Innovation generation in the presence of unknown inputs : application to robust failure detection

Ramine Nikoukhah

INRIA

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Innovation generation in the presence of unknown inputs : application to robust failure detection

Ramine NIKOUKHAH

N° 1914
Mai 1993

PROGRAMME 5

Traitement du signal,
automatique et
productique

Rapport de recherche

1993

Innovations generation in the presence of unknown inputs: application to robust failure detection

Ramine Nikoukhah

INRIA, Rocquencourt

The first step in innovations-based failure detection is the construction of an innovations generator, i.e., a filter which, in the absence of failures, from the inputs and the outputs of the system, generates a zero-mean white process with known covariance called innovations. Decision on whether a failure has occurred is then made by monitoring and applying statistical tests to this innovations process. In this paper, we present a method for constructing innovations in the case where the model contains unknown inputs and disturbances. Our solution is complete in the sense that it covers all "singular" cases.

Génération de l'innovation en présence des entrées inconnues et son application à la détection de pannes

Dans le cadre du problème de détection de pannes pour les systèmes linéaires, on considère la construction d'un générateur d'innovation, c'est à dire un filtre qui, en absence de panne, génère un signal centré, blanc avec une covariance connue à partir d'une connaissance partielle des entrées et des sorties du système. La solution qu'on propose est complète dans le sens où elle couvre tous les cas singuliers.

1 Introduction

In recent years, a large variety of methods have been proposed for solving the failure detection problem (see e.g. [1, 2, 3, 7, 9]). The problem of failure detection consists in detecting failures in a physical system by monitoring its inputs and outputs. If no failure has occurred, the inputs and outputs of the system follow a "normal behavior", and thus the problem of failure detection is one of testing this normality.

Theoretically, we could imagine constructing the set of inputs and outputs that constitute normal behaviors in which case, detecting failures reduces to testing whether or not the inputs and outputs of the system under consideration belong to this set. Construction of such a set of time trajectories however is not desirable (or even possible). Instead, the membership test is done by constructing a filter which has zero output if fed with inputs and outputs of the physical system that belong to the set of normal inputs and outputs and a non-zero output otherwise, i.e., when a failure has occurred. The output of this filter is usually referred to as residual and the filter, the residual generator.

In practice, the situation is somewhat more complicated. First of all, the residual generator is constructed from a deterministic model describing normal behavior of the physical system which is almost never exact; the model is usually a linearized version of a non-linear model (which itself is approximate) or is obtained by some identification algorithm. Furthermore, inputs and outputs of the physical system have to be measured which means that they are subject to measurement noises and furthermore, the physical systems can be affected by unmeasured disturbances. Because of all these unmodeled uncertainties, we cannot expect the residual to be exactly equal to zero even if no failure has occurred. The simplest way to get around this problem is to replace the test of nullness on the residual by some type of treshold test, i.e., we decide a failure has occurred if some function of the residual is larger than some specified treshold. This way, instead of associating lack of failure to normal behavior, we associate it with sufficient closeness to some normal behavior.

The problem with this simple approach is that the decision does not take into account any information that may be available concerning various noises, disturbances and model uncertainties. But often, statistical information can be obtained on the noises and disturbances and even on model uncertainties. To take into account this additional information, one can use a stochastic dynamical model to describe normal behavior of the physical system. This type of model can also be obtained from statistical input-output data when physical information is not available. The residual in this case becomes a stochastic process which, in the absence of failures, has some known statistical properties.

For linear models, the innovations process (associated with the Kalman filter) has been used as residual. Nice properties of the innovations process are that it is zero-mean (decoupled from the input) and it is white with known covariance (independent of the input). Once the innovations are generated, the decision problem on whether a failure has occurred reduces to a hypothesis testing problem on the whiteness of the residual (innovations) [6].

The drawback with the stochastic approach is its lack of robustness. Often physical systems are subject to unmeasured inputs and disturbances or have unkown parameters which cannot, in any reasonable way, be modeled as random processes or variables with known statistics. In such cases, we would like the decision on whether a failure has occurred to be insensitive to these unknowns. This means that the decision should not be based on any residual which is "corrupted" by these unknowns (the residual should be decoupled from the unknowns). Residual generation for deterministic models which contain unknown inputs have been extensively studied (see for example [3, 5]).

In this paper, we assume that the normal behavior of the physical system can be modeled by a dynamical system in which we allow both stochastic processes and unknown processes and consider the problem of residual generation for such systems. In particular, we suppose that the behavior of the system can be described by the linear, time-invariant stochastic model

$$\dot{x}(t) = Ax(t) + B_1u(t) + B_2v(t) + B_3w(t) + B_4f(t) \quad (1.1)$$

$$y(t) = Cx(t) + D_1u(t) + D_2v(t) + D_3w(t) + D_4f(t) \quad (1.2)$$

where u and y represent respectively known inputs and outputs, w, a zero-mean unit covariance white noise process, v, unknown inputs and disturbances, and f possible failures (with f set to zero, (1.1)-(1.2) describes the normal (no failure) behavior of the system).

The vector process v contains both unknown inputs and unknown disturbances. In particular,

$$\begin{pmatrix} B_2 \\ D_2 \end{pmatrix} v(t) = \begin{pmatrix} B_{21} & B_{22} \\ D_{21} & D_{22} \end{pmatrix} \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} \quad (1.3)$$

where v_1 represents unknown inputs, i.e., subset of inputs to which we do not have access for failure detection, and v_2, the unknown disturbances (nuisances) which may even include effects of other types failures which we do not intend to detect (we may be interested in detecting a special class of failures only: failure isolation). As far as we are concerned, there is no point distinguishing v_1 from v_2. What matters is that no a priori assumption is made concerning either of them which means that v can take any value at any time. So any projection of v added to any signal destroys the information in the signal (because the result is completely arbitrary) and thus it is clear that the residual that we generate should be completely decoupled from v.

We do not make the assumption that A is stable. The model could very well be meaningful even if A is unstable provided System (1.1)-(1.2) is placed inside a stabilizing feedback loop (guaranteeing that the mean and covariance of x remains bounded for a bounded v). We do assume however that $\left(A, \begin{bmatrix} B_1 & B_{21} \end{bmatrix}\right)$ is stabilizable; this condition is necessary for the existence of such a stabilizing feedback loop. We do not require detectability of (C, A) since y is not necessarily the output used for the feedback.

Our objective is to construct a filter, called an *innovations filter*, which from the observed quantities u and y generates an output decoupled from v, zero-mean and white (in the absence of failure). From here on we shall refer to this residual as innovations. We have chosen this terminology because, in the absence of the unknown process v, this residual coincides with the usual innovations.

The outline of the paper is as follows. In Section 2, we define and characterize the notion of innovations filter and in Section 3, we give a complete solution for its construction. The application of innovations filter to failure detection is examined in Section 4.

We limit our development to the continuous-time case but all the results presented in this paper can trivially be extended to the discrete-time case.

2 Innovations filter

Definition 2.1 *A finite-dimensional linear time-invariant system V is called an innovations filter for System (1.1)-(1.2) if it is stable with the least number of outputs such that, in the absence of failure,*

1- its output

$$\nu(t) = V \begin{pmatrix} u(t) \\ y(t) \end{pmatrix} \tag{2.1}$$

is zero-mean, white and decoupled from u and v,

2- if V' is any finite-dimensional linear time-invariant system such that

$$\mu(t) = V' \begin{pmatrix} u(t) \\ y(t) \end{pmatrix} \tag{2.2}$$

is decoupled from u and v, then there exists a linear system L such that

$$V' = LV. \tag{2.3}$$

Condition 2- guarantees that the innovations filter does not destroy any useful information contained in u and y; it kills off only parts that are corrupted by the unknown vector v, i.e., parts that contain no information at all.

Even though generically System (1.1)-(1.2) does have innovations filters, there are simple examples of systems that do not have innovations filters. We shall completely characterize this class of systems later in this paper.

Example Consider the following very simple system in the absence of failures which contains no known inputs u and no unknown inputs and disturbances v,

$$\dot{x}(t) = -x(t) + w(t) \tag{2.4}$$

$$y(t) = -x(t) + w(t). \tag{2.5}$$

It is straightforward to verify that there is no V satisfying 1- in Definition 2.1 but V' defined by

$$V' \begin{pmatrix} u(t) \\ y(t) \end{pmatrix} = y(t) \tag{2.6}$$

satisfies 2-. This implies that (2.4)-(2.5) does not have an innovations filter. This does not mean, of course, that we cannot detect failures by observing y, but simply that the steady-state innovations-based failure detection considered in this paper cannot be used in this case.

An immediate consequence of Definition 2.1 is that, representing innovations filters by their transfer functions as we shall do from here on, if $V_1(s)$ and $V_2(s)$ are two innovations filters for System (1.1)-(1.2), there exists an invertible rational matrix $\Omega(s)$ such that

$$V_1(s) = \Omega(s)V_2(s). \tag{2.7}$$

So that, even though the innovations filter is not unique, all innovations filters (for the same system) have the same row-image.

Lemma 2.1 *$V(s)$ is an innovations filter for System (1.1)-(1.2) if and only if $V(s)$ is a full row rank, stable rational matrix such that*

$$\textit{Row-Im } V(s) = \textit{Left-Ker} \begin{pmatrix} I & 0 \\ C(sI-A)^{-1}B_1 + D_1 & C(sI-A)^{-1}B_2 + D_2 \end{pmatrix}, \tag{2.8}$$

$$H(s) = V(s) \begin{pmatrix} 0 \\ C(sI-A)^{-1}B_3 + D_3 \end{pmatrix} \tag{2.9}$$

is stable and

$$\Psi = H(s)H^{\sim}(s) \tag{2.10}$$

is constant where $H^{\sim}(s) \triangleq H^T(-s)$.

Proof In the transfer domain (in the absence of failures, $f = 0$), the output y can be expressed as follows

$$y = (C(sI-A)^{-1}B_1 + D_1)u + (C(sI-A)^{-1}B_2 + D_2)v + (C(sI-A)^{-1}B_3 + D_3)w \tag{2.11}$$

which implies that

$$\begin{aligned} \nu &= V(s)\begin{pmatrix} u \\ y \end{pmatrix} \\ &= V(s)\begin{pmatrix} I & 0 \\ C(sI-A)^{-1}B_1 + D_1 & C(sI-A)^{-1}B_2 + D_2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + H(s)w. \end{aligned} \tag{2.12}$$

Suppose that $V(s)$ satisfies conditions of the lemma. Then clearly ν is zero-mean, white, with covariance Ψ. Thus $V(s)$ fulfills the first condition in Definition 2.1. To show that it also satisfies the second condition, note that $V'(s)$ must satisfy

$$V'(s)\begin{pmatrix} I & 0 \\ C(sI-A)^{-1}B_1 + D_1 & C(sI-A)^{-1}B_2 + D_2 \end{pmatrix} = 0 \tag{2.13}$$

which implies that

$$\text{Row-Im } V'(s) \subset \text{Left-Ker} \begin{pmatrix} I & 0 \\ C(sI-A)^{-1}B_1 + D_1 & C(sI-A)^{-1}B_2 + D_2 \end{pmatrix}. \tag{2.14}$$

But rows of $V(s)$ form a basis for

$$\text{Left-Ker}\begin{pmatrix} I & 0 \\ C(sI-A)^{-1}B_1+D_1 & C(sI-A)^{-1}B_2+D_2 \end{pmatrix}$$

which implies that there exists $L(s)$ such that

$$V'(s) = L(s)V(s). \tag{2.15}$$

Now suppose that $V(s)$ is an innovations filter, then from (2.12) we can see that

$$\text{Row-Im } V(s) \subset \text{Left-Ker}\begin{pmatrix} I & 0 \\ C(sI-A)^{-1}B_1+D_1 & C(sI-A)^{-1}B_2+D_2 \end{pmatrix}, \tag{2.16}$$

$H(s)$ is stable and $H(s)H^{\sim}(s)$ is constant. So all that remains to be shown is that the inclusion in (2.16) is an equality. For that, note that we can always choose a $V'(s)$ in the second part of Definition 2.1 such that

$$\text{rank } V'(s) = \text{rank Left-Ker}\begin{pmatrix} I & 0 \\ C(sI-A)^{-1}B_1+D_1 & C(sI-A)^{-1}B_2+D_2 \end{pmatrix}. \tag{2.17}$$

But, (2.3) implies that

$$\text{rank } V(s) \geq \text{rank } V'(s) \tag{2.18}$$

which thanks to (2.16) implies (2.8) and the lemma is proved. ■

3 Innovations filter construction

Theorem 3.1 *$V(s)$ is an innovations filter for System (1.1)-(1.2) if and only if*

$$V(s) = W(s)\begin{pmatrix} -B_1 & 0 \\ -D_1 & I \end{pmatrix} \tag{3.1}$$

where $W(s)$ is a stable full row rank rational matrix such that

$$\textit{Row-Im } W(s) = \textit{Left-Ker}\begin{pmatrix} -sI+A & B_2 \\ C & D_2 \end{pmatrix} \tag{3.2}$$

and

$$\Psi = H(s)H^{\sim}(s) \tag{3.3}$$

is constant where

$$H(s) = W(s)\begin{pmatrix} B_3 \\ D_3 \end{pmatrix}. \tag{3.4}$$

Proof Let $W(s)$ be a full row rank stable rational matrix satisfying (3.2)-(3.4), and let $V(s)$ be obtained from (3.1). We need to show that $V(s)$ is stable, has full row rank and satisfies (2.8)-(2.10). Clearly $V(s)$ is stable since $W(s)$ is stable. Let

$$W(s) = \begin{pmatrix} W_1(s) & W_2(s) \end{pmatrix} \tag{3.5}$$

then thanks to (3.2) we have that

$$W_1(s) = W_2(s)C(sI - A)^{-1} \tag{3.6}$$

which since $W(s)$ has full row rank implies that $W_2(s)$ has full row rank as well. But

$$V(s) = \begin{pmatrix} -W_1(s)B_1 - W_2(s)D_1 & W_2(s) \end{pmatrix} \tag{3.7}$$

which implies that $V(s)$ has full row rank.

$$\begin{aligned}
&\text{Row-Im } W(s) = \\
&\quad \text{Left-Ker} \begin{pmatrix} -sI + A & B_2 \\ C & D_2 \end{pmatrix} \subset \\
&\quad \text{Left-Ker} \begin{pmatrix} -sI + A & B_2 \\ C & D_2 \end{pmatrix} \begin{pmatrix} (sI - A)^{-1}B_1 & (sI - A)^{-1}B_2 \\ 0 & I \end{pmatrix} = \\
&\quad \text{Left-Ker} \begin{pmatrix} -B_1 & 0 \\ -D_1 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ C(sI - A)^{-1}B_1 + D_1 & C(sI - A)^{-1}B_2 + D_2 \end{pmatrix}
\end{aligned} \tag{3.8}$$

which implies that

$$\begin{aligned}
\text{Row-Im } V(s) &= \text{Row-Im } W(s) \begin{pmatrix} -B_1 & 0 \\ -D_1 & I \end{pmatrix} \\
&\subset \text{Left-Ker} \begin{pmatrix} I & 0 \\ C(sI - A)^{-1}B_1 + D_1 & C(sI - A)^{-1}B_2 + D_2 \end{pmatrix}.
\end{aligned} \tag{3.9}$$

But

$$\begin{aligned}
\text{Row-Im } W(s) &= \text{Left-Ker} \begin{pmatrix} -sI + A & B_2 \\ C & D_2 \end{pmatrix} \\
&= \text{Left-Ker} \begin{pmatrix} I & 0 \\ -C(sI - A)^{-1} & C(sI - A)^{-1}B_2 + D_2 \end{pmatrix}
\end{aligned} \tag{3.10}$$

and thus

$$\begin{aligned}
&\dim \text{Left-Ker} \begin{pmatrix} I & 0 \\ C(sI - A)^{-1}B_1 + D_1 & C(sI - A)^{-1}B_2 + D_2 \end{pmatrix} = \\
&\quad \dim \text{Left-Ker} \begin{pmatrix} I & 0 \\ -C(sI - A)^{-1} & C(sI - A)^{-1}B_2 + D_2 \end{pmatrix} = \\
&\quad \text{rank } W(s) = \text{rank } V(s)
\end{aligned} \tag{3.11}$$

the last equality being due to (3.7). From (3.9) and (3.11) we easily obtain (2.8).

$$H(s) = V(s)\begin{pmatrix} 0 \\ C(sI-A)^{-1}B_3 + D_3 \end{pmatrix} =$$
$$W(s)\begin{pmatrix} -B_1 & 0 \\ -D_1 & I \end{pmatrix}\begin{pmatrix} 0 \\ C(sI-A)^{-1}B_3 + D_3 \end{pmatrix} =$$
$$W(s)\begin{pmatrix} -sI+A & B_3 \\ C & D_3 \end{pmatrix}\begin{pmatrix} (sI-A)^{-1}B_3 \\ I \end{pmatrix} = W(s)\begin{pmatrix} B_3 \\ D_3 \end{pmatrix} \tag{3.12}$$

which clearly shows that $H(s)$ is stable and (2.10) is satisfied.

Now let $V(s)$ be an innovations filter, i.e., let $V(s)$ be stable, full row rank and satisfy (2.8)-(2.10). What we need to show is that there exists a stable full row rank $W(s)$ satisfying (3.1)-(3.4). For that, let

$$W(s) = V(s)\begin{pmatrix} 0 & 0 \\ C(sI-A)^{-1} & I \end{pmatrix}. \tag{3.13}$$

From (2.8) we obtain

$$V(s)\begin{pmatrix} 0 \\ C(sI-A)^{-1}\begin{bmatrix} B_1 & B_2 \end{bmatrix} \end{pmatrix} = -V(s)\begin{pmatrix} I & 0 \\ D_1 & D_2 \end{pmatrix} \tag{3.14}$$

which thanks to stability of $V(s)$ implies that

$$V(s)\begin{pmatrix} 0 \\ C(sI-A)^{-1}\begin{bmatrix} B_1 & B_2 \end{bmatrix} \end{pmatrix}$$

is stable. From stability of $V(s)$ and stabilizability of $\left(A, \begin{bmatrix} B_1 & B_2 \end{bmatrix}\right)$ which follows from stabilizability of $\left(A, \begin{bmatrix} B_1 & B_{21} \end{bmatrix}\right)$, thanks to Lemma 3.1, we can show that

$$V(s)\begin{pmatrix} 0 \\ C(sI-A)^{-1} \end{pmatrix}$$

and thus $W(s)$ is stable.

Now let

$$V(s) = \begin{pmatrix} V_1(s) & V_2(s) \end{pmatrix}. \tag{3.15}$$

From (2.8), we know that

$$V(s)\begin{pmatrix} I \\ C(sI-A)^{-1}B_1 + D_1 \end{pmatrix} = V_1(s) + V_2(s)(C(sI-A)^{-1}B_1 + D_1) = 0 \tag{3.16}$$

which, since $V(s)$ has full row rank, implies that $V_2(s)$ has full row rank which, in turn, since

$$W(s) = \begin{pmatrix} V_2(s)C(sI-A)^{-1} & V_2(s) \end{pmatrix} \tag{3.17}$$

implies that $W(s)$ has full row rank.

Finally,

$$W(s)\begin{pmatrix} -sI+A & B_2 \\ C & D_2 \end{pmatrix} = V(s)\begin{pmatrix} 0 & 0 \\ 0 & C(sI-A)^{-1}B_2+D_2 \end{pmatrix} = 0, \tag{3.18}$$

$$W(s)\begin{pmatrix} B_3 \\ D_3 \end{pmatrix} = V(s)\begin{pmatrix} 0 & 0 \\ 0 & C(sI-A)^{-1}B_3+D_3 \end{pmatrix} = H(s) \tag{3.19}$$

and

$$\begin{aligned} W(s)\begin{pmatrix} -B_1 & 0 \\ -D_1 & I \end{pmatrix} &= V(s)\begin{pmatrix} 0 & 0 \\ 0 & C(sI-A)^{-1}B_3+D_3 \end{pmatrix}\begin{pmatrix} -B_1 & 0 \\ -D_1 & I \end{pmatrix} \\ &= V(s)\begin{pmatrix} 0 & 0 \\ -C(sI-A)^{-1}B_1-D_1 & I \end{pmatrix} = V(s) \end{aligned} \tag{3.20}$$

(the last equality follows (3.16)). Thus $W(s)$ satisfies (3.1)-(3.4) and the theorem is proved. ∎

Lemma 3.1 *Let $U(s)$ be a stable rational matrix and (F,G) a stabilizable pair. Then, stability of*

$$U(s)(sI-F)^{-1}G$$

implies stability of

$$U(s)(sI-F)^{-1}.$$

Proof There exists a coordinate transformation matrix T such that

$$TG = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} \tag{3.21}$$

$$TFT^{-1} = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} \tag{3.22}$$

where all eigenvalues of F_1 have negative real parts and that of F_2 positive or zero real parts. Then, thanks to the assumption that (F,G) is stabilizable, we know that (F_2,G_2) is controllable. Let

$$\begin{pmatrix} U_1(s) \\ U_2(s) \end{pmatrix} = U(s)T \tag{3.23}$$

so that

$$U(s)(sI-F)^{-1}G = U_1(s)(sI-F_1)^{-1}G_1 + U_2(s)(sI-F_2)^{-1}G_2. \tag{3.24}$$

Since $U_1(s)$ and $(sI-F_1)^{-1}$ are stable, stability of

$$U(s)(sI-F)^{-1}G$$

implies stability of

$$U_2(s)(sI-F_2)^{-1}G_2.$$

Noting that

$$s^j U_2(s)(sI - F_2)^{-1}G_2 = U_2(s)(sI - F_2)^{-1}F_2{}^j G_2, \;\; j = 0, 1, \dots \tag{3.25}$$

we can see that

$$U_2(s)(sI - F_2)^{-1}F_2{}^j G_2, \;\; j = 0, 1, \dots$$

is stable. Thus

$$U_2(s)(sI - F_2)^{-1}\begin{pmatrix} G_2 & F_2G_2 & \dots & F_2{}^{n_2-1}G_2 \end{pmatrix}$$

is stable where n_2 denotes the size of F_2. But

$$\begin{pmatrix} G_2 & F_2G_2 & \dots & F_2{}^{n_2-1}G_2 \end{pmatrix}$$

has full row rank (thanks to controllability of (F_2, G_2)) and thus has a right inverse which implies that

$$U_2(s)(sI - F_2)^{-1}$$

is stable. So,

$$U(s)(sI - F)^{-1} = U_1(s)(sI - F_1)^{-1} + U_2(s)(sI - F_2)^{-1} \tag{3.26}$$

is stable. ■

In the rest of Section 3, we present a methodology for constructing $W(s)$ in Theorem 3.1; construction of the innovations filter $V(s)$ is then done using (3.1). First, we consider the special case where the system contains no unknown inputs and disturbances (equivalently, $B_2 = 0$ and $D_2 = 0$), and give an explicit solution under a mild regularity condition. Then, we consider the case of systems with unknown inputs and disturbances and present a method for reducing them to the case of systems with no unknown inputs and disturbances. Finally, we lift the regularity condition and give a complete solution to the problem.

3.1 System with no unknown inputs and disturbances

In this section, we consider the construction of the innovations filter in case there is no unknown inputs or disturbances v (i.e. when $B_2 = 0$ and $D_2 = 0$). Thanks to Theorem 3.1, the construction of the innovations filter comes down to the problem of finding $W(s)$, a stable full row rank rational matrix such that

$$\text{Row-Im } W(s) = \text{Left-Ker} \begin{pmatrix} -sI + A \\ C \end{pmatrix} \tag{3.27}$$

and

$$\Psi = H(s)H^{\sim}(s) \tag{3.28}$$

is constant where

$$H(s) = W(s)\begin{pmatrix} B_3 \\ D_3 \end{pmatrix}. \tag{3.29}$$

We call such a $W(s)$ a *prefilter* for system (A, B_3, C, D_3).

Definition 3.1 *We say that* (A, B_3, C, D_3) *is regular if*

$$\begin{pmatrix} -j\omega I + A & B_3 \\ C & D_3 \end{pmatrix} \text{ has full row rank } \forall \omega \in \mathbb{R} \cup \{\infty\}. \quad (3.30)$$

Lemma 3.2 ([4]) *Suppose* (C, A) *is detectable and* (A, B_3, C, D_3) *is regular. Then, the Riccati equation*

$$(A - SR^{-1}C)P + P(A - SR^{-1}C)^T - PC^TR^{-1}CP + Q - SR^{-1}S^T = 0 \quad (3.31)$$

where

$$\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} = \begin{pmatrix} B_3 \\ D_3 \end{pmatrix} \begin{pmatrix} B_3{}^T & D_3{}^T \end{pmatrix}. \quad (3.32)$$

has a stabilizing solution P *(*P *is a stabilizing solution if* $A + KC$ *is stable where*

$$K = -(PC^T + S)R^{-1} \quad (3.33)$$

denotes the Kalman gain matrix).

The stabilizing solution of (3.31) can easily be constructed from the Hamiltonian matrix:

$$\mathcal{H} = \begin{pmatrix} A - SR^{-1}C & -Q + SR^{-1}S^T \\ -C^TR^{-1}C & -A^T + C^TR^{-1}S^T \end{pmatrix}. \quad (3.34)$$

In particular, if we the rows of $\begin{pmatrix} X & Y \end{pmatrix}$ form a basis for the stable left eigenspace of $\mathcal{H}$, then the stabilizing solution P of (3.31) is given by

$$P = X^{-1}Y. \quad (3.35)$$

Theorem 3.2 *Suppose* (C, A) *is detectable and* (A, B_3, C, D_3) *is regular. Then a prefilter* $W(s)$ *for* (A, B_3, C, D_3) *is*

$$W(s) = C(sI - A - KC)^{-1} \begin{pmatrix} I & K \end{pmatrix} + \begin{pmatrix} 0 & I \end{pmatrix} \quad (3.36)$$

where P *and* K *are respectively the stabilizing solution of the Riccati equation (3.31) and its associated Kalman gain matrix (3.33).*

Proof By construction, $W(s)$ in (3.36) is full row rank and stable. Moreover, it satisfies

$$\begin{aligned} W(s)\begin{pmatrix} -sI + A \\ C \end{pmatrix} &= \left(C(sI - A - KC)^{-1} \begin{pmatrix} I & K \end{pmatrix} + \begin{pmatrix} 0 & I \end{pmatrix}\right) \begin{pmatrix} -sI + A \\ C \end{pmatrix} \\ &= C(sI - A - KC)^{-1}(-sI + A + KC) + C = 0 \end{aligned} \quad (3.37)$$

which implies that (3.27) is satisfied because the dimension of the left null-space of $\begin{pmatrix} -sI + A \\ C \end{pmatrix}$ equals the number of rows of C, i.e., the number of rows of $W(s)$. So all that we need to show is that

$$\Psi = W(s) \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} W^{\sim}(s) \tag{3.38}$$

is constant. Using (3.36) we get

$$\begin{aligned} \Psi &= \left(C(sI - A - KC)^{-1} \begin{pmatrix} I & K \end{pmatrix} + \begin{pmatrix} 0 & I \end{pmatrix} \right) \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \\ &\quad \times \left(\begin{pmatrix} I \\ K \end{pmatrix} (-sI - A - KC)^{-T} C^T + \begin{pmatrix} 0 \\ I \end{pmatrix} \right) \\ &= C(sI - A - KC)^{-1}(Q + KS^T + SK^T + KRK^T)(-sI - A - KC)^{-T}C^T \\ &\quad + (S^T + RK^T)(-sI - A - KC)^{-T}C^T + C(sI - A - KC)^{-1}(S + KR) + R \end{aligned} \tag{3.39}$$

which using (3.31) and (3.33) yields

$$\begin{aligned} \Psi &= C(sI - A - KC)^{-1}(Q + KS^T + SK^T + KRK^T + AP + PA + KCP + PC^TK^T) \\ &\quad \times(-sI - A - KC)^{-T}C^T + R = C(sI - A - KC)^{-1} \\ &\quad \times((A - SR^{-1}C)P + P(A - SR^{-1}C)^T - PC^TR^{-1}CP + Q - SR^{-1}S^T)^{-T} \\ &\quad \times(-sI - A - KC)^{-T}C^T + R = R \end{aligned} \tag{3.40}$$

which completes the proof. ■

Thus, an innovations filter for a detectable system in the absence of unknown inputs and disturbances is

$$V(s) = C(sI - A - KC)^{-1} \begin{pmatrix} -KD_1 - B_1 & K \end{pmatrix} + \begin{pmatrix} -D_1 & I \end{pmatrix}. \tag{3.41}$$

In this case, the output of the innovations filter, i.e., the innovations, coincides with the standard innovations process generated by the Kalman filter.

When the pair (C, A) is not detectable, we have to discard the unstable non-observable part of the system before constructing its prefilter. The following is a standard result in linear system theory.

Lemma 3.3 *Suppose (C, A) is not detectable. Then there exists an invertible matrix*

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}. \tag{3.42}$$

such that

$$\begin{aligned} TAT^{-1} &= \begin{pmatrix} A^{(11)} & A^{(12)} \\ 0 & A^{(22)} \end{pmatrix} \\ CT^{-1} &= \begin{pmatrix} 0 & C^{(2)} \end{pmatrix} \\ TB_3 &= \begin{pmatrix} B_3{}^{(1)} \\ B_3{}^{(2)} \end{pmatrix} \end{aligned} \tag{3.43}$$

where $(C^{(2)}, A^{(22)})$ *is observable (and thus detectable).*

Moreover, if (A, B_3, C, D_3) *is regular, so is* $(A^{(22)}, B_3{}^{(2)}, C^{(2)}, D_3)$.

Lemma 3.4 *Suppose* (C, A) *is not detectable and let* T, $A^{(22)}$, $B_3{}^{(2)}$ *and* $C^{(2)}$ *be as defined in Lemma 3.4. Then* $W(s)$ *is a prefilter for* (A, B_3, C, D_3) *if and only if*

$$W(s) = W_d(s) \begin{pmatrix} T_{21} & T_{22} & 0 \\ 0 & 0 & I \end{pmatrix} \tag{3.44}$$

where $W_d(s)$ *is a prefilter for* $(A^{(22)}, B_3{}^{(2)}, C^{(2)}, D_3)$.

The proof is straightforward and is omitted.

3.2 System with unknown inputs and disturbances

In the general case, construction of $W(s)$ in Theorem 3.1 can be reduced to the construction of a prefilter thanks to the following result:

Theorem 3.3 $W(s)$ *in Theorem 3.1 exists if and only if it can be expressed as*

$$W(s) = W_r(s)\Gamma \tag{3.45}$$

where $W_r(s)$ *is a prefilter for* $(A_r, B_{3,r}, C_r, D_{3,r})$ *where*

$$\begin{pmatrix} -sI + A_r & B_{3,r} \\ C_r & D_{3,r} \end{pmatrix} = \Gamma \left(\begin{array}{cc|c} -sI + A & B_2 & B_3 \\ C & D_2 & D_3 \end{array} \right) \begin{pmatrix} \Phi & 0 \\ 0 & I \end{pmatrix} \tag{3.46}$$

for some respectively full row and full column rank matrices Γ *and* Φ.

Moreover, if

$$\begin{pmatrix} -j\omega I + A & B_2 & B_3 \\ C & D_2 & D_3 \end{pmatrix} \text{ has full row rank } \forall \omega \in \mathbb{R} \cup \{\infty\} \tag{3.47}$$

then $(A_r, B_{3,r}, C_r, D_{3,r})$ *is regular.*

The construction of $W(s)$ is thus reduced to that of $W_r(s)$, i.e., the problem with unknown inputs and disturbances is transformed into a problem without unknown inputs and disturbances. We will present a simple recursive algorithm for constructing $(A_r, B_{3,r}, C_r, D_{3,r})$ and Γ. Since the size of A_r is smaller than the size of A, the resulting innovations filter has a smaller order than the original system, i.e., we obtain a reduced order innovations filter.

Before proving this result, we present the following lemma.

Lemma 3.5 *Let $\{E, F\}$ be an arbitrary pencil. Then there exist matrices M and N, respectively full row and full column rank, such that if we let*

$$E' = MEN \tag{3.48}$$
$$F' = MFN \tag{3.49}$$

1- E' has full column rank,

2- if $\mathcal{N}$ and $\mathcal{N}'$ denote respectively left null-spaces of $sE - F$ and $sE' - F'$, then

$$\mathcal{N} = \mathcal{N}'M. \tag{3.50}$$

The proof is constructive and is given as an algorithm which is a key result of this paper:

Reduction Algorithm Let

$$E' = E \tag{3.51}$$
$$F' = F \tag{3.52}$$
$$M = I \tag{3.53}$$
$$N = I \tag{3.54}$$

While E' not full column rank do

- Find $\Sigma = (\Sigma_1 \quad \Sigma_2)$ be an orthogonal matrix column compressing E'

  $$(E_1' \quad 0) = E'(\Sigma_1 \quad \Sigma_2) \tag{3.55}$$

  where E_1' has full column rank and let

  $$(F_1' \quad F_2') = F'(\Sigma_1 \quad \Sigma_2). \tag{3.56}$$

- Find highest rank full row rank matrix L satisfying

  $$LF_2' = 0. \tag{3.57}$$

- Let

$$E' = LE_1' \tag{3.58}$$
$$F' = LF_1' \tag{3.59}$$
$$M = LM \tag{3.60}$$
$$N = N\Sigma_1 \tag{3.61}$$

end do.

Clearly the algorithm ends in a finite number of steps because at each step the number of columns of E' is reduced at least by one. Note that, at each step, the new pencil $\{E', F'\}$ is constructed from the previous $\{E', F'\}$ by premultiplication by L, postmultiplication by invertible matrix Σ and removal of zero columns. Thus, to justify the algorithm, all we need to show is that by premultiplying $(sE' - F')$ by L, we do not change the dimension of its left null-space. But this follows from the fact that L is the highest rank matrix satisfying (3.57) which implies that the the row space of L includes the left null-space of $(sE' - F')$.

Proof of Theorem 3.3 Let

$$E = \begin{pmatrix} -I & 0 \\ 0 & 0 \end{pmatrix}, \quad (3.62)$$

$$F = \begin{pmatrix} A & B_2 \\ C & D_2 \end{pmatrix}, \quad (3.63)$$

apply Lemma 3.5 to construct E', F', M and N, find an invertible matrix T such that

$$TE' = \begin{pmatrix} -I \\ 0 \end{pmatrix} \quad (3.64)$$

and let

$$\Gamma = TM \quad (3.65)$$

$$\Phi = N \quad (3.66)$$

$$\begin{pmatrix} A_r \\ C_r \end{pmatrix} = -TF' \; (= -\Gamma F \Phi) \quad (3.67)$$

$$\begin{pmatrix} B_{3,r} \\ D_{3,r} \end{pmatrix} = \Gamma \begin{pmatrix} B_3 \\ D_3 \end{pmatrix}. \quad (3.68)$$

Then clearly

$$\begin{pmatrix} -sI + A_r & B_{3,r} \\ C_r & D_{3,r} \end{pmatrix} = \Gamma \left(\begin{array}{cc|c} -sI + A & B_2 & B_3 \\ C & D_2 & D_3 \end{array} \right) \begin{pmatrix} \Phi & 0 \\ 0 & I \end{pmatrix} \quad (3.69)$$

and

$$\text{Left-Ker} \begin{pmatrix} -sI + A & B_2 \\ C & D_2 \end{pmatrix} = \left[\text{Left-Ker} \begin{pmatrix} -sI + A_r \\ C_r \end{pmatrix} \right] \Gamma. \quad (3.70)$$

If $W_r(s)$ is a prefilter for $(A_r, B_{3,r}, C_r, D_{3,r})$, i.e., $W_r(s)$ is stable, has full row rank and satisfies

$$\text{Row-Im } W_r(s) = \text{Left-Ker} \begin{pmatrix} -sI + A_r \\ C_r \end{pmatrix} \quad (3.71)$$

and

$$\Psi_r = H_r(s) H_r^{\sim}(s) \quad (3.72)$$

is constant where

$$H_r(s) = W_r(s)\begin{pmatrix} B_{3,r} \\ D_{3,r} \end{pmatrix}. \tag{3.73}$$

Then,

$$W(s) = W_r(s)\Gamma \tag{3.74}$$

satisfies conditions in Theorem 3.1: clearly $W(s)$ is stable and has full row rank (because Γ has full row rank),

$$\begin{aligned} \text{Row-Im } W(s) &= \text{Row-Im } (W_r(s)\Gamma) \\ &= (\text{Row-Im } W_r(s))\Gamma \\ &= \left[\text{Left-Ker } \begin{pmatrix} -sI + A_r \\ C_r \end{pmatrix}\right]\Gamma \\ &= \text{Left-Ker } \begin{pmatrix} -sI + A & B_2 \\ C & D_2 \end{pmatrix} \end{aligned} \tag{3.75}$$

and

$$\begin{aligned} H(s) = W(s)\begin{pmatrix} B_3 \\ D_3 \end{pmatrix} &= W_r(s)\Gamma\begin{pmatrix} B_3 \\ D_3 \end{pmatrix} \\ &= W_r(s)\begin{pmatrix} B_{3,r} \\ D_{3,r} \end{pmatrix} = H_r(s) \end{aligned} \tag{3.76}$$

which implies that

$$\Psi = H(s)H^{\sim}(s) = H_r(s)H_r^{\sim}(s) = \Psi_r \tag{3.77}$$

is constant.

Now suppose that $W(s)$ satisfies conditions of Theorem 3.1, and let

$$W_r(s) = W(s)\Gamma^{\dagger} \tag{3.78}$$

where $\Gamma^{\dagger}$ is any right inverse of Γ (i.e., $\Gamma\Gamma^{\dagger} = I$). Then, clearly $W_r(s)$ is stable. Also, since

$$\text{Row-Im } W(s) = \left[\text{Left-Ker } \begin{pmatrix} -sI + A_r \\ C_r \end{pmatrix}\right]\Gamma \tag{3.79}$$

which implies that

$$\ker\Gamma \subset \ker W(s) \tag{3.80}$$

we have that

$$W_r(s)\Gamma = W(s)\Gamma^{\dagger}\Gamma = W(s) \tag{3.81}$$

which in turn implies that $W_r(s)$ has full row rank. In addition,

$$\begin{aligned} \text{Row-Im } W_r(s) &= \text{Row-Im } (W(s)\Gamma^{\dagger}) \\ &= (\text{Row-Im } W(s))\Gamma^{\dagger} \end{aligned}$$

$$
\begin{aligned}
&= \left[\text{Left-Ker}\begin{pmatrix} -sI+A & B_2 \\ C & D_2 \end{pmatrix}\right]\Gamma^{\dagger} \\
&= \left[\text{Left-Ker}\begin{pmatrix} -sI+A_r \\ C_r \end{pmatrix}\right]\Gamma\Gamma^{\dagger} \\
&= \text{Left-Ker}\begin{pmatrix} -sI+A_r \\ C_r \end{pmatrix}
\end{aligned} \tag{3.82}
$$

and

$$
H_r(s) = W_r(s)\begin{pmatrix} B_{3,r} \\ D_{3,r} \end{pmatrix} = W(s)\Gamma^{\dagger}\Gamma\begin{pmatrix} B_3 \\ D_3 \end{pmatrix} = W(s)\begin{pmatrix} B_3 \\ D_3 \end{pmatrix} = H(s) \tag{3.83}
$$

which implies that

$$
\Psi_r = H_r(s)H_r^{\sim}(s) = H(s)H^{\sim}(s) = \Psi \tag{3.84}
$$

is constant and thus $W_r(s)$ is a prefilter for $(A_r, B_{3,r}, C_r, D_{3,r})$. This completes the proof of the first part of the theorem.

To show the second part, simply note that from the reduction algorithm, we know that there exists an invertible matrix Λ (which is just a product of Σ's in the reduction algorithm) such that

$$
\left(\begin{array}{cc|c} -sI+A_r & 0 & B_{3,r} \\ C_r & 0 & D_{3,r} \end{array}\right) = \Gamma\left(\begin{array}{cc|c} -sI+A & B_2 & B_3 \\ C & D_2 & D_3 \end{array}\right)\begin{pmatrix} \Lambda & 0 \\ 0 & I \end{pmatrix}. \tag{3.85}
$$

But Γ has full row rank which means that if (3.47) holds, the left hand side of (3.85) has full row rank for $s = j\omega$, $\forall\omega \in I\!R \cup \{\infty\}$, i.e., $(A_r, B_{3,r}, C_r, D_{3,r})$ is regular. This completes the proof. ■

Thanks to Theorem 3.3 and Lemma 3.4 we can reduce the problem of construction of $W(s)$ to the problem of construction of a prefilter for a detectable system. If this system happens to be regular, we have an explicit solution given in Theorem 3.2. In the next section, we consider the case where this system is not regular.

3.3 Prefilter construction in the irregular case

In the previous sections, we have shown that the construction of an innovations filter can be reduced to the construction of a prefilter $W(s)$ for a detectable system (A, B, C, D) to which an explicit solution was given when (A, B, C, D) was regular, in which case, the covariance of the innovations Ψ was positive definite ($= R$). In the irregular case, the covariance of the innovations process (if it exists), Ψ, may or may not be positive definite. In fact, we can distinguish three cases: the first case is the purely deterministic case, i.e., $\Psi = 0$. This happens when $B = 0$ and $D = 0$. The second case is when $B \neq 0$ and/or $D \neq 0$ but

$$
\begin{pmatrix} -sI+A & B \\ C & D \end{pmatrix}
$$

does not have full (generic) rank. In this case, the innovations process (if it exists) can be broken up into two signals, one purely deterministic (i.e. zero) and the other purely stochastic (positive definite covariance). The last case is when

$$\begin{pmatrix} -sI+A & B \\ C & D \end{pmatrix}$$

has full row rank but may loose rank for $s = j\omega$ for some ω's in $I\!R \cup \{\infty\}$. In this case, if $W(s)$ exists, Ψ is positive definite, i.e., the innovations process is purely stochastic just as in the regular case. This can be seen by noting that

$$W(s) \begin{pmatrix} -sI+A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & H(s) \end{pmatrix} \tag{3.86}$$

from which and the full row rankedness of the two matrices on the left hand side of it we can deduce that $H(s)$ has full row rank which clearly implies that $\Psi = H(s)H^{\sim}(s)$ is positive definite.

3.3.1 Purely deterministic case

When $B = 0$ and $D = 0$, the prefilter $W(s)$ is any full row rank stable matrix satisfying

$$\text{Row-Im } W(s) = \text{Left-Ker} \begin{pmatrix} -sI+A \\ C \end{pmatrix}. \tag{3.87}$$

It is straightforward to verify that such a $W(s)$ can be constructed as follows:

Lemma 3.6 *A prefilter $W(s)$ for $(A, 0, C, 0)$ is given by*

$$W(s) = C(sI - A - KC)^{-1} \begin{pmatrix} I & K \end{pmatrix} + \begin{pmatrix} 0 & I \end{pmatrix} \tag{3.88}$$

where K is any matrix such that $A + KC$ is stable (such a K exists because (C, A) is detectable).

The output of the innovations filter in this case is identically zero (in the absence of failures) and it corresponds to what is usually referred to as a parity check, analytical redundancy check, etc...

3.3.2 Semi-deterministic semi-stochastic case

In this case, the innovations process can be broken up into two processes. One purely stochastic (with invertible covariance) and the other purely deterministic ($= 0$).

Theorem 3.4 *Let $W_1(s)$ be a stable full row rank matrix and, B_c and D_c constant matrices such that*

$$\textit{Row-Im } W_1(s) = \textit{Left-Ker} \begin{pmatrix} -sI+A & B \\ C & D \end{pmatrix} \tag{3.89}$$

and

$$W_1(s)\begin{pmatrix} B_c \\ D_c \end{pmatrix} = I. \tag{3.90}$$

Then if $W_2(s)$ is a stable full row rank matrix such that

$$\textit{Row-Im } W_2(s) = \textit{Left-Ker}\begin{pmatrix} -sI+A & B_c \\ C & D_c \end{pmatrix} \tag{3.91}$$

and

$$\Psi_2 = H_2(s)H_2{}^{\sim}(s) \tag{3.92}$$

is constant where

$$H_2(s) = W_2(s)\begin{pmatrix} B \\ D \end{pmatrix}, \tag{3.93}$$

then,

$$W(s) = \begin{pmatrix} W_1(s) \\ W_2(s) \end{pmatrix} \tag{3.94}$$

is a prefilter for (A, B, C, D) .

Moreover, if no $W_2(s)$ satisfying the above conditions exists ($W_1(s)$, B_c and D_c always exist), (A, B, C, D) has no prefilter.

Proof For the first part, what we have to show is that $W(s)$ is stable, has full row rank,

$$\text{Row-Im } W(s) = \text{Left-Ker}\begin{pmatrix} -sI+A \\ C \end{pmatrix}, \tag{3.95}$$

and

$$\Psi = H(s)H^{\sim}(s) \tag{3.96}$$

is constant where

$$H(s) = W(s)\begin{pmatrix} B \\ D \end{pmatrix}. \tag{3.97}$$

Clearly $W(s)$ is stable. To show that it has full row rank, since $W_1(s)$ and $W_2(s)$ each have full row rank, all we need to show is that if

$$J_1(s)W_1(s) + J_2(s)W_2(s) = 0 \tag{3.98}$$

then $J_1(s) = J_2(s) = 0$. By post multiplying (3.98) by

$$\begin{pmatrix} -sI+A & B \\ C & D \end{pmatrix}$$

we get

$$J_2(s)W_2(s)\begin{pmatrix} -sI+A & B \\ C & D \end{pmatrix} = 0 \tag{3.99}$$

which thanks to (3.91) implies that

$$J_2(s)W_2(s)\begin{pmatrix} -sI+A & B & B_c \\ C & D & D_c \end{pmatrix} = 0. \tag{3.100}$$

But

$$\begin{pmatrix} -sI+A & B & B^c \\ C & D & D^c \end{pmatrix}$$

has full row rank because if

$$p(s)\begin{pmatrix} -sI+A & B & B^c \\ C & D & D^c \end{pmatrix} = 0 \tag{3.101}$$

then thanks to (3.89), there exists $q(s)$ such that

$$p(s) = q(s)W_1(s) \tag{3.102}$$

which implies that

$$q(s) = q(s)W_1(s)\begin{pmatrix} B^c \\ D^c \end{pmatrix} = 0 \tag{3.103}$$

which implies that $p(s) = 0$.

Thus

$$J_2(s)W_2(s) = 0. \tag{3.104}$$

So from full rankedness of $W_2(s)$ we get that $J_2(s) = 0$ and from that of $W_1(s)$ and (3.98) that $J_1(s) = 0$. Thus $W(s)$ has full row rank.

Also,

$$W(s)\begin{pmatrix} -sI+A \\ C \end{pmatrix} = \begin{pmatrix} W_1(s) \\ W_2(s) \end{pmatrix}\begin{pmatrix} -sI+A \\ C \end{pmatrix} = 0 \tag{3.105}$$

which mean that

$$\text{Row-Im } W(s) \subset \text{Left-Ker}\begin{pmatrix} -sI+A \\ C \end{pmatrix}. \tag{3.106}$$

Clearly,

$$\text{rank } W_1(s) = n + m - d \tag{3.107}$$

where n and m denote respectively the size of A and the number of rows of C, and

$$d = \text{rank}\begin{pmatrix} -sI+A & B \\ C & D \end{pmatrix} \tag{3.108}$$

and

$$\begin{aligned} \text{rank } W_2(s) &= n + m - \text{rank}\begin{pmatrix} -sI+A & B_c \\ C & D_c \end{pmatrix} \\ & \quad n + m - \left(n + \text{rank}\begin{pmatrix} B_c \\ D_c \end{pmatrix}\right) \end{aligned} \tag{3.109}$$

But,

$$\operatorname{rank}\begin{pmatrix} -sI+A & B & B_c \\ C & D & D_c \end{pmatrix} = n+m$$
$$= \operatorname{rank}\begin{pmatrix} B_c \\ D_c \end{pmatrix} + d \qquad (3.110)$$

So,

$$\operatorname{rank}\begin{pmatrix} B_c \\ D_c \end{pmatrix} = n+m-d \qquad (3.111)$$

from which we deduce that

$$\begin{aligned}\operatorname{rank} W(s) &= \operatorname{rank}\begin{pmatrix} W_1(s) \\ W_2(s) \end{pmatrix} \\ &= m \\ &= \operatorname{rank} \text{Left-Ker}\begin{pmatrix} -sI+A \\ C \end{pmatrix}\end{aligned} \qquad (3.112)$$

which thanks to (3.106) implies (3.95). Finally, it is easy to see that

$$H(s) = \begin{pmatrix} 0 \\ H_2(s) \end{pmatrix} \qquad (3.113)$$

which shows that

$$\Psi = \begin{pmatrix} 0 & 0 \\ 0 & \Psi_2 \end{pmatrix} \qquad (3.114)$$

is constant. This completes the proof of the first part of the theorem.

To show the second part, suppose that $W_2(s)$ does not exist but $W(s)$ does, i.e.,

$$\text{Row-Im } W(s) = \text{Left-Ker}\begin{pmatrix} -sI+A \\ C \end{pmatrix} \qquad (3.115)$$

and

$$\Psi = H(s)H^{\sim}(s) \qquad (3.116)$$

is constant where

$$H(s) = W(s)\begin{pmatrix} B \\ D \end{pmatrix}. \qquad (3.117)$$

Let

$$T = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} \qquad (3.118)$$

be an invertible matrix such that

$$T\Psi T^T = \begin{pmatrix} 0 & 0 \\ 0 & \Psi_2 \end{pmatrix} \qquad (3.119)$$

where $\Psi_2 > 0$. If we now let

$$W_2(s) = T_2 W(s)\left(I - \begin{pmatrix} B_c \\ D_c \end{pmatrix} W_1(s)\right) \tag{3.120}$$

it is straightforward to verify that this $W_2(s)$ satisfies all the conditions of the theorem which is a contradiction. Thus $W_2(s)$ exists. ■

Construction of $W_1(s)$, B_c and D_c is done as follows. By applying Theorem 3.3 and if necessary, Lemma 3.4, we can construct $W_1(s)$ as

$$W_1(s) = W_{r,1}\Gamma_1 \tag{3.121}$$

where Γ_1 is a full row rank constant matrix and $W_{r,1}$, a prefilter to a detectable, purely deterministic system $(A_{r,1}, 0, C_{r,1}, 0)$. Using the result of the previous section, we obtain

$$W_1(s) = \left(C_{r,1}(sI - A_{r,1} - K_1C_{r,1})^{-1}\begin{pmatrix} I & K_1 \end{pmatrix} + \begin{pmatrix} 0 & I \end{pmatrix}\right)\Gamma_1 \tag{3.122}$$

where K_1 is any matrix such that $A_{r,1} + K_1C_{r,1}$ is stable. Finally, matrices B_c and D_c must satisfy

$$\begin{pmatrix} I & K_1 \\ 0 & I \end{pmatrix}\Gamma_1\begin{pmatrix} B_c \\ D_c \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix} \tag{3.123}$$

which means that we can let

$$\begin{pmatrix} B_c \\ D_c \end{pmatrix} = {\Gamma_1}^{\dagger}\begin{pmatrix} -K_1 \\ I \end{pmatrix} \tag{3.124}$$

where ${\Gamma_1}^{\dagger}$ is any right inverse of Γ_1.

The problem of construction of $W_2(s)$ as formulated in Theorem 3.4, thanks to Theorem 3.3 and if necessary, Lemma 3.4, reduces to the problem of constructing a prefilter $W_{r,2}$ for a detectable system $(A_{r,2}, B_{r,2}, C_{r,2}, D_{r,2})$.

Lemma 3.7 *System $(A_{r,2}, B_{r,2}, C_{r,2}, D_{r,2})$ is purely stochastic.*

Proof Since $(A_{r,2}, B_{r,2}, C_{r,2}, D_{r,2})$ is obtained from the reduction of Problem (3.91)-(3.93), there exist a full row rank matrix Γ_2 and an invertible matrix Λ such that

$$\left(\begin{array}{cc|c} -sI + A_{r,2} & 0 & B_{r,2} \\ C_{r,2} & 0 & D_{r,2} \end{array}\right) = \Gamma_2\left(\begin{array}{cc|c} -sI + A & B_c & B \\ C & D_c & D \end{array}\right)\begin{pmatrix} \Lambda & 0 \\ 0 & I \end{pmatrix}. \tag{3.125}$$

But

$$\begin{pmatrix} -sI + A & B^c & B \\ C & D^c & D \end{pmatrix}$$

has full row rank, thus, so does

$$\begin{pmatrix} -sI + A_{r,2} & B_{r,2} \\ C_{r,2} & D_{r,2} \end{pmatrix}$$

which means that $(A_{r,2}, B_{r,2}, C_{r,2}, D_{r,2})$ is purely stochastic. ■

If in addition $(A_{r,2}, B_{r,2}, C_{r,2}, D_{r,2})$ is regular, we can use Theorem 3.2 to construct $W_{r,2}$; construction of prefilters for purely stochastic systems in the irregular case is the subject of the next section.

3.3.3 Purely stochastic case

In this section, we consider the problem of prefilter construction in the irregular purely stochastic case, i.e., when

$$\begin{pmatrix} -sI + A & B \\ C & D \end{pmatrix}$$

has full row rank but looses rank for $s = j\omega_0$ for some ω_0 in $\mathbb{R} \cup \{\infty\}$. In this case, we say that (A, B, C, D) has a zero at $j\omega_0$.

It turns out that existence of finite and infinite zeros yield completely different situations and thus each case must be analyzed separately. We consider the case of infinite ω_0 first.

System with infinite zeros (A, B, C, D) has an infinite zero means that D does not have full row rank. Let us consider an example of such a system.

Example Consider the problem of constructing an innovations filter for the following system

$$\dot{x}(t) = -x(t) + u(t) + w(t) \tag{3.126}$$

$$y(t) = x(t). \tag{3.127}$$

It is not difficult to see that

$$\nu(t) = -u(t) + \dot{y}(t) + y(t) = w(t) \tag{3.128}$$

is an innovation which means that

$$V(s) = \begin{pmatrix} -1 & s+1 \end{pmatrix} \tag{3.129}$$

is an innovations filter. This result suggests that unlike in the regular case, the prefilter and the innovations filter are not proper when (A, B, C, D) has an infinite zero as confirmed by the following result.

Lemma 3.8 *Suppose (A, B, C, D) has an infinite zero. Then, every prefilter for (A, B, C, D) (if any exists) is improper.*

Proof Let

$$W(s) = \begin{pmatrix} W'(s) & W''(s) \end{pmatrix} \tag{3.130}$$

denote a prefilter for (A, B, C, D). Then,

$$\begin{pmatrix} W'(s) & W''(s) \end{pmatrix} \begin{pmatrix} -sI + A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & H(s) \end{pmatrix} \tag{3.131}$$

which implies that

$$\begin{aligned} W'(s)(sI - A) &= W''(s)C \\ W'(s)B + W''(s)D &= H(s) \end{aligned} \tag{3.132}$$

and thus

$$W''(s)\left[C(sI - A)^{-1}B + D\right] = H(s). \tag{3.133}$$

If we assume that $W''(s)$ is proper then by taking limit as s goes to infinity in (3.133) we obtain

$$W''(\infty)D = H(\infty). \tag{3.134}$$

Since $W''(s)$ is square and D does not have full row rank, $H(\infty)$ does not have full row rank which implies that the covariance of the innovations process

$$\Psi = H(s)H^{\sim}(s) = H(\infty)H^{\sim}(\infty) \tag{3.135}$$

is not positive definite. But this is a contradiction and thus, $W''(s)$ and consequently $W(s)$ is not proper. ■

Thus, if we do not accept improper innovations filters, if in the construction of our innovations filter we encounter infinite zeros we say that no innovations filter exists. But if we do accept improper systems we should go ahead and construct them using the method given below.

Lemma 3.9 *Let*

$$\begin{pmatrix} -sI + A & B \\ C & D \end{pmatrix}$$

have full row rank but loose rank at $s = \infty$, *i.e.,* (A, B, C, D) *has an infinite zero. Then, there exists an invertible polynomial matrix* $\Pi(s)$ *with no unstable zero such that*

$$C'(sI - A)^{-1}B + D' = \Pi(s)(C(sI - A)^{-1}B + D) \tag{3.136}$$

where (A, B, C', D') *has no infinite zero, i.e.,* D' *has full row rank.*

Proof If we admitted that $\Pi(s)$ had finite zeros at zero, then C', D' and $\Pi(s)$ would be obtained directly from Silverman's structure algorithm [8], but we require that all the finite zeros of $\Pi(s)$ be stable. For that, consider the system $(A - \alpha I, B, C, D)$ for some $\alpha > 0$ and find C', D' and $\hat{\Pi}(s)$ such that

$$C'(sI - A + \alpha I)^{-1}B + D' = \hat{\Pi}(s)(C(sI - A + \alpha I)^{-1}B + D) \tag{3.137}$$

where $(A+\alpha I, B, C', D')$ has no infinite zero. The existence of C', D' and $\hat{\Pi}(s)$, having zeros at zero, is shown in [8]. It is now easy to see that C', D' and

$$\Pi(s) = \hat{\Pi}(s+\alpha) \tag{3.138}$$

satisfy (3.136), (A, B, C', D') has no infinite zero and $\Pi(s)$ has all of its finite zeros at $-\alpha$. ∎

Construction of the polynomial matrix $\Pi(s)$ and constant matrices C' and D' can be done very effectively using a slight variant of Silverman's structure algorithm.

Structure Algorithm Pick an $\alpha > 0$ and let

$$C' = C \tag{3.139}$$

$$D' = D \tag{3.140}$$

$$\Phi(s) = I \tag{3.141}$$

While D' not full row rank do

- Find $L = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}$ be an orthogonal matrix row compressing D':

$$\begin{pmatrix} D'_1 \\ 0 \end{pmatrix} = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} D' \tag{3.142}$$

  where D'_1 has full row rank, and let

$$\begin{pmatrix} C'_1 \\ C'_2 \end{pmatrix} = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} C'. \tag{3.143}$$

- Let

$$C' = \begin{pmatrix} C'_1 \\ C'_2(\alpha I + A) \end{pmatrix} \tag{3.144}$$

$$D' = \begin{pmatrix} D'_1 \\ C'_2 B \end{pmatrix} \tag{3.145}$$

$$\Phi(s) = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & L_1 \\ C'_2 & (s+\alpha)L_2 \end{array}\right) \Phi(s). \tag{3.146}$$

end do.

The structure algorithm gives us C', D' and the polynomial matrix $\Phi(s)$ which satisfies

$$\begin{pmatrix} -sI+A & B \\ C' & D' \end{pmatrix} = \Phi(s) \begin{pmatrix} -sI+A & B \\ C & D \end{pmatrix}; \tag{3.147}$$

$\Pi(s)$ is nothing but the $(2,2)$-block of $\Phi(s)$.

Theorem 3.5 *Let C', D' and $\Phi(s)$ be obtained from the structure algorithm. Then, $W(s)$ is a prefilter for (A, B, C, D) if and only if*

$$W(s) = W'(s)\Phi(s) \tag{3.148}$$

where $W'(s)$ is a prefilter for (A, B, C', D').

The result follows the fact that $\Phi(s)$ has a stable inverse. The proof is straightforward and is omitted.

System with finite zeros

Theorem 3.6 *Let (A, B, C, D) be detectable, purely stochastic and have finite zeros on the imaginary axis. Then (A, B, C, D) has no prefilter.*

Proof Suppose a prefilter $W(s)$ exists and let

$$\begin{pmatrix} W_1(s) & W_2(s) \end{pmatrix} = W(s) \begin{pmatrix} I & -K \\ 0 & I \end{pmatrix} \tag{3.149}$$

where K is such that $A + KC$ is stable. Then

$$\begin{pmatrix} W_1(s) & W_2(s) \end{pmatrix} \begin{pmatrix} I & K \\ 0 & I \end{pmatrix} \begin{pmatrix} -sI + A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & H(s) \end{pmatrix} \tag{3.150}$$

which implies that

$$W_1(s)(sI - A - KC) = W_2(s)C \tag{3.151}$$

and thus $W_2(s)$ is invertible ($W_2(s)$ is square and full row rank) which means that we can rewrite (3.151)

$$C = W_2{}^{-1}(s)W_1(s)(sI - A - KC). \tag{3.152}$$

Now let s_0 denote a zero of (A, B, C, D) on the imaginary axis. Then

$$\begin{pmatrix} -s_0I + A + KC & B + KD \\ C & D \end{pmatrix} = \begin{pmatrix} I & K \\ 0 & I \end{pmatrix} \begin{pmatrix} -s_0I + A & B \\ C & D \end{pmatrix} \tag{3.153}$$

does not have full row rank which means that there exist a and b such that

$$\begin{pmatrix} a^T & b^T \end{pmatrix} \begin{pmatrix} -s_0I + A + KC & B + KD \\ C & D \end{pmatrix} = 0 \tag{3.154}$$

which implies that

$$a^T(-s_0I + A + KC) + b^TC = 0 \tag{3.155}$$

$$a^T(B + KD) + b^TD = 0 \tag{3.156}$$

which thanks to (3.152) and invertibility of $-s_0I + A + KC$ implies that

$$b^T W_2^{-1}(s_0)W_1(s_0) = a^T \tag{3.157}$$

if $W_2(s_0)$ is invertible. Thus, thanks to (3.156) we get

$$b^T W_2^{-1}(s_0)(W_1(s_0)(B + KD) + W_2(s_0)D) = 0. \tag{3.158}$$

But $b \neq 0$ (otherwise

$$a^T(-s_0I + A + KC) = 0 \tag{3.159}$$

which contradicts invertibility of $-s_0I + A + KC$) and so

$$H(s_0) = W_1(s_0)(B + KD) + W_2(s_0)D \tag{3.160}$$

does not have full row rank. If $W_2(s_0)$ is not invertible, thanks to (3.151) and invertibility of $-s_0I + A + KC$, we can see that

$$\begin{pmatrix} W_1(s_0) & W_2(s_0) \end{pmatrix}$$

does not have full row rank which again implies that $H(s_0)$ does not have full row rank.

But if $H(s_0)$ does not have full row rank, the covariance of the innovations process

$$\Psi = H(s)H^{\sim}(s) = H(s_0)H^{\sim}(s_0) \tag{3.161}$$

is not positive definite which is a contradiction. Thus, $W(s)$ does not exist. ∎

3.4 Summary of the innovations filter construction method

The procedure for the construction of an innovations filter for System (1.1)-(1.2) can be summerized as follows:

a- First consider the problem of constructing $W(s)$ in Theorem 3.1.

1- If B_2 or D_2 are not zero, use Theorem 3.3 to construct Γ and system $\mathcal{S}$ such that $W(s) = W'(s)\Gamma$ where $W'(s)$ is a prefilter for $\mathcal{S}$, if not let $\Gamma = I$ and $\mathcal{S} = (A, B_3, C, D_3)$.

2- If $\mathcal{S}$ is not detectable, use Lemma 3.4 to construct Γ' and $\mathcal{S}'$ such that $W'(s) = W''(s)\Gamma'$ where $W''(s)$ is a prefilter for $\mathcal{S}'$, if not let $\Gamma' = I$ and $\mathcal{S}' = \mathcal{S}$.

3- If $\mathcal{S}$ is regular, construct $W''(s)$ using Theorem 3.2 and let $W(s) = W''(s)\Gamma'\Gamma$. If not

i- If $\mathcal{S}'$ is purely deterministic, use Lemma 3.6 to construct $W''(s)$ and let $W(s) = W''(s)\Gamma'\Gamma$.

ii- If $\mathcal{S}'$ is semi-deterministic semi-stochastic, break up the problem of construction of $W''(s)$ into two problems. In particular, consider constructing $W''(s)$ as

$$W''(s) = \begin{pmatrix} W_1''(s) \\ W_2''(s) \end{pmatrix}. \tag{3.162}$$

For construction of $W_1''(s)$ and $W_2''(s)$ go to 1-. Their constructions never get back to this point because they lead to a purely deterministic and a purely stochastic system. Once you have $W_1''(s)$ and $W_2''(s)$ construct $W''(s)$ and let $W(s) = W''(s)\Gamma'\Gamma$.

iii- If $\mathcal{S}'$ is purely stochastic and contains finite imaginary axis zeros, no innovations filter exists.

iv- If $\mathcal{S}'$ is purely stochastic and contains infinite zeros, construct $\mathcal{S}''$ and the polynomial matrix $\Phi(s)$ using Theorem 3.5 such that $W''(s) = W'''(s)\Phi(s)$ where $W'''(s)$ is a prefilter for $\mathcal{S}''$ where $\mathcal{S}''$ has no zeros at infinity. Then construct $W'''(s)$ using Theorem 3.2 and let $W(s) = W'''(s)\Phi(s)\Gamma'\Gamma$.

b- Let

$$V(s) = W(s) \begin{pmatrix} -B_1 & 0 \\ -D_1 & I \end{pmatrix}. \tag{3.163}$$

Note that innovations filter $V(s)$ exists only if during its construction step iii- (and iv- if we do not accept improper filters) is never reached, i.e., we never encounter the problem of prefilter construction for a system with finite zeros on the imaginary axis (and infinite zeros if we do not accept improper filters).

Also note that if a minimal innovations filter is desired, Lemma 3.4 should systematically be applied in step 2.

4 Failure sensitivity

Even if an innovations filter does exist, it is not necessarily useful in detecting failures. Let us consider the following example:

$$\dot{x}(t) = -x(t) + u(t) + w(t) \tag{4.1}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} v(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} f(t). \tag{4.2}$$

This system does have an innovations filter, but the output of this filter, since it has to be decoupled from v, ignores y_2 which is the only way through which f can affect this output. In this case, the output of the innovations filter is not only decoupled from v but also from f which makes it useless in detecting the failure f.

Definition 4.1 *An innovations filter for system (1.1)-(1.2) is called failure sensitive if its output is not decoupled from f.*

It is not difficult to see that $V(s)$ is not failure sensitive if and only if

$$V(s) \begin{pmatrix} 0 \\ C(sI - A)^{-1}B_4 + D_4 \end{pmatrix} = 0 \tag{4.3}$$

from which we get the following result.

Lemma 4.1 *System (1.1)-(1.2) has a failure sensitive innovations filter if and only if every innovations filter for this system is failure sensitive.*

Thus failure sensitivity is a property of the system and does not depend on the choice of the innovations filter.

Lemma 4.2 *System (1.1)-(1.2) has failure sensitive innovations filters if and only if*

$$W(s)\begin{pmatrix} B_4 \\ D_4 \end{pmatrix} \neq 0 \tag{4.4}$$

where

$$W(s) = V(s)\begin{pmatrix} 0 & 0 \\ C(sI-A)^{-1} & I \end{pmatrix} \tag{4.5}$$

(which is just a $W(s)$ in Theorem 3.1) and $V(s)$ is any innovations filter.

Proof The output of the innovations filter is

$$\begin{aligned} \nu = V(s)\begin{pmatrix} u \\ y \end{pmatrix} &= \mu + V(s)\begin{pmatrix} 0 \\ C(sI-A)^{-1}B_4 + D_4 \end{pmatrix} f \\ &= \mu + W(s)\begin{pmatrix} B_4 \\ D_4 \end{pmatrix} f \end{aligned} \tag{4.6}$$

where μ is the innovations in the absence of failures (it is zero-mean white with covariance Ψ). ∎

If we let (A_w, B_w, C_w, D_w) be the computed realization of $W(s)$, the output of the innovations filter can be expressed as follows

$$\nu(t) = C_w z(t) + \bar{D}_w f(t) + \mu(t) \tag{4.7}$$

$$\dot{z}(t) = A_w z(t) + \bar{B}_w f(t) \tag{4.8}$$

where

$$\bar{B}_w = B_w \begin{pmatrix} B_4 \\ D_4 \end{pmatrix} \tag{4.9}$$

$$\bar{D}_w = D_w \begin{pmatrix} B_4 \\ D_4 \end{pmatrix} \tag{4.10}$$

and $\mu(t)$ is zero mean white with covariance Ψ. Using (4.9)-(4.10) and a-priori assumptions on possible failures $f(t)$, it is now possible to develop necessary statistical test to develop the second step of the failure detection algorithm.

5 Conclusion

The results we have presented in this paper generalize and unify the innovations approach and the robust deterministic approach to residual generation for failure detection. These two approaches can be obtained as special cases in our problem formulation. We have given a complete state-space solution to this problem and a solution method which is based solely on robust matrix manipulations.

References

[1] Basseville, M., Detecting changes in signals and systems- a survey, *Automatica*, vol. 24, 1990.

[2] Basseville, M. and Benveniste, A., (eds), Detection of abrupt changes in signals and dynamical systems, *Lecture notes in Control and Information Science*, vol. 77, Springer-Verlag, 1985.

[3] Frank, P. M., Fault detection in dynamic systems using analytical redundancy: a survey of some new results, *Automatica*, vol. 26, 1990.

[4] Kucera, V., Algebraic Riccati equations: Hermitian and definite solutions, in *The Riccati Equation*, S. Bittanti, A. J. Laub and J. C. Willems (eds), Springer-Verlag, 1991.

[5] Massoumnia, M., A geometric approach to the synthesis of failure detection filters, *IEEE Trans. Aut. Control*, vol. 31, 1986.

[6] Mehra, R. K. and Peschon, J., An innovations approach to fault detection and diagnosis in dynamic systems, *Automatica*, vol. 7, 1971.

[7] Patton, R. J., Frank, P. M. and Clark, R. N., (eds), *Fault diagnosis in dynamic systems*, Prentice Hall, 1989.

[8] Silverman, L. M., Inversion of multivariable linear systems, *IEEE Trans. Aut. Control*, vol. 14, 1969.

[9] Willsky, A. S., A survey of design methods for failure detection in dynamics systems, *Automatica*, vol. 12, 601-611, 1976.

Unité de Recherche INRIA Rocquencourt
Domaine de Voluceau - Rocquencourt - B.P. 105 - 78153 LE CHESNAY Cedex (France)
Unité de Recherche INRIA Lorraine Technopôle de Nancy-Brabois - Campus Scientifique
615, rue du Jardin Botanique - B.P. 101 - 54602 VILLERS LES NANCY Cedex (France)
Unité de Recherche INRIA Rennes IRISA, Campus Universitaire de Beaulieu 35042 RENNES Cedex (France)
Unité de Recherche INRIA Rhône-Alpes 46, avenue Félix Viallet - 38031 GRENOBLE Cedex (France)
Unité de Recherche INRIA Sophia Antipolis 2004, route des Lucioles - B.P. 93 - 06902 SOPHIA ANTIPOLIS Cedex (France)

EDITEUR
INRIA - Domaine de Voluceau - Rocquencourt - B.P. 105 - 78153 LE CHESNAY Cedex (France)

ISSN 0249 - 6399

RR-1914