View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by INRIA a CCSD electronic archive server

A Unifying look at d-dimensional periodicities ans space covering

Mireille Regnier, L. Rostami

To cite this version:

Mireille Regnier, L. Rostami. A Unifying look at d-dimensional periodicities ans space covering. [Research Report] RR-2021, INRIA. 1993. inria-00074650

HAL Id: inria-00074650
https://hal.inria.fr/inria-00074650

Submitted on 24 May 2006

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

INRIA

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

A Unifying Look at d-dimensional Periodicities and Space Coverings

Mireille RÉGNIER
Ladan ROSTAMI

N° 2021
Septembre 1993

PROGRAMME 2

Calcul symbolique,
programmation
et génie logiciel

Rapport de recherche

1993

A Unifying Look at d-dimensional Periodicities and Space Coverings

Mireille Régnier
INRIA-78 153 Le Chesnay,
France
regnier@margaux.inria.fr *

Ladan Rostami
LITP-2 place Jussieu
75 251 Paris Cedex 05-FRANCE
rostami@litp.ibp.fr

Abstract

We propose a formal characterization of d-dimensional periodicities. We show first that any periodic pattern has a canonical decomposition and a minimal generator, generalizing the 1D property. This allows to classify the d-dimensional patterns in $d+1$ classes, according to their periodicities, each class having subclasses. A full classification of the coverings of a 2-dimensional space by a pattern follows. These results have important algorithmic issues in pattern matching. First, the covering classification allows an efficient use of the now classical "duel" paradigm. Second, d-dimensional pattern matching complexity is intrinsically different for each class.

Périodes des motifs multidimensionnels et recouvrement de l'espace

Resumé

Nous proposons une caractérisation formelle des periodicités dans les motifs à d dimensions. Nous montrons d'abord que tout motif périodique admet une décomposition canonique et un générateur minimal, généralisant ainsi la propriété établie en dimension 1. Ceci permet en fonction des périodicités, une classification des motifs d-dimensionnels en $d+1$ classes, chacune possédant des sous-classes. Une classification exhaustive des recouvrements de l'espace à 2 dimensions en découle. Ces résultats ont d'importantes applications en recherche de motifs. D'une part, la classification des recouvrements permet une utilisation efficace du paradigme maintenant classique du "duel". D'autre part, la complexité dans le cas le pire de la recherche de motifs d-dimensionnels est différente pour chaque classe.

*This research was supported by ESPRIT III program No. 7141 (ALCOM II).

A Unifying Look at d-dimensional Periodicities and Space Coverings

Mireille Régnier[1] and Ladan Rostami[2]

[1] INRIA, 78153 Le Chesnay, France,
regnier@margaux.inria.fr
[2] LITP, 2 place Jussieu,
75 251 Paris Cedex 05, France,
rostami@litp.ibp.fr

Abstract. We propose a formal characterization of d-dimensional periodicities. We show first that any periodic pattern has a canonical decomposition and a minimal generator, generalizing the 1D property. This allows to classify the d-dimensional patterns in $d+1$ classes, according to their periodicities, each class having subclasses. A full classification of the coverings of a 2-dimensional space by a pattern follows. These results have important algorithmic issues in pattern matching. First, the covering classification allows an efficient use of the now classical "duel" paradigm. Second, d-dimensional pattern matching complexity is intrinsically different for each class.

1 Introduction and State of the Art

A lot of attention has been given in the last decade to string searching in a text. String searching can be generalized to "multidimensional search" or "multidimensional pattern matching". A multidimensional pattern, p, most often an array and usually connex and convex, is searched in a multidimensional array, the text, t. It remains a widely open research area, although a strong interest appeared recently [ZT89, BYR90, ABF92, GP92]. It is interesting to notice that the now classical "duel" paradigm [Vis85] allows a drastic improvement on string searching average efficiency [Gal92] while a refined analysis of possible repetitions and periods in a word allowed to derive the worst case complexity, $(1+O(\frac{1}{m})).n$, **and** to achieve it [CH92].

It seems reasonable to expect that a classification of repetitions in d-dimensional patterns will allow to derive the theoretical complexity bound and achieve it. To support this strong assertion, we point out that, as extensively studied for string searching, lower bounds on the worst-case complexity depend on the **maximal** number of occurrences of a given pattern that can be found in a text. That is, in a d-dimensional space, possible space coverings. Also, if such a compact representation of p is not a tiling, certainly p overlaps with himself: some subpattern is repeated inside p. Notice that preliminary results, as well as algorithmic issues, are presented in [NR92].

Let us turn now to the state of the art and a definition of repetitions or periods. In dimension 1, the problem has been extensively studied by various mathematicians, whose results are grouped in [Lot83]. Notably, it is proved that a word w is

* This research was supported by ESPRIT III program No. 7141 (ALCOM II).

selfoverlapping iff exists a word x such that

$$w = s(x).x^k = x^k p(x) \quad (1)$$

where $p(x)$ and $s(x)$ are a prefix and a suffix of x. Equivalently, w is a truncation of some word of $x^* = \cup_m \{x^m\}$. When $w \neq x$, w is called *periodic*, and x is a *period* of w. Additionnally, a unicity property holds. A *primitive* word w being a word that is not the power of any other word, we have:

Theorem 1. *Given a word w, let x_1, x_2 be two primitive words such that:*

$$w = x_1^{k_1}.p(x_1) = x_2^{k_2} p(x_2), k_1 > 1, k_2 > 1 \ . \quad (2)$$

Then $x_1 = x_2$ and $k_1 = k_2$.

In that case, the word w satisfies the (more restrictive) periodicity definition: a self-overlap accross the middle point. We prefer to maintain the definition of period of [Lot83], and distinguish non-degenerated periodicities ($|x| < |w|/2$) and degenerated periodicities. Notice that degenerated periodicities do have consequences on the average case of string searching algorithms. A study can be found for Knuth-Morris-Pratt in [Rég89]. Also, they provide the variations on the second order term, $O(\frac{1}{m})$ of 1D complexity [CH92] (although this is not stated explicitely in the paper). A hint on their importance in d-dimension. As a matter of fact, they appear in the first order [NR92]. In the following, we use equivalently the terms of self-overlapping patterns and periodic patterns. A recent work [AB92] started the study of 2-dimensional periods. In that work, a pattern is periodic iff it selfoverlaps **and** this overlapping includes the center. This is one natural extension of 1D non-degeneracy notion, and will be called in the following simple periodic patterns. [AB92] introduces the notion of *sources*, the locations where an overlap can originate. The main result is a classification of pattern periodicities according to such locations: non-periodic, line periodic, radiant periodic and lattice periodic. A geometric regularity of this pattern of "candidates to be a source" is claimed, but actual sources only form a subset of that location pattern (the difference occurs essentially for line and radiant periodicities). That classification is incomplete, as the characterization of such subsets is not provided. We will give it. Also, it is claimed that the 1D property in Theorem period", does not generalize to 2 dimensions. We invalidate that claim, showing that a periodic pattern actually is the repetition of a smaller pattern. This leads to provide the lacking criterium cited above, and to fully describe the characters repetitions in all periodic patterns. This result also provides a (natural) classification of space coverings in 3 classes (3 stands for $2^{d-1}+1$ for a dimension d equal to 2). We discuss in where non-simple and degenerated periodicities are essential, and the relationship to the previous classification. Notably, we discuss the algorithmic consequences and the extension to d-dimensional spaces.

2 Formalism

Basic Notations A d-dimensional pattern p is a d-dimensional array $p[[1 \ldots l_1], \ldots, [1 \ldots l_d]]$ where l_i is some integer, called the i-th dimension. We denote $\bar{j} = j \bmod d$ and call

$(\mathbf{e}_1, \ldots, \mathbf{e}_d)$ an euclidean basis of the d-dimensional space. In the following, we only consider the points with integer coordinats. Such a set is called a lattice. Most results on p will be proved on the vectorial subsets P and T defined below. To express our results, we need the fundamental notion of direction:

Definition 2. Let $\mathcal{R}$ be the equivalence relation:

$$\mathbf{u}\mathcal{R}\mathbf{v} \Leftrightarrow \forall i,\ \mathbf{u}_i\mathbf{v}_i \geq 0 \ . \tag{3}$$

A *direction* is an equivalence class. Given a vector $\mathbf{u}$ with all coordinates $(\mathbf{u}_i)_{1\leq i\leq d}$ non-zero, we can characterize its direction by a sequence

$$(\epsilon_1, \ldots, \epsilon_{d-1}) \in \{0,1\}^{d-1} \ ,$$

defined as follows:

$$\begin{array}{l} \text{if } u_1 > 0 \text{ then } \epsilon_i = \begin{cases} 0 \text{ if } u_{i+1} > 0 \\ 1 \text{ otherwise} \end{cases} \\ \text{else } dir(\mathbf{u}) = dir(-\mathbf{u}) \end{array} \tag{4}$$

The 2^{d-1} directions so-defined can be numbered from 1 to 2^{d-1} by the application:

$$(\epsilon_1, \ldots, \epsilon_{d-1}) \rightarrow n = 1 + \sum_{i=1}^{d-1} \epsilon_i 2^{i-1} \ . \tag{5}$$

Also, when $d = 2$, a vector $\mathbf{u}$ is said *positive* (respectively *negative*) if $\mathbf{u}_1 \geq 0$ (respectively $\mathbf{u}_1 \leq 0$).

Remark. A vector with some coordinates equal to 0, say k, belongs to 2^k different directions. For example, vectors $\mathbf{e_i}$ belong to 2^{d-1} directions.

We can also define a canonical numbering on the corners of p.

Definition 3. Given two opposite corners in direction i, we number C^i (respectively $C^{2^{d-1}+i}$) the one with smaller (respectively greater) 1-coordinate.

Vectorial formalism for the periodicities

Definition 4. Given a pattern p, let P be the subset:

$$\{\mathbf{v}; \mathbf{v} = \sum_i v_i \mathbf{e_i}, (v_1, \ldots, v_d) \in \prod_1^d [0 \ldots l_i - 1]\} \ . \tag{6}$$

Let T be the set of *translation* vectors:

$$\{\mathbf{v}; \mathbf{v} = \sum_i v_i \mathbf{e_i}, (|v_1|, \ldots, |v_d|) \in \prod_1^d [0 \ldots l_i - 1]\} \ . \tag{7}$$

In the following, we note, for $\mathbf{u} \in P$, $p(\mathbf{u})$ the element $p[u_1 + 1, \ldots, u_d + 1]$ of p. Intuitively, any point C in the pattern is naturally associated to 2^d vectors $C - C^i, i = 1 \ldots 2^d$, lying in T. P is the set of representatives $C - C^1$. In Figure 2, $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ and $\mathbf{w}_4$ have $\mathbf{w}_1$ as a common representative.

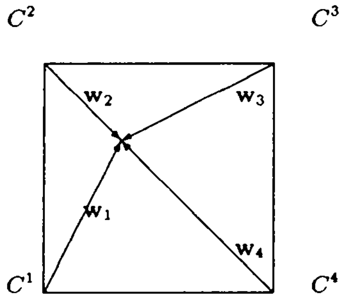


Fig. 1. Translation vectors

Definition 5. Given a pattern p and a vector $\mathbf{u}$, one notes:

$$P_{\mathbf{u}} = \{\mathbf{v}; \mathbf{v} = \mathbf{w} + \mathbf{u}, \mathbf{w} \in P\} \ . \tag{8}$$

Proposition 6. *A vector* $\mathbf{u}$ *is a* translation vector *iff* $P_{\mathbf{u}} \cap P \neq \emptyset$.

Intuitively, $P_{\mathbf{u}}$ points in the 2-dimensional space to a copy of pattern p shifted by $\mathbf{u}$. A translation vector is associated to two overlapping copies. We are interested in shifts such that the two copies are consistent in the overlapping area. That is:

Definition 7. A translation vector $\mathbf{u}$ is an *invariance* vector for p iff:

$$\mathbf{v} \in P, \mathbf{v} + \mathbf{u} \in P \Rightarrow p(\mathbf{u} + \mathbf{v}) = p(\mathbf{v}) \ . \tag{9}$$

An invariance vector $\mathbf{u}$ is *simple* if $2\mathbf{u}$ is an invariance vector. A couple $(\mathbf{u}, \mathbf{v})$ of non-colinear invariance vectors is *non-degenerated* iff

$$\forall j : \sum_j |u_j| + |v_j| \leq l_j \tag{10}$$

A vector $\mathbf{u}$ is *non-degenerated* iff it belongs to a non-degenerated couple.

We note I and $\tilde{I}$ the set of invariance vectors and the set of non-degenerated invariance vectors. They generate

$$\bar{I} = \{\lambda\mathbf{u} + \mu\mathbf{v}; \mathbf{u} \in \tilde{I}, \mathbf{v} \in \tilde{I}\} \ . \tag{11}$$

Finally, we note G the set of *non-degenerated invariance couples.*

Note 8. An invariance vector is associated to a "source" in [AB92] terminology. In [AB92], a source must also satisfy the condition: $2\mathbf{u} \in T$, i.e. be simple. For us, two simple non-colinear invariance vectors always form a non-degenerated couple. But simple vectors can be degenerated and non-simple vectors are not always degenerated.

In order to derive properties of patterns with several periods and to define a minimal representation of the set of periods, we define an **order** on invariance vectors:

Definition 9. Let $\leq$ be the order on the d-dimensional vectorial space:

$$\mathbf{u} \leq \mathbf{v} \Leftrightarrow dir(\mathbf{u}) = dir(\mathbf{v}) \text{ and } \mathbf{v} \in P_{\mathbf{u}} \ . \tag{12}$$

Analytically, this is equivalent to:

$$|u_i| \leq |v_i|, u_i.v_i \geq 0, \ 1 \leq i \leq d \ .$$

Note 10. This notion coincides with the "monotone ordering" of sources in [AB92]. Also, when $dir(\mathbf{u}) = dir(\mathbf{v})$, we have $\mathbf{u} \leq \mathbf{v} \Leftrightarrow dir(\mathbf{u} - \mathbf{v}) = dir(\mathbf{u})$.

Definition 11. Given a pattern p, and $\mathbf{u}$ a vector, the $\mathbf{u}$-strip for that pattern is:

$$S_{\mathbf{u}} = S_{\mathbf{u}}^{+} \cup S_{\mathbf{u}}^{-} \ , \tag{13}$$

where $S_{\mathbf{u}}^{+} = \{\mathbf{v} \in P; \mathbf{u} + \mathbf{v} \notin P\}$ and $S_{\mathbf{u}}^{-} = \{\mathbf{v} \in P; \mathbf{v} - \mathbf{u} \notin P\}$.

The $\mathbf{u}$-dead zone is:

$$DZ_{\mathbf{u}} = \{\mathbf{v} \in P; \mathbf{v} + \mathbf{u} \notin P \text{ and } \mathbf{v} - \mathbf{u} \notin P\} \ = \ S_{\mathbf{u}}^{+} \cap S_{\mathbf{u}}^{-} \ . \tag{14}$$

Let us define:

$$F_{\mathbf{u}}^{+} = \{\mathbf{w} \in P_{-\mathbf{u}} \cap P; p(\mathbf{w} + \mathbf{u}) \neq p(\mathbf{w}))\}, F_{\mathbf{u}}^{-} = \{\mathbf{w} \in P_{\mathbf{u}} \cap P; p(\mathbf{w} - \mathbf{u}) \neq p(\mathbf{w}))\} \ . \tag{15}$$

A pattern is $\mathbf{u}$-selfoverlapping iff $F_{\mathbf{u}}^{+} \cup F_{\mathbf{u}}^{-}$ is empty. We define the **u-free zone** as:

$$F_{\mathbf{u}} = (F_{\mathbf{u}}^{+} \cap F_{\mathbf{u}}^{-}) \cup (F_{\mathbf{u}}^{+} - P_{\mathbf{u}}) \cup (F_{\mathbf{u}}^{-} - P_{-\mathbf{u}}) \ , \tag{16}$$

Intuitively, $DZ_{\mathbf{u}}$ is the subset of the pattern that is not constrained by a $\mathbf{u}$-period. One always has: $DZ_{\mathbf{u}} \cap F_{\mathbf{u}} = \emptyset$. $F_{\mathbf{u}}^{+} \cup F_{\mathbf{u}}^{-}$ is the zone of non-invariance. The free zone is its subset; one excludes points $p(\mathbf{t} + \mathbf{u}) = p(\mathbf{t}) \neq p(\mathbf{t} - \mathbf{u})$ or $p(\mathbf{t} + \mathbf{u}) \neq p(\mathbf{t}) = p(\mathbf{t} - \mathbf{u})$. These domains are represented on Figure 2

Basic definitions on lattices Finally, we recall the basic definitions for lattices, as defined for elliptic functions (see for instance [Lan87]).

Definition 12. Given two non-colinear vectors $\boldsymbol{\omega_1}$ and $\boldsymbol{\omega_2}$, we note $L_{\omega_1}, \boldsymbol{\omega_2}$ the sub-lattice:

$$\{\lambda\boldsymbol{\omega_1} + \mu\boldsymbol{\omega_2}; \lambda \in Z, \mu \in Z\} \cap T \ . \tag{17}$$

$(\boldsymbol{\omega_1}, \boldsymbol{\omega_2})$ is said a basis of the lattice. The *vectorial fundamental parallelogram* with respect to this basis is:

$$VFP_{\omega_1,\omega_2} = \{\alpha\boldsymbol{\omega_1} + \beta\boldsymbol{\omega_2}; 0 \leq \alpha < 1, 0 \leq \beta < 1\} \ . \tag{18}$$

The *fundamental parallelogram* located at some point C is:

$$FP_{\omega_1,\omega_2} = \{C + \mathbf{w}; \mathbf{w} \in VFP_{\omega_1,\omega_2}\} \ . \tag{19}$$

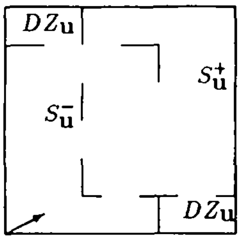


Fig. 2. Special zones

A lattice admits several basis. Among them, some have specific properties, the so-called *fundamental basis.*

Definition 13. A basis $(\mathbf{E}, \mathbf{F})$ is said *fundamental* iff

$$\begin{cases} dir(\mathbf{E}+\mathbf{F}) = dir(\mathbf{E}) & \neq dir(\mathbf{F}) \\ \text{all angles in triangle} < \mathbf{E}, \mathbf{F}, \mathbf{E}+\mathbf{F} > & \leq \frac{\pi}{2} \\ \mathbf{E}_1, \mathbf{F}_1 & \geq 0 \end{cases} \quad (20)$$

Also, we will make use of the basic mapping (also used in [GP92] and called "lattice congruence").

Definition 14. Let ϕ_{ω_1,ω_2} be the mapping $p \longrightarrow FP_{\omega_1,\omega_2}$:

$$\alpha \longrightarrow \phi_{\omega_1,\omega_2}(\alpha) \;=\; C^i + x \;, \quad (21)$$

where x is the only point in FP_{ω_1,ω_2} such that $(\alpha - C^i) - x \in L_{\omega_1,\omega_2}$.

In Figure 4, two fundamental parallelograms are represented. Both satisfy conditions of Definition 13. Remark that point $(3,7)$ is lattice congruent to $(1,2)$ with respect to $(\mathbf{u}, \mathbf{v})$ and to $(5,3)$ with respect to $(\mathbf{E}, \mathbf{F})$.

In the whole complex plane, any point is lattice congruent to a point of the fundamental parallelogram. For a finite pattern, border effects appear. We are led to define:

Definition 15. Given ω_1, ω_2 two non-colinear vectors, we define the border B_{ω_1,ω_2} as :

$$B_{\omega_1,\omega_2} \;=\; \{w; \phi_{\omega_1,\omega_2}(P \cap P_{\mathbf{W}}) \neq FP_{\omega_1,\omega_2}\} \;. \quad (22)$$

Intuitively, some points in FP_{ω_1,ω_2} have no ϕ_{ω_1,ω_2}-image in $P \cap P_{\mathbf{W}}$. In Figure 4, the interior dotted line is the limit of $B_{\mathbf{E},\mathbf{F}}$.

Definition 16. Given a set S of vectors, a S-path from a vector $\mathbf{w}$ to an other vector $\mathbf{w}'$ is a sequence $(\mathbf{w}_i)_{0\ldots n}$ such that:

$$\mathbf{w}_0 = \mathbf{w}, \mathbf{w}_n = \mathbf{w}', \mathbf{w}_{i+1} - \mathbf{w}_i \in S \cup (-S) \ . \tag{23}$$

It is *monotonic* with the restriction:

$$\mathbf{w}_i \leq \mathbf{w}_{i+1} \ . \tag{24}$$

Given a pattern p, a S-path is said *p-valid* iff all its elements lie in P.

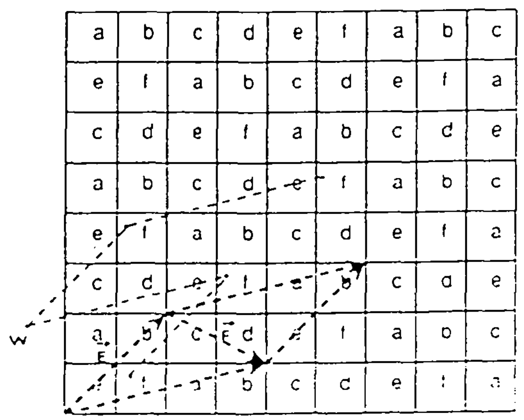


Fig. 3. Lattice congruence and path.

In Figure 2, we represent a $(\mathbf{E}, \mathbf{F})$-pathfrom $(6,5)$ to $(2,1)$, where $\mathbf{E} = 2\mathbf{e}_1 + 2\mathbf{e}_2$ and $\mathbf{F} = 2\mathbf{e}_1 - \mathbf{e}_2$. This path is not p-valid, as it runs out of the pattern.

3 Combining periods

This section is devoted to the study of patterns having several invariance vectors. One would expect the sum of invariance vectors is an invariance vector. Unfortunately, it is not always the case. Nevertheless, basic equality $p((\mathbf{u}+\mathbf{v})+\mathbf{w}) = p(\mathbf{w})$ is satisfied *almost everywhere*, and we may characterize this zone of non-invariance.

We exhibit first constraints on the patterns, that will be extended in the following section by a study of possible generating subpatterns. Our first lemma states general conditions under which the sum of invariance vectors is an invariance vector, formalizing analytically and generalizing the result in [AB92].

Lemma 17. *Let p be a d-dimensional "rectangular" array. If $\mathbf{u}$ is an invariance vector, then: $k\mathbf{u} \in T, k \in Z$ is an invariance vector. Let $\mathbf{u}, \mathbf{v}$ be two invariance vectors in the same direction. If $\mathbf{u} \leq \mathbf{u}+\mathbf{v}$ and $\mathbf{v} \leq \mathbf{u}+\mathbf{v}$, then $k_1\mathbf{u}+k_2\mathbf{v}, k_1, k_2 \in N$, is an invariance vector if it lies in T.*

Proof. The property for P is a direct consequence of the definition (see [AB92] for dimension 2). It is enough to remark that, for a (convex) rectangular array, whenever $\mathbf{t} \in P$ and $\mathbf{t} + (\mathbf{u} + \mathbf{v}) \in P$, either $\mathbf{t} + \mathbf{u} \in P$ or $\mathbf{t} + \mathbf{v} \in P$.

Our next theorems will rely on a very simple but useful lemma:

Lemma 18. *Let* $\mathbf{u}, \mathbf{v}$ *be two invariance vectors. Let* $\mathbf{t} = \mathbf{u} + \mathbf{v}$. *Then,*

$$F^+_{\mathbf{t}} \subseteq S^+_{\mathbf{u}} \cap S^+_{\mathbf{v}} \cap P_{-\mathbf{t}} \ , F^-_{\mathbf{t}} \subseteq S^-_{\mathbf{u}} \cap S^-_{\mathbf{v}} \cap P_{\mathbf{t}} \ . \tag{25}$$

That is, $\mathbf{t}$ *is an invariance vector in* $P - (F^+_{\mathbf{t}} \cup F^-_{\mathbf{t}})$.

Proof. Let $\mathbf{w}$ be a vector in $P_{-\mathbf{t}} \cap P$. Then:

$$\mathbf{w} \in P_{-\mathbf{u}} \cap P \Rightarrow \left\{ \begin{array}{c} p(\mathbf{w} + \mathbf{u}) =_{\mathbf{u}} p(\mathbf{w}) \\ p(\mathbf{w} + \mathbf{t}) =_{\mathbf{v}} p(\mathbf{w} + \mathbf{u}) \end{array} \right\} \Rightarrow p(\mathbf{w}) = p(\mathbf{w} + \mathbf{t}) \Rightarrow \mathbf{w} \notin F^+_{\mathbf{t}} \ . \tag{26}$$

Hence, $F^+_{\mathbf{t}} \subseteq S^+_{\mathbf{u}} = P - S^+_{\mathbf{u}}$; similarly, $F^+_{\mathbf{t}} \subseteq S^+_{\mathbf{v}}$, which yields our inclusion. The same reasoning applies for $F^-_{\mathbf{t}}$.

This proposition shows that only some borders of the pattern are not constrained, while the $\mathbf{u}+\mathbf{v}$ invariance holds in the interior. These borders are precised below and in the next section, where a canonical decomposition of periodicities is proposed.

Proposition 19. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be two invariance vectors such that:*

$$\forall i \ \mathbf{u}_i.\mathbf{v}_i \geq 0$$

or

$$\exists j \ s.t. \left\{ \begin{array}{ll} \mathbf{u}_j \mathbf{v}_j \leq 0 & |\mathbf{u}_j| + |\mathbf{v}_j| \leq l_j \\ \forall i \neq j : \mathbf{u}_i \mathbf{v}_i \geq 0 & \end{array} \right.$$

Then $\mathbf{u} + \mathbf{v}$ *is an invariance vector.*

Note 20. The first case is equivalent to the conditions of Lemma 17, for which we already gave a simple proof.

Proof. Let $\mathbf{t} = \mathbf{u} + \mathbf{v}$ and $\mathbf{w} \in P_{-\mathbf{t}}$. For any i such that $\mathbf{u}_i.\mathbf{v}_i \geq 0$, we have $\mathbf{w}_i + (\mathbf{u}_i + \mathbf{v}_i) \in [0 \ldots l_i]$. Hence $\mathbf{w}_i + \mathbf{u}_i, \mathbf{w}_i + \mathbf{v}_i \in [\min(\mathbf{w}_i + (\mathbf{u}_i + \mathbf{v}_i), \mathbf{w}_i), \max(\mathbf{w}_i + (\mathbf{u}_i + \mathbf{v}_i), \mathbf{w}_i)] \subseteq [0, l_i]$. Also, $|(\mathbf{w}_j + \mathbf{u}_j) - (\mathbf{w}_j + \mathbf{v}_j)| = |\mathbf{u}_j - \mathbf{v}_j| = |\mathbf{u}_j| + |\mathbf{v}_j| \leq l_j$. As $\mathbf{u}_j.\mathbf{v}_j \leq 0$ and $\mathbf{w}_j \in [0 \ldots l_j]$, this implies that either $\mathbf{w}_j + \mathbf{u}_j$ or $\mathbf{w}_j + \mathbf{v}_j \in [0 \ldots l_j]$, that is either $\mathbf{w} + \mathbf{u}$ or $\mathbf{w} + \mathbf{v}$ is in T. Hence, $F^+_{\mathbf{t}} = \emptyset$. Similarly, one proves that $F^-_{\mathbf{t}} = \emptyset$.

We get as an immediate corollary the fundamental result, also generalizing the one in [AB92]:

Theorem 21. *Assume* $d = 2$. *Let* $(\mathbf{u}, \mathbf{v})$ *be a non-degenerated couple such that* $dir(\mathbf{u}) \neq dir(\mathbf{v})$. *Then,* $\mathbf{u} + \mathbf{v}$ *and* $\mathbf{u} - \mathbf{v}$ *are invariance vectors.*

Remark. It is interesting to see that the implication: $dir(\mathbf{u}) \neq dir(\mathbf{v}) \Rightarrow \mathbf{u} - \mathbf{v}$ *is an invariance vector* does not hold in higher dimensions. Also, improvement on [AB92], and [GP92] as well, is that $\mathbf{u}$ and $\mathbf{v}$ need not be both simple.

4 Canonical Decomposition

The aim of this section is twofold. First, we define a generator notion for 2D patterns, based on words. Second, we extend the primality notion to dimension 2. This will allow to define minimal generators. These results are the basis for the next sections. Notably, we will define a basic set of generators for I, $(\mathbf{E}, \mathbf{F})$, and characterize a subpattern that generates P; also, we will characterize I as a subset of $L_{\mathbf{E},\mathbf{F}}$ plus a set of degenerated invariance vectors. Section 5 is devoted to the case where exist non-colinear invariance vectors. In 6, we consider sets of colinear invariance vectors. All results are grouped to get the pattern periodicities classification in 7, as well as the maximal coverings in 8.

Definition 22. A sequence $S = \{s_j\}_{1 \leq j \leq \delta}$ of primitive words is said a (L, δ)-linear generator of a bidimensional pattern in direction i iff:

$$p((j-1+k\delta)\mathbf{e}_{\overline{\mathbf{i+1}}} + (-1)^{i-1}\alpha \mathbf{e}_{\mathbf{i}}) = s_j[(\alpha - kL) \pmod{|s_j|}] \,. \tag{27}$$

If this equation holds out of a subpattern A, S is said a A-linear generator.

Intuitively, rows i, $i \in \{1 \ldots \delta\}$ are linear concatenations of s_i^* and row $j+\delta$ is equal to row j shifted by L. Notably, we have:

$$p((j-1+k\delta)\mathbf{e}_{\overline{\mathbf{i+1}}} + (-1)^{i-1}\alpha \mathbf{e}_{\mathbf{i}}] = p((j-1+(k+1)\delta)\mathbf{e}_{\overline{\mathbf{i+1}}} + (\alpha - L)\mathbf{e}_{\mathbf{i}})$$

For example, in Figure 5, $\{abcdefgh\}$ is a $(1,1)$-linear A-generator in direction 1, where A is the union of the upper left and lower right corners. In Figure 4, $\{eac, fbd\}$ is a $(2,2)$-linear generator in direction 2.

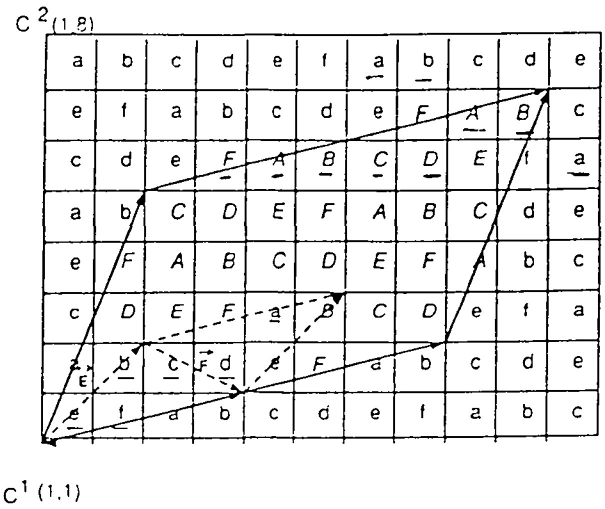
Clearly, any pattern has a trivial linear generator, where S is the set of its m rows (or columns). We now consider the non trivial generators, as well as the minimal generators characterization.

Proposition 23. *A pattern p has a non trivial linear generator iff it is monooverlapping, i.e. iff it has a translation vector. More precisely, the sequence of primitive words s_j is determined from the invariance vector $L\mathbf{e}_i + \delta \mathbf{e}_{\overline{\mathbf{i+1}}}$ by:*

$$\prod_{k=1}^{\lfloor (l_{i+1}-j)/\delta \rfloor} \prod_{\alpha=0}^{L-1} p((j-1+k\delta)\mathbf{e}_{\overline{\mathbf{i+1}}} + (-1)^{i-1}\alpha \mathbf{e}_{\mathbf{i}}) . \prod_{\alpha=0}^{l_i-1} p((j-1)\mathbf{e}_{\overline{\mathbf{i+1}}} + (-1)^{i-1}\alpha \mathbf{e}_{\mathbf{i}}) = s_j^{\beta_j} \tag{28}$$

Proof. The definition 27 implies that $(-1)^{i-1}L\mathbf{e}_{\mathbf{i}} + \delta \mathbf{e}_{\overline{\mathbf{i+1}}} \in I$. Also, equation above is easily checked.

In 1D, minimal generators and periods are based on primality notion on words. We first define a primality notion for invariance vectors in 26. Then, in Proposition 28, we establish a relationship between primitive vectors and minimal linear generators.



$$\left.\begin{array}{ll} |FP_{\mathbf{u},\mathbf{v}}| & = \mathbf{v}_1\mathbf{u}_2 - \mathbf{v}_2\mathbf{u}_1 = 18 \\ s & = efabcd \\ (\mathbf{u},\mathbf{v}) & = ((2,5);(8,2))(\text{continuous parallelogram}) \\ (\mathbf{E},\mathbf{F}) & = ((2,2);(2,-1))(\text{dotted parallelogram}) \\ \sigma = (1,6,4,2,7,5,3) & \end{array}\right\}$$

Fig. 4. Fundamental parallelograms.

Definition 24. Let $(\mathbf{u},\mathbf{v})$ be a non-degenerated couple of invariance vectors. Define $\psi^{[i]}_{\mathbf{u},\mathbf{v}}$ on any fundamental parallelogram $FP_{\mathbf{u},\mathbf{v}}$ by:

$$\psi^{[i]}_{\mathbf{u},\mathbf{v}} : \alpha \longrightarrow \phi_{\mathbf{u},\mathbf{v}}(\alpha + \mathbf{e}_i) \tag{29}$$

Intuitively, when applying $\psi^{[1]}$, we follow a row j to the right, then jump to the leftist element of FP that is in row $\sigma(j)$, where σ is defined by:

$$j \longrightarrow j - min(u_2, v_2) \pmod{u_2 + v_2} \tag{30}$$

More generally:

Proposition 25. *The shift functions $\psi^{[i]}$ satisfy the fundamental property*

$$(\psi^{[i]})^j(C) = \phi(C + j\mathbf{e}_{\mathbf{i}}) \tag{31}$$

Also, the sequences $x(j)$ of $i\bar{+}1$-coordinates satisfy:

$$\begin{cases} x(j+1) = x(j) & \textit{if } \phi(C + (j+1)\mathbf{e}_{\mathbf{i}}) = \phi(C + j\mathbf{e}_{\mathbf{i}}) + \mathbf{e}_{\mathbf{i}} \\ x(j+1) = \sigma(x(j)) & \textit{otherwise} \end{cases}$$

where σ is the permutation:

$$j \longrightarrow j - \min(\mathbf{u}_{i\bar{+}1}, \mathbf{v}_{i\bar{+}1}) \pmod{\mathbf{u}_{i\bar{+}1} + \mathbf{v}_{i\bar{+}1}} \tag{32}$$

In Figure 4, permutation in direction 1 is $\sigma : j \to j - 2 \bmod 7$. Hence $\sigma = (1,6,4,2,7,5,3)$. Let us consider sequences $p((\psi^{[i]})^j(C)$ where C is the origin of the parallelogram. Indices encountered are $[(1,1);(4,6);\ldots;(8,6)]$ (capital and underlined letters). Next positions (capital letters) are [...]. Notice that $(\psi^{[1]})^{(37)}(C) = C$, that is 36 is a period of ψ. Also, $\prod_{j=1}^{36}[p((\psi^{[i]})^j(C)] = (efabcd)^6$. Notice now that $\psi^{[1]}_{\mathbf{E},\mathbf{E}+\mathbf{F}}$ has period 6, the length of primitive word $efabcd$. To formalize this minimality property of $(\mathbf{E},\mathbf{F})$, we state the definition:

Definition 26. $(\mathbf{u},\mathbf{v})$ is said p-primitive iff, in one direction i:

$$\exists C \in FP_{\mathbf{u},\mathbf{v}} : \prod_{j=1}^{|FP_{\mathbf{u},\mathbf{v}}|} [p((\psi^{[i]})^j(C)] \in PW \tag{33}$$

where PW is the set of primitive words.

Note 27. When σ is a cycle, property 29 is independent of the point C that was chosen, as the circular permutation of a string is primitive iff this string is primitive. Otherwise, we can chose any point C that belong to the same orbit. Using the existence of a p-valid $(\mathbf{u},\mathbf{v})$-path, we also get independence on the parallelogram chosen.

Proposition 28. *Let $(\mathbf{u},\mathbf{v})$ be a primitive couple. Then p admits in each direction i a (L_i,δ_i) linear generator. Moreover, in each direction:*

$$\begin{cases} \delta_i &= \mathbf{u}_i \wedge \mathbf{v}_i, \\ GCD(|s_j|) = \frac{|FP_{\mathbf{u},\mathbf{v}}|}{\delta_i} &= \min\{k; k\mathbf{e}_i \in L_{\mathbf{u},\mathbf{v}}\}, \\ L_i &= \min\{k; k(-1)^{i-1}\mathbf{e}_i + \delta_i \mathbf{e}_{\overline{i+1}} \in \tilde{I}\}. \end{cases} \tag{34}$$

Note 29. Computing the size of a parallelogram, we see that, when $\mathbf{u}$ and $\mathbf{v}$ lie in the same direction, (L_1,δ_1) and (L_2,δ_2) satisfy the relationship:

$$s = L_1 L_2 + \delta_1 \delta_2$$

Proof. σ defined above is the product of δ_i cycles. Hence, ψ has δ_i orbits of equal length $\frac{|FP_{\mathbf{u},\mathbf{v}}|}{\delta_i}$, that define δ_i primitive words s_j. ¿From 27, we see that $|s_j|$ and $\frac{|FP_{\mathbf{u},\mathbf{v}}|}{\delta_i}$ are periods on each orbit. Hence, $GCM(|s_j|)\mathbf{e_i}$ and $\frac{|FP_{\mathbf{u},\mathbf{v}}|}{\delta_i}\mathbf{e_i}$ are invariance vectors. Assume now that $s = \min\{k; k\mathbf{e_i} \in L_{\mathbf{u},\mathbf{v}}\} \neq GCM(|s_j|)$. Then, exists s_j such that $s \neq k|s_j|$. Iterating ψ and using 27, one may rewrite

$$s_j = u.v = v.u \quad ,$$

a contradiction with Conjugacy Theorem [Lot83]. Finally, by primality property, $\frac{|FP_{\mathbf{u},\mathbf{v}}|}{\delta_i}$ is equal to some s_j, which achieves the proof of second equality in direction i.

Also, by σ definition and Property 31,rows 1 and $1+\delta$ have identical sets of ϕ-images. That is, exists k such that $\phi(C + k\mathbf{e}_1 + \delta\mathbf{e}_2) = C$, which establishes that $\{k; k\mathbf{e}_1 + \delta\mathbf{e}_2 \in \tilde{I}\}$ is non empty, hence the existence of L_2. The same reasoning on columns yields the existence of L_2.

In Figure 4, we have $\mathbf{E} = 2\mathbf{e}_1 + 2\mathbf{e}_2, \mathbf{F} = 2\mathbf{e}_1 - \mathbf{e}_2$ and $\mathbf{E} + \mathbf{F} = 4\mathbf{e}_1 + \mathbf{e}_2$. Hence, $\delta_1 = 1, \delta_2 = 2$. In direction 1, $\sigma_1 : j \rightarrow j - 1 \bmod 3$ is the cycle $(1,3,2)$. This defines (small and underlined letters) the set $S = \{efabcd\}$ associated to indices $[(1,1);(4,3);(5,3);(2,2);(3,2);(4,2)]$, and $6\mathbf{e}_1 \in I$. Also, $4\mathbf{e}_1 + \mathbf{e}_2 \in I$, hence $L_1 = 4$. In direction 2, $\sigma_2 : j \rightarrow j - 2 \bmod 6$ is the product of two cycles $(1,5,3)(2,6,4)$ associated to the two sequences of indices: $[(1,1);(5,3);(3,2)]$ and $[(2,2);(4,2);((4,3)]$. They define the set $S = \{eac, bdf\}$. $GCD(|s_j|) = 3$, and one can check that $3\mathbf{e}_2 \in I$. Here, $\mathbf{e}_1 + 2\mathbf{e}_2 \in I$,i.e. $L_2 = 1$.

5 Non-colinear Invariance Vectors

We turn now to the case where non-colinear multiplicities occur. Basically, we show that all non-degenerated invariance vectors are generated by a 2-set, as well as a minimality property for the associated linear generators. We first state two technical lemma:

Lemma 30. *Let $\{\mathbf{u},\mathbf{v}\}$ be a non-degenerated couple. Then T contains at least one fundamental parallelogram $FP_{\mathbf{u},\mathbf{v}}$. If T contains two of them, $FP^1_{\mathbf{u},\mathbf{v}}$ and $FP^2_{\mathbf{u},\mathbf{v}}$, any point in $FP^1_{\mathbf{u},\mathbf{v}}$ has a p-valid $(\mathbf{u},\mathbf{v})$-path to $FP^2_{\mathbf{u},\mathbf{v}}$. Additionally, if $dir(\mathbf{u}) \neq dir(\mathbf{v})$, then any point in p has a p-valid $(\mathbf{u},\mathbf{v})$-path to $FP_{\mathbf{u},\mathbf{v}}$.*

Lemma 31. *Let p be a pattern admitting a non-degenerated couple of invariance vectors $\mathbf{u}$ and $\mathbf{v}$ in different directions. Then exists a primitive couple of invariance vectors $(\mathbf{E},\mathbf{F})$ such that any non-degenerated invariance vector belong to $L_{\mathbf{E},\mathbf{F}}$.*

Proof. We first prove the additive property:

$$\forall (l,j) \in N^2 : \ \psi^{j+l}(C) - [\psi^l(C) + [\psi^j(C) - C]] \in L_{\mathbf{u},\mathbf{v}} \tag{35}$$

where C is the origin of the parallelogram. Let us set $\mathbf{w} = \psi^j(C) - \psi(C)$, and rewrite below, where computations are made modulo the elements of $L_{\mathbf{u},\mathbf{v}}$:

$$\begin{aligned}\psi^{j+l}(C) - \psi^l(C) &= \psi(\psi^{l-1}(C+\mathbf{w})) - \psi(\psi^{l-1}(C)) = \psi^{l-1}(C+\mathbf{w}) + \mathbf{e}_1 - (\psi^{l-1}(C) + \mathbf{e}_1)\\ &= \psi^{l-1}(C+\mathbf{w}) - \psi^{l-1}(C) \qquad = \ldots\\ &= C + \mathbf{w} - C \qquad\qquad\qquad = \psi^j(C) - C\end{aligned}$$

We note $S = \prod_{j=1}^{|FP_{\mathbf{u},\mathbf{v}}|} p[\psi^j(C)]$, and s the associated primitive word. Then we have:

$$\forall l \in N \quad p[\psi^{|s|+l}(C)] = p[\psi^l(C)] \ . \tag{36}$$

Let us show that the *set of invariance vectors in $FP_{\mathbf{u},\mathbf{v}}$* is the set $\{\mathbf{w}_i\}$ defined as:

$$\mathbf{w}_i = \psi^{i|s|}(C) - C = \psi^{|s^i|}(C) - C \ . \tag{37}$$

Let $\mathbf{w} = \mathbf{w}_i$. For any point α such that $\alpha + \mathbf{w} \in p$, we have, by $\mathbf{u}$ and $\mathbf{v}$-invariance and Lemma 30:

$$\begin{aligned}p[\alpha + \mathbf{w}] &= p[\phi(\alpha + \mathbf{w})]\\ p[\alpha] &= p[\phi(\alpha)]\end{aligned}$$

As $\phi(\phi(\alpha)+\mathbf{w})=\phi(\alpha)$, it is enough to prove $\mathbf{w}$-invariance for points α in $FP_{\mathbf{u},\mathbf{v}}$. Defining l by equation:$\alpha=\psi^{l}(C)$, we get, by $\phi(\alpha+\mathbf{w})=\psi^{|s|+l}(C)$. Hence, using 36, we get $p[\alpha+\mathbf{w}]=p[\psi^{l}(C)]=p[\alpha]$.

Applying Theorem 21, all linear combinations of $\mathbf{w}_i$, $\mathbf{u}$ and $\mathbf{v}$ also are.

Now, let us consider a basis $(\mathbf{E},\mathbf{F})$ of the lattice generated by $\{\mathbf{w}_i\}$ and show that it is primitive. By $\mathbf{u},\mathbf{v},\mathbf{E}$ and $\mathbf{F}$-invariance, we have:

$$p[\phi_{\mathbf{u},\mathbf{v}}(x)]=p[\phi_{(\mathbf{E},\mathbf{F})}(x)]$$

By 35, we also have $\mathbf{u},\mathbf{v}\in L_{\mathbf{E},\mathbf{F}}$. Hence, $|FP_{\mathbf{u},\mathbf{v}}|=\lambda|FP_{\mathbf{E},\mathbf{F}}|, \lambda\in Z$ and:

$$\prod_{1}^{|FP_{\mathbf{u},\mathbf{v}}|} p[\psi^{j}_{\mathbf{u},\mathbf{v}}(C)]=\prod_{1}^{\lambda|FP_{\mathbf{E},\mathbf{F}}|} p[\psi^{j}_{\mathbf{E},\mathbf{F}}(C)]=[\prod_{1}^{|FP_{\mathbf{E},\mathbf{F}}|} p[\psi^{j}_{\mathbf{E},\mathbf{F}}(C)]]^{\lambda}$$

which implies,

$$\prod_{1}^{|FP_{\mathbf{E},\mathbf{F}}|} p[\psi^{j}_{\mathbf{E},\mathbf{F}}(C)]\in s^{*}$$

Now, $|FP_{\mathbf{E},\mathbf{F}}|=k>|s|$ implies that $\psi^{k}(C)$ has an image interior to $FP_{\mathbf{E},\mathbf{F}}$. As $(\mathbf{E},\mathbf{F})$ Z-generates $\{\mathbf{w}_i\}$, this is a contradiction.

Finally, let us consider the case $u_2\wedge v_2=\delta$, with $\delta\neq 1$. Then σ is the product of δ cycles of equal length $\max(u_2,v_2)/\delta$. This defines δ words $\prod_p\psi^{p}(D^{j})$, where $D^{j}=\phi_{\mathbf{u},\mathbf{v}}(C+j\mathbf{e}_1)$. If the associated primitive words s_j satisfy $\theta=\vee|s_j|\leq|FP_{\mathbf{u},\mathbf{v}}|/\delta$, they have θ as a common period, and the set of invariance vectors in $FP_{\mathbf{u},\mathbf{v}}$ is $\{\mathbf{w}_i=(C,\psi^{i\theta}(C))\}$. If not, no invariance vector lies inside the parallelogram, and the lattice basis is $(\mathbf{u},\mathbf{v})$.

We are now ready to give an analytic expression of the border.

Definition 32. Let $(\mathbf{E},\mathbf{F})$ be any primitive couple. Assme, say, $dir(\mathbf{E})=dir(\mathbf{E}+\mathbf{F})=i$. Denote $\delta_i=\mathbf{E}_i\wedge\mathbf{F}_i$, $\sigma=|FP_{\mathbf{E},\mathbf{F}}|$. We define

$$Q=\{(-1)^{i-1}\frac{\sigma}{\delta_i}\mathbf{e}_i+\lambda\mathbf{F})\}_{\lambda\in Z}\cup\{(-1)^{i-1}\frac{\sigma}{\delta_i}\mathbf{e}_i+\lambda\bar{\mathbf{F}})\}_{\lambda\in Z} \tag{38}$$

where $\bar{\mathbf{F}}$ is the symmetric of $\mathbf{F}$ with respect to $\mathbf{e_i}$. Then:

$$\begin{aligned}B_{\mathbf{E},\mathbf{F}}&=(\cap_{\mathbf{x}\in Q}S^{+}_{\mathbf{x}})\cup(\cap_{\mathbf{x}\in Q}S^{-}_{\mathbf{x}})\\&=\cup_{\mathbf{x}\in Q}\{\mathbf{w}\in T;(C^{i+2^{d-1}}-C^{i})-\mathbf{x}<\mathbf{w}\}\end{aligned}$$

Lemma 33. *Let $\mathbf{w}$ be some translation vector, $(\mathbf{E},\mathbf{F})$ be any primitive couple, $FP_{\mathbf{E},\mathbf{F}}$ be any fundamental parallelogram. Then:*

$$\phi_{(\mathbf{E},\mathbf{F})}(P\cap P_{\mathbf{w}})\neq FP_{(\mathbf{E},\mathbf{F})}\Leftrightarrow\mathbf{w}\in B_{\mathbf{E},\mathbf{F}}\ .$$

Proof. No vector $\mathbf{w}$ in $B_{\mathbf{E},\mathbf{F}}$ satisfies $\phi(\mathbf{w})=\phi((C^{i+2^{d-1}}-C^{i})-(-1)^{i-1}\frac{\sigma}{\delta_i}\mathbf{e_i})$.

We get as an immediate corollary the main result of this section.

Theorem 34. *Let p be a pattern admitting a non-degenerated couple of invariance vectors $\mathbf{u}$ and $\mathbf{v}$ in different directions. Then exists a unique fundamental basis $(\mathbf{E}, \mathbf{F}) \in I^2$ satisfying :*

$$\begin{cases} \tilde{I} & = L_{\mathbf{E},\mathbf{F}} \cap T \\ I - \tilde{I} \subseteq B_{\mathbf{E},\mathbf{F}} & . \end{cases} \tag{39}$$

Moreover, $(\mathbf{E}, \mathbf{F})$ is p-primitive. Such a basis is said the canonical basis.

Remark. The unicity properties follow from the unicity conditions for fundamental basis defined for elliptic functions. Properties of such a basis, and algorithms to derive them, may be found in [Lan87, Val91]. Intuitively, $(\mathbf{E}, \mathbf{F})$ represents the "smallest" invariance vectors, and defines set of sequences $S_{\mathbf{E}}$ and $S_{\mathbf{F}}$ that repeat indefinitely in the pattern.

Proof. It is enough to show that an invariance vector $\mathbf{w}$ is in $L_{(\mathbf{E},\mathbf{F})}$. More precisely, $p \cap p_{\mathbf{W}}$ contains the linear generator $\{s_j\}$.

From our assumptions, exists C in $P \cap P_{\mathbf{W}}$ that relies on the lattice. We work on the $\phi_{\mathbf{E},\mathbf{F}}$images in $FP_{\mathbf{E},\mathbf{F}}$. We define (l, j) by: $C + \mathbf{w} = \psi^l(C)$ and $j|s| \leq l < (j+1)|s|$, and we may rewrite:

$$\prod_{j|s|}^{k+|s|-1} \psi^i(C) = \begin{cases} \prod_{j|s|}^{(j+1)|s|-1} \psi^i(C) \prod_{(j+1)|s|}^{k+|s|-1} \psi^i(C) = s.a \\ \prod_{j|s|}^{k-1} \psi^i(C) \prod_{k}^{k+|s|-1} \psi^i(C) \qquad = a.s \end{cases} \tag{40}$$

where a is the prefix of s of length $k - j|s|$. By Conjugacy Theorem [Lot83] and s primality, a is some power of s, hence is the empty word. Hence, $\mathbf{w}$ relies on $L_{\mathbf{E},\mathbf{F}}$.

Note 35. Reasoning as in [GO81], one may prove that one can always exhibit a binary linear generator such that $\mathbf{w}$ is an invariance vector.

Corollary 36. *In each direction, the set of linear generators associated to non-degenerated invariance vectors admits a minimal element: the linear generator associated to the primitive fundamental basis.*

We provide below a few easy results when basic vectors are not simple, that we believe useful for algorithmic purposes.

Proposition 37. *With the notations above , if $\mathbf{E}$ and $\mathbf{F}$ are non-simple, then:*

$$G = \{(\mathbf{E}, \mathbf{F})\} \ .$$

Notably, $((\mathbf{E}, \mathbf{F}))$ is unique, I has no simple vector in the direction of $\mathbf{E}$ and $\mathbf{F}$.

If a simple vector $\mathbf{E}$ belongs to such a basis, it belongs to any other fundamental basis.

Proof. There is no simple vector, as such a vector would be interior to $FP_{(\mathbf{E},\mathbf{F})}$. Also the condition $\mathbf{E}+\mathbf{F} \in P$ implies that one and only one among the two has its second (respectively first) coordinate greater than $\frac{m}{2}$. Then, choose $\mathbf{Z}$ such that $\mathbf{Z}_1$ be maximal. As:

$$\left\{\begin{array}{l} \mathbf{F}_1 \leq \mathbf{Z}_1 \\ \mathbf{F}_2 \leq \frac{m}{2} \\ \mathbf{E}_2 \geq \frac{m}{2} \end{array}\right\} \mathbf{F} \in FP_{\mathbf{E},\mathbf{Z}}$$

which is a contradiction.

A similar reasoning yields the unicity of a simple vector.

We turn now to the case of an invariance couple with vectors in the same direction.

Theorem 38. *Let p be a pattern admitting a non-degenerated couple $(\mathbf{u},\mathbf{v})$ such that $dir(\mathbf{u}) = dir(\mathbf{v})$. Then, exists a unique fundamental basis $(\mathbf{E},\mathbf{F})$ such that:*

(i) the set of non-degenerated invariance vectors satisfies:

$$\tilde{I} \subseteq L_{\mathbf{E},\mathbf{F}} \cap T \ .$$

More precisely, exists $(\alpha_1, \alpha_2) \in N^{*2}$ such that:

$$\tilde{I} = \{\mathbf{u} \in L_{(\mathbf{E},\mathbf{F})}\} \cap T_{\alpha_1,\alpha_2} \ . \tag{41}$$

where $T_{\alpha_1,\alpha_2} = \{\mathbf{u} \in T; |u_1| \geq \alpha_1, |u_2| \geq \alpha_2\}$.

(ii) The set of degenerated invariance vectors is a subset of $B_{\mathbf{E},\mathbf{F}}$.

Moreover, $(\mathbf{E},\mathbf{F})$ satifies property 33 for any fundamental parallelogram included in $T - T_{\alpha_1,\alpha_2}$.

Proof. As $\mathbf{u}+\mathbf{v} \in T$, P contains fundamental parallelogram $FP_{\mathbf{u},\mathbf{v}}$ in the origin, and we may define ψ on it. For a sake of clarity, assume again that $\delta = 1$. The word $S = \prod_j \psi^j(C)$ defines again a single primitive word s, and we get as before to a canonical basis $(\mathbf{E},\mathbf{F})$. Remark that $\mathbf{E}$ and $\mathbf{F}$ are not invariance vectors on P, but only out of $F_{\mathbf{u}-\mathbf{v}}$. Still, weaker property (i) can be established. We know from the reasoning above that non-degenerated vectors lie on $L_{\mathbf{E},\mathbf{F}}$ while degenerated ones belong to $B_{\mathbf{E},\mathbf{F}}$. Now, 41 is trivially achieved with the assumptions of the previous theorem with $\alpha_1 = \alpha_2 = 0$. Assume now that $\mathbf{u}-\mathbf{v} \notin I$. The s repetition must be interrupted somewhere at some points in $F_{\mathbf{u}-\mathbf{v}}$. Then, any point $\mathbf{w} \in L_{\mathbf{E},\mathbf{F}}$ is not in I iff $P_{\mathbf{w}}$ includes such a point. $F_{\mathbf{u}-\mathbf{v}}$ is made of two non-connex parts adjacent to segments $< C^1, C^2 >$ or $< C^3, C^4 >$. We set α_1 (respectively α_2) the highest first (respectively second) coordinate of such points in the first (respectively second) part.

Note 39. As stated in [AB92], all basic sources "radiate" from the same corner. The equidistribution appearing on the figures of that work is now precised. Only points falling in a canonical segment (bolded characters in Figure 5) define actual periodicities. The other ones (italic characters) do not, due to the intersection of the shifted copies so-defined with the free zone. In our Figure, all points on that segment define periodicities, but this is not general).

b	g	c	d	e	f	g	h	a	b	c	d	e
c	d	b	c	d	e	f	g	h	a	b	c	d
g	h	*a*	b	c	d	e	f	g	h	a	b	c
f	g	h	a	b	c	d	e	f	g	h	a	b
e	f	g	h	a	b	c	d	e	f	g	h	a
d	e	f	g	h	a	b	c	d	e	f	g	h
c	d	e	f	g	h	a	b	c	d	e	f	g
b	c	d	e	f	g	h	*a*	b	c	a	b	g
a	b	c	d	e	f	g	h	*a*	b	a	c	d

Fig. 5. Radiant biperiodic pattern

Note 40. Again, the set of linear generator admits, in each direction, a minimum. This minimum does not necessarily belong to the set. This minimal periodicity search appears very closed to the problem of finding the generator of the intersection of ideals over Z.

Note 41. In an independent work, [GP92] proved point (i) of Theorem, when exists a basis of simple invariance vectors.

6 Colinear Invariance Vectors

From the definition of an invariance vector, it appears that $S_{\mathbf{u}}^{+}$ (or $S_{\mathbf{u}}^{-}$) is a generator of the pattern, as:

$$\mathbf{w} \in P \Leftrightarrow \exists \mathbf{t} \in S_{\mathbf{u}}^{+}, \exists k \in Z : \mathbf{w} = \mathbf{t} + k\mathbf{u} \ . \tag{42}$$

Then, $p(\mathbf{w}) = p(\mathbf{t})$. A similar property holds for $S_{\mathbf{u}}^{-}$. We want a *minimal representation* of generators. We define characteristic parameters for a set of colinear invariance vectors. Then, in Theorem 43, we characterize positions of sources. We prove additional constraints and results in Corollary Finally, in Proposition 46, we characterize border effects.

Then, we address the case when two non-colinear invariance vectors exist, and show that the generating strip reduces to a (linear) word. We state below the associated definitions:

Definition 42. Let Δ be a set of colinear invariance vectors. A vector $\mathbf{v}$ in Δ is said minimal iff the only solution in I of

$$v = k\mathbf{w}, k \in Z \tag{43}$$

is $\mathbf{v}$ itself.

The *pseudo-period* associated to Δ is

$$\mathbf{u} = GCD\{\mathbf{v}; \mathbf{v} \in \Delta\} = GCD\{\mathbf{v}; \mathbf{v} \in \Delta, \mathbf{v}\ minimal\} \tag{44}$$

The *quasi-period* associated to Δ is

$$\mathbf{u} = GCD\{\mathbf{v}; \mathbf{v} \in \Delta\} = GCD\{\mathbf{v}; \mathbf{v} \in \Delta, \mathbf{v} \text{ minimal } and \text{ simple}\} \tag{45}$$

A vector $\mathbf{u}$ is the *period* of Δ iff

$$\{k\mathbf{u}; k \in Z\} \cap T = \Delta \tag{46}$$

Theorem 43. *Let Δ be a set of colinear vectors. Then any quasi-period is a multiple of the pseudo-period $\mathbf{u}$. Two cases may occur:*

(i) Δ has some period $\mathbf{u}$. This case occurs iff it has only one minimal element, $\mathbf{u}$. This period also is the pseudo-period as well as the unique quasi-period.
(ii) Δ is quasi-periodic, but has no period. This occurs iff Δ has several minimal elements.

Then, exists $\lambda_1 < \ldots < \lambda_k = \lambda$, a sequence (δ_i) and a set $\{\mu_i\}$ such that:

$$\begin{cases} \lambda_k \mathbf{u} & \text{are simple} \\ \mu_k \mathbf{u} & \text{are not simple} \\ GCD(\lambda_1, \ldots, \lambda_i) = \delta_i & \\ \Delta = \{\nu\lambda_1\mathbf{u}; \nu \in N^*\} \cup_{i=2}^{k} \{(\lambda_i + \nu\delta_i)\mathbf{u}; \nu \in N^*\} \cup \{\mu_i \mathbf{u}\} & , \end{cases}$$

and δ_k is the quasi-period.

Proof. Let us prove (i). It is clear that condition 46 implies the unicity of the minimal element. The reverse is true, as, from Lemma vectors also are in I. Also such $\mathbf{u}$ satisfies pseudo-period definition, as well as quasi-period definition.

Let us prove (ii). Assume now that exist two minimal elements. Note $\mathbf{w} = GCD\{\mathbf{v}; \mathbf{v} \in \Delta, \mathbf{v} \text{ simple}\}$. Minimal elements are defined as a sequence $(\lambda_i)\mathbf{w}$. Let $\lambda_1 = \min(\lambda_i)$ and $\lambda_2 = \min_{\lambda_i > \lambda_1} \lambda_i$. Then for any $k \in Z$, $\lambda_2\mathbf{w} + k(\lambda_2 - \lambda_1)\mathbf{w} \in I$. To prove it, consider any point α such that $\alpha + \lambda_2\mathbf{u} + k(\lambda_2 - \lambda_1)\mathbf{u} \in p$. Then, by convexity, we have:

$$\alpha + \lambda_2\mathbf{u} + (k-1)(\lambda_2 - \lambda_1)\mathbf{u} \in p, \alpha + (k-1)(\lambda_2 - \lambda_1)\mathbf{u} \in p$$

Hence:

$$\begin{aligned} p[\alpha + \lambda_2\mathbf{u} + k(\lambda_2 - \lambda_1)\mathbf{u}] = p[\alpha + k(\lambda_2 - \lambda_1)\mathbf{u}] &= p[\alpha + \lambda_2\mathbf{u} + (k-1)(\lambda_2 - \lambda_1)\mathbf{u}] \\ = \ldots \qquad &= p[\alpha + \lambda_1\mathbf{u}] = p[\alpha] \end{aligned}$$

Remark. In 2D, the pseudo-period is not always a period.

Corollary 44. *Given a set $(\lambda_i\mathbf{u})$ of colinear invariance vectors, the sequence $(|\lambda_i - \lambda_{i1}|)$ is decreasing.*

It is also of interest to determine conditions, notably arithmetic conditions, on colinear invariance vectors in order to have a period. We first have, from the previous section:

Corollary 45. *Assume p is biperiodic. Let Δ be a set of colinear invariance vectors.*

(i) If p is lattice periodic, then the quasi-period is in I.
(ii) If p is radiant periodic, then either the quasi-period is in I, or

$$\Delta = \{(\lambda + \nu\delta)\mathbf{u}; \nu \in N\} \cap \{\mu_i\mathbf{u}\}$$

Finally, we state:

Proposition 46. *Let $\Delta_{\mathbf{u}}$ be the set of invariance vectors colinear to some vector* $\mathbf{u}$. *Let $(\lambda_i \mathbf{u})_{1 \leq i \leq k}, k \geq 2$ be a subset of $\Delta_{\mathbf{u}}$ such that:*

$$\begin{cases} GCD(\lambda_i) & = r \\ (\lambda_k + \lambda_{k-1} - 1)\mathbf{u} \in T \end{cases} \tag{47}$$

Then $r\mathbf{u}$ *is an invariance vector out of* $DZ_{(\lambda_k+\lambda_{k-1}-1)\mathbf{u}}$.

Proof. Let $\mathbf{w}$ be a translation vector. Assume $\mathbf{w} \notin DZ_{(\lambda_k+\lambda_{k-1}-1)\mathbf{u}}$, say $\mathbf{w} + (\lambda_k + \lambda_{k-1} - 1)\mathbf{u} \in T$. We define the mapping $\sigma : \{0, \ldots, (\lambda_k + \lambda_{k-1} - 1)\} \rightarrow \{0, \ldots, (\lambda_k + \lambda_{k-1} - 1)\}$ by $\sigma(j) = j + \lambda_{k-1} \pmod{\lambda_k}$. σ is a cycle of length $\lambda_k + \lambda_{k-1}$. More precisely, let $l = (\lambda_k + \lambda_{k-1})/\lambda_k - 1$. Then, for any $i \in [0 \ldots r-1]$:

$$\{\sigma^{il+j}; 0 \leq j < r\} = \{n; 0 \leq n \leq (\lambda_k+\lambda_{k-1}-1), n \pmod{\lambda_{k-1}} = -i\lambda_k \pmod{\lambda_{k-1}}\}$$

Hence, we have a p-valid $(\lambda_{k-1}\mathbf{u}, \lambda_k\mathbf{u})$-path form $\mathbf{w}$ to himself including $\{\mathbf{w} + \lambda r\mathbf{u}\}_{0 \leq \lambda \leq (\lambda_k+\lambda_{k-1}-1)r}\mathbf{u}$, which achieves the proof.

Corollary 47. *If condition 47 is satisfied by the set of minimal elements, then* $r = 1$ *and* $\mathbf{u}$ *is a quasi-period of* $\Delta_{\mathbf{u}}$.

To summarize, we have been able to characterize the set of sources in the general case, and to point out the simplification that occurs when exists a non-degenerated invariance couple. Also, we have characterized the borders -i.e. dead zones- that prevent form the existence of a period. Corollary 44 is also proved in [GP92].

7 Periodicities and Coverings Classification

We are now ready to give our classification of periodicities and coverings.

Definition 48. We classify periodicities in three classes. A pattern is:

(1) **non-periodic:** neither non-degenerated nor simple invariance vectors.
(2) **monoperiodic:** all invariance vectors are degenerated, and exists one simple vector. Let $\mathbf{u}$ be the pseudoperiod.
(3) **biperiodic:** exist one non-degenerated invariance couple. It divides into two subclasses:
   (a) **lattice periodic:** p admits a fundamental basis of invariance vectors
   (b) **radiant periodic:** all non-degenerated invariance vectors are in the same direction.

We will discuss in 8 the relationship to the definition the relationship to [AB92] classification. *Lattice, radiant and line periodic* patterns as defined in [AB92] are *lattice biperiodic , radiant biperiodic and monoperiodic* according to our classification. Our *non-periodic* patterns form a strict subset of [AB92] non-periodic patterns. Also, when p has a unique invariance couple, $(\mathbf{u}, \mathbf{v})$, none being simple, it is biperiodic in our classification. Depending of the invariance or not of $\mathbf{u}-\mathbf{v}$, it is either lattice or radiant periodic.

But our proposed classification takes into account all periodicities (e.g includes degenerated periodicities), which is essential to a space covering classification. Also, it allowed a full characterization of the relative position of sources (see above). We consider three classes instead of the 4 in [AB92, GP92]. As a matter of fact, lattice and radiant periodicities share the major property of the existence of a non-degenerated couple, with, consequently, a minimal linear generator. Difference appear minor, i.e. proceed from border effect. Also, this reasoning allows to classify biperiodic patterns with non simple vectors in the fundamental basis. Finally, this generalizes to higher dimensions.

To study the maximal coverings, we will need a second classification. That is:

Definition 49. A pattern is k-self overlapping if it may overlap with himself on exactly k directions.

As integer k ranges in $[0, 2^{d-1}]$, this determines $2^{d-1}+1$ classes. In dimension 2, more precisely studied in this paper, we get:

Definition 50. We classify 2-dimensional patterns in 3 disjoint classes:

(1) A pattern is *non-overlapping* iff it has no invariance vector.
(2) A pattern is *mono-overlapping* if all its invariance vectors lie in the same direction.
(3) A pattern is *bi-overlapping* iff exist invariance vectors in both directions.

8 Maximal Coverings Classification

We classified the possible coverings as a function of the number of directions where the pattern p may overlap. Also, all periods, including the "degenerated" ones are to be taken into account. Our study of maximal coverings relies on the fundamental notion of *mutual consistency* of overlapping copies of p.

Definition 51. Given a pattern p and two vectors $\mathbf{u}, \mathbf{v}$ in P, two sets $P_{\mathbf{u}}$ and $P_{\mathbf{v}}$ are said mutually consistent iff:

$$\mathbf{w} \in P_{\mathbf{u}} \cap P_{\mathbf{v}} \Rightarrow p(\mathbf{w}-\mathbf{u}) = p(\mathbf{w}-\mathbf{v}) \tag{48}$$

Intuitively, this means that whenever a copy of p is shifted by $\mathbf{u}$ and $\mathbf{v}$, the two copies can coincide in the text. Remark that, whenever $P_{\mathbf{u}} \cap P_{\mathbf{v}} = \emptyset$, $P_{\mathbf{u}}$ and $P_{\mathbf{v}}$ are mutually consistent. We get the following lemma, easy to prove, but fundamental for algorithmic issues [BYR90, NR92]:

Lemma 52. *Given a pattern p, let $\mathbf{u}$ and $\mathbf{v}$ be two invariance vectors. $P_{\mathbf{u}}$ and $P_{\mathbf{v}}$ overlap iff $\mathbf{u}-\mathbf{v} \in T$. Also, two overlapping sets $P_{\mathbf{u}}$ and $P_{\mathbf{v}}$ are mutually consistent iff $\mathbf{u}-\mathbf{v}$ is an invariance vector.*

Proof. It is enough to remark that two copies of p shifted by $\mathbf{u}$ and $\mathbf{v}$ overlap between themselves as p and a copy shifted by $\mathbf{u}-\mathbf{v}$. That is, mutual consistency is equivalent to have $\mathbf{u}-\mathbf{v}$ as an invariance vector, or to have $P_\mathbf{u} \cap P_\mathbf{v} = \emptyset$.

The following theorem steadily follows:

Theorem 53. *Let $P, P_\mathbf{u}, P_\mathbf{v}$ be three overlapping copies of a pattern. They are mutually consistent in either one of the three cases (and no other):*

(a) p is lattice periodic. *Then* $\mathbf{u}$ *and* $\mathbf{v}$ *lie on the canonical lattice with its origin in a corner of* p.
(b) p is radiant periodic. *Then* $\mathbf{u}$ *and* $\mathbf{v}$ *form a monotonic $\tilde{I}$-path.*
(c) p is monoperiodic. *Then,* $\mathbf{u}, \mathbf{v} \in L_{\alpha\mathbf{G},\beta\mathbf{G}}$ *where* $\mathbf{G}$ *is the pseudo-period, and* $\alpha\mathbf{G}, \beta\mathbf{G}$ *are two minimal elements.*

We first state the definitions:

Definition 54. Given S a set of vectors in the same direction, all positive or all negative, a S-overlapping sequence is a set of copies $(p_i)_{i\in I}$ of p, such that two copies p_i and p_{i+1} are shifted by some vector $\mathbf{x}$ in S.

A S-diagonal covering is a tiling of S-overlapping sequences.

A $(\mathbf{u}, \mathbf{v})$-lattice covering is a set of interleaved $\mathbf{u}$-overlapping sequences where two neighbouring sequences are shifted by $\mathbf{v}$. Given a pattern p, it is *regular* if $\mathbf{u}+\mathbf{v} \in T$, else it is said *extended.*

These coverings are presented in Figure 8 (a), (b), (c) and (d).
It follows from Theorem 53:

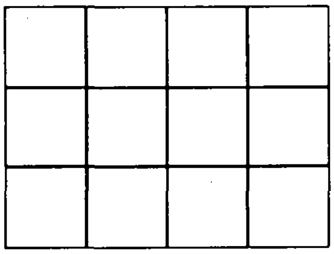
Theorem 55. *A maximal covering of the 2-dimensional space by a pattern p is either of the three following.*

(1) tiling,
(2) S-diagonal covering, *where S is a set of invariance vectors.*
(3) $(\mathbf{u}, \mathbf{v})$*-lattice coverings, where* $(\mathbf{u}, \mathbf{v})$ *is a basis of the canonical lattice.*

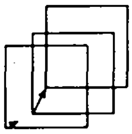
Additionnally, p has a maximal covering in class I iff p is in overlapping class I. Also, only lattice periodic patterns have regular lattice coverings. Finally, elements in S satisfy minimality condition: $|P_\mathbf{x} \cap P| = \min\{|P_\mathbf{u} \cap P|\}$.

Remark that extended lattice coverings are an extension of the covering notion, where some "holes" appear in the representation. Nevertheless, this is pertinent for algorithmic issues as it allows to determine the **maximum** number of occurrences of a given pattern. (This number may not be given by the maximal covering, when degeneracies appear). This parameter is clearly related to the worst-case complexity. This is extensively studied in a companion paper [NR92]. Notably, one proves a complexity $1.n$ for non-overlapping patterns and a complexity $\alpha.n, \alpha > 1$ for bi-overlapping patterns. We believe that the refinements of periodicity classification presented here will allow to extend the results to mono-overlapping patterns.

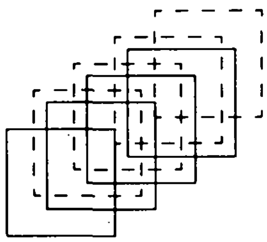
Finally, note this covering classification follows from our overlapping classification in Definition 50, while it is "orthogonal" to the one in [AB92], and to our



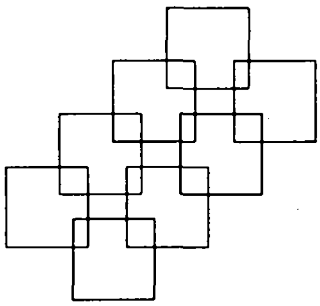
(a) Tiling.



(b) Diagonal covering.



(c) Regular lattice covering.



(d) Extended lattice covering.

Fig. 6. Four classes of coverings

periodicities classification as well. Class 3 contains patterns with periods in two directions. A radiant periodic pattern (periodicity class (3)(b)) with its pseudoperiods in some direction i lies either in class (3) (with an extended lattice covering) or in class (2). This depends of the existence (or not) of a degenerated invariance vector in the other direction. In any case, its maximal covering is a diagonal covering (class (2)).

9 Conclusion

We exhibited here the relationship between the periods of a pattern and the possible space coverings by the same pattern. This is relevant both to the derivation of the theoretical complexity of d-dimensional pattern matching and to algorithmic issues. Notably, combining the duel paradigm with knowledge on periods should improve average and worst-case complexity of pattern matching. This is treated in a companion paper. We proved here that a periodic pattern is generated by a subpattern, and exhibit the subpattern as well as the generating law. This considerably refines and achieves the previous classification by [AB92] and allows for a classification of space coverings. Additionnally, it provides tools for a generalization to any dimension. Notably, the number of periodicity classes appear linear in the dimension. Finally, it provides knowledge to derive efficient pattern preprocessing.

Acknowledgements

We wish to thank Pierre Nicodème, Xavier Gourdon and Gaston Gonnet for many fruitful discussions.

References

[AB92] A. Amir and G. Benson. Two-dimensional periodicity and its application. In *SODA'92*, 1992. Proc. 3-rd Symposium on Discrete Algorithms, Orlando,FL.

[ABF92] A. Amir, G. Benson, and M. Farach. Alphabet independent two dimensional matching. In *STOC'92*, pages 59–67, 1992. Victoria,BC.

[BYR90] R. Baeza-Yates and M. Régnier. Fast algorithms for two dimensional and multiple pattern matching. In *SWAT'90*, volume 447 of *Lecture Notes in Computer Science*, pages 332–347. Springer-Verlag, 1990. Preliminary draft in Proc. Swedish Workshop on Algorithm Theory, Bergen, Norway. To appear in IPL.

[CH92] R. Cole and R. Hariharan. Tighter Bounds on the Exact Complexity of String Matching. In *FOCS'92*. IEEE, 1992. Proc. 33-rd IEEE Conference on Foundations of Computer Science, Pittsburgh, USA.

[Gal92] Z. Galil. Hunting Lions in the Desert Optimally or a Constant-Time Optimal Parallel String-Matching Algorithm. In *STOCS'92*, 1992. Victoria,BC.

[GO81] L. Guibas and A.M. Odlyzko. Periods in Strings. *Journal of Combinatorial Theory*, Series A, 30:19–42, 1981.

[GP92] Z. Galil and K. Park. Truly Alphabet Independent Two-Dimensional Pattern Matching. In *FOCS'92*. IEEE, 1992. Proc. 33-rd IEEE Conference on Foundations of Computer Science, Pittsburgh, USA.

[Lan87] S. Lang. *Elliptic Functions*. Springer-Verlag, New-York, 1987.

[Lot83] Lothaire. *Combinatorics on Words*. Addison-Wesley, Reading, Mass., 1983.

[NR92] P. Nicodème and M. Régnier. Towards 2D Pattern Matching Complexity, 1992. in preparation.

[Rég89] M. Régnier. Knuth-Morris-Pratt algorithm: an analysis. In *MFCS'89*, volume 379 of *Lecture Notes in Computer Science*, pages 431–444. Springer-Verlag, 1989. Proc. Mathematical Foundations for Computer Science 89, Porubka, Poland.

[Val91] B. Vallée. Gauss Algorithm Revisited. *JALG*, 12:556–572, 1991.

[Vis85] U. Vishkin. Optimal Parallel Pattern Matching in Strings. *Information and Control*, 67:91–113, 1985.

[ZT89] R.F. Zhu and T. Takaoka. A technique for two-dimensional pattern matching. *CACM.*, 32(9):1110–1120, 1989.

This article was processed using the LaTeX macro package with LLNCS style

Unité de Recherche INRIA Rocquencourt
Domaine de Voluceau - Rocquencourt - B.P. 105 - 78153 LE CHESNAY Cedex (France)
Unité de Recherche INRIA Lorraine Technopôle de Nancy-Brabois - Campus Scientifique
615, rue du Jardin Botanique - B.P. 101 - 54602 VILLERS LES NANCY Cedex (France)
Unité de Recherche INRIA Rennes IRISA, Campus Universitaire de Beaulieu 35042 RENNES Cedex (France)
Unité de Recherche INRIA Rhône-Alpes 46, avenue Félix Viallet - 38031 GRENOBLE Cedex (France)
Unité de Recherche INRIA Sophia Antipolis 2004, route des Lucioles - B.P. 93 - 06902 SOPHIA ANTIPOLIS Cedex (France)

EDITEUR
INRIA - Domaine de Voluceau - Rocquencourt - B.P. 105 - 78153 LE CHESNAY Cedex (France)

ISSN 0249 - 6399

RR_2021