On fluid approximation for stable networks

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On Fluid Approximation for Stable Networks.

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Abstract

In this paper the fluid dynamics of stable work-conserving networks is studied. On the fluid level these networks behave in a similar manner to Jackson networks with Poisson arrivals and exponential service times. We develop the analysis of the fluid dynamics made in [3], where it was described in terms of the oblique reflection mapping. We use another approach and describe the dynamics in terms of the so-called second vector field in an orthant $\mathbf{R}_+^N$, where N is the number of nodes in the network [15, 16]. Our goal here is to construct the fluid dynamics in an explicit form and provide an efficient algorithm to calculate it. In particular, we calculate the exact time when all nonbottlenecks become empty or, for an ergodic network, the time when the system becomes empty. Properties of the dynamics are also studied. We prove that the fluid dynamics of each stable work-conserving network is *strictly acyclic* in the sense of [16].

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Approximation fluide des réseaux stables.

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Résumé

Dans ce papier, nous étudions la dynamique fluide des réseaux conservatifs stables. Pour le modèle fluide, ces réseaux se comportent d'une manière similaire aux réseaux de Jackson avec arrivées Poisson et services exponentiels. Nous développons l'analyse du système dynamique fluide traité dans [16], où il était décrit par la procédure dite *oblique reflection mapping* . Nous utilisons une autre approche et décrivons la dynamique en termes de second champ de vecteurs dans l'orthant $\mathbf{R}_+^N$, où N est le nombre de files dans le réseau [15, 16]. Notre but est de construire les trajectoires sous une forme explicite et de fournir une méthode efficace pour les calculs. En particulier, nous calculons l'instant exact où toutes les files stables se vident et, dans le cas d'ergodicité, l'instant où le système se vide. Nous étudions également les propriétés des trajectoires et prouvons que les trajectoires fluides des réseaux conservatifs stables sont strictement acycliques au sens de [16].

1 Introduction

In this paper we develop the analysis of the fluid approximation for the class of so-called stable work-conserving networks made in [3] (see also the review of the related results therein). In particular, Jackson networks belong to this class of networks. In [3], for stable work-conserving networks, functional strong laws-of-large-numbers was established, i.e. the *Euler queue length-time scaling limit* almost surely exists:

$$\lim_{n\to\infty} \frac{Z([nz_0], nt)}{n} = z(t)$$

for each vector $z_0 \in \mathbf{R}_+^N$ and all $t \geq 0$ uniformly on compact intervals, where N is the number of nodes in the network, $z : [0, \infty) \to \mathbf{R}_+^N$ is a continuous function with $z(0) = z_0$, $Z_i(L, s)$ denotes the number of jobs in the ith queue at time $s \geq 0$ with the initial number $L_i \geq 0$ of jobs in the ith queue, $Z(L, s) = (Z_1(L, s), \ldots, Z_N(L, s))$, $L = (L_1, \ldots, L_N)$. Moreover, the fluid dynamics $\{z(t) = (z_1(t), \ldots, z_N(t)),\ t \geq 0\}$ can be described in terms of the *oblique reflection mapping* [3] (see also **Theorem 1** in the next Section).

But, such a description of the fluid dynamics is rather implicit and does not give a simple procedure to calculate it. Here we use another approach and describe the fluid dynamics in terms of the so-called *second vector field* [15, 16]. This approach presents the opportunity to construct the fluid dynamics in an explicit form and calculate the queue length at each time $t \geq 0$ explicitly. We also provide efficient algorithms to calculate the fluid dynamics and study its properties. In particular, the exact time to the moment when all nonbottleneck nodes will become empty can be found. In the ergodic case this is the time to approach 0 by the dynamics and it is used to estimate the time of simulations for networks with large initial queues to approach a stationary regime.

On the level of a fluid approximation, all stable work-conserving networks behave in a similar manner to classical Jackson networks with Poisson arrivals and exponential service times. Thus, it sufficient only to calculate the fluid dynamics for Jackson networks. As we shall see later, the fluid dynamics $\{z(t) = (z_1(t), \ldots, z_N(t)) \in \mathbf{R}_+^N,\ t \geq 0\}$ of such networks is reasonably simple. It is piecewise linear with at most N the number of pieces and, moreover, *strictly acyclic* in the sense of [16] (see **Theorem 5**). This means that if the initial point $z(0) \in \mathbf{R}_+^N$ has a strictly positive ith coordinate, i.e. $z_i(0) > 0$, and at some time t' the ith node becomes empty, i.e. $z_i(t') = 0$, then it remains empty forever, i.e. $z_i(t) = 0$ for all $t \geq t'$.

Jackson networks are a particular case of continuous time random walks in $\mathbf{Z}_+^N$. The behaviour of the underlying fluid dynamical system for general random walks in $\mathbf{Z}_+^N$ can be more complicated (see e.g. [16], [15]). In particular, it can be non-acyclic or the so-called *scattering phenomena* [16] can occur. Moreover, generally, for $N \geq 4$, the law of large numbers does not govern the deterministic large behaviour of N-dimensional random walk in Z_+^N. In [1] we prove the existence of the Euler space-time scaling limit for some classes of random walks which can be not acyclic. In [1] also we show that the Euler space-time scaling limit leads to deterministic motion which can be described in terms of the second vector field.

It is important to note that the second vector field is uniquely defined only on ergodic faces (see the definition in Section 4). On non-ergodic faces it is not uniquely defined. But in the case of stable work-conserving networks it is possible to define it in a unique way on all faces. It is not straightforward and a large part of this paper is devoted to this problem (see the proof of **Theorems 2-4** in Section 7). For this we must carefully analyse the relations between ergodic faces and the second vector field.

Also, we note the explicit calculation of the second vector field is possible only in some special cases. In particular, this is possible to do in the case of Jackson networks (or, more generally, stable work-conserving networks). This vector field appeared for the first time in [15] in the context of random walks in $\mathbf{Z}_+^N$. It has a natural probabilistic interpretation (see **Proposition 1** in Section 4). General properties of the underlying dynamical systems defined by this vector field are studied in [16].

In this paper we consider only the case of open networks. Closed networks are considered in [2]. The paper is organised as follows. In Section 2 we recall the result from [3]. We discuss the flow-balance equations for open Jackson networks in Section 3. The second vector field for random walks in $\mathbf{Z}_+^N$ and notions related to it are defined in Section 4. In Section 5 we formulate our main results: **Theorems 2-5**. In Section 6 efficient algorithms for finding all bottlenecks and the calculation of the second vector field are presented. **Theorems 2-4** and **Theorem 5** are proved in Sections 7 and 8, respectively.

In the paper, we shall use the following notations. For a matrix A and a vector x we say that $A \geq 0$ or $x \geq 0$ ($A > 0$ or $x > 0$) if all elements of A or x are nonnegative (strictly positive). By definition $A_1 \geq A_2$, $x_1 \geq x_2$ ($A_1 > A_2$, $x_1 > x_2$) mean that $A_1 - A_2 \geq 0$, $x_1 - x_2 \geq 0$ ($A_1 - A_2 > 0$, $x_1 - x_2 > 0$), respectively. For vectors $x = (x_1, \ldots, x_N)$, $y = (y_1, \ldots, y_N)$ we shall denote

$$\min(x, y) \equiv (\min(x_1, y_1), \ldots, \min(x_N, y_N)).$$

2 Oblique Reflection Mapping and Fluid Approximation

Here we recall some definitions and results from [3].

Let us denote the space of functions $x : [0, \infty) \to \mathbf{R}^N$ by D_0^N which are right continuous, have left limits and $x(0) \in \mathbf{R}_+^N$, endowed with the topology of uniform convergence on compact intervals. We say that function $y : [0, \infty) \to \mathbf{R}$ *increases at time* $t \geq 0$ if $y(t + \epsilon) > y_i(t-)$ for all $\epsilon > 0$, with the convention $y(0-) \equiv y(0)$.

Let P be a nonnegative matrix with spectral radius strictly less than 1. We shall denote the unit $N \times N$-matrix by I .

Definition. A *regulator* of an element $x \in D_0^N$ is an element $y \in D_0^N$ such that

(i) for all $t \geq 0$,

$$z(t) = x(t) + y(t)(I - P) \geq 0;$$

(ii) each y_i is nondecreasing for all $i = 1, \ldots, N$;

(iii) each y_i increases only at those times t when $z_i(t) = 0$ for all $i = 1, \ldots, N$.

The *oblique reflection mapping* ψ_P is defined on elements $x \in D_0^N$ for which a regulator y exists and is unique. More precisely, in [8] it was shown that, if the spectral radius of P is strictly less than 1, then ψ_P is correctly defined and is continuous on D_0^N. In this case we write $\psi_P(x) = y$.

In this paper we consider communication networks and describe them in terms of servers and jobs (see e.g. [3] for other interpretations of networks under consideration). A network consists of N nodes, indexed by $1, \ldots, N$. We add a node 0 to denote the *outside world.* The network works as follows. Jobs arrive at node i to join the ith queue. After a job is served the job either leaves the network or is transferred to some other node j.

For $t \geq 0$, we denote the quantity of work arrived at node j from the outside world during the time interval $[0, t]$ by $G_j(t)$, $j = 1, \ldots, N$. Let $F_{ij}(t)$ and $F_{i0}(t)$ be the cumulative *potential* work transferred from node i to node j and the cumulative potential work from node i to the outside world after node i has been busy for a total of t units of time, $t \in \mathbf{R}_+$. We shall denote the number of jobs in the ith queue at time t by $Z_i(t)$, $Z(t) = (Z_1(t), \ldots, Z_N(t))$ and the total time the ith queue is nonempty during the time interval $[0, t]$ is denoted by $B_i(t)$, $B(t) = (B_1(t), \ldots, B_N(t))$. In particular, $Z(0)$ denotes the initial state of the network.

Definition. The network is *work conserving* if the relations

$$Z_i(t) = Z_i(0) + G_i(t) + \sum_{j=1}^{N} F_{ji}(B_j(t)) - \sum_{j=0}^{N} F_{ij}(B_i(t)) \tag{2.1}$$

and

$$B_i(t) = \int_0^t 1_{(0,\infty)}[Z_i(u)]du \tag{2.2}$$

are satisfied for all $t \geq 0$ and $j = 1, \ldots, N$.

Definition. The network is *stable* with parameters $\lambda = (\lambda_1, \ldots, \lambda_N)$ and

$$Q = (\alpha_{i,j})_{i=1,\ldots,N; j=0,\ldots,N}, \quad \alpha_{ij} \in \mathbf{R}_+$$

if, for each $t \geq 0$, the following limits exist:

$$\lim_{R\to\infty} \frac{1}{R} G_j(Rt) = \lambda_j t, \qquad j = 1, \ldots, N, \qquad (u.o.c.) \tag{2.3}$$

and

$$\lim_{R\to\infty} \frac{1}{R} F_{ij}(Rt) = \alpha_{ij} t, \qquad i = 1, \ldots, N,\ j = 0, \ldots, N, \qquad (u.o.c.) \tag{2.4}$$

where the abbreviation (u.o.c.) means that the limit exists almost surely and uniformly on compacts.

Now let us consider a sequence $\{\mathcal{L}_n\}$ of work-conserving stable networks with N nodes indexed by $n = 1, 2, \ldots$. We shall denote the corresponding values of work arrived from the outside world and the cumulative *potential* flow transferred from node i to node j or to the outside world in the nth network by $G_j^n(t)$ and $F_{ij}^n(t)$, respectively.

We shall need the following assumptions for the random variables $G_j^n(t)$, $F_{ij}^n(t)$ [3].

Assumption 1. Each network $\mathcal{L}_n$ is stable with some parameters $\lambda = (\lambda_1, \ldots, \lambda_N)$ and $Q = (\alpha_{i,j})_{i=1,\ldots,N; j=0,\ldots,N}$.

Assumption 2. We assume that $\mu_i > 0$ for all $i = 1, \ldots, N$, where the *processing capacity* μ_i of node i is defined by

$$\mu_i = \sum_{j=0}^{N} \alpha_{ij}, \quad i = 1, \ldots, N. \tag{2.5}$$

Assumption 3. For each $t \geq 0$ and $n \in \mathbf{N}$, the following limits exist:

$$\lim_{n\to\infty} \frac{1}{n} G_j^n(nt) = \hat{\lambda}_j t, \qquad j = 1, \ldots, N, \qquad (u.o.c.) \tag{2.6}$$

for some $\hat{\lambda}_j \in \mathbf{R}_+$ and

$$\lim_{n\to\infty} \frac{1}{n} F_{ij}^n(nt) = \hat{\alpha}_{ij} t, \qquad i = 1, \ldots, N,\ j = 0, \ldots, N. \qquad (u.o.c.) \tag{2.7}$$

for some $\hat{\alpha}_{ij} \in \mathbf{R}$.

Theorem 1 ([3]). *Let a sequence of networks satisfy* **Assumptions 1-3** *and $\lambda_j = \hat{\lambda}_j$ for all $j = 1, \ldots, N$ and $\alpha_{ij} = \hat{\alpha}_{ij}$ for all $i = 1, \ldots, N$, $j = 0, 1, \ldots, N$. Suppose the following limit almost surely exists*

$$\lim_{n\to\infty} \frac{1}{n} Z_j^n(0) = z_j(0), \qquad j = 1, \ldots, N, \tag{2.8}$$

for some $z_j(0) \in \mathbf{R}_+$, where $Z^n(0) = (Z_1^n(0), \ldots, Z_N^n(0)) \in \mathbf{R}_+^N$ is the initial state of the nth network. Then, for each $t \geq 0$, the following limits exist

$$\lim_{n\to\infty} \frac{1}{n} Z_j^n(nt) = z_j(t), \qquad j = 1, \ldots, N, \qquad (u.o.c.) \tag{2.9}$$

$$\lim_{n\to\infty} \frac{1}{n} B_j^n(nt) = b_j(t), \qquad j = 1, \ldots, N, \qquad (u.o.c.) \tag{2.10}$$

for some $z = (z_1, \ldots, z_N) \in D_0^N$, $b = (b_1, \ldots, b_N) \in D_0^N$, where $Z^n(s) = (Z_1^n(s), \ldots, Z_N^n(s))$ and $Z_i^n(s)$ is the number of jobs at the ith queue in the nth network at time s. Moreover, for each $t \geq 0$, the vectors $z(t) = (z_1(t), \ldots, z_N(t))$ and $b(t) = (b_1(t), \ldots, b_N(t))$ are uniquely defined by

$$z(t) = x(t) + y(t)(I - P) \tag{2.11}$$

and

$$b_i(t) = t - \mu_i^{-1} y_i(t), \qquad i = 1, \ldots, N \tag{2.12}$$

where $P = (\frac{\alpha_{ij}}{\mu_i})_{i,j=1,\ldots,N}$, $y = \psi_P(x)$ and the function $x \in D_0^N$ is given by

$$x(t) = z(0) + \theta t \tag{2.13}$$

where

$$\theta = \lambda - \mu(I - P).$$

Remark. From **Theorem 1** it follows that the fluid dynamics $\{z(t), t \geq 0\}$ depends only on the vectors λ and μ and matrix P. So, in order to calculate the fluid dynamics we can only consider the case of an open Jackson network with Poisson arrivals and exponentially distributed service times. For this we take each network $\mathcal{L}_n$ as an open Jackson network with parameters λ, μ and P. The vectors of the initial queue lengths $\{Z^n(0), n = 1, \ldots\}$ are defined as follows. Let $z(0) \in \mathbf{R}_+^N$ be fixed. We put

$$Z_i^n(0) = [nz_i(0)]$$

for each node i and $n \in \mathbf{N}$, where $[a]$ denotes the integer part of $a \in \mathbf{R}_+$. Clearly, the sequence of stable work-conserving networks $\{\mathcal{L}_n, n = 1, \ldots\}$ satisfies the conditions of **Theorem 1**.

3 Open Jackson network: the flow-balance equation and bottlenecks

Let us introduce open Jackson networks. They are of special interest to us here. Let a Jackson network $\mathcal{L}_J$ have N nodes and $\mathcal{S} = \{1, \ldots, N\}$ denotes the set of nodes. Arrivals at node i from the outside world form a Poisson process of rate λ_i. A job at node i requires an exponentially distributed service time with parameter μ_i. After service completion at node i a job is immediately transferred with probability p_{ij}, $0 \leq p_{ij} \leq 1$, to node j, $j = 1, \ldots, N$,

$$\sum_{j=1}^{N} p_{ij} \leq 1$$

and with probability

$$p_{i0} = 1 - \sum_{j=1}^{N} p_{ij} \tag{3.1}$$

it leaves the network. So the Jackson network can be described by the triplet (λ, μ, P), where $\lambda = (\lambda_1, \ldots, \lambda_N)$, $\mu = (\mu_1, \ldots, \mu_N)$, $P = \{p_{ij}\}_{i,j=1}^N$.

Remark. It is obvious that a Jackson network is work-conserving and stable with parameters $\lambda = (\lambda_1, \ldots, \lambda_N)$ and

$$Q = (\mu_i p_{ij})_{i=1,\ldots,N; j=0,\ldots,N}.$$

We shall consider only open Jackson networks, i.e. there is at least one node $i' \in \mathcal{S}$ such that $p_{i'0} > 0$. Moreover, without loss of generality we can assume that the matrix P is *primitive*, i.e., there exists $k \in \mathbf{N}$ such that $P^k > 0$. We notice then that, $p_{ii} < 1$ for all $i \in \mathcal{S}$.

Next, an open Jackson network (λ, μ, P) with feedback, i.e. $p_{ii} > 0$ for some node i, is equivalent to an open Jackson network $(\lambda, \hat{\mu}, \hat{P})$ without feedback, where $\hat{\mu}_i = (1 - p_{ii})\mu_i$ for $i \in \mathcal{S}$, $\hat{P} = (\hat{p}_{ij})$, $\hat{p}_{ij} = \frac{p_{ij}}{1-p_{ii}}$ for $i, j \in \mathcal{S}$, $i \neq j$, and $\hat{p}_{ii} = 0$ for $i \in \mathcal{S}$, where $p_i \equiv \sum_{j \in \mathcal{S}, j \neq i} p_{ij}$. Hence without loss of generality we can also assume $p_{ii} = 0$ for all $i \in \mathcal{S}$.

We now introduce the *flow-balance* equations [17]

$$\nu_j = \lambda_j + \sum_{i=1}^{N} \min(\nu_i, \mu_i)\, p_{ij}, \quad j = 1, \ldots, N. \tag{3.2}$$

If the network is in a stationary regime then ν_i is the intensity of total flow from outside the network and from other nodes into node i. We notice in general the equations (3.2) are nonlinear.

Remark. For an ergodic open Jackson network $\nu_j < \mu_j$ for all $i \in \mathcal{S}$ and, in this case, the flow-balance equations are the usual Jackson's equations [11]:

$$\nu_j = \lambda_j + \sum_{i=1}^{N} \nu_i p_{ij} , \quad j = 1, \ldots, N. \tag{3.3}$$

Thus in the case of an ergodic open Jackson network the balance equations (3.3) are linear.

It is convenient to rewrite (3.2) in the matrix form

$$\nu = \lambda + F_\mu(\nu), \tag{3.4}$$

where $\nu = (\nu_1, \ldots, \nu_N)$ and the operator

$$F_\mu : R_+^N \to [0, \mu P] \equiv [0, d_1] \times \ldots \times [0, d_N]$$

is defined by

$$F_\mu(x) = \min(x, \mu)P, \quad x \in \mathbf{R}_+^N, \tag{3.5}$$

$$d_j = \sum_{i=1}^{N} \mu_i \; p_{ij}, \quad j = 1, \ldots, N.$$

Since the spectral radius of matrix P is strictly less than one, then the operator F_μ in $\mathbf{R}_+^N$ is a contraction [5, 3]. Hence, the nonnegative solution of (3.4) exists and is unique. The solution of (3.4) can be written as

$$\nu = \lim_{n \to \infty} \nu^{(n)}, \tag{3.6}$$

where the *nth iteration*, $\nu^{(n)}$, of the solution ν is given by

$$\nu^{(n)} = \lambda + \underbrace{F_\mu(\lambda + F_\mu(\lambda + F_\mu(\lambda + \ldots + F_\mu(\lambda + F_\mu(\lambda))\ldots)))}_{n \quad \text{times}}. \tag{3.7}$$

The solution of (3.2) can also be written in the following form [5, 3]. First, we introduce the following notations. For each vector $x = (x_1, \ldots, x_N) \in \mathbf{R}^N$ and each subsets $A, B \subset \mathcal{S}$, we denote by $x_A = (x_i, \; i \in A) \in \mathbf{R}^{|A|}$ and by P_{AB}, the $|A| \times |B|$-matrix,

$$P_{AB} = \{p_{ij}\}_{i \in A, j \in B},$$

which is obtained from matrix P by cancelling rows $i \in \overline{A}$ and columns $j \in \overline{B}$, where $\overline{A} = \mathcal{S} \setminus A$, $\overline{B} = \mathcal{S} \setminus B$. We shall also denote the matrix which is obtained from the matrix $I = (\delta_{ij})_{i,j \in \mathcal{S}}$ by cancelling columns and rows $i \in \overline{A}$ by I_A. Moreover, we shall sometimes identify the vector $x_A = (x_i, \; i \in A) \in \mathbf{R}^{|A|}$ with the projection of $x = (x_1, \ldots, x_N) \in \mathbf{R}^N$

$$\hat{x}_A = \begin{cases} x_i, & i \in A \\ 0, & i \neq A \end{cases}$$

onto the subspace $\mathbf{R}^{|A|}$.

Definition ([3]). Let $\nu = (\nu_1, \ldots, \nu_N)$ be the (unique) solution of (3.2). Node i is called a *bottleneck*, if $\nu_i \geq \mu_i$ and a *nonbottleneck*, if $\nu_i < \mu_i$.

Let $\mathcal{B}$ and $\mathcal{N}$ be sets of all bottleneck and nonbottleneck nodes, respectively. Clearly,

$$\mathcal{B} \cup \mathcal{N} = \mathcal{S}, \qquad \mathcal{B} \cap \mathcal{N} = \emptyset.$$

Then equation (3.2) can be written as

$$\nu_{\mathcal{N}} = \lambda_{\mathcal{N}} + \nu_{\mathcal{N}} P_{\mathcal{N}\mathcal{N}} + \mu_{\mathcal{B}} P_{\mathcal{B}\mathcal{N}}, \tag{3.8}$$

$$\nu_{\mathcal{B}} = \lambda_{\mathcal{B}} + \nu_{\mathcal{N}} P_{\mathcal{N}\mathcal{B}} + \mu_{\mathcal{B}} P_{\mathcal{B}\mathcal{B}}. \tag{3.9}$$

¿From (3.8) it follows that

$$\nu_{\mathcal{N}} = (\lambda_{\mathcal{N}} + \mu_{\mathcal{B}} P_{\mathcal{B}\mathcal{N}})(I_{\mathcal{N}} - P_{\mathcal{N}\mathcal{N}})^{-1}. \tag{3.10}$$

Substituting $\nu_{\mathcal{N}}$ into (3.9) we obtain [3]

$$\nu_{\mathcal{B}} = \lambda_{\mathcal{B}} + \lambda_{\mathcal{N}}(I_{\mathcal{N}} - P_{\mathcal{N}\mathcal{N}})^{-1} P_{\mathcal{N}\mathcal{B}} + \mu_{\mathcal{B}}(P_{\mathcal{B}\mathcal{B}} + P_{\mathcal{B}\mathcal{N}}(I_{\mathcal{N}} - P_{\mathcal{N}\mathcal{N}})^{-1} P_{\mathcal{N}\mathcal{B}}). \tag{3.11}$$

Thus whenever the sets $\mathcal{B}$ and $\mathcal{N}$ are known, the solution of the nonlinear balance equations (3.2) can be written in the form of (3.10) and (3.11).

Remark. Thus, the problem of finding the sets $\mathcal{B}$ and $\mathcal{N}$ is of practical interest. One of our goals here is to provide an efficient algorithm for this problem.

4 Random Walks in $\mathbf{Z}_+^N$ Second Vector field

In this section we introduce continuous time analogs of some notions from [15, 16] which will be necessary in the sequel.

An open Jackson network is a particular case of continuous time random walks in $\mathbf{Z}_+^N$. Let $\mathcal{L}$ be an open Jackson network. Denote the queue length at the ith node at time $t \in \mathbf{R}_+$ by $\phi_i(t)$. Then the open Jackson network is represented by the continuous time random walk

$$\phi(t) = (\phi_1(t), \phi_2(t), \ldots, \phi_N(t)) \in \mathbf{Z}_+^N$$

with transition intensities μ_{xy} from state $x \in \mathbf{Z}_+^N$ to state $y \in \mathbf{Z}_+^N$ given by

$$\mu_{xy} = \begin{cases} \lambda_i, & \text{if } y - x = e_i \\ \mu_i p_{i0}, & \text{if } y - x = -e_i \\ \mu_i p_{ij}, & \text{if } y - x = -e_i + e_j \end{cases} \tag{4.1}$$

where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with the ith coordinate equal to 1.

It is more convenient to define the *second vector field* for a more general class of communication networks, namely, for random walks in $\mathbf{Z}_+^N$, where $\mathbf{Z}_+^N = \{(x_1, \ldots, x_N),\ x_i \in \mathbf{Z}_+\}$. The continuous time random walk (or random walk) $\mathcal{L}$ in $\mathbf{Z}_+^N$ is a continuous time Markov chain on the state space $\mathbf{Z}_+^N$ with transition intensities $\{\mu_{xy} \geq 0,\ x, y \in \mathbf{Z}_+^N\}$. We shall assume that $\mathcal{L}$ is irreducible and

$$\sum_{y \in \mathbf{Z}_+^N} \mu_{xy} < \infty$$

for all $x \in \mathbf{Z}_+^N$. The vector field of mean jumps (or the *first vector field* [16]) $M(x) = (M_1(x), \ldots, M_N(x))$ of the random walk $\mathcal{L}$ is defined by

$$M(x) = \sum_{y \in \mathbf{Z}_+^N} (y - x)\mu_{xy}. \tag{4.2}$$

Let $\mathcal{S} = \{1, \ldots, N\}$. For $\Lambda \subset \mathcal{S}$, we denote the face of $\mathbf{R}_+^N$ by $B(\Lambda)$:

$$B(\Lambda) = \{(r_1, \ldots, r_N) \in \mathbf{R}^N : \ r_i > 0, i \in \Lambda, r_i = 0, i \notin \Lambda\}.$$

We shall assume the random walk $\mathcal{L}$ satisfies the following conditions [15, 16].

Condition B (*Boundedness of jumps*). There exists a constant $d > 0$ such that

$$\mu_{xy} = 0 \quad \text{if} \quad \max_i |x_i - y_i| > d.$$

Condition H (*Space homogeneity*). For each $\Lambda \subset \mathcal{S}$ and each $a \in B(\Lambda) \cap Z_+^N$,

$$\mu_{xy} = \mu_{x+a, y+a}$$

for all $x \in B(\Lambda) \cap Z_+^N$, $y \in Z_+^N$.

In order to define the second vector field [15, 16] we have to introduce the important concept of an induced Markov chain.

Induced Markov chains. For each $\Lambda \subset \mathcal{S}$, $\Lambda \neq \mathcal{S}$, we choose an arbitrary point $a \in B(\Lambda) \cap \mathbf{Z}_+^N$ and consider a plane $C(\Lambda)$ of dimension $N - \dim(B(\Lambda))$ perpendicular to $B(\Lambda)$ and containing a. The *induced Markov chain* $\mathcal{L}^{(\Lambda)}$ is defined as a continuous time homogeneous Markov chain on the state space $C(\Lambda) \cap \mathbf{Z}_+^N$ with transition intensities $\{{}_\Lambda\mu_{xy},\ x \in C(\Lambda) \cap \mathbf{Z}_+^N\}$ defined by

$${}_\Lambda\mu_{xy} = \mu_{xy} + \sum_{z \neq y} \mu_{xz}, \tag{4.3}$$

where the summation $\sum_{z \neq y}$ is over all $z \in \mathbf{Z}_+^N$, $z \neq y$ such that the straight line connecting z, y is perpendicular to $C(\Lambda)$. We observe that the definition of an induced Markov chain does not depend on point a due to **Condition H**.

Condition I. For each $\Lambda \subset \mathcal{S}$ the induced chain $\{{}_\Lambda\mu_{xy},\ x \in C(\Lambda) \cap \mathbf{Z}_+^N\}$ is irreducible.

Let a random walk $\mathcal{L}$ be fixed. Further in this section we shall assume that **Conditions B,H,I** are satisfied for the random walk $\mathcal{L}$.

Definition. A face $B(\Lambda)$ is said to be *ergodic* if the corresponding induced chain $\mathcal{L}^{(\Lambda)}$ is ergodic.

Let $\pi^{(\Lambda)}(x)$ be the stationary probabilities of an ergodic induced Markov chain $\mathcal{L}^{(\Lambda)}$. We introduce the *induced vector*

$$v^{(\Lambda)} = (v_1^{(\Lambda)}, \ldots, v_N^{(\Lambda)})$$

corresponding to the face $B(\Lambda)$ by setting

$$v_i^{(\Lambda)} = \begin{cases} \sum_{x \in C(\Lambda) \cap \mathbf{Z}_+^N} \pi^{(\Lambda)}(x) M_i(x), & \text{if } i \in \Lambda \\ 0, & \text{if } i \notin \Lambda \end{cases} \tag{4.4}$$

The face $B(\mathcal{S})$ is ergodic by definition and

$$v^{(\mathcal{S})} = M(x),$$

for all $x \in B(\mathcal{S}) \cap \mathbf{Z}_+^N$.

Definition. We consider faces Λ, Λ_1 such that $\Lambda \supset \Lambda_1$. Let $B(\Lambda)$ be ergodic and $v^{(\Lambda)}$ is defined. The face $B(\Lambda)$ is said to be *ingoing (outgoing)* if all the coordinates $v_i^{(\Lambda)}$ for $i \in \Lambda \setminus \Lambda_1$ are nonpositive (nonnegative). Otherwise, the face $B(\Lambda)$ is said to be *neutral.*

Second vector field. To each point $x \in R_+^N$ we assign a vector $v(x)$. For each ergodic face $B(\Lambda)$ we put

$$v(x) = v^{(\Lambda)},$$

if $x \in B(\Lambda)$, where $v^{(\Lambda)}$ is the induced vector for the face $B(\Lambda)$. For nonergodic faces the second vector field can be multivalued. If face $B(\Lambda_1)$ is nonergodic then $v(x)$ takes all values $v^{(\Lambda)}$ such that $B(\Lambda)$ is an outgoing face for $B(\Lambda_1)$. From the definition of outgoing faces it follows that, for x belonging to nonergodic faces,

$$x + v(x) \in R_+^N$$

for each value $v(x)$, when x is sufficiently far from 0. If there are no outgoing faces for some nonergodic face, we put $v(x) = 0$. Points $x \in R_+^N$, where the second vector field is multivalued, are said to be *branch points.* This vector field $\{v(x),\ x \in \mathbf{R}_+^N\}$ is called the *second vector field* [16].

The idea to implement induced vectors to characterise the large scale behaviour of random walks comes from the following proposition. Let $\xi(x,t)$ be the position of continuous random walk $\mathcal{L}$ at time t starting at point x.

Proposition 1. *Let $\mathcal{L}$ be a random walk satisfying* **Conditions B,H,I**. *Let face $B(\Lambda)$ be ergodic. Then, for all $x \in B(\Lambda)$ and $t \geq 0$, such that $x + v^\Lambda t \in B(\Lambda)$,*

$$\frac{\xi([nx], nt)}{n} \to x + v^\Lambda t, \tag{4.5}$$

almost surely, as $n \to \infty$.

Proof of Proposition. See Lemma 2.2 in [15], where it is proved for discrete time random walks and in the sense of convergence in probability. Generalisation on our case is straightforward. ■

5 Main results

In this section we formulate our main results. For each $\Lambda \subset \mathcal{S}$ we shall denote $\overline{\Lambda}$ by $\mathcal{S} \setminus \Lambda$.

First, let us calculate the *induced vectors* for ergodic faces.

Proposition 2. *Let $B(\Lambda)$, $\Lambda \subset \mathcal{S}$, be a face for an open Jackson Network.*

i) *The face $B(\Lambda)$ is ergodic if and only if*

$$\nu^{\overline{\Lambda}} \equiv \lambda^{\overline{\Lambda}}(I_{\overline{\Lambda}} - P_{\overline{\Lambda}\,\overline{\Lambda}})^{-1} \; < \; \mu_{\overline{\Lambda}}, \tag{5.1}$$

where

$$\lambda^{\overline{\Lambda}} \equiv \lambda_{\overline{\Lambda}} + \mu_{\Lambda} P_{\Lambda\,\overline{\Lambda}}.$$

ii) *(Induced vectors for an open Jackson network) If the face $B(\Lambda)$ is ergodic, then the induced vector $v^{\Lambda} = (v_1^{\Lambda}, \ldots, v_N^{\Lambda}) \in \mathbf{R}_+^N$ is given by*

$$v_i^{\Lambda} = \begin{cases} \lambda_i + \sum_{j\in\Lambda} \mu_j p_{ji} + \sum_{j\in\overline{\Lambda}} \nu_j^{\overline{\Lambda}} p_{ji} - \mu_i, & \text{if} \quad i \in \Lambda \\ 0, & \text{if} \quad i \in \overline{\Lambda} \end{cases} \tag{5.2}$$

or in the matrix form

$$v^{\Lambda} = \lambda_{\Lambda} + \mu_{\Lambda} P_{\Lambda\Lambda} + \nu^{\overline{\Lambda}} P_{\overline{\Lambda}\Lambda} - \mu_{\Lambda}, \tag{5.3}$$

where $\{\nu_j^{\overline{\Lambda}}, j \in \overline{\Lambda}\}$ are (uniquely) defined by (5.1).

Proof of Proposition 2. i) First, we note that the induced chain corresponding to the ergodic face $B(\Lambda)$ is an ergodic open Jackson subnetwork with the set of nodes $\overline{\Lambda}$. In this network the intensity of the external flow at node $j \in \overline{\Lambda}$ is given by

$$\lambda_j + \sum_{i\in\Lambda} \mu_i p_{ij}$$

and $P_{\overline{\Lambda}\,\overline{\Lambda}}$ is the transition matrix. Service intensities are exactly the same to those in the initial network. By using the ergodicity criteria for an open Jackson network [11] we immediately obtain (5.1).

ii) Let $\{\pi^{\Lambda}(x),\ x \in C(\Lambda) \cap Z_+^N\}$ be stationary probabilities of the induced Markov chain $\mathcal{L}^{(\Lambda)}$. Using the formulae (4.2) and (4.4) we have

$$\begin{aligned} v_i^{\Lambda} &= \sum_{x\in C(\Lambda)\cap Z_+^N} \pi^{\Lambda}(x) \left(\lambda_i + \sum_{j\in\Lambda} \mu_j p_{ji} + \sum_{j\in\overline{\Lambda}} I_{\{x_j>0\}} \mu_j p_{ji} - \mu_i \right) \\ &= \left(\lambda_i + \sum_{j\in\Lambda} \mu_j\ p_{ji} + \sum_{j\in\overline{\Lambda}} \mu_j p_{ji} \right) \sum_{x\in C(\Lambda)\cap Z_+^N} I_{\{x_j>0\}} \pi^{\Lambda}(x) - \mu_i. \end{aligned} \tag{5.4}$$

But

$$\sum_{x\in C(\Lambda)\cap Z_+^N} I_{\{x_j>0\}} \pi^{\Lambda}(x) = \sum_{\substack{x\in C(\Lambda)\cap Z_+^N: \\ x_j>0}} \pi^{\Lambda}(x) = \pi(x_j > 0) = \rho_j^{\overline{\Lambda}}$$

is the stationary probability for the jth node in the induced subnetwork to be busy. Therefore ([11])

$$\rho_j^{\overline{\Lambda}} = \frac{\nu_j^{\overline{\Lambda}}}{\mu_j} \tag{5.5}$$

and substituting (5.5) into (5.4) we obtain (5.2). ■

Remark. The formula (5.2) defines the second vector field on ergodic faces.

Remark. In the following theorem we establish some properties of induced vectors related to bottlenecks. In particular, we give an alternative criteria for the set $\mathcal{B}$ of all bottlenecks in terms of the induced vectors.

Theorem 2. (*Ergodic faces, induced vectors and bottlenecks*) *Let $\mathcal{L}_J$ be an open Jackson network with N nodes and parameters $\lambda = (\lambda_1, \ldots, \lambda_N) \geq 0$, $\mu = (\mu_1, \ldots, \mu_N) > 0$ and matrix P. Let $\mathcal{B}$ and $\mathcal{N}$ be the sets of all bottleneck and all nonbottleneck nodes, respectively. Let $\Lambda \subset \mathcal{S}$. Then:*

i) *The face $B(\mathcal{B})$ is ergodic;*

ii) *If a face $B(\Lambda)$ is ergodic, then $\mathcal{B} \subseteq \Lambda$ and $v_i^{\Lambda} \geq 0$ for all $i \in \mathcal{B}$.*

iii) *If a face $B(\Lambda)$ is ergodic and $\Lambda \neq \mathcal{B}$, then there is $i \in \Lambda \setminus \mathcal{B}$ such that $v_i^{\Lambda} < 0$.*

iv) *If a face $B(\Lambda)$ is ergodic and $v_i^{\Lambda} \geq 0$ for all $i \in \Lambda$, then $\Lambda = \mathcal{B}$. In particular, $v^{\Lambda} = 0$ only for $\Lambda = \mathcal{B}$.*

Remark. In **Theorem 3** we establish sufficient conditions for ergodic faces and nonergodic faces in terms of the induced vectors.

Theorem 3 (*Ergodic faces and induced vectors*). *Let $\mathcal{L}_J$ be an open Jackson network with N nodes and parameters $\lambda = (\lambda_1, \ldots, \lambda_N) \geq 0$, $\mu = (\mu_1, \ldots, \mu_N) > 0$ and matrix P. Let $\Lambda \subset \mathcal{S}$. Then:*

i) *Let $B(\Lambda)$ be an ergodic face and $\Lambda' \subset \Lambda$. Then the face $B(\Lambda')$ is ergodic, if $v_l^{\Lambda} < 0$ for each $l \in \Lambda \setminus \Lambda'$.*

ii) *Let $B(\Lambda)$ be an ergodic face and $\Lambda' \subset \Lambda$. The face $B(\Lambda')$ is not ergodic, if $v_l^{\Lambda} \geq 0$, where $l = \Lambda \setminus \Lambda'$.*

Definition. We introduce the partial order between faces. For $\Lambda_1, \Lambda_2 \subset \mathcal{S}$, we say that $\Lambda_1 \leq \Lambda_2$, if $\Lambda_1 \subset \Lambda_2$.

Let $\Lambda \subset \mathcal{S}$. We denote a set of all $\Lambda' \subset \mathcal{S}$ such that $\Lambda \subseteq \Lambda'$ and $B(\Lambda')$ is ergodic by $\mathcal{E}(\Lambda)$. In the following theorem we prove that, for each $\Lambda \subset \mathcal{S}$, the minimal element of $\mathcal{E}(\Lambda)$ exists.

Remark. It is evident that if $B(\Lambda)$ is an ergodic face then the minimal element of $\mathcal{E}(\Lambda)$ is Λ.

Remark. In **Theorem 4** we prove that, for each nonergodic $\Lambda \subset \mathcal{S}$, the minimal ergodic face exists and is unique. Some useful properties of the minimal faces in terms of the induced vectors are also established.

Theorem 4 (*Minimal ergodic faces and induced vectors*). *Let $\mathcal{L}_J$ be an open Jackson network with N nodes and parameters $\lambda = (\lambda_1, \ldots, \lambda_N) \geq 0$, $\mu = (\mu_1, \ldots, \mu_N) > 0$ and matrix P. Then:*

i) *(Existence of minimal ergodic face). For each $\Lambda \subset \mathcal{S}$, the (unique) minimal ergodic face $\Lambda_{\min} \in \mathcal{E}(\Lambda)$, in the sense that $\Lambda_{\min} \leq \Lambda'$ for all $\Lambda' \in \mathcal{E}(\Lambda)$, exists.*

ii) *For each nonergodic $\Lambda \subset \mathcal{S}$,*

$$v_i^{\Lambda_{\min}} \geq 0 \tag{5.6}$$

for all $i \in \Lambda_{\min} \setminus \Lambda$.

iii) *If $\Lambda' \in \mathcal{E}(\Lambda)$ and $v_i^{\Lambda'} < 0$ for some $i \in \Lambda' \setminus \Lambda$, then $\Lambda'' = \Lambda' \setminus \{i\} \in \mathcal{E}(\Lambda)$.*

iv) *For each nonergodic face $B(\Lambda)$ only one outgoing face $B(\Lambda')$, $\Lambda' \supset \Lambda$ exists and, moreover, $\Lambda' = \Lambda_{\min}$, i.e. if $\Lambda' \in \mathcal{E}(\Lambda)$ and $v_i^{\Lambda'} \geq 0$ for all $i \in \Lambda' \setminus \Lambda$, then $\Lambda_{\min} = \Lambda'$.*

Second vector field of an open Jackson network. To each point $x \in R_+^N$ we assign a vector $V(x)$. For each face $B(\Lambda)$ we assign the induced vector of the minimal ergodic face, i.e. we put

$$v(x) = v^{\Lambda_{\min}},$$

if $x \in B(\Lambda)$. Thus, by **Theorem 4**, the vector field $\{V(x),\ x \in \mathbf{R}_+^N\}$ is well defined and, in particular, branch points are not present. This vector field $\{V(x),\ x \in \mathbf{R}_+^N\}$ is called the *second vector field of an open Jackson network.*

Remark. Theorems 2-4 will be proved in Section 7.

Now we can construct a dynamical system described by the second vector field $\{V(x),\ x \in \mathbf{R}_+^N\}$. It will follow from **Theorem 5** that the deterministic large scaling behaviour of an open Jackson network is described by this vector field.

Let us consider a path $\Gamma_x = \Gamma_x(t)$, i.e. a continuous mapping

$$\Gamma : [0, \infty) \longrightarrow R_+^N, \quad \Gamma_x(0) = x,$$

which can be constructed as follows. Assume the starting point is $x \in B(\Lambda)$. If Λ is an ergodic face, set $\Lambda_0^{(x)} = \Lambda$. If Λ is an nonergodic face we set $\Lambda_0^{(x)} = \Lambda_{\min}$, where $\Lambda_{\min}$ is the minimal ergodic face containing face Λ. Consider the induced vector $v^{\Lambda_0^{(x)}}$ corresponding to face $\Lambda_0^{(x)}$. If all components of this vector are nonnegative we define

$$\Gamma_x(t) = x + v^{\Lambda_0^{(x)}} t$$

for all $t \geq 0$. If there is at least one strictly negative component, then we define the moment

$$t_1 = \inf\{t :\ x + v^{\Lambda_0^{(x)}} t \in \overline{B(\Lambda)}\},$$

where $\overline{B(\Lambda)}$ is the closure of $B(\Lambda)$. So, if a particle moves with the speed $v^{\Lambda_0^{(x)}}$ starting at point x, then it will hit the boundary of $B(\Lambda)$ at time t_1. We define

$$\Gamma_x(t) = x + v^{\Lambda_0^{(x)}} t$$

for all $t \in [0, t_1]$.

Let $\Lambda_1^{(x)} \subset \Lambda_0^{(x)}$ be the face hit at moment t_1, i.e. $x + v^{\Lambda_0^{(x)}} t_1 \in B(\Lambda_1^{(x)})$. By the proposition **i)** of **Theorem 3** this face is ergodic. If all components of the vector $v^{\Lambda_1^{(x)}}$ are nonnegative then we put

$$\Gamma_x(t) = \Gamma_x(t_1) + v^{\Lambda_1^{(x)}}(t - t_1)$$

for $t > t_1$. If there is at least one strictly negative component then moving along face $B(\Lambda_1^{(x)})$ another ergodic face is hit and so on. It follows from proposition **i)** of **Theorem 3** that on each step an ergodic face will be hit. If on the nth step we find face $\Lambda_n^{(x)}$ and moment t_n, then we define $\Gamma_x(t)$ by

$$\Gamma_x(t) = \Gamma_x(t_{n-1}) + v^{\Lambda_{n-1}^{(x)}}(t - t_{n-1})$$

for all $t_{n-1} < t \leq t_n$.

Clearly, this procedure has to stop for a finite number of steps since on each step the dimension of an ergodic face decreases by at least 1. We note that if, at the kth step, $v^{\Lambda_k^{(x)}} = 0$ then, by proposition **iv)** of **Theorem 2**, $\Lambda_k^{(x)} = \mathcal{B}$. Hence, by **Theorem 2**, we have to hit the ergodic face $\mathcal{B}$ on the last step. In particular, for an ergodic open Jackson network, $\mathcal{B} = \emptyset$, i.e. we always hit 0. Let it be the kth step.

Thus, we have a finite sequence of ergodic faces

$$\Lambda_0^{(x)} \supset \Lambda_1^{(x)} \supset \Lambda_2^{(x)} \supset \ldots \supset \Lambda_k^{(x)} = \mathcal{B}. \tag{5.7}$$

Also we have a finite sequence of moments

$$t_0 \equiv 0 < t_1 < t_2 < \ldots < t_k. \tag{5.8}$$

Now one can define path $\Gamma_x(t)$ as follows:

$$\Gamma_x(t) = \sum_{i=1}^{n}(t_i - t_{i-1})v^{\Lambda_{i-1}^{(x)}} + (t - t_n)v^{\Lambda_n} \tag{5.9}$$

for all $t \geq 0$, where n is uniquely defined by the condition $t_n < t \leq t_{n+1}$ and $t_{k+1} \equiv \infty$.

We define dynamical system T_t on R_+^N by

$$T_t x = \Gamma_x(t) \tag{5.10}$$

for each $x \in \mathbf{R}_+^N$ and all $t \geq 0$.

Theorem 5. *Let $\mathcal{L}_J$ be an open Jackson network with N nodes and parameters $\lambda = (\lambda_1, \ldots, \lambda_N) \geq 0$, $\mu = (\mu_1, \ldots, \mu_N) > 0$ and matrix P. Then the dynamics $\{T_t,\ t \geq 0\}$ defined by the vector field $\{V^\Lambda,\ \Lambda \subset \mathcal{S}\}$ is strictly acyclic and coincides with the fluid dynamics $\{z(t),\ t \geq 0\}$ defined by the oblique reflection mapping (2.11) in* **Theorem 1**, *i.e.*

$$z(t) = T_x(t), \qquad z(0) = x, \tag{5.11}$$

for each initial vector $x \in \mathbf{R}_+^N$ and all $t \geq 0$.

Remark. During construction of the dynamical system $\{T_t,\ t \geq 0\}$ we, in fact, have proved that it is strictly acyclic. Hence, we must only prove that both dynamical systems are equivalent. This will be proved in Section 8.

6 Algorithms

6.1 Finding All Bottlenecks

This algorithm is very similar to the algorithm for finding all bottlenecks in [3]. We describe it in terms of the second vector field. The algorithm is based on **Theorems 2** and **3**. It consists of at most $N+1$ steps numbered by $0, 1, \ldots, N$.

At step $k = 0$ we check if the network is ergodic or not. If it is ergodic, then $\mathcal{B} = \emptyset$ and the algorithm stops. Otherwise, at step $k = 1$ we take $\Lambda_1 = \mathcal{S}$ and consider $\Lambda_2 = \{i \in \Lambda_1 : v_i^{\Lambda_1} \geq 0\}$. If $\Lambda_1 = \Lambda_2$, then, by proposition **iv)** of **Theorem 2**, $\Lambda_2 = \mathcal{B}$. So $\mathcal{B} = \mathcal{S}$ and the algorithm stops. Otherwise, by proposition **i)** of **Theorem 3** the face $B(\Lambda_2)$ is ergodic and at step $k = 2$ we consider $\Lambda_3 = \{i \in \Lambda_2 : v_i^{\Lambda_2} \geq 0\}$ and so on.

The algorithm stops at step k, if $\Lambda_{k+1} = \Lambda_k$, where $\Lambda_{k+1} = \{i \in \Lambda_k : v_i^{\Lambda_k} \geq 0\}$. Then, $\mathcal{B} = \Lambda_k$ by proposition **iv)** of **Theorem 2** and the algorithm stops.

Remark. Thus in order to find $\mathcal{B}$ we calculate only the induced vectors for some finite sequence of ergodic faces $\Lambda_1 \subset \Lambda_2 \subset \ldots \subset \Lambda_k = \mathcal{B}$, such that $|\Lambda_i \setminus \Lambda_{i+1}| \geq 1$ for all $i = 1, \ldots, k-1$, where $k \leq N$.

6.2 Calculation of Second Vector Field

In order to calculate the second vector field on $\mathbf{R}_+^N$ we have to find, for each $\Lambda \subset \mathcal{S}$, its minimal ergodic face $\Lambda_{\min}$. This algorithm slightly differs from the previous one. It is based on **Theorems 3** and **4** and consists of at most $N+1$ steps numbered by $0, 1, \ldots, N$.

Let $\Lambda \subset \mathcal{S}$ be fixed. At step $k = 0$ we check if the face $B(\Lambda)$ is ergodic or not. If it is ergodic, then $\Lambda_{\min} = \Lambda$ and the algorithm stops. Otherwise, at step $k = 1$ we take $\Lambda_1 = \mathcal{S}$ and consider

$$\Lambda_2 = \{i \in \Lambda_1 : v_i^{\Lambda_1} \geq 0\} \cup \Lambda.$$

If $\Lambda_1 = \Lambda_2$ then, by proposition **iv)** of **Theorem 4**, $\Lambda_2 = \Lambda_{\min}$. and the algorithm stops.

Otherwise, by proposition **i)** of **Theorem 3**, the face $B(\Lambda_2)$ is ergodic and at step $k = 2$ we consider

$$\Lambda_3 = \{i \in \Lambda_2 : v_i^{\Lambda_2} \geq 0\} \cup \Lambda$$

and so on. We stop at step k, if we have $\Lambda_{k+1} = \Lambda_k$, where

$$\Lambda_{k+1} = \{i \in \Lambda_k : v_i^{\Lambda_k} \geq 0\} \cup \Lambda.$$

Then, by proposition **iv)** of **Theorem 4**, $\Lambda_{\min} = \Lambda_k$ and the algorithm stops.

Remark. Thus as in the previous algorithm, in order to find $\Lambda_{\min}$ we calculate only the induced vectors for some finite sequence of ergodic faces $\Lambda_1 \subset \Lambda_2 \subset \ldots \subset \Lambda_k = \Lambda_{\min}$ such that $|\Lambda_i \setminus \Lambda_{i+1}| \geq 1$ for all $i = 1, \ldots, k-1$, where $k \leq N - |\Lambda|$.

7 Proof of Theorems 2-4

Lemma 1. *Let $B(\Lambda)$, $\Lambda \subset \mathcal{S}$, be an ergodic face for an open Jackson network and $\mathcal{B} \subset \Lambda$. Then*

i) *The face* $B(\mathcal{B})$ *is ergodic;*

ii) *Let* $\Lambda' \subset \Lambda$. *Then* $B(\Lambda')$ *is ergodic, if* $v_l^{\Lambda} < 0$ *for each* $l \in \Lambda \setminus \Lambda'$;

iii) *Let* $\Lambda' \subset \Lambda$. *Then* $B(\Lambda')$ *is not ergodic, if* $\dim B(\Lambda') = \dim B(\Lambda) - 1$ *and* $v_l^{\Lambda} \geq 0$, *where* $l = \Lambda \setminus \Lambda'$.

Proof of Lemma 1. i) This proposition follows immediately from the definitions of ergodic faces, induced vectors and the sets $\mathcal{B}$ and $\mathcal{N}$.

ii) Clearly, $\overline{\Lambda'} = \overline{\Lambda} \cup (\Lambda \setminus \Lambda')$. By the conditions of the lemma from **Proposition 2** it follows that

$$v_l^{\Lambda} = \lambda_l + \sum_{j \in \Lambda} \mu_j p_{jl} + \sum_{i \in \overline{\Lambda}} \nu_i^{\overline{\Lambda}} p_{il} - \mu_l \ < \ 0 \tag{7.1}$$

for each $l \in \Lambda \setminus \Lambda'$. Let us denote $\hat{\lambda}_l^{\overline{\Lambda'}}$ by

$$\mu_l - \sum_{i \in \overline{\Lambda}} \nu_i^{\overline{\Lambda}} p_{il} - \sum_{j \in \Lambda \setminus \Lambda'} \mu_j \, p_{jl} \tag{7.2}$$

for $l \in \Lambda \setminus \Lambda'$. Then, in particular, in case **ii)** of the lemma we have

$$\lambda_l^{\overline{\Lambda'}} \equiv \lambda_l + \sum_{j \in \Lambda'} \mu_j p_{jl} < \hat{\lambda}_l^{\overline{\Lambda'}} \tag{7.3}$$

for each $l \in \Lambda \setminus \Lambda'$ and in case **iii)** of the lemma

$$\lambda_l^{\overline{\Lambda'}} \geq \hat{\lambda}_l^{\overline{\Lambda'}} \tag{7.4}$$

for $l = \Lambda \setminus \Lambda'$.

Let us consider a vector $\hat{\nu}^{\overline{\Lambda'}} = (\hat{\nu}_i^{\overline{\Lambda'}},\ i \in \overline{\Lambda'})$ defined by the equation

$$\hat{\nu}^{\overline{\Lambda'}} = \hat{\lambda}^{\overline{\Lambda'}} + \hat{\nu}^{\overline{\Lambda'}} P_{\overline{\Lambda'}\,\overline{\Lambda'}}, \tag{7.5}$$

where $\hat{\lambda}^{\overline{\Lambda'}} = (\hat{\lambda}_i^{\overline{\Lambda'}},\ i \in \overline{\Lambda'})$ and

$$\hat{\lambda}_i^{\overline{\Lambda'}} = \begin{cases} \lambda_i^{\overline{\Lambda'}}, & \text{if } \ i \in \overline{\Lambda} \\ \hat{\lambda}_i^{\overline{\Lambda'}}, & \text{if } \ i \in \Lambda \setminus \Lambda' \end{cases}$$

and $\hat{\lambda}_i^{\overline{\Lambda'}}$ is also given by (7.2). We shall check that the vector $\hat{\nu}^{\overline{\Lambda'}}$ given by

$$\hat{\nu}_i^{\overline{\Lambda'}} = \begin{cases} \nu_i^{\overline{\Lambda}}, & \text{if } \ i \in \overline{\Lambda} \\ \mu_i, & \text{if } \ i \in \Lambda \setminus \Lambda' \end{cases}$$

is the (unique) solution to equation (7.5). For $j \in \overline{\Lambda}$, we have

$$\sum_{i \in \overline{\Lambda}} \nu_i^{\Lambda} (\delta_{ij} - p_{ij}) + \sum_{l \in \Lambda \setminus \Lambda'} \mu_l (\delta_{lj} - p_{lj}) =$$

$$\lambda_j^{\overline{\Lambda}} - \sum_{l \in \Lambda \setminus \Lambda'} \mu_l p_{lj} = \lambda_j + \sum_{i \in \Lambda} \mu_i p_{ij} - \sum_{l \in \Lambda \setminus \Lambda'} \mu_l p_{lj} = \lambda_j^{\overline{\Lambda'}}$$

and, for $j \in \Lambda \setminus \Lambda'$, we have

$$\sum_{i\in\overline{\Lambda}} \nu_i^{\Lambda}(\delta_{ij} - p_{ij}) + \sum_{l\in\Lambda\setminus\Lambda'} \mu_l(\delta_{lj} - p_{lj}) =$$
$$= -\sum_{i\in\overline{\Lambda}} \nu_i^{\Lambda} p_{il} + \mu_l - \sum_{l\in\Lambda\setminus\Lambda'} \mu_l p_{lj} = \hat{\lambda}_j^{\overline{\Lambda'}}.$$

In case **ii)** of the lemma we can now prove

$$\nu_i^{\overline{\Lambda'}} \leq \nu_i^{\overline{\Lambda}} \tag{7.6}$$

for each $i \in \overline{\Lambda}$ and

$$\nu_l^{\overline{\Lambda'}} < \mu_l \tag{7.7}$$

for each $l \in \Lambda \setminus \Lambda'$. Indeed, since the matrix

$$(m_{ij}^{\overline{\Lambda'}})_{i,j\in\overline{\Lambda'}} \equiv (I_{\overline{\Lambda'}} - P_{\overline{\Lambda'}\,\overline{\Lambda'}})^{-1}$$

has nonnegative elements, then from (7.3) it follows that

$$\nu_i^{\overline{\Lambda'}} = \sum_{j\in\overline{\Lambda'}} \lambda_j^{\overline{\Lambda'}} m_{ji}^{\overline{\Lambda'}} \leq \sum_{j\in\overline{\Lambda'}} \hat{\lambda}_j^{\overline{\Lambda'}} m_{ji}^{\overline{\Lambda'}} = \hat{\nu}_i^{\overline{\Lambda'}} \tag{7.8}$$

for each $i \in \overline{\Lambda'}$. But, since

$$\lambda_i^{\overline{\Lambda'}} < \hat{\lambda}_i^{\overline{\Lambda'}}$$

and $m_{ii}^{\overline{\Lambda'}} > 0$ for all $i \in \Lambda \setminus \Lambda'$, then for $i \in \Lambda \setminus \Lambda'$, the inequality (7.8) is rigourous, i.e.

$$\nu_i^{\overline{\Lambda'}} < \hat{\nu}_i^{\overline{\Lambda'}} = \mu_i$$

and (7.6), (7.7) are proved.

Next, since the induced Markov chain corresponding to the face $B(\Lambda)$ is ergodic, then this chain is an ergodic open Jackson subnetwork and

$$\nu_i^{\overline{\Lambda}} < \mu_i,$$

for all $i \in \overline{\Lambda}$. Therefore, from (7.6) and (7.7), it follows that $\nu_i^{\overline{\Lambda'}} < \mu_i$ for each $i \in \overline{\Lambda'}$. This means that the induced Markov chain corresponding to the face $B(\Lambda')$ is also ergodic.

iii) This case can be considered similar the previous one. Since $v_l^{\Lambda} \geq 0$, we have

$$\lambda_l^{\overline{\Lambda'}} \geq \hat{\lambda}_l^{\overline{\Lambda'}}, \text{ for } l \in \Lambda \setminus \Lambda'$$

and, therefore,

$$\nu_i^{\overline{\Lambda'}} \geq \hat{\nu}_i^{\overline{\Lambda'}} \tag{7.9}$$

for each $i \in \overline{\Lambda'}$ and, in particular,

$$\nu_l^{\overline{\Lambda}} \geq \hat{\nu}_l^{\overline{\Lambda'}} = \mu_l$$

for $l = \Lambda \setminus \Lambda'$. So the induced Markov chain corresponding to the face $B(\Lambda')$ is not ergodic.
Lemma 1 is proved. ■

Remark. Theorem 3 follows immediately from **Lemma 1**. ■

Lemma 2.
i) *If a face $B(\Lambda)$ is ergodic then $\mathcal{B} \subset \Lambda$ and there exists a sequence of faces*

$$\mathcal{B} = \Lambda_k \subset \ldots \subset \Lambda_1 \subset \Lambda_0 \equiv \Lambda \tag{7.10}$$

such that each $B(\Lambda_i)$ is ergodic and

$$|\Lambda_{i-1} \setminus \Lambda_i| = 1$$

for each $i = 1, \ldots, k$.

ii) *If a face $B(\Lambda)$ is ergodic, then $v_i^\Lambda \geq 0$ for each $i \in \mathcal{B}$. But, if $\mathcal{N} \cap \Lambda \neq \emptyset$, then there exists $i \in \Lambda$ such that $v_i^\Lambda < 0$.*

iii) *If for some $\Lambda \subset \mathcal{S}$, the face $B(\Lambda)$ is ergodic and $v_i^\Lambda \geq 0$ for all $i \in \Lambda$, then $\Lambda = \mathcal{B}$. In particular, $v^\Lambda = 0$ only for $\Lambda = \mathcal{B}$.*

iv) *If a face $B(\Lambda)$, $\Lambda \subset \mathcal{S}$, is ergodic and $\Lambda \neq \mathcal{B}$, then there exists $i \in \Lambda \setminus \mathcal{B}$ such that $v_i^\Lambda < 0$.*

Remark. From **Lemma 2** it follows that each ergodic open Jackson subnetwork consists only of nonbottleneck nodes.

Proof of Lemma 2. i) Let a face $B(\Lambda)$ be ergodic. If $v_i^\Lambda < 0$ for some $i \in \Lambda$, then, by the proposition **ii)** of **Lemma 1**, the face

$$\Lambda_1 \equiv \Lambda \setminus \{i\} \subset \Lambda$$

is also ergodic and $\Lambda_1 \subset \Lambda$. Hence, there exists a sequence of ergodic faces

$$\Lambda_k \subset \ldots \subset \Lambda_1 \subset \Lambda \tag{7.11}$$

such that

$$|\Lambda_{i-1} \setminus \Lambda_i| = 1$$

for each $i = 1, \ldots, k$ and all coordinates of the vector v^{Λ_k} are nonnegative (if Λ_k is not empty), where

$$v^{\Lambda_k} = \lambda_{\Lambda_k} + \mu_{\Lambda_k} P_{\Lambda_k \Lambda_k} + \nu^{\overline{\Lambda_k}} P_{\overline{\Lambda_k} \Lambda_k} - \mu_{\Lambda_k} \geq 0. \tag{7.12}$$

Let $\mathcal{B}' \equiv \Lambda_k$, and $\mathcal{N}' \equiv \mathcal{S} \setminus \mathcal{B}'$. Then

$$\nu_{\mathcal{B}'} \equiv \lambda_{\mathcal{B}'} + \mu_{\mathcal{B}'} P_{\mathcal{B}'\mathcal{B}'} + \nu_{\mathcal{N}'} P_{\mathcal{N}'\mathcal{B}'}, \tag{7.13}$$

where $\nu_{\mathcal{N}'} \equiv \nu^{\overline{\Lambda_k}}$ is given by (5.1):

$$\nu_{\mathcal{N}'} = (\lambda_{\mathcal{N}'} + \mu_{\mathcal{B}'} P_{\mathcal{B}'\mathcal{N}'})(I_{\mathcal{N}'} - P_{\mathcal{N}'\mathcal{N}'})^{-1} < \mu_{\mathcal{N}'}, \tag{7.14}$$

which satisfies the equation

$$\nu_{\mathcal{N}'} = \lambda_{\mathcal{N}'} + \nu_{\mathcal{N}'} P_{\mathcal{N}'\mathcal{N}'} + \mu_{\mathcal{B}'} P_{\mathcal{B}'\mathcal{N}'} \tag{7.15}$$

and by ergodicity of the face $B(\mathcal{B}')$

$$\nu_{\mathcal{N}'} < \mu_{\mathcal{N}'}.$$

¿From (7.12) it follows that

$$\nu_{\mathcal{B}'} \geq \mu_{\mathcal{B}'}. \tag{7.16}$$

We notice that equations (7.15), (7.13) are exactly the same as (3.8), (3.9). By taking into account that

$$\nu_{\mathcal{B}'} \geq \mu_{\mathcal{B}'}, \quad \nu_{\mathcal{N}'} < \mu_{\mathcal{N}'},$$

$(\nu_{\mathcal{B}'}, \nu_{\mathcal{N}'})$ is found to be a solution to the equation (3.2). ¿From the uniqueness of the solution to equation (3.2) it follows that $\mathcal{B}' = \mathcal{B}$, $\mathcal{N}' = \mathcal{N}$ and

$$(\nu_{\mathcal{B}'}, \nu_{\mathcal{N}'}) = (\nu_{\mathcal{B}}, \nu_{\mathcal{N}}).$$

So,

$$\mathcal{B} = \Lambda_k \subset \Lambda_{k-1} \subset \ldots \subset \Lambda_1 \subset \Lambda, \tag{7.17}$$

is obtained, where faces $\Lambda_k, \ldots, \Lambda_1$ satisfy the conditions of proposition **ii)** of the lemma and $\mathcal{B} \subset \Lambda$.

ii) In order to prove this proposition we assume that there exists $i \in \mathcal{B}$ such that $v_i^\Lambda < 0$. Then, by proposition **ii)** of **Lemma 1**, the face

$$\Lambda_1 \equiv \Lambda \setminus \{i\} \subset \Lambda$$

is also ergodic and the set $\mathcal{B} \setminus \Lambda_1$ is not empty. Hence, there exists a set

$$\Lambda' \subset \Lambda_1 \subset \Lambda \tag{7.18}$$

such that $B(\Lambda')$ is ergodic and all coordinates of the vector $v^{\Lambda'}$ are nonnegative (if Λ' is not empty). Moreover, $\mathcal{B} \setminus \Lambda'$ is not empty.

But, by using the uniqueness argument as in the proof of proposition **ii)** of the lemma, we obtain

$$\mathcal{B} = \Lambda'.$$

So $\mathcal{B} \setminus \Lambda'$ is empty and we obtain a contradiction. This means that $v_i^\Lambda \geq 0$ for all $i \in \mathcal{B}$.

The proof of the fact, if $\mathcal{N} \cap \Lambda \neq \emptyset$, there exists $i \in \Lambda$ such that $v_i^\Lambda < 0$, is along the same lines.

iii) This proposition follows from the previous one and proposition **ii)** of **Lemma 1**.

iv) This proposition follows from propositions **i,ii)** of the lemma.

Lemma 2 is proved. ■

Proof of Theorem 2. Proposition **i)** of the theorem follows from proposition **i)** of **Lemma 1**. Propositions **ii,iii,iv)** follow from propositions **ii,iii,iv)** of **Lemma 2.** ■

Lemma 3. i) *(Existence of minimal element of $\mathcal{E}(\Lambda)$) For each nonergodic $\Lambda \subset \mathcal{S}$, there exists $\Lambda_{\min} \in \mathcal{E}(\Lambda)$ such that $\Lambda_{\min} \leq \Lambda'$ for all $\Lambda' \in E(\Lambda)$.*

ii) *Let $\Lambda \subset \mathcal{S}$. Then, for each $\Lambda' \in \mathcal{E}(\Lambda)$, there exists a sequence of faces*

$$\Lambda_{\min} = \Lambda_k \subset \ldots \subset \Lambda_1 \subset \Lambda_0 \equiv \Lambda', \tag{7.19}$$

such that each $\Lambda_i \in \mathcal{E}(\Lambda)$ and

$$|\Lambda_{i-1} \setminus \Lambda_i| = 1$$

for all $i = 1, \ldots, k$. Moreover,

$$v_i^{\Lambda_{\min}} \geq 0 \tag{7.20}$$

for $i \in \Lambda_{\min} \setminus \Lambda$.

iii) *If $\Lambda' \in \mathcal{E}(\Lambda)$ and $v_i^{\Lambda'} < 0$ for some $i \in \Lambda' \setminus \Lambda$, then $\Lambda'' = \Lambda \setminus \{i\} \in \mathcal{E}(\Lambda)$.*

iv) *If $\Lambda' \in \mathcal{E}(\Lambda)$ and $v_i^{\Lambda'} \geq 0$ for all $i \in \Lambda' \setminus \Lambda$, then $\Lambda_{\min} = \Lambda'$.*

Proof of Lemma 3. i-ii) Consider an open Jackson network $\mathcal{L}_{\overline{\Lambda}}$ with $N - |\Lambda|$ nodes and transition matrix $P_{\overline{\Lambda}\,\overline{\Lambda}}$. Arrivals at node $j \in \overline{\Lambda}$ from outside the network form a Poisson process with parameter

$$\lambda_j + \sum_{i \in \Lambda} \mu_i p_{ij}$$

and the service rate at node j is μ_j. The flow balance equation for this open Jackson network can be written as

$$\nu_{\overline{\Lambda}} = \lambda_{\overline{\Lambda}} + \mu_\Lambda P_{\Lambda\,\overline{\Lambda}} + \min(\nu_{\overline{\Lambda}}, \mu_{\overline{\Lambda}}) P_{\overline{\Lambda}\,\overline{\Lambda}}. \tag{7.21}$$

Since $P_{\overline{\Lambda}\,\overline{\Lambda}}$ has the spectral radius strictly less than 1, then (7.21) has a unique nonnegative solution. Let $\mathcal{B}_{\overline{\Lambda}}, \mathcal{N}_{\overline{\Lambda}}$ be the sets of bottleneck and nonbottleneck nodes of $\mathcal{L}_{\overline{\Lambda}}$, respectively. We shall prove that

$$\Lambda_{\min} = \Lambda \cup \mathcal{B}_\Lambda.$$

Denote a face corresponding to a set $\Lambda' \subset \overline{\Lambda}$ of the network $\mathcal{L}_{\overline{\Lambda}}$ by $B_{\overline{\Lambda}}(\Lambda')$. ¿From the definitions of an induced chain and an induced vector, it follows that the face $B_{\overline{\Lambda}}(\Lambda')$, for $\Lambda' \subset \overline{\Lambda}$, is ergodic in the network $\mathcal{L}_{\overline{\Lambda}}$ if and only if $B(\Lambda \cup \Lambda')$ is ergodic in $\mathcal{L}$. Moreover, if $B_{\overline{\Lambda}}(\Lambda')$, $\Lambda' \subset \overline{\Lambda}$, is ergodic, then

$$v_i^{\Lambda \cup \Lambda'} = v_i^{\Lambda'}(\overline{\Lambda})$$

for all $i \in \overline{\Lambda}$, where $v^{\Lambda'}(\Lambda')$ is the induced vector corresponding to the ergodic face $B_{\overline{\Lambda}}(\Lambda')$. In particular, the face $B(\Lambda \cup \mathcal{B}_{\overline{\Lambda}})$ is ergodic, since face $B_{\overline{\Lambda}}(\mathcal{B}_{\overline{\Lambda}})$ is ergodic in the network $\mathcal{L}_{\overline{\Lambda}}$.

Let $\Lambda'' \in \mathcal{E}(\Lambda)$. We observe that if $i \in \Lambda'' \setminus \Lambda$ such that $v_i^{\Lambda''} < 0$, then, by proposition ii) of **Lemma 1**, the face

$$\Lambda_1 \equiv \Lambda'' \setminus \{i\}$$

is also ergodic and $\Lambda_1 \in \mathcal{E}(\Lambda)$. Hence, there exists a sequence

$$\Lambda \subset \Lambda_k \subset \ldots \subset \Lambda_1 \tag{7.22}$$

of ergodic faces such that $\Lambda_i \in \mathcal{E}(\Lambda)$,

$$|\Lambda_{i-1} \setminus \Lambda_i| = 1,$$

for each $i = 1, \ldots, k$ and $v_j^{\Lambda_k} \geq 0$ for each $j \in \Lambda_k \setminus \Lambda$ (if $\Lambda_k \setminus \Lambda$ is not empty).

Hence, $v_j^{\Lambda_k \setminus \Lambda}(\Lambda) \geq 0$ for each $j \in \Lambda_k \setminus \Lambda$. Since equation (7.21) has a unique solution, then, by using similar arguments as in the proof of proposition **i)** of **Lemma 2**, we obtain

$$\mathcal{B}_{\overline{\Lambda}} = \Lambda_k \setminus \Lambda.$$

iii) This proposition follows immediately from propositions **ii)** of **Lemma 1** and **ii)** of the lemma.

iv) This proposition can be proven along the same lines as propositions **i,ii)** of the lemma. So **Lemma 3** is proved. ■

Remark. Theorem 4 follows from **Lemma 3.** ■

8 Proof of Theorem 5

In this section we prove that, in fact, for each $x \in \mathbf{R}_+^N$, the function $x(\cdot)$ defined by

$$x(t) = x + vt, \quad v = \lambda - \mu(I - P)$$

for all $t \geq 0$ and its (unique) regulator $y(\cdot)$ satisfy the equation

$$\Gamma_x(t) = x(t) + y(t)(I - P) \tag{8.1}$$

for all $t \geq 0$, where $\Gamma_x(t)$ is the path with initial point x, as was defined in Section 5. ¿From the uniqueness of representation (8.1) it will follow **Theorem 5.**

Let an open Jackson network be defined by the triplet (λ, μ, P) and

$$v = \lambda - \mu(I - P),$$

that is, $v = v^{(S)}$ is the second vector field inside $\mathbf{R}_+^N$. Next we find a regulator for the linear function $x(t) = x + vt$ with $x \geq 0$. Without loss of generality we can suppose that at least one component of v is negative. Otherwise, if $v \geq 0$, then $y(t) = 0$ for all $t \geq 0$.

Let us consider path $\Gamma_x(t)$ with initial point x. Then, as was described in Section 5, one can associate with any path a sequence of times,

$$0 = t_0 < t_1 < t_2 < \ldots < t_k,$$

and a sequence of ergodic faces

$$\Lambda_0 \supset \Lambda_1 \supset \Lambda_2 \supset \ldots \supset \Lambda_k = \mathcal{B},$$

for some $k \leq N$. Let $y(\cdot)$ be the function defined by

$$y(t) = \sum_{j=0}^{i-1} (\mu_{\overline{\Lambda_j}} - \nu^{\overline{\Lambda_j}})(t_{j+1} - t_j) + (\mu_{\overline{\Lambda_i}} - \nu^{\overline{\Lambda_i}})(t - t_i), \tag{8.2}$$

for all $t \geq 0$, where i is such that $t_i < t \leq t_{i+1}$ and $t_{k+1} \equiv \infty$. For $\Lambda \neq \emptyset$, the vector $\nu^{\overline{\Lambda_j}}$ is a (unique) solution of the system

$$\nu^{\overline{\Lambda_j}}(I_{\overline{\Lambda_j}} - P_{\overline{\Lambda_j}\overline{\Lambda_j}}) = \lambda_{\overline{\Lambda_j}} + \mu_{\Lambda_j} P_{\Lambda_j \overline{\Lambda_j}}$$

and, for $\Lambda = \emptyset$, we put $\mu_\Lambda = 0$, $\nu^\Lambda = 0$.

Lemma 4. *Let $x \in \mathbf{R}_+^N$. For $x(t) = x + vt$, let $y(t) = \phi_P(x(t))$. Then $y(t)$ can also be defined by (8.2) and*

$$\Gamma_x(t) = x(t) + y(t)(I - P)$$

for all $x \in \mathbf{R}_+^N$ and $t \geq 0$.

Proof of Lemma 4. We shall prove this lemma by induction in moments t_i, $i = 0, 1, \ldots, k$. First, the initial point $x_0 = x \in B(\Lambda)$ for some $\Lambda \subset \mathcal{S}$, where the face $B(\Lambda)$ can be either ergodic or nonergodic. Let us consider both the cases.

Case when Λ is ergodic. The first step of induction is evident, since $y(t_0) \equiv 0$, $\Gamma_x(0) = x$. Next suppose we have already proved

$$\Gamma_x(t_i) = x(t_i) + y(t_i)(I - P). \tag{8.3}$$

So we must show that equation (8.3) holds for $t_i < t \leq t_{i+1}$. First we note that, by proposition **ii)** of **Lemma 1**, the face $B(\Lambda_i)$ is also ergodic. By taking into account that

$$\Gamma_x(t) = \Gamma_x(t_i) + v^{\Lambda_i}(t - t_i),$$

$$x(t) = x(t_i) + v(t - t_i),$$

$$y(t) = y(t_i) + (\mu_{\overline{\Lambda_i}} - \nu^{\overline{\Lambda_i}})(t - t_i)$$

we find that for $t_i < t \leq t_{i+1}$, equation (8.3) is equivalent to the equation:

$$v^{\Lambda_i} = v + (\mu_{\overline{\Lambda_i}} - \nu^{\overline{\Lambda_i}})(I - P). \tag{8.4}$$

Using formula (5.2) for vector v^{Λ_i}, equation (8.4) can be written as follows:

$$\lambda_{\Lambda_i} + \mu_{\Lambda_i} P_{\Lambda_i \Lambda_i} + \nu^{\overline{\Lambda_i}} P_{\overline{\Lambda_i}\Lambda_i} - \mu_{\Lambda_i} = v + (\mu_{\overline{\Lambda_i}} - \nu^{\overline{\Lambda_i}})(I - P). \tag{8.5}$$

But

$$v = \lambda + \mu P - \mu$$

$$\lambda = \lambda_{\Lambda_i} + \lambda_{\overline{\Lambda_i}}, \quad \mu = \mu_{\Lambda_i} + \mu_{\overline{\Lambda_i}},$$

$$\nu^{\overline{\Lambda_i}} P = \nu^{\overline{\Lambda_i}} P_{\overline{\Lambda_i}\overline{\Lambda_i}} + \nu^{\overline{\Lambda_i}} P_{\overline{\Lambda_i}\Lambda_i},$$

and, therefore,

$$\lambda_{\Lambda_i} + \mu_{\Lambda_i} P_{\Lambda_i \Lambda_i} + \nu^{\overline{\Lambda_i}} P_{\overline{\Lambda_i}\Lambda_i} - \mu_{\Lambda_i} =$$

$$\lambda_{\Lambda_i} + \lambda_{\overline{\Lambda_i}} + \mu P - \mu_{\Lambda_i} - \mu_{\overline{\Lambda_i}} + \mu_{\overline{\Lambda_i}} - \mu_{\overline{\Lambda_i}} P - \nu^{\overline{\Lambda_i}}(I_{\overline{\Lambda_i}} - P_{\overline{\Lambda_i}\overline{\Lambda_i}}) + \nu^{\overline{\Lambda_i}} P_{\overline{\Lambda_i}\Lambda_i}.$$

After simple algebra we arrive at the equation

$$\nu^{\overline{\Lambda_i}}(I_{\overline{\Lambda_i}} - P_{\overline{\Lambda_i}\overline{\Lambda_i}}) = \lambda_{\overline{\Lambda_i}} + \mu_{\Lambda_i} P - \mu_{\Lambda_i} P_{\Lambda_i \Lambda_i}$$

or

$$\nu^{\overline{\Lambda_i}}(I_{\overline{\Lambda_i}} - P_{\overline{\Lambda_i}\overline{\Lambda_i}}) = \lambda_{\overline{\Lambda_i}} + \mu_{\Lambda_i} P_{\Lambda_i \overline{\Lambda_i}}.$$

Thus, we obtain the equation

$$\nu^{\overline{\Lambda_i}}(I_{\overline{\Lambda_i}} - P_{\overline{\Lambda_i \Lambda_i}}) = \lambda^{\overline{\Lambda_i}}. \tag{8.6}$$

But (8.6) is the definition for $\nu^{\overline{\Lambda_i}}$.

Case when Λ is nonergodic. The difference from the previous case is as follows. If we prove (8.3) is true for $t = t_1$, then we shall be in the situation of the previous case since the face $B(\Lambda_1)$ must be ergodic. So let us prove (8.3) for $t = t_1$.

Let $\Delta > t_1$ be fixed. Put $d = (1, \ldots, 1) \in \mathbf{R}_+^N$ and consider $x_\epsilon = x + \epsilon d$ for $\epsilon > 0$ to be sufficiently small.

Then, by **Theorem 1**, $z^{x_\epsilon}(t) \to z^x(t)$ as $\epsilon \to 0$ uniformly on $[0, \Delta]$, where $z^{x_\epsilon}(t)$ is defined by

$$z^{x_\epsilon}(t) = x_\epsilon(t) + \phi_P(x_\epsilon)(t)$$

with $x_\epsilon(t) = x_\epsilon + vt$.

We note that x_ϵ belongs to the ergodic face $B(\mathcal{S})$. Then, by the previous case of the lemma,

$$z^{x_\epsilon}(t) = \Gamma_{x_\epsilon}(t)$$

for all $t \geq 0$. Hence, if we prove that

$$\Gamma_{x_\epsilon}(t) \to \Gamma_x(t) \tag{8.7}$$

as $\epsilon \to 0$ uniformly on $[0, \Delta]$, then we prove the lemma.

Indeed, we note that after time $s(\epsilon) = O(\epsilon)$, $\Gamma_{x_\epsilon}(t)$ must hit the ergodic face $B(\Lambda_{\text{min}})$ at some point $\hat{x}_\epsilon$. This easily follows from **Theorem 3** since each ergodic face Λ', hit by $\Gamma_{x_\epsilon}(t)$ before the face $B(\Lambda_{\text{min}})$, satisfies the following conditions: $\Lambda_{\text{min}} \subset \Lambda'$ and there exists $i' \in \Lambda' \setminus \Lambda_{\text{min}}$ such that $v_{i'}^{\Lambda'} < 0$. Clearly, $\|x - \hat{x}_\epsilon\| \leq C\epsilon$ for some $C < \infty$. ¿From this fact (8.7) easily follows. In particular, we proved (8.3) for $t = t_1 < \Delta$.

Lemma 4 is proved. ■

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