x_2))\|^2 \, ds \leq \left(K_1 \, A(t_0(t))^2 + K_2 \, \alpha(t_0(t))^2 \, (t - t_0(t))\right) \, \|x_1 - x_2\|^2,$$

using (4), (9) for the integral over $[0, t_0(t)]$ and (4), (7) for the integral over $[t_0(t), t]$. Then again the same argument as in [1] proposition 2.1 applies to give P_0–p.s.:

$$E_B \left[\sup_{x \in \Theta_a^t} \left| \int_0^t \left(h(\Phi_s(x)) - h(\Phi_s(X_0))\right) \, dB_s \right| \right]$$
$$\leq K_3 \, \left(K_1 \, A(t_0(t))^2 + K_2 \, \alpha(t_0(t))^2 \, (t - t_o(t))\right)^{\frac{1}{2}} \tag{17}$$
$$\leq K_3 \, K_1^{\frac{1}{2}} \, A(t_0(t)) + (K_2 \, (t - t_0(t)))^{\frac{1}{2}} \, \alpha(t_0(t))$$

for some $K_3 > 0$, which clearly implies the lemma, using (10) and (11). □

3 Main results

Recall that f_t is given by (2).

Proposition 3.1 *Assume (A1), (A2) and (A3) are fulfilled. Then the following convergence holds:*

$$\frac{\int_{\Phi^{-1}_{t_0(t)}\left(\mathcal{C}^{(\delta),c}\right)} f_t(x)\,dx}{\int_{\Phi_t^{-1}(B^t_a)} f_t(x)\,dx} \xrightarrow{P} 0 \text{ as } t \to +\infty, \tag{18}$$

for any $a > 0$.

Proof Using the characterisation of the convergence in probability in temrs of a.s. convergence of sub–sequences, this will follow from the convergence P–a.s. along a sequence s_n such that $\lim_{n\to+\infty} s_n = +\infty$ and for which the convergences (13) and (14) take place a.s. We consider a small and s_n large such that $B^{t_0(s_n)}_{\frac{a}{C}} \subset \mathcal{C}^{(\delta)}$, where C is given by lemma 2.2. This is possible from lemma 2.4. Then we have:

$$\begin{aligned}
&\int_{\Phi^{-1}_{t_0(s_n)}\left(\mathcal{C}^{(\delta),c}\right)} f_{s_n}(x)\,dx \\
&\leq \exp\left[\sup_{x\in D}\left|\int_0^{s_n}\left(h(\Phi_s(x)) - h(\Phi_s(X_0))\right)\,dB_s\right|\right]\int_{\Phi^{-1}_{t_0(s_n)}\left(\mathcal{C}^{(\delta),c}\right)} p_0(x)\,dx \\
&\leq K\exp\left[-\Delta(t_0(s_n)) + \sup_{x\in D}\left|\int_0^{s_n}\left(h(\Phi_s(x)) - h(\Phi_s(X_0))\right)\,dB_s\right|\right] \\
&\leq K\exp\left[-\frac{1}{2}\Delta(t_0(s_n))\right]
\end{aligned}$$

for n large enough, using (A3) (i), and the definition of (s_n). On the other hand, τ_1 being given by (15) with $\frac{a}{C}$ instead of a, notice that, on the set $\{\tau_1 > t\}$:

$$B^0_{\frac{a}{\alpha(\tau_1)C}} \subset \Phi^{-1}_{\tau_1}\left(B^{\tau_1}_{\frac{a}{C}}\right) \subset \Phi^{-1}_t\left(B^t_a\right).$$

Then using (4), (7), (9):

$$\forall x \in B^0_{\frac{a}{\alpha(\tau_1)C}},\quad \int_0^t \left\|h\left(\Phi_s(x)\right) - h(X_s)\right\|^2\,ds \leq K_h^2\int_0^{\tau_1}\alpha(s)^2\frac{a^2}{\alpha(\tau_1)^2C_2}\,ds + K_h^2\int_{\tau_1}^t a^2\,ds.$$

So

$$\begin{aligned}
&\int_{\Phi^{-1}_{s_n}(B^{s_n}_a)} f_{s_n}(x)\,dx \\
&\geq \int_{B^0_{\frac{a}{\alpha(\tau_1)C}}} f_{s_n}(x)\,dx
\end{aligned}$$

$$\geq \exp\left[-K_1 A(\tau_1)^2 \frac{a^2}{\alpha(\tau_1)^2\, C^2} - K_2\, a^2\,(s_n - \tau_1)\right.$$

$$\left.- \sup_{x\in B^0_{\frac{a}{C\,\alpha(\tau_1)}}} \left|\int_0^{s_n} \left(h(\Phi_s(x)) - h(\Phi_s(X_0))\right)\, dB_s\right|\right] \int_{B^0_{\frac{a}{C\,\alpha(\tau_1)}}} p_0(x)\, dx$$

$$\geq K_4 \exp\left[-K_5\, a^2\, s_n\right] \tag{19}$$

for n large enough, using the definition of (s_n), and where all the K_i are strictly positive and independent of s_n (but possibly random, note also that K_5 does not depend on a). Then we obtain:

$$\frac{\int_{\Phi^{-1}_{t_0(s_n)}\left(\mathcal{C}^{(\delta),c}\right)} f_{s_n}(x)\, dx}{\int_{\Phi^{-1}_{s_n}(B_a^{s_n})} f_{s_n}(x)\, dx} \leq K_6 \exp\left[-\frac{1}{2}\Delta(t_0(s_n)) + K_5\, a^2\, s_n\right]$$

which tends to 0 by (12) and $\lim_{t\to+\infty}\Delta(t) = +\infty$. □

Proposition 3.2 *Assume (A1), (A2) and (A3) are fulfilled. Then the following convergence holds:*

$$\frac{\int_{\Phi^{-1}_{t_0(t)}\left(\mathcal{C}^{(\delta)}\cap\Phi^{-1}_{t-t_0(t)}\left(B_a^{t,c}\right)\right)} f_t(x)\, dx}{\int_{\Phi_t^{-1}(B_a^t)} f_t(x)\, dx} \xrightarrow{P} 0 \text{ as } t\to+\infty \tag{20}$$

for any $a > 0$.

Proof Let

$$\Theta_a^t = \Phi^{-1}_{t_0(t)}\left(\mathcal{C}^{(\delta)} \cap \Phi^{-1}_{t-t_0(t)}\left(B_a^{t,c}\right)\right),$$

where the superscript c denotes the complementary. For all $x \in \mathcal{C}^{(\delta)}$ let us denote by $\tilde{x}$ the asymptotic phase of x on $\mathcal{C}$. Then we have for all $x \in \Theta_a^t$, $\|\Phi_{t_0(t)}(x) - X_{t_0(t)}\| > \frac{a}{C}$, from lemma 2.2. So for $s \in]\frac{1}{2}(t+t_0(t)), t]$,

$$\|\Phi_s(x) - X_s\| \leq 2L\, e^{-\frac{1}{2}(t+t_0(t))\sigma} + \|\widetilde{\Phi_s(x)} - \tilde{X}_s\|$$

by (6) ($\tilde{X}_s$ exists a.s. by lemma 2.4). Notice the following inclusions:

$$\forall u \in [0, t-t_0(t))],\ \ \Phi^{-1}_{t-t_0(t)-u}\left(B_a^{t,c}\right) = \Phi_u\left(\Phi^{-1}_{t-t_0(t)}\left(B_a^{t,c}\right)\right) \subset B^{t_0(t)+u,c}_{\frac{a}{C}},$$

so

$$\|\Phi_s(x) - X_s\| \geq \frac{a}{C},$$

and thus using (10)

$$\|\widetilde{\Phi_s(x)} - \tilde{X}_s\| \geq \frac{a}{2C}$$

for s large enough. We get, by a similar argument:

$$\begin{aligned}
&\int_0^t \|h(\Phi_s(x)) - h(\Phi_s(X_0))\|^2 \, ds \\
&\geq \int_{\frac{1}{2}(t+t_0(t))}^t \|h(\Phi_s(x)) - h(\Phi_s(X_0))\|^2 \, ds \\
&\geq \frac{1}{4}\int_{\frac{1}{2}(t+t_0(t))}^t \|h(\widetilde{\Phi_s(x)}) - h(\tilde{X}_s)\|^2 \, ds - K_0 \int_{\frac{1}{2}(t+t_0(t))}^t e^{-2\sigma s} \, ds \\
&\geq K_1 \, t
\end{aligned}$$

for t large enough, using also (A1), and for some K_0, $K_1 > 0$. Then:

$$\int_{\Theta_a^t} f_t(x) \, dx \leq \exp\left[-K_2 \, t + \sup_{x \in \Theta_a^t} \left| \int_0^t (h(\Phi_s(x)) - h(X_s)) \; dB_s \right| \right].$$

Let s_n such that $\lim_{n \to +\infty} s_n = +\infty$ and for which the convergences (14) and (16) take place a.s. Then for n large enough we obtain:

$$\int_{\Theta_a^{s_n}} f_{s_n}(x) \, dx \leq K \exp\left[-K_3 \, s_n\right]$$

for some $K_3 > 0$. Assume $a_0 < \frac{\delta}{4}$ to have $B_{a_0}^t \subset \mathcal{C}^{(\delta)}$ for lage t, and let $a_0 \leq a$. Recall (19):

$$\int_{\Phi_{s_n}^{-1}(B_a^{s_n})} f_{s_n}(x) \, dx \geq \int_{\Phi_{s_n}^{-1}\left(B_{a_0}^{s_n}\right)} f_{s_n}(x) \, dx \geq K_4 \, \exp\left[-K_5 \, a_0^2 \, s_n\right]$$

for n large enough, where K_5 does not depend on a_0. Thus we obtain:

$$\frac{\int_{\Theta_a^{s_n}} f_{s_n}(x) \, dx}{\int_{\Phi_{s_n}^{-1}(B_a^{s_n})} f_{s_n}(x) \, dx} \leq K_6 \, \exp\left[K_5 \, a_0^2 \, s_n - K_3 \, s_n\right],$$

which tends to 0 provided we choose a_0 small enough. We conclude using the characterisation of the convergence in probability in terms of a.s. convergence of sub–sequences. □

Finaly we can state the main result:

Theorem 3.3 *Assume that (A1), (A2) and (A3) are fulfilled. Then for all $a > 0$ the following convergence holds:*

$$\int_{\{\|x - X_t\| \leq a\}} \mu_t(x) \, dx \xrightarrow{P} 1 \;\; as \; t \to +\infty,$$

where μ_t is the conditional density of X_t given $\sigma(Y_s, \, s \leq t)$.

Proof It is equivalent to show the following convergence:

$$\frac{\int_{\Phi_t^{-1}(B_a^{t,c})} f_t(x)\,dx}{\int_{\Phi_t^{-1}(B_a^t)} f_t(x)\,dx} \xrightarrow{P} 0 \text{ as } t \to +\infty.$$

Then we get:

$$\frac{\int_{\Phi_t^{-1}(B_a^{t,c})} f_t(x)\,dx}{\int_{\Phi_t^{-1}(B_a^t)} f_t(x)\,dx} \leq \frac{\int_{\Phi_{t_0(t)}^{-1}\left(\mathcal{C}^{(\delta),c}\right)} f_t(x)\,dx}{\int_{\Phi_t^{-1}(B_a^t)} f_t(x)\,dx} + \frac{\int_{\Phi_{t_0(t)}^{-1}\left(\mathcal{C}^{(\delta)}\cap\Phi_{t-t_0(t)}^{-1}\left(B_a^{t,c}\right)\right)} f_t(x)\,dx}{\int_{\Phi_t^{-1}(B_a^t)} f_t(x)\,dx},$$

which tends to 0 in probability, using propositions 3.1 and 3.2. □

4 Conclusion

We have shown on a particular case how the tools of dynamical systems theory may be used to give some results concerning the long time behaviour of non linear filtering problems. We hope that these results will be generalized to other types of attractors with asymptotic phases, using for example the results presented in [3].

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References

[1] F. CÉROU. Long time asymptotics for some dynamical noise free non linear filtering problems. Rapport de Recherche 2446, INRIA, December 1994.

[2] P. HARTMAN. *Ordinary differential equations.* Birkhäuser, 1982.

[3] M. W. HIRSCH. Asymptotic phase, shadowing and reaction–diffusion systems. In K. D. ELWORTHY, W. N. EVERITT, and E. B. LEE, editors, *Differential equations, dynamical systems, and control science*, pages 87–99. Dekker, 1994.

[4] D. JI. *Asymptotic analysis of nonlinear filtering problems.* PhD thesis, Brown University, 1988.

[5] H. KUNITA. Asymptotic behavior of the nonlinear filtering errors of Markov processes. *Journal of Multivariate Analysis*, 1:365–393, 1971.

[6] H. KUNITA. Ergodic properties of nonlinear filtering processes. In K.L. Alexander and J.C. Watkins, editors, *Spatial Stochastic Processes : Festschrift in honor of T.E. Harris*, volume 19 of *Progress in Probability*, pages 233–256. Birkhäuser, Boston, 1991.

[7] D. OCONE and E. PARDOUX. Asymptotic stability of the optimal filter with respect to its initial condition. To appear in SIAM Journal on Control and Optimization.

[8] E. PARDOUX. Filtrage non linéaire et équations aux dérivées partielles stochastiques associées. In P.L. Hennequin, editor, *Ecole d'Eté de Probabilités de Saint–Flour XIX – 1989*, volume 1464 of *Lecture Notes in Mathematics*, pages 67–163. Springer Verlag, Berlin, 1991.

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