Lyapunov exponents of controlled SDE's and stabilizability property : Some examples

Fabien Campillo, Abdoulaye Traore

INRIA

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Lyapunov exponents of controlled SDE's and stabilizability property : Some examples

Fabien Campillo and Abdoulaye Traoré

N° 2397

Novembre 1994

PROGRAMME 5

Traitement du signal,

automatique

et productique

Rapport de recherche

1994

ISSN 0249-6399

Lyapunov exponents of controlled SDE's and stabilizability property : Some examples

Fabien Campillo* and Abdoulaye Traoré**

Programme 5 — Traitement du signal, automatique et productique
Projet Mefisto

Rapport de recherche n° 2397 — Novembre 1994 — 15 pages

Abstract: We consider a stochastic differential equation with linear feedback control :

$$dX_t = (A + B\,K)\,X_t\,dt + \sum_{k=1}^{r}(A_k + B_k\,K)\,X_t\,\circ dW_k(t)$$

where K is the feedback gain matrix. For each value of K, let λ_K be the Lyapunov exponent associated with the solution of the SDE. The set of λ_K, as K describe the set of matrices, is a connected interval of $\mathbb{R}$. We present some examples where $-\infty$ is the lower bound of this set. For these cases, we say that the corresponding EDS is stabilizable.

Key-words: Stochastic differential equation, stabilizability, Lyapunov exponent.

(Résumé : tsvp)

*campillo@sophia.inria.fr
**traore@sophia.inria.fr

Unité de recherche INRIA Sophia-Antipolis
2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex (France)
Téléphone : (33) 93 65 77 77 – Télécopie : (33) 93 65 77 65

Exposants de Lyapunov d'EDS contrôlées et propriété de stabilisabilité : Quelques exemples

Résumé : On considère une équation différentielle stochastique avec contrôle linéaire en boucle fermée :

$$dX_t = (A + B\,K)\,X_t\,dt + \sum_{k=1}^{r}(A_k + B_k\,K)\,X_t\;\circ dW_k(t)$$

où K est une matrice de gain. Pour chaque valeur de K, soit λ_K l'exposant de Lyapunov associé à la solution de cette EDS. L'ensemble de valeurs de λ_K lorsque K parcours l'espace des matrices est un intervalle connexe de $\mathbb{R}$. On présente quelques exemples où $-\infty$ est la borne inférieure de cet ensemble. Pour ces exemples, nous dirons que l'EDS correspondante est stabilizable.

Mots-clé : Équation différentielle stochastique, stabilisabilité, exposant de Lyapunov.

1 Preliminaries

We consider the following linear stochastic differential equation in $\mathbb{R}^d$

$$dX_t = A\,X_t\,dt + \sum_{k=1}^{r} A_k\,X_t \circ dW_k(t)\ ,\ \ X_0 = x_0 \in \mathbb{R}^d\ ,\ \ x_0 \neq 0\ , \tag{1}$$

where $A, A_1, \ldots, A_k$ are $d \times d$ matrices, $W_1, \ldots, W_r$ are independent standard Wiener processes. Here "$\circ dW$" (resp. "dW") refer to the Stratonovich (resp. Itô) stochastic integral.

We define the Lyapunov exponent of the solution of (1) starting at x_0

$$\lambda(x_0) \triangleq \lim_{t\to\infty} \frac{1}{t} \log \|X_t\|\ . \tag{2}$$

Oseledec's multiplicative ergodic theorem states that the limit (2) exists with probability one and that there are d fixed numbers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d$ – called the Lyapunov exponents of (1) – such that the random variable $\lambda(x_0)$ takes on only these values (see [1] for a review). Moreover, (1) is exponentially stable with probability one if and only if $\lambda_1 < 0$.

Let $S^{d-1} = \{x \in \mathbb{R}^d; \|x\| = 1\}$ denote the unit sphere of $\mathbb{R}^d$. We can define the projection of X_t onto the sphere by

$$U_t \triangleq \|X_t\|^{-1}\ X_t\ .$$

U_t is the solution of the following SDE on S^{d-1}

$$dU_t = h(A, U_t)\,dt + \sum_{k=1}^{r} h(A_k, U_t) \circ dW_k(t)\ ,\quad U_0 = u_0 \triangleq \|x_0\|^{-1}\,x_0 \tag{3}$$

where

$$h(C, u) \triangleq C\,u - (C\,u, u)\,u\ . \tag{4}$$

Here (x, y) is the scalar product on $\mathbb{R}^d$ and $\|x\|^2 = (x, x)$.

Moreover,

$$\|X_t\| = \|x_0\|\ \exp\left\{\int_0^t q(U_s)\,ds + \sum_{k=1}^{r}\int_0^t p_k(U_s)\,dW_k(s)\right\} \tag{5}$$

with

$$
\begin{aligned}
q_0(u) &\stackrel{\triangle}{=} (A\,u,u)\ , \quad p_k(u) \stackrel{\triangle}{=} (A_k\,u,u)\ , \quad k=1,\dots,r\ ,\\
q_1(u) &\stackrel{\triangle}{=} \frac{1}{2}\sum_{k=1}^{r}[(A_k^2\,u,u)+\|A_k\,u\|^2-2(A_k\,u,u)^2]\ ,\\
q(u) &\stackrel{\triangle}{=} q_0(u)+q_1(u)\ .
\end{aligned}
$$

For each matrix M, $h(M,-u)=-h(M,u)$, so that $h(M,\cdot)$ can be viewed as a vector field on the projective space P^{d-1} (obtained from S^{d-1} by identifying u and $-u$). Therefore (3) can be considered as a stochastic differential equation on P^{d-1} and (5) is still valid with this definition. We make the following

Hypothesis 1.1 *For all $u \in P^{d-1}$*

$$
\dim \mathsf{Lie\ Algebra}\{h(A,.),h(A_k,.),k=1,\dots,r\}(u)=d-1\ .
$$

Theorem 1.2 *Under Hypothesis 1.1*

(i) *The diffusion process U_t admits a unique invariant probability measure μ. Moreover μ has a C^∞ density p with respect to the Lebesgue measure on P^{d-1} which solves the Fokker–Planck equation $L^*p=0$, where L is the infinitesimal generator associated with equation (3).*

(ii) *The number*

$$
\lambda \stackrel{\triangle}{=} \int_{P^{d-1}} q(u)\,\mu(du)
$$

is equal to the top Lyapunov exponent λ_1.

(iii) *For all $x_0 \in \mathbb{R}^d$, $x_0 \neq 0$, $\lambda(x_0)=\lambda$ with probability one. When $\lambda<0$, the system (1) is exponentially stable with probability one.*

2 Main result

We consider the controlled SDE

$$
dX_t=(A\,X_t+B\,u_t)\,dt+\sum_{k=1}^{r}(A_k\,X_t+B_k\,u_t)\circ dW_k(t)
$$

where A, A_k, $k=1,\ldots,r$ are $d\times d$ matrices, B, B_k, $k=1,\ldots,r$ are $d\times p$ matrices. We suppose that these matrices are given.

We restrict ourselves to feedback controls, i.e. $u_t = K\,X_t$, where K is a $p\times d$ matrix. The resulting SDE is

$$dX_t = (A+B\,K)\,X_t\,dt + \sum_{k=1}^{r}(A_k+B_k\,K)\,X_t\circ dW_k(t)\ ,\ X_0 = x_0\in \mathbb{R}^d\ ,\ x_0\neq 0\ . \tag{6}$$

The problem is to choose the feedback gain matrix K so as to stabilize the system (6).

The projection of X_t onto P^{d-1}, which now depends on K, satisfies

$$dU_t = h(A+B\,K,U_t)\,dt + \sum_{k=1}^{r} h(A_k+B_k\,K,U_t)\circ dW_k(t)\ , \tag{7}$$

with $U_0 = u_0 \triangleq \|x_0\|^{-1}\,x_0$, and

$$\|X_t\| = \|x_0\|\ \exp\left\{\int_0^t q(K,U_s)\,ds + \sum_{k=1}^{r}\int_0^t p_k(K,U_s)\,dW_k(s)\right\} \tag{8}$$

with

$$\begin{aligned}
q_0(K,u) &\triangleq ((A+B\,K)\,u,u)\ ,\\
q_1(K,u) &\triangleq \frac{1}{2}\sum_{k=1}^{r}\Big\{((A_k+B_k\,K)^2u,u) + |(A_k+B_k\,K)\,u|^2\\
&\qquad\qquad -2((A_k+B_k\,K)\,u,u)^2\Big\}\ ,\\
q(K,u) &\triangleq q_0(K,u)+q_1(K,u)\ ,\\
p_k(K,u) &\triangleq ((A_k+B_k\,K)\,u,u)\ ,\quad k=1,\ldots,r\ .
\end{aligned}$$

2.1 First case : $B_k = 0$, $k=1,\ldots,r$

Here we suppose that only the drift coefficient is controlled, i.e. $B_k = 0$, $k = 1,\ldots,r$. The equation for U_t reduce to U_t is solution of the following equation

$$dU_t = h(A+B\,K,U_t)\,dt + \sum_{k=1}^{r} h(A_k,U_t)\circ dW_k(t)\ . \tag{9}$$

We make the following

Hypothesis 2.1 *For all $u \in P^{d-1}$ and $K \in M(p \times d)$*

$$\dim \mathsf{Lie\ Algebra}\{h(A+BK,.), h(A_k,.),\, k=1,\dots,r\}(u) = d-1 \ ,$$

where $M(p \times d)$ is the set of $p \times d$ matrices.

Under Hypothesis 2.1, the Theorem 1.2 states that, for all $K \in M(p \times d)$ the diffusion process U_t admits a unique invariant probability measure μ_K. Let λ_K denote the Lyapunov exponent associated with Equation (6)

$$\lambda_K \triangleq \int_{P^{d-1}} q(K,u)\, \mu_K(du) \ .$$

Proposition 2.2 *Under Hypothesis 2.1, $K \mapsto \lambda_K$ define a continuous function defined on $M(p \times d)$.*

Since $M(p \times d)$ is a connected set, we have the following

Corollary 2.3 *Under Hypothesis 2.1,*

$$\mathcal{D} \triangleq \{\lambda_K; K \in M(p \times d)\}$$

is a connected interval of $\mathbb{R}$.

In order to prove Proposition 2.2 we need the following

Lemma 2.4 *For all $t \geq 0$, there exist $C_t < \infty$, such that for all K_1, $K_2 \in M(p \times d)$*

$$\sup_{u \in S^{d-1}} E \left\| U_t^{1,u} - U_t^{2,u} \right\|^2 \leq C_t \, \|K_1 - K_2\| \ ,$$

where $U_t^{i,u}$ denote the solution of (9) with control matrix K_i and starting at point u.

Proof Let U_t^u denote the solution of Equation (9) with control matrix K and starting at point u. U_t^u is solution of the following Itô equation

$$dU_t^u = h(A+B\,K, U_t^u)\,dt + \frac{1}{2}\sum_{k=1}^r h'(A_k, U_t^u)\,h(A_k, U_t^u)\,dt + \sum_{k=1}^r h(A_k, U_t^u)\,dW_k(t)\ ,$$

with $h'(C,u)v = C\,v - (C\,u,u)\,v - (C\,u,v)\,u - (C\,v,u)\,u$. We get

$$dU_t^u = b(K, U_t^u)\,dt + \sum_{k=1}^r \sigma_k(U_t^u)\,dW_k(t)\ ,\quad U_0^u = u\ ,$$

where

$$b(M,u) \triangleq h(A + B\,M, u) + \frac{1}{2}\sum_{k=1}^r h'(A_k,u)\,h(A_k,u)\ ,\quad \sigma_k(u) \triangleq h(A_k,u)\ .$$

The drift coefficient $b(M,\cdot)$ and diffusion coefficients $\sigma_k(\cdot)$ are polynomial functions of u, so they are locally Lipschitz. But they are also globally Lipschitz because S^{d-1} is a compact set. Hence, there exist $L > 0$ such that for all $u, v \in S^{d-1}$ and $k = 1, \ldots, r$

$$\|b(M,u) - b(M,v)\| + \|\sigma_k(u) - \sigma_k(v)\| \le L\ \|u - v\|\ .$$

Also, for all $K, K' \in M(p \times d)$ and $u \in S^{d-1}$

$$\|b(K,u) - b(K',u)\| \le \|B\|\ \|K - K'\|\ .$$

Now we go back to the proof of the lemma :

$$\begin{aligned}
&\left\|U_t^{1,u} - U_t^{2,u}\right\|^2 \\
&\le 2\left\|\int_0^t [b(K_1, U_s^{1,u}) - b(K_2, U_s^{2,u})]\,ds\right\|^2 \\
&\qquad + 2\left\|\sum_{k=1}^r \int_0^t [\sigma_k(U_s^{1,u}) - \sigma_k(U_s^{2,u})]\,dW_k(s)\right\|^2 \\
&\le 2\,t\int_0^t \left\|b(K_1, U_s^{1,u}) - b(K_2, U_s^{2,u})\right\|^2 ds \\
&\qquad + 2\,r\sum_{k=1}^r \left\|\int_0^t [\sigma_k(U_s^{1,u}) - \sigma_k(U_s^{2,u})]\,dW_k(s)\right\|^2\ .
\end{aligned}$$

So

$$\begin{aligned}
E\left\|U_t^{1,u}-U_t^{2,u}\right\|^2 &\leq 4\,t\int_0^t E\left\|b(K_1,U_s^{1,u})-b(K_2,U_s^{1,u})\right\|^2\,ds\\
&\quad+4\,t\int_0^t E\left\|b(K_2,U_s^{1,u})-b(K_2,U_s^{2,u})\right\|^2\,ds\\
&\quad+2\,r\sum_{k=1}^r\int_0^t E\left\|\sigma_k(U_s^{1,u})-\sigma_k(U_s^{2,u})\right\|^2\,ds\ .
\end{aligned}$$

So we get

$$\begin{aligned}
E\left\|U_t^{1,u}-U_t^{2,u}\right\|^2 &\leq (4\,t+2\,r)\,L^2\int_0^t E\left\|U_s^{1,u}-U_s^{2,u}\right\|^2\,ds\\
&\quad+4\,t^2\,\|B\|^2\,\|K_1-K_2\|^2\ .
\end{aligned}$$

From this last inequality and Gronwall's inequality we prove the lemma. □

Proof of Proposition 2.2 Let $K_n\to K$ as $n\to\infty$. We want to prove that $\lambda_{K_n}\to\lambda_K$ as $n\to\infty$. Let U_t^u (resp. $U_t^{n,u}$) denote the solution of Equation (9) with control matrix K (resp. K_n) and starting at point u.

From Lemma 2.4,

$$\lim_{n\to\infty}\sup_{u\in S^{d-1}}E\,\|U_t^{n,u}-U_t^u\|^2=0\ . \tag{10}$$

Now we show that the sequence $\mu_n\triangleq\mu_{K_n}$ admits a weak limit μ and that $\mu=\mu_K$. First, it is clear that the sequence $\{\mu_n\}$ is tight because S^{d-1} is a compact set. So there exist a sub–sequence, denoted $\{\mu_{n'}\}$, and a probability measure μ defined on S^{d-1} such that $\mu_{n'}\Rightarrow\mu$.

Now we prove that $\mu=\mu_K$. Let f be Lipschitz on S^{d-1}, then there exists $\alpha>0$ such that $|f(u)-f(v)|\leq\alpha\,\|u-v\|$ for all $u,v\in S^{d-1}$, and from (10), we have

$$\begin{aligned}
\sup_{u\in S^{d-1}}|Ef(U_t^{n',u})-Ef(U_t^u)|^2 &\leq \sup_{u\in S^{d-1}}E|f(U_t^{n',u})-f(U_t^u)|^2\\
&\leq \alpha^2\sup_{u\in S^{d-1}}E\left\|U_t^{n',u}-U_t^u\right\|^2\to 0\ . \qquad (11)
\end{aligned}$$

From the fact that $\mu_{n'} \Rightarrow \mu$ and that $\mu_{n'}$ is an invariant probability measure for $U_t^{n',u}$, we have

$$\int_{P^{d-1}} Ef(U_t^{n',u})\,\mu_{n'}(du) = \int_{P^{d-1}} f(u)\mu_{n'}(du) \to \int_{P^{d-1}} f(u)\,\mu(du)\ .$$

Furthermore

$$\begin{aligned}
&\left|\int_{P^{d-1}} Ef(U_t^{n',u})\,\mu_{n'}(du) - \int_{P^{d-1}} Ef(U_t^{u})\,\mu(du)\right| \\
&\quad\le \int_{P^{d-1}} |Ef(U_t^{n',u}) - Ef(U_t^{u})|\,\mu_{n'}(du) \\
&\qquad + \left|\int_{P^{d-1}} Ef(U_t^{u})\,[\mu_{n'}(du) - \mu(du)]\right| \\
&\quad\le \sup_{u\in S^{d-1}} |Ef(U_t^{n',u}) - Ef(U_t^{u})| + \left|\int_{P^{d-1}} Ef(U_t^{u})\,[\mu_{n'}(du) - \mu(du)]\right| \\
&\quad\to 0\ .
\end{aligned}$$

Indeed, the first term tends to 0 because of (11) and the second term tend to 0 because the function $u \mapsto Ef(U_t^u)$ is continuous and $\mu_{n'} \Rightarrow \mu$.

At last, we get

$$\int_{P^{d-1}} Ef(U_t^{u})\,\mu(du) = \int_{P^{d-1}} f(u)\,\mu(du)\ , \quad \forall t \ge 0$$

that is $\mu = \mu_K$.

The invariant measure μ_K is unique, so the whole sequence $\{\mu_n\}$ converge to μ_K.

Finally

$$\begin{aligned}
|\lambda_{K_n} - \lambda_K| &\le \left|\int_{P^{d-1}} q(K_n,u)\,\mu_n(du) - \int_{P^{d-1}} q(K,u)\,\mu(du)\right| \\
&\le \sup_{u\in S^{d-1}} |q(K_n,u) - q(K,u)| + \left|\int_{P^{d-1}} q(K,u)\,[\mu_n(du) - \mu(du)]\right|
\end{aligned}$$

which tends to 0. □

2.2 General case

We go back to the general set up (6)–(8). We make the following

Hypothesis 2.5 *For all $u \in P^{d-1}$ and $K \in M(p \times d)$*

$$\dim \text{Lie Algebra}\, \{h(A + B\,K, .),\, h(A_k + B_k\,K, .)\,;\; k = 1, \dots, r\}\,(u) = d - 1\ .$$

Under this hypothesis, U_t admit a unique invariant measure μ_K and the Lyapunov exponent is given by

$$\lambda_K = \int_{P^{d-1}} q(K, u)\, \mu_K(du)\ .$$

Proposition 2.6 *Under Hypothesis 2.5, $K \mapsto \lambda_K$ defines a continuous function defined on $M(p \times d)$.*

Since $M(p \times d)$ is a connected set, we have the following

Corollary 2.7 *Under Hypothesis 2.5,*

$$\mathcal{D} \stackrel{\triangle}{=} \{\lambda_K; K \in M(p \times d)\}$$

is a connected interval of $\mathbb{R}$.

Proof of Proposition 2.6 The proof is equivalent to Proposition 2.2. □

3 Examples

In these examples, we consider feedback gain matrices $K(\alpha)$ parametrized by a one dimensional parameter $\alpha \in \mathbb{R}$. We suppose that $\{K(\alpha); \alpha \in \mathbb{R}\}$ is a connected subset of $M(p \times d)$. Let λ_α (resp. μ_α) denote the Lyapunov exponent (resp. the invariant measure) associated with $K(\alpha)$ and

$$\lambda_\star = \inf_{\alpha \in \mathbb{R}} \lambda_\alpha$$

We provide in this section two examples where $\lambda_\star = -\infty$ and one where $\lambda_\star = 1$.

In all the examples, $d = 2$ and $r = 1$. For any 2×2 matrix C, $h(C, \cdot)$ defined in (4) is a vector field on P^1, and

$$h(C, \theta) = (-c_{11} + c_{22})\, \cos\theta\, \sin\theta - c_{12}\, \sin^2\theta + c_{21}\, \cos^2\theta$$

for all $\theta \in P^1$.

3.1 Example 1

We identify P^1 to $[-\pi/2, \pi/2]$. Let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \quad B_1 = 0 .$$

We consider feedback gain matrices K of the form

$$K(\alpha) = \begin{pmatrix} \alpha & -1 \\ -1 & 2\alpha \end{pmatrix}$$

parametrized by $\alpha \in \mathbb{R}$.

The Hypothesis 2.1 is satisfied, since

$$h(A + BK(\alpha), \theta) = \alpha \sin\theta \cos\theta , \quad h(A_1, \theta) = 1 , \quad \forall \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] .$$

We have

$$q_0(K(\alpha), \theta) = \alpha + \alpha \sin^2\theta , \quad q_1(\theta) = 0 , \quad \forall \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] .$$

Finally we get

$$\lambda_\alpha = \alpha + \alpha \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2(\theta) \, \mu_\alpha(d\theta) \leq \alpha .$$

So the system is stabilizable because

$$\lim_{\alpha \to -\infty} \lambda_\alpha = -\infty .$$

3.2 Example 2

We identify P^1 and $[0, \pi]$, and we take

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} , \quad B = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} , \quad A_1 = 0 , \quad B_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

and

$$K(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} .$$

So

$$h(A+BK(\alpha),\theta)=\alpha\,\sin\theta\,\cos\theta\ ,\quad h(A_1+B_1K(\alpha),\theta)=\alpha\ ,\quad \forall\theta\in[0,\pi]\ ,$$

and

$$q_0(K(\alpha),\theta)=-1+\alpha\,\sin^2\theta\ ,\quad q_1(K(\alpha),\theta)=0\ ,\quad \forall\theta\in[0,\pi]\ .$$

The Lyapunov exponent is

$$\lambda_\alpha=-1+\alpha\int_0^\pi \sin^2(\theta)\,\mu_\alpha(d\theta)\ . \tag{12}$$

We want to compute the limit of λ_α as $\alpha\to-\infty$. We can check that the projected process U_t is solution of

$$dU_t=h(A,U_t)\,dt+\alpha\,h(A',U_t)dt+\alpha\,h(A'',U_t)\circ dW(t)$$

with

$$A'=\begin{pmatrix}0&0\\0&1\end{pmatrix}\ ,\quad A''=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\ .$$

We make the following time scale transformation $\tilde U^\alpha_t\triangleq U_{t/\alpha^2}$. $\tilde U_t$ is solution of

$$\tilde U^\alpha_t=U_0+\frac{1}{\alpha^2}\int_0^t h(A,\tilde U^\alpha_s)\,ds+\frac{1}{\alpha}\int_0^t h(A',\tilde U^\alpha_s)\,ds+\int_0^t h(A'',\tilde U^\alpha_s)\circ d\tilde W^\alpha_s \tag{13}$$

where $\tilde W^\alpha_t\triangleq\alpha W_{t/\alpha^2}$ is a standard Wiener process.

When $\alpha\to-\infty$, we get the following limit equation

$$\tilde U_t=U_0+\int_0^t h(A'',\tilde U_s)\circ d\tilde W_s$$

where $\tilde W_t$ is a standard Wiener process. Let $\tilde U_t=(\cos\tilde\theta_t,\sin\tilde\theta_t)$. Because $h(A'',\theta)=1$, we get

$$\tilde\theta_t=\theta_0+\tilde W_t\ . \tag{14}$$

Proposition 3.1

$$\mu_\alpha\Rightarrow\mathcal{U}[0,\pi]\quad as\quad \alpha\to-\infty$$

where $\mathcal{U}[0,\pi]$ is the uniform law on $[0,\pi]$.

Proof It is clear that $\mu_\alpha \Rightarrow \mu_{-\infty}$, where $\mu_{-\infty}$ satisfies the following Fokker–Planck equation $L^*\mu_{-\infty} = 0$ and L is the infinitesimal generator associated with equation (14). Moreover, $\mu_{-\infty}$ is the uniform law on $[0, \pi]$. □

Then using this proposition and (12) we have

Corollary 3.2

$$\lambda_\alpha \to -\infty \quad as \quad \alpha \to -\infty \; .$$

3.3 Example 3

Now we present an example which is not stabilizable. We consider the same coefficients as in the previous section except for matrices A, B_1 :

$$A = I \; , \quad B_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \; .$$

Let $X_t = (X_t^1, X_t^2)$, we get the following system

$$\begin{aligned} dX_t^1 &= X_t^1 \, dt - \alpha \, X_t^2 \circ dW_t \; , \quad X_0^1 = x_0^1 \; , \\ dX_t^2 &= (1+\alpha) \, X_t^2 \, dt \; , \quad X_0^2 = x_0^2 \; , \quad x_0 \neq 0 \; , \end{aligned}$$

whose solution is $X_t^2 = e^{(1+\alpha)t} \, x_0^2$, and

$$\begin{aligned} X_t^1 &= e^t \, x_0^1 - \alpha \int_0^t e^{t-s} \, X_s^2 \circ dW_s \\ &= e^t \, x_0^1 - \alpha \, x_0^2 \int_0^t e^{t+\alpha s} \, dW_s \\ &= e^t \, (x_0^1 - Y_t) \end{aligned}$$

with

$$Y_t \triangleq \alpha \, x_0^2 \int_0^t e^{\alpha s} \, dW_s \; .$$

If $x_0^2 \neq 0$, we deduce from

$$\|X_t\| \geq |X_t^2| = e^{(1+\alpha)t} \, |x_0^2|$$

that $\lambda_\alpha \geq 1 + \alpha$. If $x_0^2 = 0$ then $x_0^1 \neq 0$ (because, $x_0 \neq 0$) and $\lambda_\alpha = 1$. So, for $\alpha \geq 0$, $\lambda_\alpha \geq 1$.

Let us consider the case $\alpha < 0$. By the theorem of convergence of martingales, $Y_t \to Y_\infty \triangleq \alpha\, X_0^2 \int_0^\infty e^{\alpha s}\, dW_s$ as $t \to \infty$ and this convergence holds a.s. and in L^2.

We deduce from

$$||X_t||^2 = e^{2t} \left\{ e^{2\alpha t} |x_0^2|^2 + |x_0^1 - Y_t|^2 \right\}$$

and from (2) that

$$\lambda_\alpha = 1 + \lim_{t \to +\infty} \frac{1}{2t} \log \left[e^{2\alpha t} |x_0^2|^2 + |x_0^1 - Y_t|^2 \right] = 1 \quad a.s.$$

(this limit is valid whether $x_0^2 = 0$ or not). Which proves that the system is not stabilizable.

References

[1] L. ARNOLD. Stabilization by noise revisited. *ZAMM, Z. angew. Math. Mech.*, 70(7):235–246, 1990.

[2] L. ARNOLD, H. CRAUEL, and V. WIHSTUTZ. Stabilization of linear systems by noise. *SIAM Journal on Control and Optimization*, 21(3):451–461, 1983.

[3] L. ARNOLD, E. OELJEKLAUS, and E. PARDOUX. Almost sure and moment stability for linear itô equations. In L. Arnold and V. Wihstutz, editors, *Lyapunov Exponents, Bremen–1984*, volume 1186 of *Lecture Notes in Mathematics*, pages 129–159, Berlin, 1986. Springer Verlag.

[4] P.H. BAXENDALE. Moment stability and large deviations for linear stochastic differential equations. In N.Ikeda, editor, *Probabilistic Methods in Mathematical Physics, Katata and Kyoto 1985*, pages 31–54, Tokyo, 1987. Kinokuniya.

[5] N. IKEDA and S. WATANABE. *Stochastic Differential Equations and Diffusion Processes.* North–Holland/Kodansha, Amsterdam, 1981.

[6] E. PARDOUX and V. WIHSTUTZ. Lyapunov exponent of linear stochastic systems with large diffusion term. *Stochastic Processes and their Applications*, 40(2):289–308, 1992.

Unité de recherche INRIA Lorraine, Technopôle de Nancy-Brabois, Campus scientifique,
615 rue du Jardin Botanique, BP 101, 54600 VILLERS LÈS NANCY
Unité de recherche INRIA Rennes, Irisa, Campus universitaire de Beaulieu, 35042 RENNES Cedex
Unité de recherche INRIA Rhône-Alpes, 46 avenue Félix Viallet, 38031 GRENOBLE Cedex 1
Unité de recherche INRIA Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex
Unité de recherche INRIA Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 SOPHIA-ANTIPOLIS Cedex

Éditeur
INRIA, Domaine de Voluceau, Rocquencourt, BP 105, 78153 LE CHESNAY Cedex (France)
ISSN 0249-6399