# Acceleration of the path-following method for optimization over the cone of positive semidefinite matrices

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# ACCELERATION OF THE PATH-FOLLOWING METHOD FOR OPTIMIZATION OVER THE CONE OF POSITIVE SEMIDEFINITE MATRICES

# ACCÉLÉRATION DE LA MÉTHODE DU SUIVI DE CHEMIN POUR L'OPTIMISATION DANS LE CÔNE DES MATRICES SEMI-DÉFINIES POSITIVES

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ABSTRACT

The paper is devoted to acceleration of the path-following interior point polyomial time method for optimization over the cone of positive semidefinite matrices, with applications to quadratically constrained problems and extensions onto the general self-concordant case. In particular, we demonstrate that in a problem involving $m$ of general type $\mu \times \mu$ Linear Matrix Inequalities with $n \geq \mu$ scalar control variables the Conjugate-Gradient-based acceleration allows to reduce the arithmetic cost of an $\epsilon$-solution by a factor of order of $\max\{n^{1/3}\mu^{-1/6}, n^{1/5}\}$: for the Karmarkar-type acceleration this factor is of order of $\min\{n, m^{1/2}\}$. The Conjugate-Gradient-Based acceleration turns out to be efficient also in the case of several specific "structured" problems coming from applications in Control and Graph Theory.

**Key words:** interior-point polynomial time methods; path-following methods; Linear Matrix Inequalities; eigenvalue minimization

RÉSUMÉ

Cet article est consacré à l'accélération de la méthode polynômiale de point intérieur avec suivi de chemin, pour l'optimisation dans le cône des matrices semi-définies positives, avec application aux problèmes d'optimisation quadratique et extension au cas général de fonctions self-concordantes. Nous démontrons en particulier que, dans un problème ayant $m$ inégalités linéaires sur des matrices générales $\mu \times \mu$ dépendant de $n \geq \mu$ variables de commande, l'accélération de type gradient conjugué permet de réduire par un facteur de l'ordre de $\max\{n^{1/3}\mu^{-1/6}, n^{1/5}\}$ le coût d'obtention d'une $\varepsilon$-solution; pour une accélération de type Karmarkar, ce facteur est de l'ordre de $\min\{n, m^{1/2}\}$. L'accélération utilisant le gradient conjugué se trouve être efficace également dans le cas de plusieurs problèmes spécifiques "structurés" provenant d'applications en commande optimale et en théorie des graphes.

**Mots-Clefs:** méthodes polynômiales de points intérieurs; méthodes de suivi de chemin; inégalités linéaires matricielles; minimisation de valeurs propres.

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# 1 Introduction

This paper is devoted to acceleration of the path-following interior point polynomial time method for optimization over the cone of positive semidefinite matrices, with applications to quadratically constrained problems and extensions onto general self-concordant case.

## 1.1 The problem

Let $\mathcal{M} = \{\mu_1, ..., \mu_m\}$ be a collection of positive integers; we call such a collection a *structure* and define its *size* as

$$|\mathcal{M}| = \sum_{i=1}^{m} \mu_i.$$

We associate with a structure $\mathcal{M}$ the space $S_{\mathcal{M}}$ comprised of all $|\mathcal{M}| \times |\mathcal{M}|$ symmetric block-diagonal matrices with $m$ diagonal blocks of the sizes $\mu_1, ..., \mu_m$, and denote by $S^+_{\mathcal{M}}$ the cone comprised of positive semidefinite matrices from $S_{\mathcal{M}}$. In what follows $S_{\mathcal{M}}$ is regarded as a Euclidean space with the standard inner product

$$\langle X, Y \rangle = \mathrm{Tr}\{XY\},$$

Tr being the trace. The corresponding norm will be denoted $\|\cdot\|_2$.

The problem we will deal with – the problem of optimization over the cone of positive semidefinite matrices (PD problem for short) – is as follows:

$$\mathcal{P}_{n,\mathcal{M}}: \quad \textit{minimize} \quad c^T x \quad \textit{subject to} \quad \mathcal{A}(x) \in S^+_{\mathcal{M}}, \tag{1}$$

where $\mathcal{A}(x)$ is an *affine* mapping from $\mathbf{R}^n$ into $S_{\mathcal{M}}$.

This is the natural form for many essentially nonlinear convex programs arising in various applications, from Graph Theory and Combinatorics (computing Lovasz capacity number of a graph [Lo 79]; nonlinear Lagrangean relaxation of a Boolean Programming problem [SD 85]) to Control Theory (numerous problems of Robust Stability Analysis: [BBl 90, BBr 91, BBK 89, BG 92, BY 89, DPZ 91, Do 82, FN 91, FT 86, FT 88, FT 91, FTD 91, GB 86, KR 91, PZPB 91, PZPLB 91]). At the same time it was realized recently that in some aspects PD problems are very similar to LP programs (note that an LP problem with $n$ variables and $m$ inequality constraints is precisely problem $\mathcal{P}_{n,\mathcal{M}}$ associated with $\mu_i = 1$, $i = 1, ..., m$). In particular, polynomial time interior point methods known for LP can be naturally generalized to handle PD programs (see [NN 88, NN 89, NN 91a, Al 91a, Al 91b, Al 92, Ja 91, BG 92]). In this paper we focus on the path-following method for $\mathcal{P}_{n,\mathcal{M}}$ which extends onto PD problems the barrier LP-method of Gonzaga ([Go 87]).

## 1.2 Basic barrier method for PD

The method can be applied to a *strictly feasible* problem $\mathcal{P}_{n,\mathcal{M}}$, i.e., with the image of $\mathcal{A}(\cdot)$ intersecting int $S^+_{\mathcal{M}}$; besides this, one should assume the feasible domain

$$G = \{x \in \mathbf{R}^n \mid \mathcal{A}(x) \in S_{n,\mathcal{M}}\}$$

of the problem to be bounded, which immediately implies that the homogeneous part of $\mathcal{A}$ is of full column rank.

The method is associated with the barrier

$$F(x) = -\ln \mathrm{Det}(\mathcal{A}(x)) \tag{2}$$

for the feasible domain $G$ of the problem and at its *main stage* follows the path

$$x^*(t) = \mathrm{argmin}\,\{F_t(x) = tc^T x + F(x) \mid x \in \mathrm{int}\, G\} \tag{3}$$

as $t \to +\infty$. Namely, it generates sequential pairs $(x^i, t_i)$ satisfying, for certain fixed positive $\kappa < 1$ (*path tolerance*, the first parameter of the method) the predicate

$$\mathcal{C}_\kappa(x,t)\colon \{x \in \mathrm{int}\, G,\ t > 0\}\ \&\ \left\{\left((F_t'(x))^T(F''(x))^{-1}(F_t'(x))\right)^{1/2} < \kappa\right\}. \tag{4}$$

The pairs $(x^i, t_i)$ are updated as follows. First, we update the penalty parameter $t$:

$$t_{i+1} = \exp\left\{\frac{\vartheta}{|\mathcal{M}|^{1/2}}\right\} t_i; \tag{5}$$

the *penalty rate* $\vartheta > 0$ is the second parameter of the method.

To describe the rule for updating $x$, denote by $e(x, F_t)$ the Newton direction of $F_t$ at $x$, i.e.,

$$e(x, F_t) = (F''(x))^{-1}F_t'(x) \quad (= (F_t''(x))^{-1}F_t'(x));$$

this direction is well-defined, since $F''$ is nondegenerate on $\mathrm{int}\, G$ (as we already have mentioned, the homogeneous part of $\mathcal{A}$ is of full column rank). Let the *Newton decrement* $\lambda(x, F_t)$ be defined as

$$\lambda(x, F_t) = \{(e(x, F_t))^T F''(x) e(x, F_t)\}^{1/2}.$$

To update $x^i$, we minimize $F_t$ by the following Newton-type procedure:

$$y^0 = x^i; \quad y^{s+1} = y^s - \frac{1}{1 + \lambda(y^s, F_{t_{i+1}})}(F''(y^s))^{-1}F_{t_{i+1}}'(y^s) \tag{6}$$

Process (6) is terminated when the pair $(y^s, t_{i+1})$ satisfies $\mathcal{C}_\kappa(\cdot,\cdot)$, or, which is the same, when $\lambda(y^s, F_{t_{i+1}})$ becomes less than $\kappa$, and the corresponding $y^s$ is taken as $x^{i+1}$.

It is known (see [NN 88, NN 89, NN 91a]) that

**B.1.** Process (6) terminates after $O(1)$ steps, provided that $(x^i, t_i)$ satisfies $\mathcal{C}_\kappa(\cdot,\cdot)$;

**B.2.** For any pair $(x, t)$ satisfying $\mathcal{C}_\kappa(\cdot,\cdot)$ one has

$$c^T x - c^T x^* \le O(1)|\mathcal{M}|t^{-1}$$

where $x^*$ is the optimal solution to $P_{\cdot|\mathcal{M}}$

(in this section all $O(1)$ are positive cosnatnts depending only on the parameters $\kappa$ and $\vartheta$ of the method).

Thus, the presented method $\mathcal{B}$ (we shall refer to it as to the *basic* one) improves the accuracy of current approximate solution $x$ by an absolute constant factor, say, 2, at the cost of no more than

$$O(1)|\mathcal{M}|^{1/2}$$

steps of the type

$$y \mapsto y - \gamma(F''(y))^{-1}F'(y). \tag{7}$$

Of course, one should find an appropriate initial pair to start the method, i.e., a pair satisfying $\mathcal{C}_\kappa(\cdot,\cdot)$; this can be done by the same technique, provided that we are given in advance a strictly feasible initial solution $w$ (i.e., a point $w \in \operatorname{int} G$). Namely, it suffices to follow in the above manner the auxiliary path

$$z_*(\tau) = \operatorname{argmin}\{\Phi_\tau(z) = -\tau z^T F'(w) + F(z)\}$$

maintaining the predicate

$$\mathcal{C}'_{\kappa/2}(z,\tau):$$

$$\{z \in \operatorname{int} G,\ \tau > 0\}\ \&\ \left\{\left((\Phi'_\tau(z))^T(F''(z))^{-1}\Phi'_\tau(z)\right)^{1/2} < \kappa/2\right\}.$$

To follow this auxiliary path, one starts with the pair $(w,1)$ (note that $w = z_*(1)$, so that $(w,1)$ satisfies $\mathcal{C}'_0(\cdot,\cdot)$ and now there is no problem with a proper starting pair). When following the path $z_*(\cdot)$ we, instead of increasing penalty parameter $\tau$, decrease it at the same rate:

$$\tau_{s+1} = \exp\left\{-\frac{\vartheta}{|\mathcal{M}|^{1/2}}\right\}\tau_s,$$

so that the path $z_*$ is followed as $\tau \to 0$. This *preliminary stage* is terminated at the first step when it turns out that the current iterate $z$ satisfies, say, $\mathcal{C}_{2\kappa/3}(\cdot,0)$, and this $z$ is taken as $x^0$. Now it is not difficult to choose $t_0 > 0$ such that $(x^0,t_0)$ is an appropriate starting pair for the main stage.

The main result on this method is as follows:

**Theorem 1.1** *Let $\varepsilon \in (0,1)$. The described two-stage path-following method associated with path tolerance $\kappa < 1$ and penalty rate $\vartheta > 0$, being applied to $\mathcal{P}_{n,\mathcal{M}}$, finds an $\varepsilon$-solution to the problem, i.e., a feasible solution $x_\varepsilon$ such that*

$$c^T x_\varepsilon - c^T x^* \le \varepsilon\{\max_G c^T x - \min_G c^T x\},$$

*in no more than*

$$O(1)|\mathcal{M}|^{1/2}\ln\left\{\frac{2|\mathcal{M}|}{\varepsilon\gamma}\right\}$$

*Newton-type steps (7) of the both stages. Here the quantity*

$$\gamma = \max\{t \mid w + t(w - G) \subset G\}$$

*is responsible for the quality of the starting point $w$ (this is the "asymmetry coefficient" of $G$ with respect to $w$). The constant factor denoted by $O(1)$ depends only on the path tolerance $\kappa$ and the penalty rate $\vartheta$.*

This statement is an immediate consequence of

(a) the fact that barrier (2) is $|\mathcal{M}|$-*self-concordant* (see [NN 88, NN 89]), and

(b) the general result on path-following methods associated with self-concordant barriers established in the latter paper.

## 1.3 Acceleration: motivation and summary of results

In the above theorem the complexity of the method is measured in total number $N(\varepsilon)$ of Newton-type steps (6); the arithmetic complexity $M_{\mathcal{B}}(\varepsilon)$ of the method (i.e., the total number of arithmetic operations) is $N(\varepsilon)$ times the arithmetic cost of updating (7). We see that

$$M_{\mathcal{B}}(\varepsilon) = M_{\mathcal{B}} \ln \left\{ \frac{2|\mathcal{M}|}{\varepsilon\gamma} \right\},$$

where $M_{\mathcal{B}}$ is the arithmetic cost "per twicening the penalty parameter", i.e., the cost at which one can solve the following Basic Subproblem

(BSP): given a pair $(x, T)$ satisfying $\mathcal{C}_{O(\kappa)}(\cdot,\cdot)$, transform it into a pair $(x', T')$ of the same type with $T' = 2T$

(we are speaking about the main stage; at the preliminary stage $\mathcal{C}$ should be replaced by $\mathcal{C}'$ and the relation $T' = 2T$ by $T' = T/2$).

For the basic method $\mathcal{B}$ one clearly has

$$M_{\mathcal{B}} = |\mathcal{M}|^{1/2} \mathcal{N}_{\mathcal{P}}, \tag{8}$$

where $\mathcal{N}_{\mathcal{P}}$ is the arithmetic cost at which, given $x$, $b$ and $t$, one can form and solve the Newton system

$$F''(x)h = F'(x) + b. \tag{9}$$

It is well-known that in the case of LP the path-following method admits acceleration, i.e., there are possibilities to solve (BSP) at the cost less (in order of dependence of the sizes $n$ and $m$) than $M_{\mathcal{B}}$. All known acceleration schemes in LP are based on the following two observations:

(i) when solving (BSP), one can use "proper" approximate solutions to systems (9) instead of exact ones;

(ii) since systems (9) arising at neighbouring steps are close to each other, there is a possibility to handle them in such a way that the average (per system) arithmetic cost of finding "proper" approximate solutions will be less in order than that one of finding exact solutions.

In the case of LP there are several ways to use these observations. Most of researchers (starting from Vaidya [Va 87] and Gonzaga [Go 87]) used various forms of the strategy

originating from Karmarkar [Ka 84]: approximate solutions $\xi_i$ to the sequential Newton systems

$$H_i\xi = b_i \tag{10}$$

are computed as

$$\xi_i = (H_i')^{-1}b_i, \tag{11}$$

where symmetric positive definite matrices $H_i'$ are compatible with the precise Hessians $H_i$ within a factor $\rho$ close to 1:

$$\rho^{-1}H_i \leq H_i' \leq \rho H_i \tag{12}$$

(throughover the paper inequality $A \leq B$ between matrices means that $B - A$ is positive semidefinite). It turns out that in an LP problem with $n$ variables and $m = O(n)$ inequality constraints there is a possibility to maintain (12) by updatings

$$\{H_i', (H_i')^{-1}\} \mapsto \{H_{i+1}', (H_{i+1}')^{-1}\}$$

based on rank 1 corrections, with $O(n^{1/2})$ corrections (at average) per Newton step. This allows to solve (BSP) in $O(n^3)$ operations (*if the traditional linear algebra technique is used; this assumption is used throughover the paper*).

A somewhat different acceleration strategy for LP was developed in [NN 89, NN 91b]. The main difference with the Karmarkar scheme is that now "proper" $\xi_i$ is computed by a prescribed number $K = K(n, m)$ of steps of the Conjugate Gradient method, the method being associated with the Euclidean structure defined by the inner product $\langle x, y\rangle_i = y^T H_i' x$. Since now $K$ is much greater than 1 ($K = 1$ corresponds to the Karmarkar acceleration), we can relax the "compatibility condition" (12) by replacing fixed and close to 1 parameter $\rho$ by "large" (tending to $\infty$ as $n, m$ grow) $\rho = \rho(n, m)$; this allows to make rank 1 corrections more rare. The total cost of solving (BSP) in the case of $m = O(n)$, anyhow, is the same as above, i.e., $O(n^3)$ (the advantages of the latter scheme in the LP case are seen only when "fast multiplication" of matrices is used).

As far as we know, all developed so far acceleration schemes for path-following methods deal with LP (or, which is basically the same for our topic, with linearly constrained quadratic problems). The main goal of the paper is to demonstrate that the Conjugate-Gradient-based acceleration scheme close to that one developed in [NN 89, NN 91b] can be naturally extended onto PD programming (Sections 2 and 3), and in a sense even onto general nonlinear problems (Section 6). In the case of "general" PD problems (no specific sparsity-type stucture) the effect of acceleration varies from $n^{1/3}$ (case of $\mu_i \backsim \mu \backsim 1$) to $n^{1/5}$ (case of $\mu_i \backsim \mu \backsim n$). Besides this "unstructured" case, we consider several specific "structured" PD problems coming from applications; for all these structures, Conjugate-Gradient-based acceleration also is efficient.

We demonstrate also that PD programming admits a Karmarkar-type acceleration, provided that all $\mu_i$ are of the same order of magnitude and the number $m$ of diagonal blocks in the image space $S_{\mathcal{M}}$ is large (Sect. 4); the effect of acceleration here is $\min\{n, m^{1/2}\}$. Section 5 is devoted to acceleration in quadratically constrained quadratic programming.

The background for this paper is formed by theory of self-concordant functions which takes its origin in [Ne 88a, Ne 88b] and is developed in [NN 88, NN 89, NN 91a]. As far as the path-following methods are concerned, a self-contained summary of the latter studies (with complete proofs) can be found in 20-page appendix to [B-TN 92]; at the moment this summary might be more available for the reader than the original papers.

# 2 Conjugate-Gradient-Based Acceleration

## 2.1 Preliminary remarks

Let us first indicate what are "proper" approximate solutions to Newton systems arising at sequential steps of the method. We say that a vector $\xi$ is a $\nu$-solution to a linear system

$$Vz = b$$

associated with a symmetric positive definite matrix $V$, if

$$(z - z_*)^T V(z - z_*) \leq \nu^2 (0 - z_*)^T V(0 - z_*) \quad (= \nu^2 b^T V^{-1} b)$$

where $z_*$ is the exact solution to the system (i.e., $z$ should be $1/\nu$ times closer to $z_*$ than the trivial approximate solution 0, the distances being measured in the norm associated with the inner product $u^T V v$).

Now assume that we modify the basic path-following method as follows: given certain $\nu > 0$, we replace exact Newton steps (6) with the process

$$y^0 = x^i; \quad y^{s+1} = y^s - \frac{1}{1 + \lambda_s} \xi_s, \quad \lambda_s = \{(\xi_s)^T F''(y^s) \xi_s\}^{1/2}, \tag{13}$$

where $\xi_s$ is a $\nu$-solution to the system

$$F''(y^s)\xi = F'_{t_{i+1}}(y^s). \tag{14}$$

We terminate process (13) at the first step where $\lambda_s < \kappa/4$ and take the corresponding $y^s$ as $x^{i+1}$.

In what follows we refer to the whole process (13) as to an *iteration*, and we say that this iteration is associated with value $t_{i+1}$ of the penalty parameter; updatings $y^s \mapsto y^{s+1}$ will be called *Newton steps*. The above modification of $\mathcal{B}$ will be referred to as $\mathcal{B}_\nu$ (in fact, of course, this is not a *single* modification, but a family of them differing in the manner of finding $\nu$-solutions to the Newton systems; more precise wording would be "a method of the type $\mathcal{B}_\nu$").

Following the line of argument from [NN 89], one can prove

**Proposition 2.1** *For any pair $\kappa \in (0,1)$, $\vartheta > 0$ one can find positive $\nu = \nu(\kappa, \vartheta)$ such that*

(i) *Each iteration of $\mathcal{B}_{\nu(\kappa,\vartheta)}$ takes $O(1)$ Newton steps, and the pair $(x^{i+1}, t_{i+1})$ produced at $i$-th iteration satisfies the predicate required by the stage in question (i.e., either $\mathcal{C}'_{\kappa/2}$,*

*or $\mathcal{C}_\kappa$). In particular, $\mathcal{B}_{\nu(\kappa,\vartheta)}$ solves (BSP) in no more than $O(1)|\mathcal{M}|^{1/2}$ Newton steps, with $O(1)$ depending on $(\kappa,\vartheta)$ only.*

(ii) *There exists some (depending on $\kappa,\vartheta$ only) constant $\rho_0$ such that for all iterates $y^s$ generated by $\mathcal{B}_{\nu(\kappa,\vartheta)}$ at an iteration associated with certain value $t$ of the penalty one has*

$$\rho_0^{-1}F''(x^*(t)) \leq F''(y^s) \leq \rho_0 F''(x^*(t)) \tag{15}$$

(iii) *There exists some (depending on $\kappa,\vartheta$ only) constant $\rho_1$ such that for all iterates $y^s$ generated by $\mathcal{B}_{\nu(\kappa,\vartheta)}$ at an iteration associated with certain value $t$ of the penalty one has*

$$\rho_1^{-1}\mathcal{A}(x^*(t)) \leq \mathcal{A}(y^s) \leq \rho_1\mathcal{A}(x^*(t)). \tag{16}$$

Since the complexity bound from Theorem 1.1 in fact depends only on the amount of Newton steps spent to solve sequential problems (BSP), it follows that the method $\mathcal{B}_{\nu(\kappa,\vartheta)}$ possesses the efficiency estimate (in terms of the number of steps) indicated in Theorem 1.1.

From now on let us fix $\kappa \in (0,1)$ and $\vartheta > 0$, and let $\nu = \nu(\kappa,\vartheta)$; by default all $O(1)$ below depend on these parameters only.

## 2.2 Acceleration strategy: outline

Our goal is to find a strategy for implementing $\mathcal{B}_\nu$ with better arithmetic cost of solving (BSP) than that one for the basic path-following method $\mathcal{B}$, where we solve exactly the sequential Newton systems. For the sake of definiteness, let us speak about problem (BSP) associated with the main stage (the case of the preliminary stage is completely similar).

Thus, we are given a pair $(x,T)$ satisfying $\mathcal{C}_\kappa$ and wish to transform it into a pair $(x',T'=2T)$ with the same property.

Our strategy is as follows: we split the whole segment $[T,2T]$ of values of the penalty parameter $t$ into sequential segments $\Delta_q = [T_q, T_{q+1}]$, of length $\delta T$ each (the last segment may be shorter); the quantity $\delta \leq 1$ will be specified later. Now we have $O(1/\delta)$ subproblems ($\mathrm{BSP}_q$) of tracing the path along the segments $\Delta_q$. We solve ($\mathrm{BSP}_q$) as follows: for $t = T_q$ we, same as in the basic method $\mathcal{B}$, compute the Hessian $\mathcal{H}_q = F''(x(T_q))$ and then compute the inverse of the Hessian; here $x(t)$ denotes the point at which we start process (13) at the iteration associated with value $t$ of the penalty, i.e., $x(t)$ is the result of the preceding iteration.

Then we sequentially increase the penalty parameter according to (5) and update $x$'s according to (13). It remains to explain how we compute approximate Newton directions $\xi_s$. This is done by applying to the corresponding system (14) a prescribed number $K$ of steps of the Conjugate Gradient method "prescaled" by the matrix $\mathcal{H}_q$ (see below).

In what follows we refer to the presented scheme as to the *Conjugate-Gradient-based* (CG-based for short). Note that the scheme itself is independent of the concrete analytical nature of the PD-problem.

## 2.3 Prescaled Conjugate Gradients: description

Let us indicate exactly what is the "prescaled Conjugate Gradient method". Let $U$ and $V$ be positive definite symmetric matrices of the same size. Assume that we know $U^{-1}$ (more precisely, know how to multiply vectors by this matrix) and wish to solve the equation

$$Vz = b. \tag{17}$$

To this end let us rewrite the equation as

$$W\zeta \equiv (U^{-1/2}VU^{-1/2})\zeta = \beta \equiv U^{-1/2}b, \quad z = U^{-1/2}\zeta \tag{18}$$

and apply to the latter equation the usual Conjugate Gradient method:

$$\zeta_0 = 0, \ \pi_0 = 0;$$

$$\rho_s = W\zeta_s - \beta, \ \pi_{s+1} = \rho_s - \frac{\rho_s^T W\pi_s}{\pi_s^T W\pi_s}\pi_s, \ \zeta_{s+1} = \zeta_s - \frac{\rho_s^T W\pi_{s+1}}{\pi_{s+1}^T W\pi_{s+1}}\pi_{s+1}$$

(here a fraction with zero denominator by definition is zero). When rewritten in $z$-variables, the process becomes

$$z_0 = 0, \ p_0 = 0;$$

$$r_s = Vz_s - b, \ p_{s+1} = U^{-1}r_s - \frac{r_s^T U^{-1}Vp_s}{p_s^T Vp_s}p_s, \ z_{s+1} = z_s - \frac{r_s^T U^{-1}Vp_{s+1}}{p_{s+1}^T Vp_{s+1}}p_{s+1}.$$

This latter process will be called the Conjugate Gradient method *prescaled* by matrix $U$. Note that each step of this process requires two matrix-vector multiplications involving $U^{-1}$ and two more involving $V$ (and $O(\dim z)$ additional arithmetic operations); in fact one can make only one multiplication by $V$ and one by $U^{-1}$ per step using recurrent formulae for $r_s$ and $U^{-1}r_s$.

Now let us formulate the only property of the prescaled Conjugate Gradients we need. To this end let us introduce a simple notion as follows.

**Definition 2.1** Let $U$ and $V$ be a pair of symmetric positive definite matrices of the same size and let $\rho_r, \rho_l \geq 1$ be a pair of reals. We say that $V$ is $[\rho_l, \rho_r]$-compatible with $U$ if

$$\rho_l^{-1}U \leq V \leq \rho_r U.$$

We say that $V$ is $[\rho_l, \rho_r]$-compatible with $U$ up to a correction of rank $k$, if there exists a symmetric *positive semidefinite* matrix $V'$ of rank $k$ such that $V - V'$ is $[\rho_l, \rho_r]$-compatible with $U$.

$k$-rank matrix $V'$ mentioned in Definition 2.1 will be called *correction* to $V$.

**Remark 2.1** *It is clearly seen that*

(i) *$V$ is $[\rho_l, \rho_r]$-compatible with $U$ up to a correction of rank $k$ if and only if $W \equiv U^{-1/2}VU^{-1/2}$ is $[\rho_l, \rho_r]$-compatible with the unit matrix $I$ up to a correction of the same*

*rank $k$, and if it is the case and $W'$ is a correction to $W$, then $V' = U^{1/2}W'U^{1/2}$ is a correction to $V$.*

(ii) *A symmetric positive definite matrix $W$ is $[\rho_l, \rho_r]$-compatible with $I$ if and only if all eigenvalues of $W$ are $\geq \rho_l^{-1}$ and at most $k$ eigenvalues of $W$ are $> \rho_r$ (from now on, all egienvalues are counted with their multiplicities). If it is the case, then a correction to $W$ can be chosen to have the same eigenvectors as $W$.*

In view of the latter remark, if $U$ and $V$ are of certain common block-diagonal structure and $V$ is $[\rho_l, \rho_r]$-compatible with $U$ up to a correction of rank $k$, then this correction can be chosen to have the same block-diagonal structure.

Now let us formulate the aforementioned property of the prescaled Conjugate Gradients.

**Proposition 2.2** *Let system (17) associated with positive definite $V$ be solved by the Conjugate Gradient method prescaled by a positive definite matrix $U$. Assume that, for some $\rho_l, \rho_r, k$, $V$ is $[\rho_l, \rho_r]$-compatible with $U$ up to a correction of rank $k$. Then for any $\nu \in (0, 1/2)$ the method finds a $\nu$-solution to the system in no more than*

$$k + C_{\mathrm{CG}}(\rho_l\rho_r)^{1/2}\ln(\frac{2}{\nu})$$

*steps, $C_{\mathrm{CG}}$ being an absolute constant (one can set $C_{\mathrm{CG}} = 1$). All further approximate solutions found by the method also are $\nu$-solutions to the system.*

This statement should be regarded as well-known; to make the paper more self-contained, we present the proof in the Appendix.

In the outlined acceleration strategy the prescaled by $\mathcal{H}_q$ Conjugate Gradients should solve to the accuracy $\nu(\kappa, \vartheta)$ all Newton systems arising as the penalty passes through the segment $\Delta_q$. To this end we should properly choose the number $K$ of steps of Conjugate Gradients. According to Propsition 2.2, in order to choose $K$ properly it suffices to understand how "far" could be the corresponding Hessians from the matrix $\mathcal{H}_q$. This is what we are coming to.

## 2.4 The Main Inequality

What follows extends straightforwardly to the case of PD problems the "main inequality" from [NN 89, Chpt.5].

Let us write explicitly the expressions for the gradient and the Hessian of the barrier $F$ at a point $x \in \mathrm{int}\, G$. Let

$$\mathcal{A}(x) = Ax + a \tag{19}$$

be the representation of the affine mapping $\mathcal{A}$ as the sum of a homogeneous mapping and a constant. One evidently has

$$F'(x) = -A^*(\mathcal{A}^{-1}(x)), \tag{20}$$

$$F''(x)h = A^*(\mathcal{A}^{-1}(x)(Ah)\mathcal{A}^{-1}(x)); \tag{21}$$

here $A^*$ denotes the conjugate to $A$ linear operator from $S_{\mathcal{M}}$ into $\mathbf{R}^n$ (the usual notation $A^T$ would be inconvenient, since the range space of $A^*$ is comprised of matrices, and expressions of the type $A^T X$ would be confusing: $A^*X$ is not a product of matrices, but the image of a vector (which occasionaly is a matrix itself) under a linear mapping).

From (21) it follows that

$$h^T F''(x)h = \| \mathcal{A}^{-1/2}(x)(Ah)\mathcal{A}^{-1/2}(x) \|_2^2 \tag{22}$$

(recall that $\| \cdot \|_2$ denotes the Euclidean norm on $S_{\mathcal{M}}$ associated with the inner product $\langle X, Y \rangle = \mathrm{Tr}(XY)$).

Let us start with the following key (although very simple) statement.

**Lemma 2.1** *Let $t$ and $t_+$ be two values of the penalty parameter, and let*

$$X = \mathcal{A}(x^*(t)), \; X_+ = \mathcal{A}(x^*(t_+))$$

*be the images in $S_{\mathcal{M}}$ of the corresponding points of the path. Set*

$$Z = (X^{-1/2} X_+ X^{-1/2})^{1/2}.$$

*Then*

$$\| \sqrt{t_+} Z - \sqrt{t} Z^{-1} \|_2^2 = |\mathcal{M}| \frac{(t - t_+)^2}{(\sqrt{t} + \sqrt{t_+})^2}. \tag{23}$$

**Proof.** By definition of the path $x^*$ (see (3)) and in view of (20) we have

$$A^*(X^{-1}) = tc, \quad A^*(X_+^{-1}) = t_+ c.$$

Multipluing these equalities from the left by $(x_+ - x)^T$ and taking into account that $A(x_+ - x) = X_+ - X$, we obtain

$$\left\langle X^{-1}, X_+ - X \right\rangle = (t/t_+) \left\langle X_+^{-1}, X_+ - X \right\rangle,$$

whence, in view of $\langle Y^{-1}, Y \rangle \equiv \mathrm{Tr}(Y^{-1}Y) = |\mathcal{M}|$,

$$\mathrm{Tr}(X^{-1}X_+ + (t/t_+)X_+^{-1}X) = \left\langle X^{-1}, X_+ \right\rangle + (t/t_+) \left\langle X_+^{-1}, X \right\rangle = |\mathcal{M}|(1 + \frac{t}{t_+}). \tag{24}$$

The left hand side of the latter relation clearly equals to $\mathrm{Tr}(Z^2 + (t/t_+)Z^{-2})$, which, in turn, is nothing but $\mathrm{Tr}((Z - \sqrt{t/t_+}Z^{-1})^2) + 2|\mathcal{M}|(t/t_+)^{1/2}$, and (23) immediately follows from (24). ∎

**Lemma 2.2** *Let $t$, $t_+$ be two values of the penalty parameter, and let $x = x^*(t)$, $X = \mathcal{A}(x)$, $Y = X^{-1}$, $x_+ = x^*(t_+)$, $X_+ = \mathcal{A}(x_+)$, $Y_+ = X_+^{-1}$. Set*

$$\rho_l = \max \left\{ 4 \frac{\max\{t, t_+\}}{\min\{t, t_+\}}, |\mathcal{M}| \frac{(t - t_+)^2}{\min^2\{t, t_+\}} \right\} \tag{25}$$

*Then for every $\rho_r$ such that*

$$\rho_r \geq 4\frac{\max\{t, t_+\}}{\min\{t, t_+\}} \tag{26}$$

*the matrix $Y_+$ is $[\rho_l, \rho_r]$-compatible with $Y$ up to a rank $k$-correction with*

$$k = \lfloor \frac{1}{\rho_r} |\mathcal{M}| \frac{(t - t_+)^2}{\min^2\{t, t_+\}} \rfloor \tag{27}$$

*(from now on, $\lfloor a \rfloor$ denotes the largest integer not exceeding $a$).*

**Proof.** Let $Z$ be the same as in Lemma 2.1, and let $\lambda_1, ..., \lambda_{|\mathcal{M}|}$ be the eigenvalues of $Z$ taken with their multiplicities. Relation (23) means that

$$\sum_{i=1}^{|\mathcal{M}|} \left( \sqrt{t_+}\lambda_i - \sqrt{t}\frac{1}{\lambda_i} \right)^2 = |\mathcal{M}| \frac{(t - t_+)^2}{(\sqrt{t} + \sqrt{t_+})^2}$$

In view of this inequality and (26) at most $k$ of $\lambda_i$, with $k$ given by (27), can be less than $\rho_r^{-1/2}$. Indeed, each $\lambda_i$ with the latter property corresponds to a left hand side term which is at least $\frac{1}{4}\rho_r \min\{t, t_+\}$, while the right hand side of the inequality is at most

$$|\mathcal{M}| \frac{(t - t_+)^2}{4 \min\{t, t_+\}}.$$

By similar reasoning, no $\lambda_i$ can be greater than $\rho_l^{1/2}$. Consequently, no more than $k$ eigenvalues of $Z^{-2} = Y^{-1/2} Y_+ Y^{-1/2}$ (counted with their multiplicities) can be to the right of the segment $[\rho_l^{-1}, \rho_r]$ and no one of them can be to the left of the segment. This observation, in view of Remark 2.1, completes the proof. ∎

**Proposition 2.3** *Let $t$, $t_+$ be two values of the penalty parameter, let $x = x^*(t)$, $x_+ = x^*(t_+)$ be the corresponding points of the path, and let $\rho_l$ be defined by (25) and $\rho_r$ satisfy (26). Then the matrix $F''(x_+)$ is $[\rho_l^2, \rho_r^2]$-compatible with $F''(x)$ up to a correction of rank $k$, with $k$ given by*

$$k = \max\{\mu_1 + 1, ..., \mu_m + 1\} \lfloor \frac{1}{\rho_r} |\mathcal{M}| \frac{(t - t_+)^2}{\min^2\{t, t_+\}} \rfloor. \tag{28}$$

**Proof. 1.** We start with the following simple statement.

**Lemma 2.3** *Let for a positive definite matrix $W \in S_{\mathcal{M}}$ the $n \times n$ matrix $P_W$ be defined by the relation*

$$P_W h = A^*(WA(h)W) \tag{29}$$

*(note that $P_W$ evidently is symmetric positive definite: strict positivity follows form the fact that $A$ is of full column rank, see Sect. 1.2).*

*Now, let $U$, $V$ be a pair of positive definite matrices from $S_{\mathcal{M}}$ such that $V$ is $[a, b]$-compatible with $U$:*

$$a^{-1}U \leq V \leq bU. \tag{30}$$

*Then $P_V$ is $[a^2, b^2]$-compatible with $P_U$:*

$$a^{-2}P_U \leq P_V \leq b^2 P_U. \tag{31}$$

*In particular, if $U \leq V$, then $P_U \leq P_V$.*

**Proof.** Relation (30) means that the matrix

$$Z = V^{1/2}U^{-1/2}$$

satisfies the inequalities

$$a^{-1}w^T w \leq (Zw)^T(Zw) \leq bw^T w, \quad w \in \mathbf{R}^{|\mathcal{M}|}. \tag{32}$$

We have

$$V^{1/2} = ZU^{1/2}, \quad V^{1/2} = U^{1/2}Z^T \tag{33}$$

(the first equality - by definition, the second - by taking the transposes). One clearly has

$$h^T P_W h = \mathrm{Tr}(W^{1/2}A(h)WA(h)W^{1/2}); \tag{34}$$

since $a^{-1}U \leq V \leq bU$, we conclude that

$$a^{-1}Q(h) \equiv a^{-1}\mathrm{Tr}(V^{1/2}A(h)UA(h)V^{1/2}) \leq h^T P_V h \leq bQ(h). \tag{35}$$

Further (see (33)),

$$Q(h) = \mathrm{Tr}(V^{1/2}A(h)UA(h)V^{1/2}) = \mathrm{Tr}(ZU^{1/2}A(h)UA(h)U^{1/2}Z^T). \tag{36}$$

Now,

$$R_h \equiv U^{1/2}A(h)UA(h)U^{1/2}$$

is a symmetric positive semidefinite matrix; let

$$R_h = \sum_{i=1}^{|\mathcal{M}|} f_i f_i^T$$

be its representation as a sum of rank 1 positive semidefinite matrices. We have (see (36))

$$Q(h) = \mathrm{Tr}(ZR_h Z^T) = \sum_{i=1}^{|\mathcal{M}|}(Zf_i)^T(Zf_i),$$

whence, in view of (32),

$$a^{-1}\sum_{i=1}^{|\mathcal{M}|} f_i^T f_i \leq Q(h) \leq b\sum_{i=1}^{|\mathcal{M}|} f_i^T f_i,$$

or, in view of origin of $f_i$ and (34)

$$a^{-1}h^T P_U h \equiv a^{-1}\mathrm{Tr}\left(U^{1/2}A(h)UA(h)U^{1/2}\right) \leq Q(h) \leq bh^T P_U h.$$

This inequality combined with (35) leads to

$$a^{-2}h^T P_U h \le h^T P_V h \le b^2 h^T P_U h,$$

as required in (31). The "In particular" part of the conclusion is nothing but (31) for the case of $a = 1$. ∎

**2.** Now, in view of (21) and (29) we have

$$F''(x) = P_Y, \quad F''(x_+) = P_{Y_+}. \tag{37}$$

From Lemma 2.2 we know that

$$\rho_l^{-1} Y \le Y_+,$$

which in view of (37) and Lemma 2.3 leads to

$$\rho_l^{-2} F''(x) \le F''(x_+). \tag{38}$$

Further, same Lemma 2.2 says that there exists a positive definite matrix $Y' \le Y_+$ of the same block-diagonal structure as $Y_+$ which, on one hand, differs from $Y_+$ by a $k'$-rank matrix, $k'$ being given by (27), and on the other hand satisfies the relation

$$Y' \le \rho_r Y.$$

The latter inequality in view of Lemma 2.3 implies

$$P_{Y'} \le \rho_r^2 F''(x),$$

and since $Y' \le Y_+$, we also have (same Lemma 2.3) $P_{Y'} \le P_{Y_+} = F''(x_+)$. We see that all we need in order to complete the proof is to demonstrate that.

$$\mathrm{rank}(P_{Y_+} - P_{Y'}) \le k' \max\{\mu_1 + 1, ..., \mu_m + 1\}. \tag{39}$$

(compare the definition of $k'$ and the right hand side of (28)). This is an immediate consequence of the following

**Lemma 2.4** *If $U$, $V \in S_{\mathcal{M}}$ are such that* $\mathrm{rank}(U - V) = 1$, *then*

$$\mathrm{rank}(P_U - P_V) \le \max\{\mu_1 + 1, ..., \mu_m + 1\}.$$

**Proof.** Without loss of generality we may assume that $U - V = ff^T$, where $f \in \mathbf{R}^{\mu_1} \times ... \times \mathbf{R}^{\mu_m}$. Since $U$ and $V$ are of the same block-diagonal structure $\mathcal{M}$, all but one projections of $f$ onto the direct factors in the product $\mathbf{R}^{\mu_1} \times ... \times \mathbf{R}^{\mu_m}$ are zeros. That evidently means that we can reduce our considerations to the case when $m = 1$, where $U$ and $V$ are $\mu \times \mu$ matrices and $f \in \mathbf{R}^{\mu}$, $\mu$ being one of $\mu_i$. We have

$$P_U - P_V = A^*(\mathcal{R}\mathcal{S}_f + \mathcal{T}_f),$$

where:

$\mathcal{S}_f$ is the mapping from $\mathbf{R}^n$ into the space $L_\mu$ of $\mu \times \mu$ matrices which maps a vector $h \in \mathbf{R}^n$ into the matrix $f(VA(h))^T$;

$\mathcal{R}$: $X \mapsto X + X^T$ is the "symmetrization mapping" from $L_\mu$ into the space $S_\mu$ of symmetric $\mu \times \mu$ matrices;

$\mathcal{T}_f$ is the mapping from $\mathbf{R}^n$ into $S_\mu$ which maps $h \in \mathbf{R}^n$ into the matrix $(f^T A(h) f) f f^T$.

Since evidently $\operatorname{rank} \mathcal{S}_f \leq \mu$ and $\operatorname{rank} \mathcal{T}_f \leq 1$, we conclude that $\operatorname{rank}(P_U - P_V) \leq \mu + 1$, as claimed. ∎

To prove (39), note that one can evidently "link" $Y'$ and $Y_+$ by a sequence of matrices $Y' \equiv Y_0, Y_1, ..., Y_{k'} \equiv Y_+$ from $S_{\mathcal{M}}$ in such a way that any two neighbours in the sequence will differ by a rank 1 matrix. From Lemma 2.4 it follows than any two neighbours in the associated sequence $P_{Y'} = P_{Y_0}, P_{Y_1}, ..., P_{Y_{k'}} = P_{Y_+}$ differ by a matrix of rank at most $\max\{\mu_1 + 1, ..., \mu_m + 1\}$, and (39) follows. ∎

## 2.5 Prescaled Conjugate Gradients: # of steps

The results of Propositions 2.1, 2.2 and 2.3 can be summarized as follows.

**Proposition 2.4** *Consider the Conjugate-Gradient-based acceleration strategy from Sect. 2.2; let the strategy be associated with some $\delta$ satisfying the relations*

$$|\mathcal{M}|^{-1/2} \leq \delta \leq 1. \tag{40}$$

*and let the number $K$ of steps of the prescaled Conjugate Gradients at each Newton step be defined as*

$$K = \lfloor \min\{K_1, K_2\} \rfloor. \tag{41}$$

*where*

$$K_1 = C_{\mathrm{CG}} \rho_0^2 (\rho_l^*)^2 \ln(\frac{2}{\nu}) \tag{42}$$

*and*

$$K_2 = 128 \rho_0^2 C_{\mathrm{CG}} \ln(\frac{2}{\nu}) |\mathcal{M}| \delta^2 \sqrt{\mu + 1}; \tag{43}$$

*here*

$\mu = \max\{\mu_1, ..., \mu_m\}$;

$\rho_0$ *is the (depending on $\kappa, \vartheta$ only) constant from Proposition 2.1 (ii);*

$C_{\mathrm{CG}}$ *is the constant from Proposition 2.2;*

$\rho_l^* = \max\{8, |\mathcal{M}| \delta^2\}$ *is an upper bound for the right hand side of (25) for the case of $t, t_+ \in \Delta_q$.*

*Then the resulting method does find $\nu$-solutions to all Newton systems arising in course of solving (BSP) and therefore can be regarded as method $\mathcal{B}_\nu$ from Proposition 2.1.*

**Proof.** From Proposition 2.1 it is easily seen (induction) that what in fact should be proved is the following:

(*): assume that we perform a Newton step from a point $y$ at an iteration associated with a value $t_+ \in \Delta_q = [T_q, T_{q+1}]$ of the penalty and it is known that, first, $(y,t)$ satisfies (15):

$$\rho_0^{-1} F''(x^*(t_+)) \leq F''(y) \leq \rho_0 F''(x^*(t_+)). \tag{44}$$

and, second, that

$$\rho_0^{-1} F''(x^*(T_q)) \leq F''(x(T_q)) \leq \rho_0 F''(x^*(T_q)) \tag{45}$$

(recall that $x(\tau)$ is the point where we start (13) at the iteration associated with penalty value $\tau$).

Then $K$ steps of the Conjugate Gradients prescaled by the matrix $\mathcal{H}_q = F''(x(T_q))$ do result in a $\nu$-solution to the Newton system corresponding to $y$.

To verify the latter statement, let us act as follows. Set

$$H_y = F''(y), \quad H_{t_+} = F''(x^*(t_+)), \quad H_t = F''(x^*(T_q)).$$

The relations between these matrices are as follows: $H_{t_+}$ is $[\rho_0, \rho_0]$-compatible with $H_y$, same as $H_t$ is $[\rho_0, \rho_0]$-compatible with $\mathcal{H}_q$ (premise of the statement we are proving). Further, from Proposition 2.3 as applied to $t = T_q$ and our $t_+$ it follows that for any $\rho \geq 8$ the matrix $H_{t_+}$ is $[(\rho_l^*)^2, \rho^2]$-compatible with $H_t$ up to a correction of rank $k(\rho)$, where

$$k(\rho) = (\mu + 1)\lfloor \frac{1}{\rho} |\mathcal{M}| \delta^2 \rfloor \tag{46}$$

Let us derive from these observations that

(C): the matrix $H_y$ (which is the matrix of the Newton system we are solving at the step in question) is $[\rho_0^2(\rho_l^*)^2, \rho_0^2\rho^2]$-compatible with the matrix $\mathcal{H}_q$ (this matrix prescales the Conjugate Gradients at the step) up to a correction of rank $2k(\rho)$.

Indeed, consider the Rayleigh ratio $R(h)$ of the quadratic forms $h^T H_y h$ and $h^T \mathcal{H}_q h$. We have

$$R(h) \equiv \frac{h^T H_y h}{h^T \mathcal{H}_q h} = \left\{\frac{h^T H_y h}{h^T H_{t_+} h}\right\} \left\{\frac{h^T H_{t_+} h}{h^T H_t h}\right\} \left\{\frac{h^T H_t h}{h^T \mathcal{H}_q h}\right\}.$$

As we just have mentioned, the first and the third factors in the right hand side of this equality take their values in $[\rho_0^{-1}, \rho_0]$. Now, the second factor is the Rayleigh ratio of quadratic forms associated with a pair of matrices with the first being $[(\rho_l^*)^2, \rho^2]$-compatible with the second up to a correction of rank $k(\rho)$; it follows immediately that

$$R(h) \geq \rho_0^{-2}(\rho_l^*)^{-2}, \quad h \neq 0,$$

and that there exists a subspace $L$ of codimension $k(\rho)$ such that the ratio is $\leq \rho_0^2\rho^2$ whenever $h \in L$.

Now (C) can be immediately obtained from the following evident statement:

**Lemma 2.5** *Let $U$ and $V$ be two positive definite symmetric matrices of the same size, and let the Rayleigh ratio $(h^TVh)/(h^TUh)$ of the associated quadratic forms be $\geq \psi^{-1}$, and being restricted onto a subspace $L$ of a codimension $k$ let it be $\leq \pi$. Then $V$ is $[\psi, \pi]$-compatible with $U$ up to a correction of rank $2k$.*

**Proof.** The statement clearly can be reduced to the similar one for the matrices $U_+ = I$ and $V_+ = U^{-1/2}VU^{-1/2}$, so that we from the very beginning may assume $U$ to be the unit matrix. All that we should prove is that there exists a positive semidefinite matrix $Z$ of rank $2k$ such that $V - Z \leq \pi I$. Let $I - P$ be the orthoprojector on $L$, and let $V' = (I-P)V(I-P) + P$. The Rayleigh ratio of the forms associated with the matrices $V'$ and $I$ evidently is $\leq \pi$, so that $V' \leq \pi I$. On the other hand, the matrix $Z' = V - V'$ is of rank at most $2k$. Now let $Z$ be the "nonnegative part" of $Z'$, i.e., the matrix with the same eigenvectors and the eigenvalues being positive parts $\max\{\cdot, 0\}$ of those of $Z'$; we clearly have $V - Z \leq V - Z' \leq \pi I$, and $Z$ is the required $2k$-rank correction to $V$. ∎

Now we are ready to prove (*). Assume, first, that $K = K_1$. Since $k(\rho) = 0$ whenever $\rho > \rho_l^*$, (C) implies that $H_y$ is $[\rho_0^2(\rho_l^*)^2, \rho_0^2(\rho_l^*)^2]$-compatible with $\mathcal{H}_q$, and in view of Proposition 2.2 $K_1$ Conjugate Gradients steps do result in a $\nu$-solution to the system in question.

Now let $K = K_2$. We set

$$\eta(\rho) = (\mu+1)\frac{1}{\rho}\,|\mathcal{M}|\,\delta^2,$$

(so that $k(\rho) \leq \lfloor \eta(\rho) \rfloor$, see (46)),

$$\rho_r = \operatorname{argmin}\{2\eta(\rho) + \rho_0^2\rho_l^*\rho C_{\mathrm{CG}}\ln(\frac{2}{\nu}) \mid \rho \geq 8\} = \max\left\{8, \left(\frac{2(\mu+1)\,|\mathcal{M}|\,\delta^2}{\rho_0^2\rho_l^* C_{\mathrm{CG}}\ln(\frac{2}{\nu})}\right)^{1/2}\right\},$$

which in view of $\rho_l^* = \max\{8, |\mathcal{M}|\delta^2\}$, $|\mathcal{M}|\delta^2 \geq 1$ results in

$$2k(\rho_r) + C_{\mathrm{CG}}\rho_0^2\rho_l^*\rho_r\ln(\frac{2}{\nu}) \leq 128\rho_0^2 C_{\mathrm{CG}}\ln(\frac{2}{\nu})\,|\mathcal{M}|\,\delta^2\sqrt{\mu+1}.$$

In view of (C) and Proposition 2.2, the first quantity in this chain is the number of steps in which the Conjugate Gradient method prescaled by the matrix $\mathcal{H}_q$ finds a $\nu$-solution to the Newton system in question, and the last quantity is $K_2 = K$, so that in the case under consideration, same as in the previous one, $K$ steps of the Conjugate Gradients are sufficient to find the required approximate solution to the Newton system. ∎

# 3 Efficiency of CG-based Acceleration

In this section we investigate when the Conjugate-Gradient-based strategy, as compared to the basic one, does result in acceleration in orders of dependence of the sizes of problem $\mathcal{P}_{n,\mathcal{M}}$. Since we are interested only in orders, we do not take care of constant factors; all these factors depend on the parameters $\kappa, \vartheta$ of the path-following procedure only.

To simplify notation, we do not write constant factors even as $O(1)$; in order to express that certain quantity $A$ coincides with another quantity $B$ up to a factor which is bounded from above, same as bounded away from zero, by something depending on $\kappa, \vartheta$ only, we write $A \backsim B$; this convention is extended in the evident way onto expressions of the type $A \gtrsim B$.

## 3.1 Efficiency: general relations

Assume that when solving $\mathcal{P}_{n,\mathcal{M}}$ by the basic method $\mathcal{B}$ we act as follows: to perform the Newton step from a current iterate $y$, we compute the gradient $F'(y)$ (which is required to form the right hand side of the Newton system) and then somehow assemble and invert the Hessian $F''(y)$ (independently of intermediate results of previous steps). Let $Q_e$ denote the arithmetic cost of these computations; we assume that after they are finished, we obtain a possibility to multiply an arbitrary vector $h \in \mathbf{R}^n$ by the inverted Hessian $(F''(y))^{-1}$ at certain arithmetic cost $M_e$. Of course, we assume that $M_e \geq n$ (we should spend at least $n$ operations simply to store the result which is an $n$-dimensional vector). It should be stressed that we does not assume that the Hessian and its inverse are computed in their standard form (as $n \times n$ matrices). All which is assumed is that we use certain procedure which, given $y$, processes the data of the problem at the cost of $Q_e$ operations in such a way that, first, the gradient $F'(y)$ becomes known and, second, we obtain the possibility to compute, for any given $h$, the vector $(F''(y))^{-1}h$ in no more than $M_e$ operations. In what follows we refer to this procedure as to *processing of Hessian*.

According to our hypothesis, the cost of a Newton step for the basic method is $Q_e + M_e$, so that the arithmetic cost at which the method solves the basic subproblem (BSP) is

$$M_{\mathcal{B}} \backsim (Q_e + M_e)|\mathcal{M}|^{1/2} \tag{47}$$

(the first factor is the arithmetic cost of a Newton step, the second - the total amount of these steps when solving (BSP); according to our assumption that $M_e \geq n$, we may once for ever forget about the remaining operations at a step, like computing the stepsize and $y^{s+1}$ after the Newton direction is computed, since the corresponding effort is dominated already by $M_e$).

Now let us evaluate the arithmetic cost of solving (BSP) by the Conjugate-Gradient-based method $B_\nu$. We assume that here at a Newton step from current iterate $y$ we compute the gradient $F'(y)$ and, possibly, perform some additional preprocessing and then obtain a possiblity to multiply an arbitrary vector from $\mathbf{R}^n$ by $F''(y)$ at certain arithmetic cost $M \geq n$. The cost of computing $F'(y)$ and of the above preprocessing will be called $Q$. Now, once per each segment $\Delta_q$ of values of the penalty we should compute the exact Hessian and its inverse; as in the basic strategy, it is assumed that a single action of this type takes $Q_\epsilon$ operations and allows to multiply a vector form $\mathbf{R}^n$ by $\mathcal{H}_q^{-1}$ at the cost $M_e$ operations.

Sometimes we will refer to the above quantities $Q_\epsilon, M_\epsilon, Q, M$ as to *basic computational characteristics* of the problem.

Under the above assumptions the arithmetic cost $M_{\mathcal{B}_\nu}$ at which $\mathcal{B}_\nu$ solves (BSP) can be computed as follows. At each Newton step we spend $Q$ operations to compute the gradient $F'(y)$ at the current iterate $y$ and to preprocess matrix-vector multiplications involving the Hessian $F''(y)$ Then we should perform $K \backsim 1 + \mathcal{K} \equiv 1 + \min\{\mu^{1/2}|\mathcal{M}|\delta^2, (|\mathcal{M}|\delta^2)^2\}$ Conjugate Gradient steps (see (41)). Each Conjugate Gradient step costs two multiplications of a vector by the current Hessian ($\backsim M$ operations) and two multiplications of a vector by the matrix inverse to that one used for the prescaling ($\backsim M_e$ operations). Thus, the total cost at which $\mathcal{B}_\nu$ performs a single Newton step is $\backsim Q + (M + M_e)\mathcal{K}$ operations. This cost should be paid $\backsim |\mathcal{M}|^{1/2}$ times (this is the total number of Newton steps required to solve (BSP)); besides this, we should $\backsim \delta^{-1}$ times pay $Q_e$ operations to process $\mathcal{H}_q$, $q = 1, 2, ..., O(\delta^{-1})$.

Thus, we come to the following expression for the arithmetic cost of the method $\mathcal{B}_\nu$:

$$M_{\mathcal{B}_\nu} \lesssim (Q_e + M_e)\delta^{-1} + (Q + M + M_e + (M + M_e)\mathcal{K})\,|\mathcal{M}|^{1/2},$$
$$\mathcal{K} = |\mathcal{M}|\,\delta^2 \min\{\mu^{1/2}, |\mathcal{M}|\,\delta^2\}. \qquad (48)$$

This cost depends on $\delta$, and of course we should choose $\delta$ optimally, keeping in mind that $\delta$ should satisfy (40). In fact we can forget about the lower bound in (40), since when the $\delta$ is at this bound or less, then the right hand side of (48) clearly is larger than that one of (47), so that no acceleration occurs; it follows that if the value $\delta^*$ which minimizes the right hand side of (48) under the only constraint $\delta \leq 1$ does result in acceleration (and these are exactly the cases we are interested in), then this $\delta^*$ automatically satisfies the left inequality in (40) and therefore optimizes the right hand side of (48) under the whole pair of constraints (40).

To measure the acceleration, it is convenient to deal with the *acceleration ratio*

$$\mathcal{R} = \frac{\mathcal{M}_{\mathcal{B}_\nu}}{\mathcal{M}_{\mathcal{B}}};$$

the less is this ratio, the more is the efficiency of acceleration: we do have an acceleration when $\mathcal{R} \lesssim 1$.

It is easily seen the acceleration ratio corresponding to the optimal $\delta$ is as follows:

$$\mathcal{R} \backsim \frac{Q + M + M_e}{Q_e + M_e} + \frac{1}{\sqrt{|\mathcal{M}|}} + \min\{\mathcal{R}_1, \mathcal{R}_2\},$$
$$\mathcal{R}_1 = \mu^{1/6}\left(\frac{M + M_e}{Q_e + M_e}\right)^{1/3}; \quad \mathcal{R}_2 = \left(\frac{M + M_e}{Q_e + M_e}\right)^{1/5}. \qquad (49)$$

## 3.2 Efficiency: examples

Looking at expressions (49), we see that the acceleration ratio heavily depends on relations between the characteristics $Q_e, M_e, Q, M$ of our basic Linear Algebra routines. Recall that these quantities are the arithmetic costs of

computing $F'(y)$ and processing $F''(y)$ ($Q_e$) in a way which allows to multiply vectors by $(F''(y))^{-1}$ at the cost $M_e$;

computing $F'(y)$ and preprocessing $F''(y)$ ($Q$) in a way which allows to multiply vectors by $F''(y)$ at the cost $M$.

In this section we consider several typical relations between these characteristics and present the corresponding acceleration ratio. *Within the bounds of this section*, for the sake of definiteness, *we deal only with "homogeneous" structures:*

$$\mathcal{M} = \overbrace{\{\mu, ..., \mu\}}^{m \text{ times}}.$$

(of course, this is the same as to deal with the structures where all $\mu_i$ coincide with each other up to a constant factor of order of 1; we will meet this latter situation in Sect 3.2.4).

Recall that our problem $\mathcal{P}_{n,\mathcal{M}}$ (see (1)) is defined by an affine mapping $\mathcal{A}(x) = Ax + \alpha$ from $\mathbf{R}^n$ into $S_{\mathcal{M}}$; the dimension of the latter space is $\backsim m\mu^2$, and since $A$ is of full column rank (see Sect. 1.2), we have

$$n \lesssim m\mu^2; \tag{50}$$

it is convenient to assume also that

$$n \gtrsim \mu, \tag{51}$$

as it is the case in all applications known to us.

### 3.2.1 General "unstructured" case

We start with the general case where no specific structure of problem $\mathcal{P}_{n,\mathcal{M}}$ is assumed: the affine mapping $\mathcal{A}$ is represented as a collection of $n+1$ matrices (the constant term and $n$ images of the standard $n$-dimensional orths under the homogeneous part $A$ of the mapping), which can be arbitrary symmetric matrices of the block-diagonal structure $\mathcal{M}$.

In this case one evidently has

$Q \backsim nm\mu^2 + m\mu^3 \backsim nm\mu^2$

[according to (20), we should compute $\mathcal{A}(x)$, which takes $\backsim nm\mu^2$ operations, then invert $\mathcal{A}(x)$ ($\backsim m\mu^3$ operations more) and, last, multiply the result by $A^*$ (again $\backsim nm\mu^2$ operations); the term $m\mu^3$ disappears due to (51)];

$M \backsim nm\mu^2 + m\mu^3 \backsim nm\mu^2$

[according to (21), to compute $F''h$ we should compute $Ah$ ($\backsim nm\mu^2$ operations), multiply this matrix from the left and from the right by already computed $(\mathcal{A}(x))^{-1}$ ($\backsim m\mu^3$ operations) and, last, multiply the result by $A^*$ ($\backsim nm\mu^2$ operations more)]

Now, in the case in question the natural way to process Hessian is to assemble and to invert it directly, i.e., to compute its values at the standard basis orths in the range space and then straightforwardly invert the resulting $n \times n$ matrix (or find its Cholesski factorization); recall that we use the traditional Linear Algebra only. To find the value of the Hessian at a basis orth $e$ it takes the same $\backsim nm\mu^2$ operations as to compute it on an arbitrary vector (although now we should not pay for computing $Ae$, we still should pay the above cost for applying $A^*$ to $(\mathcal{A}(x))^{-1}(Ae)(\mathcal{A}(x))^{-1}$). Thus,

$Q_e \backsim n^2 m\mu^2 + n^3 \backsim n^2 m\mu^2$
[$n^3$ dissapears in view of (50)] and, of course,
$M_e \backsim n^2$.
With these dependences, (49) and (50), (51) result in

$$\mathcal{R} \backsim \min\{\mu^{1/6} n^{-1/3}, n^{-1/5}\} \tag{52}$$

We see that in the range of sizes defined by (51), (50) accceleration does occurs; its order varies from $n^{1/3}$ ($\mu \backsim 1$) to $n^{1/5}$ ($\mu \backsim n$).

Note that the case of $\mu = 1$ is precisely the case of Linear Programming; if, in addition, $n \backsim m$ (this is the standard asssumption when analysing acceleration in LP), we obtain acceleration of order $m^{1/3}$, which is by factor $m^{1/6}$ worse than the efficiency of the usual Karmarkar speed up. What should be stressed is that the acceleration of order $n^{1/3}$ is achieved not only when $\mu = 1$, but also when $\mu$ is a fixed constant (of course, then the constant factors in our equivalences $\backsim$ depend on $\mu$). On the other hand, there are important problems where $\mu$ indeed is a once for ever fixed small integer.

Now let us look what happens in the case when $\mathcal{A}$ possesses a "nontrivial structure". We do not pretend to give a general definition of a "structured" PD problem; let us simply list several real-world problems.

### 3.2.2 Robust Stability Analysis

There is a lot of PD problems in this area; from mathematical viewpoint, the "structure" here looks as follows. The control vector $x$ is a collection of symmetric matrices: $x = (x_1, ..., x_q)$, and the diagonal blocks of $\mathcal{A}(x)$ are of the type

$$\begin{pmatrix} a_{11}(x) & a_{12}(x) & \dots & a_{1k}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1}(x) & a_{k2}(x) & \dots & a_{kk}(x) \end{pmatrix},$$

where the *subblocks* $a_{uv}(x) = a_{vu}^T(x)$ are sums of products of $x_i$ and fixed matrices of appropriate sizes. The maximum over subblocks number $w$ of terms in these sums will be called the *weight* of the problem.

This description, of course, is too diffuse for a productive complexity analysis, since to the moment we did not say much; all depends on how many $x_i$ are involved in a block, how many subblocks $a_{uv}$ are there in the diagonal blocks of $\mathcal{A}$, etc. For the sake of definiteness, we restrict ourselves to two particular cases coming from applications.

**Example 1: $H_\infty$-control.** In problems of this type (see [PZPB 91, GA 92]) the numbers of control matrices, blocks and subblocks per block, same as the weight, are small integers (less than 10 in all problems we know). All control matrices are of the same row size, let it be $s$. We have, consequently,

$$n \backsim s^2, \quad \mu \backsim s, \quad m \backsim 1, \quad w \backsim 1. \tag{53}$$

Now evaluate our basic characteristics $Q, M, Q_e, M_e$. We have

computing $F'(y)$ and processing $F''(y)$ ($Q_e$) in a way which allows to multiply vectors by $(F''(y))^{-1}$ at the cost $M_e$;

computing $F'(y)$ and preprocessing $F''(y)$ ($Q$) in a way which allows to multiply vectors by $F''(y)$ at the cost $M$.

In this section we consider several typical relations between these characteristics and present the corresponding acceleration ratio. *Within the bounds of this section*, for the sake of definiteness, *we deal only with "homogeneous" structures:*

$$\mathcal{M} = \overbrace{\{\mu, ..., \mu\}}^{m \text{ times}}.$$

(of course, this is the same as to deal with the structures where all $\mu_i$ coincide with each other up to a constant factor of order of 1; we will meet this latter situation in Sect 3.2.4).

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$$n \backsim s^2, \quad \mu \backsim s, \quad m \backsim 1, \quad w \backsim 1. \tag{53}$$

Now evaluate our basic characteristics $Q, M, Q_\epsilon, M_\epsilon$. We have

$Q \backsim s^3, \quad M \backsim s^3$

[an immediate consequence of (20), (21) and our assumptions on the sizes of the problem]

Now evaluate $Q_e$. In contrast to the "unstructured" case, now we can process Hessian at a lower cost than that one corresponding to its direct assembling and inverting as an $n \times n$ matrix. Indeed, to invert in the standard manner $F''$ it would cost $\backsim n^3$ operations, not speaking of the cost of assembling $F''$; thus, to process Hessian straightforwardly it would cost at least $\backsim s^6$, and these "investments" result in $M_e \backsim n^2 \backsim s^4$. Let us demonstrate that in fact the latter possibility can be "bought" cheaper, namely, at the cost $\backsim s^5$. Indeed, since we can multiply $P = F''(x)$ by a vector at the cost $M \backsim s^3$, we can use $n$ steps of the standard Conjugate Gradients to obtain a linearly independent system of $n$ $P$-conjugate directions $p_i, i = 1, ..., n$, i.e., a system such that

$$p_j^T P p_i = 0, j \neq i,$$

and the arithmetic cost of forming this system will be $\backsim nM$. We evidently have

$$P^{-1}h = \sum_{i=1}^{n} (p_i^T P p_i)^{-1} (h^T p_i) p_i,$$

and the cost of computing the right hand side of this expression is $n^2$. The above trick is a well known general fact completely independent of the particular problem we are dealing with:

**Remark 3.1** *Given a possibility to multiply a positive definite symmetric $n \times n$ matrix $P$ by an arbitrary vector at certain cost $M$, one can obtain, at the cost of preprocessing taking $O(1)nM$ operations, a possibility to multiply an arbitrary vector by $P^{-1}$ in $O(1)n^2$ operations, both $O(1)$ being absolute constants.*

In fact this trick is more of theoretical interest, due to numerical instability of Conjugate Gradients as applied to ill-conditioned problems, but these "practical" issues are beyond the bounds of our theoretical analysis.

Thus, in our particlar problem we have

$Q_e \backsim s^5, \quad M_e \backsim s^4.$

It remains to use (49), and we come to

$$\mathcal{R} \backsim s^{-1/5} \backsim n^{-1/10}.$$

Almost nothing, but not nothing.

**Example 2: Luapunov Stability Analysis for differential inclusions.** The problem (see [BY 89, BG 92]) is of the following structure. There is a single control matrix, let its row size be $s$, and a (possibly, large) number $m$ of diagonal blocks in $\mathcal{A}(x)$ ($m$ typicaly, although not always, is the number of vertices in certain polytope in $\mathbf{R}^s$). All diagonal blocks are of the same as $x$ row size $s$, and their partition into subblocks is trivial (the

only subblock coincides with the corresponding block); the weight equals to 2. For this structure one evidently has

$$n \backsim s^2, \quad \mu = s.$$

Same as above, one can easily verify that

$$Q \backsim ms^3, \quad M \backsim ms^3.$$

As far as $Q_e, M_e$ are concerned, the situation is as follows. It is easily seen that the straightforward assembling $F''$ costs $\backsim ms^3 + nms^2 \backsim ms^4$ operations, and the standard inverting the resulting matrix costs $\backsim n^3 \backsim s^6$ operations; to process $F''$ via Conjugate Gradients (see Remark 3.1) it costs $\backsim nM \backsim ms^5$ operations; both approaches result in the same value $\backsim n^2 \backsim s^4$ of $M_e$. Which one is better, it depends on the ratio $s/m$: if it is less than 1, then the straightforward way is more profitable.

Let us start with the case $s \leq m$, which, in view of the origin of the problem, is quite realistic. In this case

$$Q_e \backsim s^4(m + s^2), \quad M_e \backsim s^4, \quad s \leq m.$$

From (49) it is easily seen that

$$\mathcal{R} \backsim \min\{n^{-1/12}\left(\frac{m}{m+n}\right)^{1/3}, n^{-1/10}\left(\frac{m}{m+n}\right)^{1/5}\} \lesssim n^{-1/10}.$$

Now consider the case of $s > m$, where it is more preferable to process Hessian via the CG. Here

$$Q_e \backsim ms^5, \quad M_e \backsim s^4, \quad s > m.$$

and the acceleration ratio now is

$$\mathcal{R} \backsim \min\{m^{-1/3}n^{-1/12}, m^{-1/5}n^{-1/10}\}.$$

### 3.2.3 Lovasz capacity number of a graph

The structure of this problem (see [Lo 79]) is as follows. $\mu$ is the number of vertices of the graph in question and $x = (\tau, u)$, where $\tau \in \mathbf{R}$ and components of $u$ are indiced by arcs of the graph. Now, $\mathcal{A}(\tau, u) = \tau I - \mathcal{G}(u)$, where $ij$-th entry of $\mathcal{G}$ is unit, if the vertices $i$ and $j$ are not adjacent in the graph; otherwise this entry is the component of $u$ indiced by the corresponding arc.

It is easily seen that in the typical case $n \backsim \mu^2$ (i.e., the case when the number of arcs in the graph is of order of squared number of vertices; we restrict ourselves to this case only) the complexity analysis is completely similar to that one from Example 1 Sect. 3.2.2, and the acceleration ratio is

$$\mathcal{R} \backsim n^{-1/10}.$$

### 3.2.4 Multiloaded Truss Topology Design

For our current goals it suffices to make the following remarks on the structure of the problem in question, let us call it TTD for short (for detailed presentation of TTD, see [B-TN 92]).

(TTD$_a$): the sizes of the problem are defined by a triple of integers $\xi$ (# of nodes in the Truss), $\eta$ (# of tentative bars) and $\lambda$ (# of loading scenarios). These numbers are as follows: $\xi$ is rather large (at least hundreds in problems of practical interest), $\eta \backsim \xi^2$ (tens of thousands) and $\lambda$ is a small integer (say, 3). Now, the block-diagonal structure $\mathcal{M}$ looks as follows: $m = 2\eta + 3\eta\lambda + \lambda \backsim \xi^2\lambda$, with $\eta\lambda$ of $\mu_i$'s being equal to 2 and the remaining $\mu_i$'s equal to 1. Now, the dimension of the control vector $x$ is $n = (\eta + \xi + 1)\lambda + \eta \backsim \xi^2\lambda$.

**Warning:** The reader should be awared that in fact the formulation of the multiloaded Truss Topology Design given in [B-TN 92] is not a PD-one; the problem there is set as

$$(*) \quad \textit{minimize} \quad c^T x \quad \textit{subject to} \quad \mathcal{E}_i(x) \in \mathcal{K}_i, i = 1, ..., m,$$

where $\mathcal{E}_i$ are affine mappings into $\mathbf{R}^{s(\mu_i)}$, with $s(1) = 1, s(2) = 3$, and $\mathcal{K}_i$ are the standard second order cones in these $\mathbf{R}^{s(\mu_i)}$ (the standard second-order cone in $\mathbf{R}^l$ is the set which in appropriate coordinates $y$ can be described as $\{y \mid y_1 \geq (y_2^2 + ... + y_l^2)^{1/2}\}$). Now, since the 1-dimensional second order cone in $\mathbf{R}^1$ is nothing but the cone of positive semidefinite $1 \times 1$ symmetric matrices and the 3-dimensional second order cone in $\mathbf{R}^3$ is nothing but the cone of positive semidefinite symmetric $2 \times 2$ matrices, the initial formulation of the problem in terms of second order cones is its PD-formulation as well.

Note that *an arbitrary* problem (*) associated with affine mappings $\mathcal{E}_i$ and second order cones $\mathcal{K}_i$ can be reformulated as a PD problem, since a $k$-dimensional second order cone can be easily represented as an intersection of the cone of positive semidefinite $k \times k$ symmetric matrices and an appropriately chosen $k$-dimensional subspace. For Truss Topology design, anyhow, this "reformulation" changes nothing but the way we look at the problem.

(TTD$_b$): although very large-scaled, at least for real-world values of $\xi$, problem TTD possesses a very specific structure which allows to perform required Linear Algebra routines at a (relatively) extremely low cost. Namely, one has (see [B-TN 92])

$$Q \backsim \lambda^2\xi^2, \quad M \backsim \lambda\xi^2,$$
$$Q_e \backsim \xi^3\lambda^3, \quad M_e \backsim \lambda^2\xi^2.$$

From (49) it follows immediately that

$$\mathcal{R} \backsim (\lambda\xi)^{-1/3},$$

which is rather encouraging, taking into account exttremely large sizes of real-world TTD problems.

# 4 Karmarkar Acceleration

In this section we demonstrate that in the case of large $m$ (we again deal with an arbitrary structure $\mathcal{M}$), the basic path-following method admits a Karmarkar-type acceleration; in the homogeneous "unstructured" case the corresponding acceleration ratio is of order of $m^{-1/2} + n^{-1}$.

## 4.1 The method

In our now scheme, same as in the previous one, we replace process (6) by (13) and find "proper" approximate solutions to the Newton systems by a prescribed number $K$ of steps of the prescaled Conjugate Gradients. The difference with the previous scheme is that now we update the prescaling matrix at every step and use low-rank corrections to maintain its $[\rho^2, \rho^2]$-compatibility with the matrix of the Newton system; here $\rho$ is a constant which depends on the path tolerance $\kappa$ and the penalty rate $\vartheta$ only. As a result, the number of Conjugate Gradient steps becomes independent of the sizes of the problem. For appropriately chosen absolute constants $\kappa, \vartheta$ the latter number could be reduced to 1, which in fact completely eliminates Conjugate Gradients and results in the "pure" Karmarkar scheme.

In what follows $y$ denotes the iterate updated at the Newton step in question, $t_y$ is the corresponding value of the penalty parameter and $B_y$ is the matrix which prescales the Conjugate Gradient method at the step. The method we are going to describe is governed by certain parameter $\rho \geq 1$ (and, of course, by $\kappa$ and $\vartheta$). To specify the method, we first explain how, given a $\rho$, we form the prescaling matrices $B_y$ and their inverses and how the number $K$ of the Conjugate Gradient steps is chosen; after this is done, we specify $\rho$ itself.

### 4.1.1 Preliminary remarks

Let us start with the following notion. Let $U$ and $V$ be a pair of positive definite matrices from $S_{\mathcal{M}}$. We associate with the pair a nonnegative $m$-dimensional *ratio vector* $r(V : U) = (r_1(V_1 : U_1), ..., r_m(V_m : U_m))$, $W_i$ being $i$-th diagonal block of $W \in S_{\mathcal{M}}$, as follows. Let $W = U^{-1/2} V U^{-1/2}$; we set

$$r_i(U_i : V_i) = \ln(\max\{1, \| W_i \|\}) + \ln(\max\{1, \| W_i^{-1} \|\}), \quad i = 1, ..., m, \qquad (54)$$

where $\| \cdot \|$ is the usual operator norm. As it is clearly seen, the ratio vector can be equivalently defined as follows. Consider the Rayleigh ratios

$$R_i(V_i : U_i) = \frac{h^T V_i h}{h^T U_i h}, \quad 0 \neq h \in \mathbf{R}^{\mu_i};$$

let $\alpha_i$ and $\beta_i$ be the maximum and the minimum values of $i$-th ratio over all nonzero $h$ and let $a_i = \max\{1, \alpha_i\}, b_i = \min\{1, \beta_i\}$. Then

$$r_i(V_i : U_i) = \ln(\frac{a_i}{b_i}).$$

From the latter description it follows that $r(V:U)$ is a "vector-valued distance":

$$r(V:U) = r(U:V), \quad r(V:U) = 0 \Leftrightarrow V = U, \quad r(W:U) \leq r(W:V) + r(V:U).$$

**Computational Remark.** Our rules for updating prescaling matrices will be based on control of ratio vectors, and therefore we should understand how to compute these vectors. Given $U$ and $V$, one can find matrices $W_i'$ orthogonally equivalent to $W_i$ at the cost of $O(\sum \mu_i^3)$ operations (it suffices to find the Cholesski decompositions $U_i = D_i D_i^T$ and set $W_i' = D_i^{-1} V_i (D_i^T)^{-1}$. Since these matrices are orthogonally equivalent to $W_i$, the question is reduced to computing norms of $W_i'$ and their inverses. To compute a norm of a symmetric $k \times k$ matrix, this is not a problem which can be solved in a finitely many arithmetic operations, although to solve it to a relative accuracy $\varepsilon$ it costs no more than $O(k^3 \ln(\frac{1}{\varepsilon}))$ operations. Fortunately, in what follows there is no necessity in exact knowledge of ratio vectors. What we need is to know the coordinates of the required ratio vectors within a bounded by an absolute constant absolute accuracy, and this problem can be solved in $O(\sum \mu_i{}^3)$ operations. The simplest way (which results in absolute accuracy $\ln 3$) is as follows: after the above matrices $W_i'$ and their inverses are computed, we find 3-diagonal orthogonal equivalents to their squares (which can be done at a cubic cost). Now, the norm of a 3-diagonal symmetric positive semidefinite matrix is at least the largest and at most 3 times the largest diagonal entry of it.

In what follows $\widehat{r}(\cdot:\cdot)$ denotes the latter "computable approximation" of $r(\cdot:\cdot)$; for this approximation one has

$$|r_i(V_i:U_i) - \widehat{r}_i(V_i:U_i)| \leq \ln 3, \quad i = 1, ..., m. \tag{55}$$

### 4.1.2 Prescaling matrices

We will form the matrices $B_y$ in a way which ensures their $[O(\rho^2), O(\rho^2)]$-compatibility with the matrices $F''(y)$ of the corresponding Newton systems; namely, we are going to ensure that

$$\frac{\rho^{-2}}{81} F''(y)) \leq B_y \leq 81 \rho^2 F''(y). \tag{56}$$

This relation in our scheme is maintained as follows. Let

$$Y_y = (\mathcal{A}(y))^{-1}. \tag{57}$$

All our $B_y$ will be produced by positive definite matrices $V_y \in S_{\mathcal{M}}$ according to the relation

$$B_y = P_{V_y} \quad (\text{i.e., } B_y h = A^*(V_y A(h) V_y)). \tag{58}$$

At the very first Newton step (let $y_0$ be the corresponding iterate) we compute $Y_{y_0}$ and set $V_{y_0} = Y_{y_0}$; then we compute $B_{y_0}$ according to (58), which results in $B_{y_0} = F''(y_0)$ (see (21)) and straightforwardly invert the latter matrix.

In order to compute $B_y$ at the remaining Newton steps we act as follows.

**1.** We successively compute $\mathcal{A}(y)$, $Y_y$ and $\widehat{r}(Y_y : V_{y_-})$, where $y_-$ denotes the preceding iterate. Then we define $V_y$ as

$$(V_y)_i = \begin{cases} (V_{y_-})_i, & \widehat{r}_i((Y_y)_i : (V_{y_-})_i) \leq \ln(3\rho); \\ (Y_y)_i, & otherwise, \end{cases} \tag{59}$$

$(\cdot)_i$ being $i$-th diagonal block of the corresponding matrix.

**2.** Now, let $k_y$ denote the number of "bad" $i$'s, i.e., those for which the second case in (59) takes place. Then $V_y$ differs from $V_{y_-}$ by a block-diagonal symmetric matrix of rank at most $k_y\mu$, where, as always, $\mu$ is the largest of $\mu_i$. It follows (see Lemma 2.4) that the matrix $B_y$ associated, according to (58), with $V_y$ differs from $B_{y_-}$ by a matrix $D_y$ of rank at most $s_y = k_y\mu(\mu+1)$. We represent $D_y$ as a sum of $s_y$ rank 1 symmetric matrices (see Sect. 4.2.2 for details) and then add these matrices one by one to $B_{y_-}$ and update the inverse matrices according to the Sherman-Morrison formula. Thus, $(B_y, B_y^{-1})$ are computed by $s_y$ sequential rank 1 corrections of $(B_{y_-}, B_{y_-}^{-1})$.

**Lemma 4.1** *The above rules for forming prescaling matrices do ensure (56).*

**Proof.** According to (59), (57), (58) and (21), we have

$$B_y = P_{V_y}, \quad F''(y)) = P_{Y_y}.$$

Further, by construction $\widehat{r}_i((Y_y)_i : (V_y)_i) \leq \ln(3\rho)$, whence, in view of (55), $r_i((Y_y)_i : (V_y)_i) \leq \ln(9\rho), i = 1, ..., m$. From the latter relation and the definition of the ratio vector it follows immediately that $Y_y$ is $[9\rho, 9\rho]$-compatible with $V_y$. This observation by virtue of Lemma 2.3 means that $B_y$ is $[81\rho^2, 81\rho^2]$-compatible with $F''(y)$, as required by (56). ∎

### 4.1.3 Prescaled Conjugate Gradients: # of steps

Let us choose $K$ as follows:

$$K = \lfloor 81 C_{CG} \rho^2 \ln(\frac{1}{\nu}) \rfloor. \tag{60}$$

**Proposition 4.1** *The outlined method does solve to the accuracy $\nu$ all Newton systems arising at sequential Newton steps and therefore can be regarded as method $\mathcal{B}_\nu$ from Proposition 2.1.*

**Proof.** By virtue of Lemma 4.1, at each Newton step the matrix $F''(y)$, $y$ being the current iterate, is $[81\rho^2, 81\rho^2]$-compatible with the matrix $B_y$ which prescales the Conjugate Gradients. It remains to apply Proposition 2.2. ∎

## 4.2 Complexity

### 4.2.1 Key estimate

Recall that updating $(B_{y_-}, B_{y_-}^{-1}) \mapsto (B_y, B_y^{-1})$ is performed via rank 1 corrections and requires at most $s_y = \mu(\mu+1)k_y$ of these corrections, $k_y$ being the number of "bad" $i$'s at the corresponding Newton step (see Sect. 4.1.2).

To evaluate arithmetic cost of the method, the following statement is the key one.

**Proposition 4.2** *Let $\rho$ be chosen according to*

$$\ln(\rho) = 4\ln(\rho_1) + 1, \tag{61}$$

*where $\rho_1$ is the (depending on $\kappa, \vartheta$ only) constant from Proposition 2.1 (iii).*

*Then the total amount*

$$s^* = \sum_y s_y$$

*of rank 1 corrections in course of solving (BSP) admits the following upper bound:*

$$s^* \leq O(1)\mu^2 m^{1/2} |\mathcal{M}|^{1/2} \tag{62}$$

*with $O(1)$ depending on $\kappa, \vartheta$ only.*

**Proof. 1.** Let us start with the following

**Lemma 4.2** *Let $y$, $y_+$ be two neighbouring iterates and let $t$, $t_+$ be the corresponding values of the penalty. Let also*

$$X = (\mathcal{A}(x^*(t)))^{-1}, \quad X_+ = (\mathcal{A}(x^*(t_+)))^{-1}.$$

*Then*

$$\| r(X_+ : X) \|_2 \leq O(1) \tag{63}$$

*and consequently*

$$\| r(X_+ : X) \|_1 \leq O(1) m^{1/2}, \tag{64}$$

*with $O(1)$ depending on $\vartheta$ only.*

**Proof.** Let $Z = (X^{-1/2} X_+ X^{-1/2})^{1/2}$. From Sect. 4.1.1 we know that

$$r_i(X_+ : X) = \ln(\max\{1, p_i^2\}) + \ln(\max\{1, q_i^{-2}\}), \tag{65}$$

where $p_i$ is the maximal, and $q_i$ is the minimal of the eigenvalues $\lambda_{ij}$, $j = 1, ..., \mu_i$ of $i$-th diagonal block of $Z$. At the same time, from Lemma 2.1 (see (23)) we immediately conclude that

$$\sum_{i=1}^{m} \left\{ \sum_{j=1}^{\mu_i} \left( \sqrt{\tau_+}\lambda_{ij} - \sqrt{\tau}\lambda_{ij}^{-1} \right)^2 \right\} \leq |\mathcal{M}|(\sqrt{\tau} - \sqrt{\tau_+})^2, \tag{66}$$

where $\tau = t/T, \tau_+ = t_+/T$. Now, since we are speaking about two neighbouring steps, one has either $t = t_+$, and then (63) is evident, or $t_+ = \exp\{\pm\vartheta|\mathcal{M}|^{-1/2}\}t$, $t, t_+ \in [T, 2T]$, whence

$$\tau, \tau_+ \in [1, 2], \quad |\mathcal{M}|(\tau - \tau_+)^2 \leq O(1) \tag{67}$$

(in this proof all $O(1)$ are positive constants depending on $\vartheta$ only). Now, let

$$\delta_{ij} = \max\{\lambda_{ij} - \tau^{1/4}\tau_+^{-1/4}, \lambda_{ij}^{-1} - \tau_+^{1/4}\tau^{-1/4}\}.$$

One evidently has $\delta_{ij} \geq 0$ and

$$(\sqrt{\tau_+}\lambda_{ij} - \sqrt{\tau}\lambda_{ij}^{-1})^2 \geq O(1)\delta_{ij}^2 \tag{68}$$

(we have taken int account that $\tau, \tau_+ \in [1,2]$, see (67)). Now, let

$$\delta_i = \max\{\delta_{ij} \mid j = 1, ..., \mu_i\}.$$

From (66), (67) and (68) it follows that

$$\sum_i \delta_i^2 \leq O(1). \tag{69}$$

On the other hand, we clearly have

$$p_i \leq \tau^{1/4}{\tau_+}^{-1/4} + \delta_i, \quad q_i^{-1} \leq {\tau_+}^{1/4}\tau^{-1/4} + \delta_i,$$

whence, in view of (67),

$$p_i \leq 1 + O(1)|\mathcal{M}|^{-1/2} + \delta_i, \quad q_i^{-1} \leq 1 + O(1)|\mathcal{M}|^{-1/2} + \delta_i.$$

Since evidently $\ln(\max\{1, a^2\}) \leq O(1)(b-1)$, $0 \leq a \leq 1+b$, $b \geq 0$, we conclude that

$$r_i^2((X_+)_i : X_i) \leq O(1)(|\mathcal{M}|^{-1} + \delta_i^2)$$

(see (65)), which combined with (69) results in desired (63); (64) is an immediate corollary of (63). ∎

**2.** Now let us estimate

$$k^* = \sum_y k_y.$$

Let the Newton steps be enumerated sequentially, and let $y_q$ be the iterate which is updated at $q$-th step and $t_q$ be the corresponding value of the parameter. Let us fix $i, 1 \leq i \leq m$. We say that at $q$-th Newton step an *event of type* $i$ occurs, if

$$\hat{r}_i((Y_{y_q})_i : (V_{y_{q-1}})_i) \geq \ln(3\rho)$$

(for notation, see (59)).

From (59) it follows that $k^*$ is exactly the total number of all events; thus, if $k_i^*$ is the total number of the events of type $i$, then

$$k^* = \sum_{i=1}^m k_i^*. \tag{70}$$

It remains to bound from above the quantity $k_i^*$. Assume that a pair of neighbouring events of type $i$ occurred at Newton steps $q$ and $q' > q$. Let us verify that then

$$r_i((Y_{y_{q'}})_i : (Y_{y_q})_i) \geq \ln(\rho). \tag{71}$$

Indeed, in view of our rules for updating $B_y$ (see Sect. 4.1.2) we know that

since at the step $q$ an event of type $i$ occurred, we have $(Y_{y_q})_i = (Y_{y_q})_i$;

since there were no events of type $i$ at the steps $q+1, ..., q'-1$, we have $(Y_{y_q})_i = (Y_{y_{q'-1}})_i$;

since there was an event of type $i$ at the step $q'$, we have $\widehat{r}_i((Y_{y_{q'}})_i : (Y_{y_{q'-1}})_i) > \ln(3\rho)$, whence, in view of (55), $r_i((Y_{y_{q'}})_i : (Y_{y_{q'-1}})_i) > \ln(\rho)$.

These observations immediately imply (71).

Now, as we already know from Proposition 4.1, our scheme results in a method which finds $\nu(\kappa, \vartheta)$-solutions to all Newton systems arising in course of solving (BSP); therefore, in view of Proposition 2.1 (iii), we have

$$\rho_1^{-1}\mathcal{A}(x^*(t_p)) \leq \mathcal{A}(y_p) \leq \rho_1\mathcal{A}(x^*(t_p)).$$

It follows that, under notation $X_p = (\mathcal{A}(x^*(t_p)))^{-1}$,

$$\rho_1^{-1}X_p \leq Y_{y_p} \leq \rho_1 X_p.$$

In particular,

$$r_i((X_q)_i : (Y_{y_q})_i) \leq 2\ln(\rho_1), \quad r_i((X_{q'})_i : (Y_{y_{q'}})_i) \leq 2\ln(\rho_1).$$

As we have seen in Sect. 4.1.1, $r_i(\cdot : \cdot)$ is a distance on the space of symmetric $\mu_i \times \mu_i$ positive definite matrices; therefore the latter inequalities combined with (71) imply that

$$r_i((X_{q'})_i : (X_q)_i) \geq \ln(\rho_1^{-4}\rho) \geq 1.$$

(the concluding inequality follows from (61)).

Again taking into account that $r_i$ is a distance, we come to

$$\sum_{p=q+1}^{q'} r_i((X_p)_i : (X_{p-1})_i) \geq 1. \tag{72}$$

It follows immediately that

$$\sum_{p=2}^{N} r_i((X_p)_i : (X_{p-1})_i) \geq k_i^* - 1, \tag{73}$$

$N$ being the total number of Newton steps. Taking sum of (73) over $i = 1, ..., m$, we come to

$$\sum_{p=2}^{N} \| r(X_p : X_{p-1}) \|_1 \geq k^* - m.$$

Now, in view of Lemma 4.2 the left hand side of the latter innequality does not exceed $O(1)m^{1/2}N = O(1)m^{1/2}|\mathcal{M}|^{1/2}$, and we obtain

$$k^* \leq O(1)m^{1/2}|\mathcal{M}|^{1/2} \tag{74}$$

with $O(1)$ depending on $\kappa, \vartheta$ only.

**3.** To conclude the proof, it remains to note that $s^* \leq \mu(\mu+1)k^*$, since $s_y \leq \mu(\mu+1)k_y$, see Sect. 4.1.2. ∎

### 4.2.2 Efficiency of Karmarkar-type acceleration

Now we are ready to estimate the arithmetic cost of the method based on Karmarkar acceleration. Of course, we should choose the parameter $\rho$ according to (61); indeed, the less is $\rho$, the less is the number $K$ of the Conjugate Gradient steps, so that we are interested in the smallest $\rho$ which still allows to bound from above the total amount of rank 1 corrections required by updating the prescaling matrices. Thus, from now on we are speaking on the method with $\rho$ given by (61). Note that under this choice of $\rho$ every Newton step requires $\backsim 1$ steps of the prescaled Conjugate Gradients, see (60).

For the sake of simplicity, in what follows we restrict ourselves to the general "unstructured" case (see Sect. 3.2.1); same as in the latter section, we deal with a homogeneous block-diagonal structure ($\mu_i \equiv \mu$, $i = 1, ..., m$) and assume that $n \geq \mu$.

The computational effort in the above method is comprised of

$\backsim |\mathcal{M}|^{1/2}$ computations of the gradient of the barrier, $\backsim mn\mu^2$ operations per each (see Sect. 3.2.1);

$\backsim |\mathcal{M}|^{1/2}$ (totally) steps of the prescaled Conjugate Gradients, with $\backsim mn\mu^2$ operations per step (see Sect. 3.2.1);

computation of the very first prescaling matrix ($\backsim n^2 m\mu^2$ operations, see Sect. 3.2.1);

updating the prescaling matrices. As we shall demonstrate in a while, a natural implementation of the updating rules described in Sect. 4.1.2 takes $\backsim n^2$ operations per each rank 1 correction (plus $\backsim m\mu^3$ preprocessing operations per Newton step; we can forget about the latter quantity, since it is dominated by the cost of computing gradient, same as by the cost of a single Conjugate Gradient step). Since the total amount of rank 1 corrections in the method is $\lesssim \mu^2 m^{1/2} |\mathcal{M}|^{1/2}$ (Proposition 4.2), all prescalings cost $\lesssim m^{1/2} n^2 \mu^2 |\mathcal{M}|^{1/2}$.

Thus, the arithmetic cost at which the Karmarkar-accelerated method solves (BSP) is

$$M_{\mathcal{B}_\nu} \backsim (mn\mu^2 + m^{1/2} n^2 \mu^2) |\mathcal{M}|^{1/2} + mn^2\mu^2$$

operations, while the cost associated with the unaccelerated method is (see Sect. 3.2.1)

$$M_{\mathcal{B}} \backsim mn^2\mu^2 |\mathcal{M}|^{1/2}.$$

The acceleration ratio is

$$\mathcal{R} \backsim n^{-1} + m^{-1/2},$$

and for large $m$ we do obtain acceleration.

It remains to verify that rank 1 corrections do cost no more than $\backsim n^2$ operations per each, plus negligible in our considerations preprocessing of $\backsim m\mu^3$ operations per Newton step. Indeed, to perform updating $(B_{y_-}, B_{y_-}^{-1}) \mapsto (B_y, B_y)^{-1})$, we can act as follows (for notation see Sect. 4.1.2):

**1.** We may assume that we already know $\mathcal{A}(y), Y_y$ (these matrices are required by the expression for $F'(y)$); besides this, we have in our disposal the matrix $Y_{y_-}$ formed

at the previous Newton step. Given $Y_y$ and $V_{y_-}$, we compute the (approximate) ratio vector $\hat{r}(Y_y : V_{y_-})$, which takes $\backsim m\mu^3$ operations (Sect. 4.1.1, Computational Remark); this vector indicates the "bad" diagonal blocks in $V_{y_-}$, i.e., the blocks which should be replaced by those of $Y_y$ in order to obtain $V_y$.

**2.** We process the "bad" blocks in turn. When processing one of the blocks, everything evidently goes on as if there were no other diagonal blocks at all, i.e., as in the case of $m = 1$. Thus, our local task is as follows: we are given a linear mapping $A$ from $\mathbf{R}^n$ into the space $S_\mu$ of $\mu \times \mu$ symmetric matrices, two positive definite matrices $V$ and $Y$ from the latter space and a symmetric positive definite matrix $B$ together with its inverse. It is also known that

$$B = B^* + B^V, \quad B^W h \equiv A^*(WA(h)W),$$

where $B^*$ is certain symmetric positive semidefinite matrix (this matrix comes from the remaining diagonal blocks of $V_{y_-}$). Our task is to update $(B, B^{-1})$ into $(B', (B')^{-1})$, with

$$B' = B^* + B^Y.$$

To this end let us straightforwardly compute $B'$ according to the latter relation, which evidently costs $\backsim n^2\mu^2$ operations (indeed, to compute the image of a standard basic orth under $B^Y$ and $B^V$ it takes $\backsim \mu^3 + n\mu^2 \backsim n\mu^2$ operations, and we need $2n$ of these images). Then let us compute the symmetric matrix $D = B' - B$; as we know from Lemma 2.4, $D$ is a symmetric matrix of rank at most $\mu(\mu + 1)$, and we can straightforwardly represent it as a sum of $\mu(\mu + 1)$ rank 1 matrices at the cost $\backsim \mu^2 n^2$ operations. Now we can add these rank 1 matrices one by one to $B$ and use the Sherman-Morrison formula to update the corresponding inverses, which takes us $\backsim \mu^2 n^2$ operations. Thus, we can replace a single "bad" block (i.e., perform $\mu(\mu + 1)$ rank 1 corrections) in $\backsim n^2\mu^2$ operations, as claimed.

Of course, this way is completely theoretical (rounding errors will for sure make $D$ a matrix of full rank). In fact there exists a slightly less trivial stable procedure which performs the required updating at the same cost.

# 5 Applications to Quadratically Constrained problems

Consider a convex quadratically constrained quadratic problem (which without loss of generality can be rewritten to have a linear objective):

$$(\mathrm{QP}): \quad \textit{minimize } c^T x \quad \textit{s.t. } f_i(x) \leq 0, \quad i = 1, ..., m, \quad x \in \mathbf{R}^n,$$

where all $f_i$ are convex quadratic forms. Assume that the constraints satisfy the Slater condition, and that the feasible domain $G$ of the problem is bounded. It is well known ([Ja 87, MS 88, NN 88, NN 89]) that under these assumptions (QP) can be solved by a polynomial time path-following method associated with the barrier

$$\Phi(x) = -\sum_1^m \ln(-f_i(x)). \tag{75}$$

To obtain the description of the method, one should simply replace $|\mathcal{M}|$ in relation (5) by $m$, which results in the rule

$$t_{i+1} = \exp\left\{\frac{\vartheta}{m^{1/2}}\right\} t_i \tag{76}$$

(of course, the rule for updating penalty at the preliminary stage should be modified in the same manner) and replace the barrier $F$ whenever it is met in the description of $\mathcal{B}$ (see Sect. 1.2) by $\Phi$. For the resulting method, let it be called $\mathcal{B}^Q$, there is a convergence statement completely similar to Theorem 1.1, again with $|\mathcal{M}|$ replaced by $m$.

What we are interested in now is to understand whether the above strategies can be used for accelerating $\mathcal{B}^Q$. It becomes clear immediately that, generally speaking, it is impossible. Indeed, consider the case when $m$ is large (say, of order of $n$ or more) and the Hessians of $f_i$ are matrices of full rank. It is easily seen that in this case computational effort in the straightforward implementation of $\mathcal{B}^Q$ is dominated by the necessity to compute gradients of $\Phi$; since, as far as we know, there are no ideas how to avoid necessity to compute the gradients of the barrier exactly, here we have no hope for acceleration. Anyhow, there are large scale real world quadratic problems which involve a lot of *low rank* quadratic constraints represented as

$$f_i(x) = x^T A_i^T A_i x + b_i^T x + c_i, \tag{77}$$

with rectangular $p_i \times n$ matrices $A_i$ and $p_i \ll n$. Here the effort required by computing and inverting Hessians of $\Phi$ may again become the bottleneck, and we might hope to accelerate the process efficiently. What we are going to do is to study the corresponding possibilities. To simplify our considerations, let us assume that the ranks of Hessians of all $f_i$ in our problem (QP) do not exceed certain $\mu - 1$, and that all constraints $f_i$ from the very beginning are represented according to (77) with $p_i \leq \mu - 1$.

Let us start with the following observation. Given a convex quadratic form $f$ with the Hessian of certain rank $\mu - 1$, we always can represent the Lebesque set $\{x \mid f(x) \leq 0\}$ of the form as the inverse image of the cone of positive semidefinite symmetric $\mu \times \mu$ matrices under an appropriately chosen affine mapping $\alpha_f$. Indeed, we can represent our form as

$$f(x) = x^T A^T A x + b^T x + c,$$

$A$ being a $(\mu - 1) \times n$ matrix, and define the affine in $x$ $\mu \times \mu$ matrix $\alpha_f(x)$ as follows:

$$\alpha_f(x) = \begin{pmatrix} -b^T x - c & (Ax)^T \\ Ax & I \end{pmatrix}.$$

It is easily seen that $\alpha_f(x)$ is positive semidefimite if and only if $f(x) \leq 0$.

Now, given a problem (QP), let us set $\mu_i = \operatorname{rank} f_i'' + 1$, $\mathcal{M} = \{\mu_1, ..., \mu_m\}$ and consider the affine mapping

$$\mathcal{A}_{\mathrm{QP}}: \mathbf{R}^n \to S_{\mathcal{M}}: \quad \mathcal{A}_{\mathrm{QP}}(x) = \alpha_{f_1}(x) \oplus ... \oplus \alpha_{f_m}(x),$$

so that $\alpha_{f_i}(x)$ is $i$-th diagonal block in $\mathcal{A}_{\rm QP}(x)$.

By construction, $\mathcal{A}_{\rm QP}(x)$ is positive semidefinite if and only if $x$ is feasible for (OP), and the PD problem $\mathcal{P}_{n,\mathcal{M}}$ generated by the mapping $\mathcal{A}_{\rm QP}$ and the objective $c$ is nothing but our initial problem (QP). Moreover, the barrier $F$ associated with this PD problem is exactly the barrier $\Phi$, since $\mathrm{Det}(\alpha_f(x)) = -f(x)$.

Thus, we conclude that $\mathcal{B}^Q$ *is nothing but* $\mathcal{B}$, *with the only (and essential) difference that now, due to very specific embedding* $\mathcal{A} \equiv \mathcal{A}_{\rm QP}$, *we may (and, of course, wish) vary the penalty at a significantly higher rate* (compare (76) and (5)).

In view of the latter conclusion, we may try to apply our acceleration schemes to the basic path-following method for (QP), keeping in mind that the only element of the schemes which we should modify is the rule for updating the penalty (our previous rule should now be replaced by (76)). Our goal in this section is to understand what are the resulting acceleration ratios.

In view of the above remarks, to obtain results on acceleration of the path-following method for quadratically constrained problems it suffices to follow step by step the reasoning of our previous sections. Not presenting this (mostly quite straightforward) work in full details, we indicate only the key points and, of course, the final results.

*Background.* Proposition 2.1 turns out to be still valid (in fact items (i) and (ii) of this proposition, same as Theorem 1.1, are completely independent of the concrete analytical structure of the barrier underlying the path-following method in question; what is crucial is self-concordance of the barrier, see Sect. 6. There are also no difficulties with more specific item (iii) of Proposition 2.1, due to the above representation of (QP) as a PD problem). Note that this Proposition contains *all* facts about the path-following scheme which were used in our reasoning.

*Main inequality*, i.e. Lemma 2.1. This was the key statement for our reasoning. Since this is a statement about the path itself, not about the manner in which we increase the penalty parameter in our method, it remains valid. In fact it can be strengthen:

**Lemma 5.1** *Let* $t$ *and* $t_+$ *be two values of the penalty parameter, and let*

$$X = \mathcal{A}_{\rm QP}(x^*(t)),\ X_+ = \mathcal{A}_{\rm QP}(x^*(t_+))$$

*be the images in* $S_{\mathcal{M}}$ *of the corresponding points of the path. Set*

$$Z = (X^{-1/2} X_+ X^{-1/2})^{1/2}.$$

*Then*

$$\| \sqrt{t_+} Z - \sqrt{t} Z^{-1} \|_2^2 = |\mathcal{M}| \frac{(t - t_+)^2}{(\sqrt{t} + \sqrt{t_+})^2}; \tag{78}$$

*moreover, if* $\lambda_i$, $i = 1, ..., m_Z$ *are (taken with their multiplicities) those of eigenvalues of* $Z$ *which differ from* 1, *then*

$$\sum_{i=1}^{m_Z} (\sqrt{t_+}\lambda_i - \sqrt{t}\lambda_i^{-1})^2 \leq 4m \frac{(t - t_+)^2}{(\sqrt{t} + \sqrt{t_+})^2}. \tag{79}$$

.

**Proof.** The only news here is (79). To prove the latter statement, denote by $N$ the multiplicity of the unit eigenvalue of $Z$ and take into account that $i$-th diagonal blocks of $X$ and $X_+$ differ from the unit matrix of the corresponding size by matrices of rank at most 2 (construction of $\mathcal{A}_{\mathrm{QP}}$). Consequently, $i$-th block of $Z$ differs from the unit matrix by a matrix of rank at most 4, so that $N \geq |\mathcal{M}| - 4m$, which immediately transforms (78) into (79). ∎

It is easily seen that whenever earlier we used (23), we would obtain better results if we could (but we could not) use (79). And now we can do it.

**Relations "rank of a correction in $S_{\mathcal{M}} \mapsto$ rank of the induced correction in $S_n$".** Besides Lemma 2.1, the basis for our analysis included only the following simple observation (Lemma 2.4): let $U$ and $V$ be two positive definite matrices from $S_{\mathcal{M}}$ which differ by a matrix of certain rank $k$, and let the $n \times n$ matrices $P_U$ and $P_V$ be defined by the relations $P_U h = A^*(UA(h)U), P_V h = A^*(VA(h)V)$. Then $P_U$ differs from $P_V$ by a matrix of rank at most $(\mu + 1)k$, where, as always, $\mu$ is the largest of components of the structure $\mathcal{M}$. We used this statement in two cases: first, when we knew nothing about the difference $U - V$, only its rank (this was the case in the Conjugate-Gradient-based part), and, second, when we knew that $U - V$ is a matrix with a single nonzero diagonal block (Karmarkar acceleration, the rule for updating prescaling matrices); in this latter case we used the estimate $\mathrm{rank}(P_U - P_V) \leq \mu(\mu + 1)$, keeping in mind that $\mathrm{rank}(U - V) \leq \mu$.

Now, the "general" estimate $\mathrm{rank}(P_U - P_V) \leq (\mu + 1)\mathrm{rank}(U - V)$ hardly might be improved even in our current specific situation. This is not the case with its consequence used in the analysis of the Karmarkar acceleration. Indeed, all blocks of all $U$ and $V$ we in fact are interested in in this analysis (see Sect. 4.1.2) are inverses to matrices from the image subspaces of the mappings $\alpha_{f_i}$. By construction, all elements from each of these subspaces differ from the unit matrix by a matrix of rank at most 2; of course, their inverses share the latter property and therefore differ from each other by matrices of rank at most 4. Thus, every "bad", in terms of Sect. 4.1.2, index $i$ causes not a $\mu(\mu + 1)$ rank 1 corrections in the prescaling matrix, but at most $4(\mu + 1)$ of these corrections (in fact, of course, no more than $\mu$ of them).

Keeping in mind the latter remarks and following the line of argument of the previous sections, one can easily come to the resluts we are about to present.

## 5.1 CG-based acceleration

As a result of possibility to use (79) instead of (23), we can ensure the conclusion of Proposition 2.4 by a better, as compared to (41), choice of the number of steps of the prescaled Conjugate Gradients:

$$K = \lfloor \min\{K_1, K_2\} \rfloor, \tag{80}$$

with

$$K_1 = O(1)m^2\delta^4, \quad K_2 = m\delta^2\mu^{1/2}. \tag{81}$$

where, same as above, all $O(1)$ depend only on the path tolerance and the penalty rate; the quantity $\delta$,

$$m^{-1/2} \leq \delta \leq 1, \tag{82}$$

is responsible for the frequency at which we compute prescaling matrices (recall that we keep these matrices fixed until the penalty parameter $t$ remains in "current" interval $\Delta_\tau$ of the length $\delta T$).

Consequently, we come to the following expression for the arithmetic cost of the CG-accelerated method associated with $\delta$ satisfying (82):

$$\mathcal{M}_\nu \backsim (Q_e + M_e)\delta^{-1} + (Q + M + M_e + (M + M_e)\mathcal{K})\, m^{1/2},$$
$$\mathcal{K} = m\delta^2 \min\{\mu^{1/2}, m\delta^2\} \tag{83}$$

(cf. (48)) and for the acceleration ratio associated with the optimal choice of $\delta$:

$$\mathcal{R} = \frac{Q + M + M_e}{Q_e + M_e} + \frac{1}{\sqrt{m}} + \min\{\mathcal{R}_1, \mathcal{R}_2\},$$
$$\mathcal{R}_1 = \mu^{1/6}\left(\frac{M + M_e}{Q_e + M_e}\right)^{1/3}; \quad \mathcal{R}_2 = \left(\frac{M + M_e}{Q_e + M_e}\right)^{1/5}. \tag{84}$$

(cf. (49)). Here $Q, M, Q_e, M_e$ are the same arithmetic costs of the basic Linear Algebra routines as in Sect. 3.

Now let us look what might be the typical values of the acceleration ration.

### 5.1.1 General "unstructured" case

Assume that, when solving (QP), we are given representations (77) of the constraints (i.e., the corresponding data $A_i$, $b_i$, $c_i$), with $\mu \times n$ matrices $A_i$, $\mu \leq n$, which may be arbitrary dense matrices of the indicated size. We also assume that we are given (or compute in advance) the $n \times n$ matrices $A_i^T A_i$. Note that since the feasible domain of the problem is assumed to be bounded, the total rank of the constraints is at least $n$:

$$\mu \leq n \leq m\mu.$$

It is easily seen that in the case in question one has

$$Q \backsim M \backsim mn\mu, \quad Q_e \backsim n^2(m + n), \quad M_e \backsim n^2.$$

With these dependences, (84) says that the acceleration does occur if, first,

$$R \equiv \frac{m\mu}{n(n + m)} \ll 1$$

(which simply means that the cost of computing gradient of $\Phi$ should be negligible as compared to the cost of assembling and inverting the Hessian of $\Phi$), and, second, if

$$1 \ll m,$$

i.e., when (QP) indeed is a "nontrivial" optimization problem. Now, under these assumptions the acceleration ratio is

$$\mathcal{R} \backsim R + \frac{1}{\sqrt{m}} + \min\{R_1, R_2\},$$

where

$$R_1 = \mu^{1/6} R^{1/3}, \quad R_2 = R^{1/5}.$$

For example, if $m \gtrsim n$, $\mu \backsim 1$, then the acceleration ratio is $\backsim n^{1/3}$.

### 5.1.2 Sparse case

Now consider the case when each of the constraints $f_i$ depends on $\mu$ of our $n$ independent variables (in our previous case $f_i$ depended on $\mu$ *linear forms* of our control variables). Now we have

$$Q \backsim M \backsim m\mu^2, \quad Q_e \backsim m\mu^2 + n^3, \quad M_e \backsim n^2.$$

Same as above, (84) says that the acceleration does occur when $m$ is large and the cost of computing gradient of $\Phi$ is negligible as compared to that one of computing the Hessian of $\Phi$; the latter means that

$$R = \frac{m\mu^2 + n^2}{m\mu^2 + n^3} \ll 1.$$

Under these assumptions the acceleration ratio is

$$\mathcal{R} \backsim R + \frac{1}{\sqrt{m}} + \min\{R_1, R_2\},$$

where

$$R_1 = \mu^{1/6} R^{1/3}, \quad R_2 = R^{1/5}.$$

Consider as example the single load Truss Topology Design in dual formulation (see [B-TB 91]). In this problem $n$ is of order of hundreds, $m = O(n^2)$ and $\mu$ is a small integer, namely, 4. Our relations result here in $\mathcal{R} \backsim n^{-1/3}$, which is not so bad!

## 5.2 Karmarkar acceleration

Following the line of argument from Sect. 4 (with (23) replaced by (79)) and keeping in mind the above remark on the number of rank 1 corrections in current prescaling matrix caused by processing a "bad" $i$, one can verify that the scheme in question is "compatible" with our new updating rule (76) for the penalty parameter, i.e., an appropriate choice of $\rho \backsim 1$ results in a method which solves all Newton systems to the prescribed accuracy $\nu$, and each system is solved in $\backsim 1$ steps of Conjugate Gradients. Besides this, the total amount of rank 1 corrections in course of solving (BSP) is $\lesssim m\mu$. Let us evaluate the arithmetic cost of solving (BSP) by the accelerated method in the case of "unstructured" problems (see Sect. 5.1.1). The total computational effort is comprised of

$\backsim m^{1/2}$ computations of the gradient of the barrier, $\backsim mn\mu$ operations per each (see Sect. 5.1.1);

$\backsim m^{1/2}$ (totally) steps of the prescaled Conjugate Gradients, with $\backsim mn\mu$ operations per step (see Sect. 5.1.1);

computation of the very first prescaling matrix ($\backsim n^2(m+n)$ operations, see Sect. 5.1.1);

updating the prescaling matrices. As we just have indicated, the total number of rank 1 corrections in course of solving (BSP) is $\backsim m\mu$. Same as in Sect. 4, it turns out that these updatings cost $\backsim n^2$ per each rank 1 correction, up to the cost of computing the matrix $Y_y$ and the ratio vector $\widehat{r}(Y_y, V_{y_-})$ (see Sect. 4.1.2) associated with the Newton step in question. Now, after the gradient of $\Phi$ at the current iterate is computed, to compute $Y$ and $r$ it costs not $\backsim m\mu^3$ operations, as in Sect. 4.1.2, but only $\backsim m\mu^2$ of them. This follows from the fact that all diagonal blocks in all our $Y$'s and $V$'s are inverse to matrices of the form $\alpha_{f_i}(\cdot)$, and these latter matrices are of size at most $\mu \times \mu$ and differ from the unit matrix by a matrix of rank at most 2. Same as in the PD case, the cost $m\mu^2$ per Newton step can be ignored - it is dominated by the cost of computing $\Phi'$.

Thus, the arithmetic cost at which the Karmarkar-accelerated method solves the QP Basic Subproblem is

$$\mathcal{M}_{\mathcal{B}_\nu} \backsim (mn\mu + m^{1/2}n^2\mu)m^{1/2} + n^2(m+n)$$

operations, while the cost of the unaccelerated method is (see Sect. 5.1.1)

$$\mathcal{M}_{\mathcal{B}} \backsim n^2(m+n)m^{1/2}.$$

The acceleration ratio is

$$\mathcal{R} \backsim \frac{m\mu}{n(m+n)} + \frac{m^{1/2}\mu}{m+n}.$$

To conclude this section we note that if we were interested in quadratically constrained quadratic problems only, we could come to the above results independently of any PD studies. Indeed, we could use the following simple observation which is fact was done in [NN 89]:

**Remark 5.1** *Let $o_1, ..., o_m$ be concave positive smooth functions on an open convex domain $Q$, and let*

$$\Phi(x) = -\sum_{i=1}^{m} \ln(o_i): Q \to \mathbf{R}.$$

*Assume that for some positive $t$ and $t_+$ the following two points are well defined:*

$$x = \operatorname{argmin}\{tc^T z + \Phi(z) \mid z \in Q\},$$
$$x_+ = \operatorname{argmin}\{t_+ c^T z + \Phi(z) \mid z \in Q\}.$$

*Then*

$$\sum_{i=1}^{m}\left(\sqrt{t}\sqrt{\frac{\phi_i(x)}{\phi_i(x_+)}}-\sqrt{t_+}\sqrt{\frac{\varphi_i(x_+)}{\varphi_i(x)}}\right)^2\le m(\sqrt{t}-\sqrt{t_+})^2. \tag{85}$$

**Proof.** By assumption, we have

$$\sum\frac{\phi_i'(x_+)}{\varphi_i(x_+)}=t_+c.$$
$$\sum\frac{\phi_i'(x)}{\varphi_i(x)}=tc.$$

Multiplying the first equality by $t_+^{-1}(x-x_+)^T$ and the second by $t^{-1}(x_+-x)^T$ and taking into account that $\phi_i$ are positive and concave on $Q$, we come to

$$\sum t_+^{-1}\left(\frac{\phi_i(x)-\phi_i(x_+)}{\phi_i(x_+)}\right)\le(x-x_+)^Tc,$$
$$\sum t^{-1}\left(\frac{\phi_i(x_+)-\phi_i(x)}{\varphi_i(x)}\right)\le(x_+-x)^Tc:$$

adding these two inequalities, we obtain

$$\sum\left(t\frac{\varphi_i(x)}{\phi_i(x_+)}+t_+\frac{\varphi_i(x_+)}{\phi_i(x)}\right)\le m(t+t_+)$$

which clearly is nothing but the desired inequality. ∎

To obtain the results of this section directly, one could apply (85) to $\varphi_i=-f_i$, $Q=\operatorname{int}G$ and use the resulting inequality instead of (23).

# 6 Generalizations

To the moment we dealt with a rather specific problems. It is time to ask ourselves whether there are any possibilities for acceleration of polynomial time path-following methods in the "general case".

Consider a convex programming problem in the so called *standard form*:

$$\mathcal{P}:\quad \textit{minimize}\quad c^Tx\quad s.t.\quad x\in G,$$

where $G$ is a closed convex set in $\mathbf{R}^n$; it is easily seen that every convex problem can be equivalently rewritten in this form. From now on we assume the feasible set $G$ of the problem to be bounded and to possess a nonempty interior. Note that a PD problem is exactly a standard problem associated with certain specific $G$, namely, an inverse image of the cone of positive semidefinite symmetric matrices under an affine mapping.

In [NN 88, NN 89, NN 91a] it is demonstrated that every standard problem can be solved by a globally linearly converging path-following method associated with a "proper" barrier for $G$. The notion of a "proper" ($\equiv$ *self-concordant*) barrier is defined as follows:

**Definition 6.1** *Let $G$ be a closed convex domain in $\mathbf{R}^n$ and $F$ be a $C^3$-smooth convex function defined on $\operatorname{int} G$ and such that $F(x)$ tends to $\infty$ along every sequence of points from $\operatorname{int} G$ converging to a boundary point of $G$. We say that $F$ is self-concordant, if*

$$|D^3(x)[h,h,h]| \leq 2\{D^2F(x)[h,h]\}^{3/2}, \quad x \in \operatorname{int} G, \quad h \in \mathbf{R}^n$$

*($D^kF(x)[h_1,...,h_k]$ denotes $k$-th differential of $F$ taken at $x$ along a collection of directions $h_1,...,h_k$). If, in addition,*

$$|DF(x)[h]| \leq \Theta^{1/2}\{D^2F(x)[h,h]\}^{1/2}, \quad x \in \operatorname{int} G, \quad h \in \mathbf{R}^n,$$

*for some constant $\Theta \geq 1$, then $F$ is called a $\Theta$-self-concordant barrier for $G$.*

Let $F$ be a $\Theta$-self-concordant barrier for $G$. One can associate with this barrier the basic path-following method solving problem $\mathcal{P}$; the description of the method looks exactly as that one for $\mathcal{B}$ (see Sect. 1.2) with the only difference that now the quantity $|\mathcal{M}|$ in the rules for updating penalty at the main and preliminary stages should be replaced by $\Theta$. As it is shown in the cited papers, this "general" path-following method satisfies certain general Convergence Theorem (which looks exactly as Theorem 1.1, up to substitution $|\mathcal{M}| \leftarrow \Theta$). In fact all convergence properties of the above path-following methods for PD and QP are immediate consequences of this general Convergence Theorem: it is easily seen that the PD-barrier (2) is a $|\mathcal{M}|$-self-concordant barrier for the feasible domain of $\mathcal{P}_{n,\mathcal{M}}$, while the QP-barrier (75) is an $m$-self-concordant barrier for the feasible domain of (QP).

Now, we may ask ourselves whether it is possible to accelerate the path-following method associated with a "general" self-concordant barrier in the same manner as it was done in the cases of PD and QP. First of all, we should understand whether it is possible to use approximate Newton directions instead of exact ones. The same general theory developed in [NN 89, NN 91a] says that it is possible: items (i) and (ii) of Proposition 2.1 in fact are completely independent of the particular analytical structure of barriers (2), (75); these statements are valid for the path-following method associated with *an arbitrary* self-concordant barrier. This is not the case with item (iii) of the latter proposition, but this item was used in the Karmarkar acceleration scheme only, and this latter scheme in any case could hardly be useful for acceleration of "general" path-following methods. Indeed, the Karmarkar acceleration is based on the fact that the "essential rank" of the difference between the Hessians occuring at a neighbouring pair of Newton steps (i.e., the dimension of the subspace where the Rayleigh ratio of the corresponding quadratic forms is greater than, say, 1.1) is a small fraction of the dimension of the control vector; this fact heavily depends on the analytical structure of the PD-barrier and, as it can be easily demonstrated, fails to be true for an arbitrary self-concordant barrier.

Thus, we might hope on CG-based acceleration only. The first feeling is that it also cannot help: our results on CG-based acceleration heavily depend on the "main inequality" (23), and it, of course, is related to the specific analytical structure of the PD-barrier (2). Nevertheless, let us look how did we use this inequality. Recall that all we need is

to understand how "far" could be the Hessian $F''(y)$ occurring at a Newton step associated with certain value $t$ of the penalty from the prescaling matrix, which is the Hessian $F''(x(T_*))$ corresponding to the first Newton step from the same segment of Newton steps (so that we know in advance that $T_*$ and $t$ are not too far from each other: $|t/T_* - 1| \leq \delta$, $\delta$ being the parameter of our acceleration scheme). Now, what does it mean "far", or, better to ask, "not far" for us? It means that the Rayleigh ratio $(h^T F''(y_+)h)/(h^T F''(y)h)$ is at least something not too small and that there is a subspace of not too large codimension where this ratio is not too large. Now, to evaluate this "distance", we used the "main ineaquality", and it told us (see Proposition 2.4) that

> (a) the Rayleigh ratio is *always* within the bounds $[\rho_0^{-2}(\rho_l^*)^{-2}, \rho_0^2(\rho_l^*)^2]$, with $\rho_0$ being some constant depending on the path tolerance and the penalty rate only and $\rho_l^* \backsim |\mathcal{M}|\,\delta^2$, and therefore prescaled Conjugate Gradients for sure find $\nu$-solution to the Newton system in question in $K_1 \backsim (|\mathcal{M}|\,\delta^2)^2$ steps (see Proposition 2.4);
>
> (b) for any $\rho \geq 8$ there is a subspace of codimension at most $2k(\rho) \backsim \rho^{-1} \max_i \mu_i\, |\mathcal{M}|\,\delta^2$ where the Rayleigh ratio is less than $\rho_0^2\rho^2$; consequently, the desired $\nu$-solution to the Newton system in question can be found in $K_2 \backsim (\max_i \mu_i)^{1/2}\, |\mathcal{M}|\,\delta^2$ steps of the prescaled Conjugate Gradients.

Of course, we have no hope to prove something like (b) for a path-following method associated with a general self-concordant barrier. Surprisingly, (a) turns out to be a completely general fact:

**Proposition 6.1** *Let $G$ be a closed and bounded convex domain in $\mathbf{R}^n$, $F$ be a $\Theta$-self-concordant barrier for $G$ and $c$ be a nonzero $n$-dimensional vector. These data define the path*

$$x^*(t) = \operatorname{argmin}\{-tc^T x + F(x) \mid x \in \operatorname{int} G\}, \quad t > 0.$$

*There exists an absolute constant $\varrho$ such that for any $t, T$, $0 < t < T$, the following inequality holds:*

$$\left\{1 + \varrho\left((\frac{T}{t} - 1)\Theta^{1/2} + (\frac{T}{t} - 1)^2\Theta\right)\right\}^2 F''(x(t)) \geq F''(x(T)) \geq$$
$$\geq \left\{1 + \varrho\left((\frac{T}{t} - 1)\Theta^{1/2} + (\frac{T}{t} - 1)^2\Theta\right)\right\}^{-2} F''(x(t)). \tag{86}$$

This proposition is proved in Appendix.

Combining the latter statement with Proposition 2.2 and items (i) and (ii) of Proposition 2.1 (as it was already indicated, these items are valid for a path-following method associated with an arbitrary self-concordant barrier), we come to the following statement

**Proposition 6.2** *Let $F$ be a $\Theta$-self-concordant barrier for the feasible set $G$ of problem $\mathcal{P}$, and let $G$ be bounded with a nonempty interior. Assume that problem $\mathcal{P}$ is solved by the*

*CG-accelerated path-following method associated with the barrier $F$, and let the number $K$ of steps of the prescaled Conjugate Gradients be chosen according to*

$$K = O(1)(\Theta\delta^2)^2.$$

*where $\delta$, $\Theta^{-1/2} \leq \delta \leq 1$, is the parameter of the CG-based acceleration scheme and $O(1)$ is an appropriately chosen constant (which depends only on the path tolerance $\kappa$ and the penalty rate $\vartheta$ used in the method). Then the method does find $\nu(\kappa,\vartheta)$-solutions to all Newton sustems arising in course of solving the Basic Subproblem (BSP) and, consequently, finds an $\varepsilon$-solution to $\mathcal{P}$ in no more than*

$$O(1)\Theta^{1/2}\ln\left(\frac{1}{\gamma\varepsilon}\right)$$

*Newton steps of the preliminary and the main stages, $\gamma$ being the asymmetry coefficient of $G$ with respect to the starting point $w$ (see Theorem 1.1), with $O(1)$ dependng on $\kappa,\vartheta$ only.*

We immediately conclude that the acceleration ratio associated with the optimal $\delta$ is as follows:

$$\mathcal{R} \backsim \frac{Q + M + M_e}{Q_e + M_e} + \frac{1}{\sqrt{\Theta}} + \mathcal{R}_1,$$
$$\mathcal{R}_1 = \left(\frac{M + M_e}{Q_e + M_e}\right)^{1/5};$$

here $Q, M, Q_e, M_e$ are our basic computational characteristics, see Sect. 3.1.

Looking at this expression, we see that the acceleration does occur (i.e., $\mathcal{R} \ll 1$), if the following two assumptions hold: first, $\Theta$ is large (i.e., $\mathcal{P}$ is a "nontrivial" optimization problem), and, second, the cost $Q_e$ of assembling and inverting $F''$ dominates significantly all other basic computational characteristics of the problem.

# 7 Concluding Remarks

The results of previous sections demonstrate that there are theoretical possibilities to accelerate the interior point path-following method for optimization over the cone of positive semidefinite matrices: similar possibilities, under appropriate relations between the sizes of the problem, exist also in quadratically constrained quadratic programming and even in the case of a path-following method associated with an arbitrary self-concordant barrier. These are theoretical possibilities; what about practical advantages of the above acceleration schemes? The very first feeling is that these schemes from practical viewpoint look not too attractive. First, the effect of acceleration is not so large (we save 1/3 or at most 1/2 in order of dependence on one of the sizes of the problem); note that in practical range of values of the sizes this effect is at least comparable with absolute constant factors which occur in our analysis. Of course, more careful analysis allows to reduce these

constants by order of magnitudes; but even then a practically oriented researcher might be dissapointed by things like a priori chosen number of steps of the Conjugate Gradients and the frequency for restoring the prescaling matrices in the CG-based acceleration scheme, same as by a priori fixed "compatibility levels" in the Karmarkar acceleration. Even with extremely carefully estimated constants, theoretical recommendations are always based on the worst-case analysis, and therefore computational procedures governed by too rigid theoretical recommendations usually are not very effective. Another, more specific source of objections could be the well-known practical instability of Conjugate Gradients as applied to ill-conditioned linear systems (so that in practice Conjugate Gradients hardly will "kill" $k$ largest eigenvalues of matrix $W$ (see (18)), as it should be according to Proposition 2.2).

We think, anyhow, that there is a possibility for very natural "practical" implementation of the CG-based acceleration. Let us fix in advance a "small" number $k$ (what is a "small" $k$, it will be explained later) and act as follows. We move along the path according to our basic scheme and at each Newton step allow the Conjugate Gradients (prescaled by certain current matrix $\mathcal{H}$) to perform at most $k$ steps. If we are successful (i.e., manage to find a "high accuracy" solution to the current Newton system), then we update the iterate and do not vary the prescaling matrix; if we fail to solve the Newton system in $k$ steps, we compute and invert the Hessian of the barrier at the current iterate and take it as our new prescaling matrix (we could also use the Karmarkar rank 1 corrections to restore "proper compatibility" of the prescaling matrix and the current Hessian of the barrier).

If we choose $k$ according to Proposition 2.4, this scheme would in fact be nothing but our CG-based acceleration, but this would again be the choice based on the worst-case theoretical analysis, and we would like to get free of the "worst case" recommendations: $k$ should, of course, be chosen in such a way that the arithmetic cost of $k$ steps of the Conjugate Gradients would be a small fraction (say, 10%) of the computational effort required by assembling and inverting the Hessian. If it is the case, than we for sure do not loose much: not more than 10% of total computational effort even in the most unpleasant case when Conjugate Gradients fail at every step. But - and this is a good news – in the case of PD problems, and consequently in the case of LP and QP problems – this unpleasant situation is "almost impossible". Let us indicate exactly what is meant. Consider the Newton step where Conjugate Gradients fail (and therefore the prescaling matrix at this step is set to $F''(y)$, $y$ being the current iterate). How long will this matrix "survive"? The answer is as follows. Let $G_y = G \cap (y + (y - G))$ be the largest symmetric with respect to $y$ subset of the feasible domain, and let $G_{y,\alpha} = y + \alpha(G_y - y)$, $0 < \alpha < 1$, be the set which is obtained by $(1/\alpha)$-times shrinking $G_y$ to $y$. An extremely simple observation is that in $G_{y,\alpha}$ the Hessian $F''(x)$ is compatible with $F''(y)$ within the factor $(1-\alpha)^{-2}$:

$$x \in G_{y,\alpha} \quad \Rightarrow \quad (1+\alpha)^{-2}F''(y) \leq F''(x) \leq (1-\alpha)^{-2}F''(y). \tag{87}$$

Indeed, let $x \in G_{y,\alpha}$, and let $h = \alpha^{-1}(x - y)$. Since $\mathcal{A}(y) \pm \mathcal{A}h = \mathcal{A}(y \pm h)$ is positive semidefinite (recall that we are speaking about a PD problem $\mathcal{P}_{n,\mathcal{A}}$), we conclude that

the matrices $Y = \mathcal{A}(y)$, $X = \mathcal{A}(x)$ are such that $Y \pm \alpha^{-1}(Y - X) \geq 0$, whence $X \geq (1-\alpha)Y$, $X \leq (1+\alpha)Y$. In view of Lemma 2.3 the latter inequalities immediately imply (87).

In view of (87) *while current iterates do not leave* $G_{y,\alpha}$, current Hessians are $[(1+\alpha)^2, (1-\alpha)^{-2}]$-compatible with $F''(y)$, and Proposition 2.2 says that if

$$k \geq \frac{1+\alpha}{1-\alpha} \ln(\frac{2}{\nu}) \tag{88}$$

then then the Conjugate Gradients are successful (recall that one can take $C_{\mathrm{CG}} = 1$), In other words, for given $k$ the Conjugate Gradients successfully follow the path until it leaves "$\alpha$-neighbourhood" of $y$, $\alpha$ being the largest real satisfying (88). For example, let $\nu = 10^{-3}$, which is more than enough for "reasonable" $\kappa$ and $\vartheta$, and let $k = 30$ (which does result in at most 10% "risk" in, say, dense PD problems with more than 1,000 variables). Then $\alpha$ is 0.59, so that the Conjugate Gradients are successful while the path remains in 60%-neighbourhood of the iterate $y$ where we compute and invert $F''$ (we mean the neighbourhood in the symmetrized domain $G_y$); while following the path in this neighbourhood, we pay per each Newton step at most 10% of the price corresponding to the basic path-following method, and the segment of the path followed in this manner looks "large".

In concern with the latter "self-tuning" CG-based scheme one might ask a reasonable question: how could we understand whether the Conjugate Gradients with $k$ steps are successful or not: given the result found by the method after $k$ steps, we, generally speaking, do not know whether it is a desired $\nu$-solution to the Newton system. There could be at least two ways to reslove this difficulty. First, we could simply define the largest $\alpha$ such that the current iterate $y_+$ belongs to $G_{y,\alpha}$ (this is a one-dimensional problem: we should find the points where the line passing through $y$ and $y_+$ intersects the boundary of $G$); if this $\alpha$ does not satisfy (88), then it is time to update the prescaling matrix; this is the strategy where we are governed not by successses or failures of CG, but by the behaviour of the path itself. Second, we could choose very small $\nu$, say, $\nu = 10^{-16}$, and ask the Conjugate Gradients to solve the system up to a relative residual at most $\nu$: if they are unable to do it in $k$ steps, we update the prescaling matrix. Note that in actual computations even "exact" Newton step results in the same level of accuracy, so that the quality of the CG-based following the path will be the same as when "exact" Newton steps are performed. On the other hand, from (88) it is seen that 100-step Conjugate Gradients do solve Newton systems to the accuracy $10^{-16}$ while current iterate is in $G_{y,\alpha}$ with $\alpha = 0.45$. Note that for problems with thousands of variables a hundred of CG steps is nothing as compared to the cost of assemling and inverting $F''$.

Of course, the above reasoning is based on the theoretical rate of convergence of the Conjugate Gradients; it is known that this method is rather sensitive to rounding errors, much more than the "non-iterative" Linear Algebra routines like Cholesski factorization. So, may we trust in the above reasoning? The answer seems to be positive. Indeed, when working within $G_{y,\alpha}$ with $\alpha$ not too close to 1, we apply the Conjugate Gradients to well-conditioned problems (for $\alpha = 0.59$ the condition number of the prescaled system is less

than 16; for $\alpha = 0.45$ it is less than 7), and there is no reason for instability of the CG. It seems that the above "self-tuning" CG-based implementation of the PD path-following method possesses certain practical potential.

# 8 Appendix A. Proof of Proposition 2.2

First note that optimal solutions to (17) ($z^*$) and to (18) ($\zeta^*$) are related as $z^* = U^{-1/2}\zeta^*$. It follows that if $z = U^{-1/2}\zeta$, then $(z - z^*)^T V(z - z^*) = (\zeta - \zeta^*)^T(U^{-1/2}VU^{-1/2})(\zeta - \zeta^*)$, so that $z$ is a $\nu$-solution to (17) if and only if $\zeta$ is a $\nu$-solution to (18). Now, the prescaled Conjugate Gradient method for (17) "in $\zeta$-variables", by construction, is the usual Conjugate Gradient method for (18). Thus, we should prove that the indicated in Proposition 2.2 number of steps is sufficient for the usual CG to find a $\nu$-solution to (18).

In view of Remark 2.1 under assumptions of the proposition in question the matrix $W$ involved into system (18) possesses the following property:

> (P): $W$ is a positive definite symmetric matrix with no eigenvalues less than $\rho_l^{-1}$ and at most $k$ of eigenvalues greater than $\rho_r$.

Let $\varepsilon(\zeta) = (\zeta - \zeta^*)^T W(\zeta - \zeta^*)$, so that the accuracy of a vector $\zeta$ regarded as an approximate solution to (18) is $\sqrt{\varepsilon(\zeta)/\varepsilon(0)}$. Thus, all we need is to demonstrate that if $\zeta_s$ is $s$-th approximate solution to (18) found by the Conjugate Gradients, then the ratio

$$\varepsilon(\zeta_s)/\varepsilon(0)$$

is at most $\nu^2$, provided that $s$ is the quantity announced in Proposition 2.2.

It is well-known that $\zeta_s$ can be defined as follows:

$$\zeta_s = p_s(W)\beta, \tag{89}$$

where $p_s$ is an optimal solution to the problem

$$\textit{minimize} \quad \varepsilon(p(W)\beta) \quad s.t. \quad p \in \mathcal{P}_s, \tag{90}$$

$\mathcal{P}_s$ being the space of all polynomials of degrees $\leq s - 1$ (the only polynomial of negative degree is $\equiv 0$).

Now, let $\Lambda$ be the spectrum of $W$, let $\zeta_\lambda$, $\lambda \in \Lambda$, be orthoprojections of the solution $\zeta^*$ to (18) onto the spectral subspaces of $W$ and let $\omega_\lambda = \zeta_\lambda^T\zeta_\lambda$. One clearly has

$$\varepsilon(p(W)\beta) = \sum_{\lambda\in\Lambda} \lambda(1 - \lambda p(\lambda))^2\omega_\lambda,$$

$$\varepsilon(0) = \sum_{\lambda\in\Lambda} \lambda\omega_\lambda. \tag{91}$$

Now, let $T_q(t) = \mathrm{ch}(q\,\mathrm{ch}^{-1}(t))$ be $q$-th Tschebyshev polynomial, and let $\Delta = [\rho_l^{-1}, \rho_r]$. We set

$$\tau(t) = \frac{\rho_l\rho_r + 1 - 2t\rho_l}{\rho_l\rho_r - 1},$$

$$\Pi_q(t) = T_q(\tau(t)),$$

$$\phi_q(t) = \Pi_q(t)/\Pi_q(0),$$

$$\pi_q(t) = \phi_q(t) \prod_{\lambda \in \Lambda \setminus \Delta} (1 - t/\lambda).$$

Since in view of (P) at most $k$ points of $\Lambda$ are outside $\Delta$, $\pi_q$ is a polynomial of degree at most $k + q$; by construction, $\pi_q(0) = 1$, so that

$$\pi_q(t) = 1 - t\psi_q(t), \quad \psi_q \in \mathcal{P}_{q+k}.$$

Now, when $t$ varies in $\Delta$, the ratio $\tau(t)$ varies in $[-1, 1]$, and therefore the first of two factors comprising $\pi_q$, i.e., the polynomial $\phi_q(t)$, in its absolute value is at most $1/\Pi_q(0)$; since in view of (P) all points in $\Lambda \setminus \Delta$ are to the right of $\Delta$, the second of the factors comprising $\pi_q$ is bounded in absolute value by 1, $t \in \Delta$. Thus, $|\pi_q(t)| \leq 1/\Pi_q(0), t \in \Delta$. When $t \in \Lambda \setminus \Delta$, $\pi_q(t) = 0$ by construction. Thus,

$$|1 - t\psi_q(t)| \leq 1/\Pi_q(0), \ t \in \Lambda$$

for some polynomials $\psi_q \in \mathcal{P}_{q+k}$, $q = 0, 1, ...$, whence, in view of (89), (90), (91),

$$\varepsilon(\zeta_{k+q}) \leq (\Pi_q(0))^{-2} \varepsilon(0), \quad q = 1, 2, ...$$

Now the conclusion of Proposition 2.2 can be obtained by straightforward estimating from below the quantity

$$\Pi_q(0) = \mathrm{ch}\left(q \, \mathrm{ch}^{-1}\left(\frac{\rho_l \rho_r + 1}{\rho_l \rho_r - 1}\right)\right).$$

# 9 Appendix B. Behaviour of Hessians along the Path

Let $G$ be a closed and bounded convex domain in $\mathbf{R}^n$, let $F$ be a $\Theta$-self-concordant barrier for $G$ and let $c$ be a nonzero linear functional on $\mathbf{R}^n$. Consider the path

$$x(t) = \mathrm{argmin}\{F(x) - tc^T x \mid x \in \mathrm{int}\, G\}, \quad t > 0, \tag{92}$$

which is well-defined and twice continuously differentiable on the whole $\mathbf{R}_{++} = \{t > 0\}$ (the proof of the latter statement, same as of all other facts from the theory of self-concordant functions which we use below, can be found in [NN 89, NN 91a]).

Our goal is to establish the following

**Theorem 9.1** *The second-order derivative of the barrier do not vary too fast along the path, namely, for certain absolute constant $\varrho$ and for any pair of $t, T$, $0 < t < T$, one has*

$$\left\{1 + \varrho\left((\frac{T}{t} - 1)\Theta^{1/2} + (\frac{T}{t} - 1)^2\Theta\right)\right\}^2 F''(x(t)) \geq F''(x(T)) \geq$$
$$\geq \left\{1 + \varrho\left((\frac{T}{t} - 1)\Theta^{1/2} + (\frac{T}{t} - 1)^2\Theta\right)\right\}^{-2} F''(x(t)). \tag{93}$$

**Proof.** It is more convenient to deal with the Legendre transformation of $F$, i.e., with the function

$$\Phi(s) = \max\{s^T x - F(x) \mid x \in \operatorname{int} G\},$$

which, as it is known from the general theory of self-concordant functions, is a self-concordant function on $\mathbf{R}^n$ and is such that

$$s^T \Phi''(s) s \leq \Theta, \quad s \in \mathbf{R}^n. \tag{94}$$

The transformation $x \mapsto F'(x)$ maps $x(t)$ onto the point $tc$, so that the image of the path is simply the ray

$$\gamma = \{tc \mid t > 0\}$$

in $\mathbf{R}^n$, and, of course,

$$F''(x(t)) = (\Phi''(tc))^{-1}.$$

We see that to prove (93) is the same as to prove that for certain absolute constant $\varrho$ and all $t, T$, $0 < t < T$, one has

$$\left\{1 + \varrho\left((\frac{T}{t} - 1)\Theta^{1/2} + (\frac{T}{t} - 1)^2\Theta\right)\right\}^2 \Phi''(tc) \geq \Phi''(Tc) \geq$$
$$\geq \left\{1 + \varrho\left((\frac{T}{t} - 1)\Theta^{1/2} + (\frac{T}{t} - 1)^2\Theta\right)\right\}^{-2} \Phi''(tc). \tag{95}$$

**1. Notation.** Let us fix $t, T$, $0 < t < T$, and let

$$\alpha = T/t - 1. \tag{96}$$

Let us provide $\mathbf{R}^n$ with the Euclidean structure

$$(s', s'') = (s')^T \Phi''(tc) s''; \tag{97}$$

the corresponding Euclidean norm will be denoted by $|\cdot|$. Now, let $e$ be the unit vector collinear to $c$. We set

$$s(r) = re, \quad r \in \mathbf{R}, \tag{98}$$

and denote by $Q(r)$ the Hessian matrix of $\Phi$ taken with respect to our Euclidean structure at the point $s(r)$. We also set

$$f(r) = \Phi(s(r)), \tag{99}$$

so that

$$f''(r) = (Q(r)e, e) = r^{-2}\, s^T(r)\Phi''(s(r))s(r) \leq \Theta r^{-2}, \quad r > 0 \tag{100}$$

(see (94)).

Last, let $\sigma > 0$ be defined by the requirement $s(\sigma) = tc$, so that

$$tc = s(\sigma), \quad Tc = s(\sigma'), \quad \sigma' = (1 + \alpha)\sigma. \tag{101}$$

Note that in view of choice of our Euclidean structure we have

$$Q(\sigma) = I, \tag{102}$$

and what we should prove is that for an appropriate *absolute constant* $\varrho$ the following matrix inequality holds:

$$\left(1 + \varrho(\alpha\Theta^{1/2} + \alpha^2\Theta)\right)^2 I \geq Q(\sigma') \geq \left(1 + \varrho(\alpha\Theta^{1/2} + \alpha^2\Theta)\right)^{-2} I. \tag{103}$$

Note that in view of (102), (100) and since $|e| = 1$ (definition of $e$) we have

$$f''(\sigma) = 1, \tag{104}$$

whence, by (100),

$$\sigma \leq \Theta^{1/2}. \tag{105}$$

In what follows $\varrho_i$ denote appropriate absolute constants.

**A.** It may happen that

$$\alpha^2\sigma^2 \leq 1/4. \tag{106}$$

Since, by construction,

$$\alpha^2\sigma^2 = (tc - Tc)^T\Phi''(tc)(tc - Tc),$$

in the case of (106) from basic inequalities on local behaviour of the second-order derivative of a self-concordant function it follows immediately that

$$(1 - \alpha\sigma)^2 Q(\sigma) \leq Q(\sigma') \leq (1 - \alpha\sigma)^{-2} Q(\sigma). \tag{107}$$

Since $\sigma \leq \Theta^{1/2}$ (see (105)), (107) means that in the case under consideration (103) is true, provided that $\varrho \geq \varrho_1$.

Thus, from now on we restrict ourselves to the case when

$$\sigma^2\alpha^2 > 1/4. \tag{108}$$

**B.** Let us start with proving that for certain absolute constant $\varrho_2$ one has

$$Q(\sigma') \geq (1 + 2\varrho_2\alpha^2\Theta)^{-2} I. \tag{109}$$

Let

$$\beta = 1/(2\sigma), \quad \sigma'' = \sigma - \beta\sigma, \quad q = \frac{\alpha + \beta}{\beta}, \tag{110}$$

$$S = s(\sigma)\ (= tc), \quad S' = s(\sigma')\ (= Tc), \quad S'' = s(\sigma''), \tag{111}$$

and let $h$ be an arbitrary vector from $\mathbf{R}^n$. Since $\Phi$ is convex, we have

$$\Phi(S'') + q\{\Phi(S + h) - \Phi(S'')\} \leq \Phi(S'' + q(S + h - S'')) = \Phi(S' + qh),$$

$$\Phi(S'') + q\{\Phi(S - h) - \Phi(S'')\} \leq \Phi(S'' + q(S - h - S'')) = \Phi(S' - qh),$$

whence

$$\Phi(S'+qh)+\Phi(S'-qh)-2\Phi(S')\geq 2\{-\frac{\alpha}{\beta}\Phi(S'')-\Phi(S')+\frac{\alpha+\beta}{\beta}\Phi(S)\}+$$
$$+\frac{\alpha+\beta}{\beta}\{\Phi(S+h)+\Phi(S-h)-2\Phi(S)\},$$

or, which is the same,

$$\Phi(S'+qh)+\Phi(S'-qh)-2\Phi(S')\geq 2\{-\frac{\alpha}{\beta}f(\sigma'')-f(\sigma')+\frac{\alpha+\beta}{\beta}f(\sigma)\}+$$
$$+\frac{\alpha+\beta}{\beta}\{\Phi(S+h)+\Phi(S-h)-2\Phi(S)\}, \qquad (112)$$

**Lemma 9.1** *There exists $\theta\in[\sigma'',\sigma']$ such that*

$$\frac{\alpha}{\beta}f(\sigma'')+f(\sigma')-\frac{\alpha+\beta}{\beta}f(\sigma)=\alpha(\alpha+\beta)\sigma^2f''(\theta)/2. \qquad (113)$$

**Proof.** We have $\sigma''=\sigma-\beta\sigma$, $\sigma'=\sigma+\alpha\sigma$. Let $g(r)=f(\sigma(1+r))$, $-\beta\leq r\leq\alpha$, and let $p(r)=g(r)-g(0)-g'(0)r$. Then the left hand side of (113) equals to

$$L[g]\equiv\frac{\alpha}{\beta}g(-\beta)+g(\alpha)-\frac{\alpha+\beta}{\beta}g(0)=\frac{\alpha}{\beta}p(-\beta)+p(\alpha)=$$
$$=\int_0^\alpha p'(r)dr-\frac{\alpha}{\beta}\int_0^\beta p'(-r)dr=\int_0^\alpha p'(r)dr-\int_0^\alpha p'(-\frac{\beta}{\alpha}r)dr=\int_0^\alpha dr\{\int_{-\frac{\beta}{\alpha}r}^{r}p''(\xi)d\xi\}.$$

We immediately conclude that if $m$ is the minimal, and $M$ is the maximal value of $f''$ on $[\sigma'',\sigma']$, or, which is the same, $\sigma^2m$ is the minimal, and $\sigma^2M$ is the maximal value of $p''$ on $[-\beta,\alpha]$, then the left hand side of (113) is between $L[\sigma^2\frac{m}{2}r^2]=m\alpha(\alpha+\beta)\sigma^2/2$ and $L[\sigma^2\frac{M}{2}r^2]=M\alpha(\alpha+\beta)\sigma^2/2$, which immediately implies (113). ∎

**Lemma 9.2** *For all $\rho\geq\sigma-1/2$ one has*

$$f''(r)\leq 16\Theta(\sigma+1)^{-2}. \qquad (114)$$

**Proof.** Assume, first, that $\sigma-1/2\leq r\leq\sigma+1/2$. The function $f$ is self-concordant on the whole axis and $f''(\sigma)=1$ (see (104)), and basic results on local behaviour of the second order derivative of a self-concordant function imply that $f''(r)\leq 4\leq 16\Theta/(\sigma+1)^2$ (we have taken into account (105)). Now, if $r>\sigma+1/2$, then (see (100)) $f''(r)\leq\Theta/r^2\leq 4\Theta/(2\sigma+1)^2$. ∎

Combining (114) (note that by construction $\sigma''=\sigma-1/2$) and (113), we can rewrite (112) as

$$\Phi(S'+qh)+\Phi(S'-qh)-2\Phi(S')\geq -16\alpha(\alpha+\beta)\Theta(\sigma/(\sigma+1))^2+$$
$$+\frac{\alpha+\beta}{\beta}\{\Phi(S+h)+\Phi(S-h)-2\Phi(S)\}, \qquad (115)$$

Now, assume that $h$ in (115) is subject to the restriction

$$|h| \geq 1; \tag{116}$$

then, in view of the origin of our Euclidean structure and basic properties of self-concordant functions,

$$\Phi(S+h)+\Phi(S-h)-2\Phi(S) \geq (4|h|-1)/6 \geq 2(|h|-1)/3 \tag{117}$$

so that for $h$ satisfying (116) relation (115) results in

$$\Phi(S'+qh)+\Phi(S'-qh)-2\Phi(S') \geq -16\alpha(\alpha+\beta)\Theta(\sigma/(\sigma+1))^2+\frac{2}{3}\frac{\alpha+\beta}{\beta}(|h|-1). \tag{118}$$

Given an arbitrary *unit* vector $w$, let us choose $t$ from the requirement that

$$32\alpha(\alpha+\beta)\Theta(\sigma/(\sigma+1))^2=\frac{2}{3}\frac{\alpha+\beta}{\beta}(t-1),$$

so that

$$t=1+48\alpha\beta\Theta(\sigma/(\sigma+1))^2, \tag{119}$$

and let us substitute into (118) $h=tw$. We obtain

$$\Phi(S'+qtw)+\Phi(S'-qtw)-2\Phi(S') \geq 16\alpha(\alpha+\beta)\Theta(\sigma/(\sigma+1))^2. \tag{120}$$

Let

$$\Omega=\Omega(w)=w^T\Phi''(S')w;$$

then either

$$q^2t^2\Omega \geq 1/32, \tag{121}$$

or

$$q^2t^2\Omega < 1/32. \tag{122}$$

Let us prove that in fact (122) is impossible. Indeed, assume that it is the case. Then the left hand side of (120), in view of basic results on local behaviour of self-concordant functions, does not exceed $8\Omega t^2q^2 \leq 1/4$, so that (120) transforms into

$$(\alpha\sigma)(\alpha\sigma+\beta\sigma)(\Theta/(\sigma+1)^2) \leq 1/64. \tag{123}$$

But $\Theta/(\sigma+1)^2 \geq \Theta/(2\sigma^2+2) \geq 1/4$ (in view of (105) and since $\Theta \geq 1$) and $\alpha\sigma \geq 1/2$ (see (108)), so that the left hand side of (123) is at least $1/16$, and (123) is impossible.

Thus, (122) is impossible and (121) is the case. Therefore

$$\Omega \geq q^{-2}t^{-2}/32=\beta^2(\alpha+\beta)^{-2}\{1+48\alpha\beta\Theta(\sigma/(\sigma+1))^2\}^{-2}/32. \tag{124}$$

Now, $\alpha\sigma \geq 1/2$, $\beta\sigma=1/2$ and, as we already have mentioned, $\Theta/(\sigma+1)^2 \geq 1/4$, whence $\{1+48\alpha\beta\Theta(\sigma/(\sigma+1))^2\} \leq \varrho_1\alpha\beta\Theta(\sigma/(\sigma+1))^2$, and (124) implies that

$$\Omega \geq \{\varrho_2(\alpha+\beta)\alpha\Theta(\sigma/(\sigma+1))^2\}^{-2}. \tag{125}$$

We have

$$(\alpha+\beta)\alpha\Theta(\sigma/(\sigma+1))^2 = \alpha^2\Theta\left\{(\frac{\sigma}{\sigma+1})^2(1+\frac{\beta}{\alpha})\right\} = \alpha^2\Theta\left\{(\frac{\sigma}{\sigma+1})^2(1+\frac{1}{2\alpha\sigma})\right\} \le 2\alpha^2\Theta$$

(we have taken into account (108)), whence

$$\Omega \ge \{2\varrho_2\alpha^2\Theta\}^{-2} \ge \{1+2\varrho_2\alpha^2\Theta\}^{-2}.$$

Thus, for any unit vector $w$ we have

$$\Omega(w) = w^T\Phi''(S')w \equiv (Q(\sigma')w,w) \ge \{1+2\varrho_2\alpha^2\Theta\}^{-2},$$

as required in (109).

**C.** It remains to prove that for certain absolute constant $\varrho_3$ one has

$$Q(\sigma') \le (1+2\varrho_3\alpha^2\Theta)^2 I. \tag{126}$$

Let

$$\beta = 1/(2\sigma), \quad \sigma'' = \sigma'+\beta\sigma, \quad q = \frac{\alpha+\beta}{\beta}, \tag{127}$$

$$S = s(\sigma)\ (=tc), \quad S' = s(\sigma')\ (=Tc), \quad S'' = s(\sigma''), \tag{128}$$

and let $h$ be an arbitrary vector from $\mathbf{R}^n$. Since $\Phi$ is convex, we have

$$\Phi(S'')+q\{\Phi(S'+h)-\Phi(S'')\} \le \Phi(S''+q(S'+h-S'')) = \Phi(S+qh),$$

$$\Phi(S'')+q\{\Phi(S'-h)-\Phi(S'')\} \le \Phi(S''+q(S'-h-S'')) = \Phi(S-qh),$$

whence

$$\Phi(S+qh)+\Phi(S-qh)-2\Phi(S) \ge 2\{-\frac{\alpha}{\beta}\Phi(S'')-\Phi(S)+\frac{\alpha+\beta}{\beta}\Phi(S')\}+$$
$$+\frac{\alpha+\beta}{\beta}\{\Phi(S'+h)+\Phi(S'-h)-2\Phi(S')\},$$

or, which is the same,

$$\Phi(S+qh)+\Phi(S-qh)-2\Phi(S) \ge 2\{-\frac{\alpha}{\beta}f(\sigma'')-f(\sigma)+\frac{\alpha+\beta}{\beta}f(\sigma')\}+$$
$$+\frac{\alpha+\beta}{\beta}\{\Phi(S'+h)+\Phi(S'-h)-2\Phi(S')\}, \tag{129}$$

**Lemma 9.3** *There exists $\theta \in [\sigma,\sigma'']$ such that*

$$\frac{\alpha}{\beta}f(\sigma'')+f(\sigma)-\frac{\alpha+\beta}{\beta}f(\sigma') = \alpha(\alpha+\beta)\sigma^2 f''(\theta)/2. \tag{130}$$

**Proof.** We have $\sigma'' = \sigma + (\alpha+\beta)\sigma$, $\sigma' = \sigma + \alpha\sigma$. Let $g(r) = f(\sigma(1+\alpha+r))$, $-\alpha \le r \le \beta$, and let $p(r) = g(r) - g(0) - g'(0)r$. Then the left hand side of (130) equals to

$$L[g] \equiv \frac{\alpha}{\beta}g(\beta) + g(-\alpha) - \frac{\alpha+\beta}{\beta}g(0) = \frac{\alpha}{\beta}p(\beta) + p(-\alpha) =$$

$$= -\int_0^\alpha p'(-r)dr + \frac{\alpha}{\beta}\int_0^\beta p'(r)dr = -\int_0^\alpha p'(-r)dr + \int_0^\alpha p'(\frac{\beta}{\alpha}r)dr = \int_0^\alpha dr\{\int_{-r}^{\frac{\beta}{\alpha}r} p''(x)dx\}.$$

We immediately conclude that if $m$ is the minimal, and $M$ is the maximal value of $f''$ on $[\sigma, \sigma'']$, or, which is the same, $\sigma^2 m$ is the minimal, and $\sigma^2 M$ is the maximal value of $p''$ on $[-\alpha, \beta]$, then the left hand side of (130) is between $L[\sigma^2\frac{m}{2}r^2] = m\alpha(\alpha+\beta)\sigma^2/2$ and $L[\sigma^2\frac{M}{2}r] = M\alpha(\alpha+\beta)\sigma^2/2$, which immediately implies (130). ∎

Combining (114) and (130), (129), we come to

$$\Phi(S+qh) + \Phi(S-qh) - 2\Phi(S) \ge -16\alpha(\alpha+\beta)\Theta(\sigma/(\sigma+1))^2 + \\ + \frac{\alpha+\beta}{\beta}\{\Phi(S'+h) + \Phi(S'-h) - 2\Phi(S')\}, \qquad (131)$$

Now, assume that $h$ in (115) is subject to the restriction

$$\pi(h) \equiv (Q(\sigma')h, h)^{1/2} \ge 1; \qquad (132)$$

then, in view of basic properties of self-concordant functions,

$$\Phi(S'+h) + \Phi(S'-h) - 2\Phi(S') \ge (4\pi(h) - 1)/6 \ge 2(\pi(h) - 1)/3 \qquad (133)$$

so that for $h$ satisfying (132) relation (131) results in

$$\Phi(S+qh) + \Phi(S-qh) - 2\Phi(S) \ge -16\alpha(\alpha+\beta)\Theta(\sigma/(\sigma+1))^2 + \frac{2}{3}\frac{\alpha+\beta}{\beta}(\pi(h)-1). \qquad (134)$$

Given an arbitrary vector $w$ with $\pi(w) = 1$, let us choose $t$ from the requirement that

$$32\alpha(\alpha+\beta)\Theta(\sigma/(\sigma+1))^2 = \frac{2}{3}\frac{\alpha+\beta}{\beta}(t-1),$$

so that

$$t = 1 + 48\alpha\beta\Theta(\sigma/(\sigma+1))^2, \qquad (135)$$

and let us substitute into (134) $h = tw$. We obtain

$$\Phi(S+qtw) + \Phi(S-qtw) - 2\Phi(S) \ge 16\alpha(\alpha+\beta)\Theta(\sigma/(\sigma+1))^2. \qquad (136)$$

Let

$$\Omega = \Omega(w) = w^T\Phi''(S)w;$$

then either

$$q^2t^2\Omega \ge 1/32, \qquad (137)$$

or

$$q^2t^2\Omega < 1/32. \tag{138}$$

Let us prove that in fact (138) is impossible. Indeed, assume that it is the case. Then the left hand side of (136), in view of basic results on local behaviour of self-concordant functions, does not exceed $8\Omega t^2q^2 \leq 1/4$, so that (136) transforms into

$$(\alpha\sigma)(\alpha\sigma + \beta\sigma)(\Theta/(\sigma+1)^2) \leq 1/64. \tag{139}$$

But $\Theta/(\sigma+1)^2 \geq \Theta/(2\sigma^2+2) \geq 1/4$ (in view of (105) and since $\Theta \geq 1$) and $\alpha\sigma \geq 1/2$ (see (108)), so that the left hand side of (139) is at least 1/16, and (139) is impossible. Thus, (138) is impossible and (137) is the case. Therefore

$$\Omega \geq q^{-2}t^{-2}/32 = \beta^2(\alpha+\beta)^{-2}\{1 + 48\alpha\beta\Theta(\sigma/(\sigma+1))^2\}^{-2}/32. \tag{140}$$

Now, $\alpha\sigma \geq 1/2$, $\beta\sigma = 1/2$ and, as we already have mentioned, $\Theta/(\sigma+1)^2 \geq 1/4$, whence

$$\{1 + 48\alpha\beta\Theta(\sigma/(\sigma+1))^2\} \leq \varrho_3\alpha\beta\Theta(\sigma/(\sigma+1))^2$$

and (140) implies that

$$\Omega \geq \left\{\varrho_3(\alpha+\beta)\alpha\Theta(\sigma/(\sigma+1))^2\right\}^{-2}. \tag{141}$$

We have

$$(\alpha+\beta)\alpha\Theta(\sigma/(\sigma+1))^2 = \alpha^2\Theta\left\{(\frac{\sigma}{\sigma+1})^2(1+\frac{\beta}{\alpha})\right\} =$$

$$= \alpha^2\Theta\left\{(\frac{\sigma}{\sigma+1})^2(1+\frac{1}{2\alpha\sigma})\right\} \leq 2\alpha^2\Theta$$

(we have taken into account (108)), whence

$$\Omega \geq \{2\varrho_3\alpha^2\Theta\}^{-2} \geq \{1 + 2\varrho_3\alpha^2\Theta\}^{-2}.$$

Thus, for any vector $w$ with $(Q(\sigma')w, w) = 1$ we have

$$\Omega(w) = w^T\Phi''(S)w \equiv (w, w) \geq \{1 + 2\varrho_3\alpha^2\Theta\}^{-2},$$

and (126) follows. ∎

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