New performance bounds and asymptotic properties of stochastic timed event graphs

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NEW PERFORMANCE BOUNDS AND ASYMPTOTIC PROPERTIES OF STOCHASTIC TIMED EVENT GRAPHS

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Bornes du Temps de Cycle et Comportements Asymptotiques des Graphes d'Evénements Stochastiques

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RESUME

Dans ce papier, nous nous intéressons à l'évaluation des performances et à l'analyse des comportements asymptotiques des graphes d'événements stochastiques. Les temps de franchissement des transitions sont générés par des variables aléatoires de distribution quelconque. Nous proposons d'abord de nouvelles bornes supérieures du temps de cycle à l'aide de la théorie des grandes déviations. A partir de ces bornes et des résultats existants, nous étudions les comportements asymptotiques en fonction de la structure du graphe d'événements, des temps de franchissement ainsi que du marquage initial. En particulier, des conditions suffisantes pour que le temps de cycle converge vers une valeur finie lorsque le nombre de transitions croît sont mises en évidence. Nous montrons que le temps de cycle converge vers celui du graphe d'événements déterministe lorsque les variances des temps de franchissement décroissent. Nous montrons également qu'en ajoutant suffisamment de jetons dans chaque place, il est possible d'approcher d'aussi près que l'on veut un temps de cycle minimal égal à la valeur maximale des temps moyens de franchissement. Nous présentons finalement l'application de ces résultats aux systèmes de production. En particulier, nous démontrons que la productivité d'une ligne de transfert décroît vers une valeur strictement positive lorsque le nombre de machines augumente.

MOTS-CLES : Graphes d'Evénements Stochastiques, Bornes, Comportements Asymptotiques

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New Performance Bounds and Asymptotic Properties of Stochastic Timed Event Graphs

Nathalie SAUER and Xiaolan XIE*

ABSTRACT

This paper addresses the performance evaluation and asymptotic properties of stochastic timed event graphs. The transition firing times are generated by random variables with general distribution. New upper bounds of the average cycle time are obtained by applying large deviation theory. Asymptotic properties with respect to the net structure, the transition firing times and the initial marking are then established on the basis of these new bounds and the existing ones. We propose in particular sufficient conditions under which the average cycle time tends to a finite positive value as the number of transitions tends to infinity. The convergence of the average cycle time, when the variances of the firing times decrease, is established. We also prove that, by putting enough tokens in all places, it is always possible to approach as close as possible to the minimum average cycle time which is equal to the maximum of the average transition firing times. Applications to manufacturing systems are presented. We prove in particular a conjecture which claims that the throughput rate of a transfer line decreases to a positive value when the number of machines increases.

KEYWORDS: Stochastic Timed Event Graphs, Performance Bounds, Asymptotic Properties

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1. INTRODUCTION

Thanks to their modeling power and the large body of mathematic tools for property checking, Petri nets have been recognized as a powerful tool for modeling and analyzing large dynamic systems. Especially, Petri nets provide a unified language for system modeling, property checking and performance evaluation for dynamic systems with synchronization, concurrency and common resources. Successful application areas include communication, computer, manufacturing systems, etc. Excellent surveys of Petri net theory and applications can be found in [14, 20].

In this paper we restrict ourselves to stochastic timed event graphs in which the firing times are generated by random variables. An event graph also called marked graph is a Petri net in which each place has exactly one input transition and one output transition. A strongly connected event graph has some important properties, specifically: (i) the number of tokens in any elementary circuit remains constant whatever the transition firings, and (ii) the system is deadlock free if and only if (or iff for short) each elementary circuit contains at least one token (see for instance [8, 9, 12]).

In the deterministic case, it was proven [8, 17] that: (i) the cycle time of an elementary circuit is given by the ratio of the sum of the firing times of the transitions of the circuit to the number of tokens in the circuit; (ii) the cycle time of a strongly connected event graph is equal to the greatest cycle time among the ones of all the elementary circuits. Furthermore, a specified cycle time being given, algorithms have been proposed in [13] to find an initial marking leading to a cycle time smaller than the specified cycle time while minimizing a linear criterion function of the initial marking.

In the stochastic case, it is impossible to take advantage of the elementary circuits to evaluate the behaviour of the event graph and to reach a given performance. Previous work mainly focused on ergodicity conditions and performance bounds. Ergodicity conditions have been obtained for timed event graphs [1], stochastic timed Petri nets [11] and max-plus algebra models of stochastic discrete event systems [18].

For a strongly connected stochastic timed event graph, it was proven that an average cycle time exists under some fairly weak conditions (see Section 2). Both upper and lower bounds have been proposed (see [2, 3, 7, 15, 16, 22]). In particular, tight upper and lower bounds were obtained by using stochastic comparison properties (see [3]), superposition properties (see [22]) and large deviation theory (see [2]).

The marking optimization problem for strongly connected stochastic timed event graphs, which consists in obtaining a specified cycle time while minimizing a linear criterion depending on the initial marking, was addressed in [15, 16, 19]. Important properties of optimal solutions can be found in [19]. Especially, it was proven that any cycle time greater than the maximal mean transition firing time can be reached provided that enough tokens are available. Heuristic algorithms for solving the stochastic marking optimization problem have been proposed.

The purpose of this paper is to provide new performance bounds and to address the asymptotic behaviour of stochastic timed event graphs. To this end, notations and assumptions are presented in Section 2 while Section 3 is a survey of the existing results.

In Section 4, upper bounds of the average cycle time are derived by using large deviation theory. These bounds are similar to but tighter than the bounds proposed in [2].

Based on these new bounds, Section 5 presents asymptotic properties with respect to the net structure. We propose in particular sufficient conditions under which the average cycle time tends to a finite positive value as the number of transitions tends to infinity.

Section 6 addresses the asymptotic behaviour of the average cycle time with respect to the transition firing times. Remind that it was proven in [3, 7] that the average cycle time reaches its minimum when the firing times become deterministic. This section establishes the convergence of the average cycle time to this minimum, the necessary and sufficient conditions under which this minimum is obtained, and the convergence of the average cycle time to the one of another timed event graph.

Section 7 is related to the asymptotic properties with respect to the initial marking. We prove that, by putting enough tokens in each place, it is always possible to approach as close as possible to the minimum average cycle time equal to the maximum of the average transition firing times. A necessary and sufficient condition under which this minimum is reachable is proposed. This section also establishes the limit value of the average cycle time when the number of tokens in a given place increases.

Section 8 presents applications to manufacturing systems. We prove in particular that when the capacity of a buffer in a transfer line increases, the throughput rate tends to the minimum of the throughput rates of two shorter transfer lines obtained by removing this buffer. We also prove the conjecture presented in [10] which claims that the throughput rate of a transfer line decreases to a positive value when the number of machines increases.

2. NOTATIONS AND ASSUMPTIONS

Let N = ($\mathcal{P}$, T, F) be the strongly connected event graph considered. $\mathcal{P}$ is the set of places, T is the set of transitions, and $F \subseteq (\mathcal{P} \times T) \cup (T \times \mathcal{P})$ is the set of directed arcs connecting places to transitions and transitions to places. We denote by $\mathcal{M}_0$ the initial marking of N.

We assume that no transition can be fired by more than one token at any time (i.e. recycled transitions). This implies that there is a self-loop place with one token related to each transition, i.e. $(t, t) \in \mathcal{P}$ and $\mathcal{M}_0((t,t)) = 1, \forall\, t \in T$ where (t, s) indicates the place connecting transition t to transition s. We further assume that, when a transition fires, the related tokens remain in the input places until the firing process ends. They then disappear and one new token appears in each output place of the transition.

As a result, the set of places $\mathcal{P}$ can be written as $\mathcal{P} = P \cup Pt$ where Pt denotes the set of self-loop places and P the other places, i.e. $Pt = \{(t, t), \forall t \in T\}$ and $P \cap Pt = \emptyset$. Furthermore, since there is always exactly one token in each place belonging to Pt, only the marking of the places belonging to P will be considered in the following.

Since N is an event graph, each place has exactly one input transition and one output transition. Without loss of generality, we assume that there exists at most one place between any two transitions. The following notations will be used :

$^\bullet t$ (resp. $t^\bullet$) : set of input (resp. output) places of transition t
$^\bullet p$ (resp. $p^\bullet$) : unique input (resp. output) transition of place p
in(t) : set of transitions immediately preceding transition t, i.e.

$$in(t) = \{s \in T / \exists\, p \in P,\ {}^\bullet p = s \text{ and } p^\bullet = t\}$$

out(t) : set of transitions immediately following transition t, i.e.

$$out(t) = \{s \in T / \exists\, p \in P,\ {}^\bullet p = t \text{ and } p^\bullet = s\}$$

(t, s) : place connecting transition t to transition s
Γ: set of elementary circuits of N
M_0: initial marking of the places belonging to P
$M_0(\gamma)$: total number of tokens contained initially in $\gamma \in \Gamma$
$R(M_0)$: the set of markings reachable from M_0

The following notations related to the transition firing times are used throughout this paper :

$X_t(k)$: non-negative random variable generating the time required for the k-th firing of transition t
$S_t(k)$: instant of the k-th firing initiation of transition t

By convention, $X_t(k) = 0, \forall\ k \le 0$ and $S_t(k) = 0, \forall\ k \le 0$. As shown in [8], the transition firing initiation instants can be determined by the following recursive equations :

$$S_t(k) = \underset{\tau \in in(t)}{\text{Max}} \left\{ S_\tau\left(k - \mathcal{M}_0((\tau,t))\right) + X_\tau\left(k - \mathcal{M}_0((\tau,t))\right) \right\} \tag{1}$$

We assume that for each transition t, the sequence of its firing times $\{X_t(k)\}_{k=1}^{\infty}$ is a sequence of independent identically distributed (i.i.d.) integrable random variables and that $\{X_t(k)\}_{k=1}^{\infty}$ for all $t \in T$ are mutually independent sequences.

It was proven in [1] that, under the foregoing assumptions, there exists a positive constant $\pi(M_0)$ such that:

$$\lim_{k\to\infty} S_t(k)/k = \lim_{k\to\infty} E\left[S_t(k)\right]/k = \pi(M_0), \quad \text{a.s.}\ \forall\ t \in T \tag{2}$$

$\pi(M_0)$ is the average cycle time of the event graph.

Since $\{X_t(k)\}_{k=1}^{\infty}$ are sequences of i.i.d. random variables, the index k is often omitted and we use X_t to denote the firing time of transition t whenever k is not necessary. We further assume that the first and second moments of X_t exist and denote by m_t its mean value and σ_t its standard deviation, i.e. $m_t = E[X_t]$ and $\sigma_t^2 = E[(X_t - m_t)^2]$.

Since any stochastic timed event graph is completely characterized by its net structure, its initial marking and the set of transition firing time sequences, it can be denoted by the triplet $(N, M_0, \{X_t(k)\})$.

Before leaving this section, we introduce some notions related to random variables which will be needed in the following. First, we need some stochastic ordering relations introduced by Stoyen [21]. In particular, the convex ordering relation and the strong ordering relation will be used. Let $\le_{icx}$ denote the convex ordering and $\le_{st}$ the strong ordering relation. Two random variables X and Y are said to satisfy the strong ordering relation (resp. the convex ordering relation), i.e. $X \le_{st} Y$ (resp. $X \le_{icx} Y$) if the following inequality

$$E[f(X)] \le E[f(Y)]$$

holds for all monotone nondecreasing (resp. convex monotone non-decreasing) function $f : \mathbb{R} \to \mathbb{R}$ provided that the expectations exist.

We say that two random variables X and Y are equivalent in law, denoted as $X =_{st} Y$, iff they have the same probability distribution, i.e.

$P\{X \le s\} = P\{Y \le s\}, \forall s \in \mathbb{R}$

We also need the notion of association of random variables (see [4, 5]). A set of random variables $\{x_1, x_2, ..., x_n\}$ is said to be associated if the following relation

$$cov(f(x_1, x_2, ..., x_n), g(x_1, x_2, ..., x_n)) \ge 0$$

holds for all non-decreasing functions h, g : $\mathbb{R}^n \rightarrow \mathbb{R}$ provided that the covariance exists.

3. BASIC PROPERTIES

This section recalls some basic properties of the average cycle time of stochastic timed event graph. These properties includes some properties with respect to the initial marking, the stochastic comparison properties, the superposition properties. Some upper bounds obtained by using superposition properties and large deviation theory are also presented.

3.1. Properties with respect to the initial marking

Property 1. ([9])
An initial marking M_0 is a live marking iff each elementary circuit contains at least one token, i.e. $M_0(\gamma) \ge 1, \forall \gamma \in \Gamma$.

Property 2. ([22])
Let M and M_0 be two markings with identical positive circuit counts, i.e. $M(\gamma) = M_0(\gamma) \ge 1, \forall \gamma \in \Gamma$. Then,

(a) M_0 and M are mutually reachable;
(b) $\pi(M) = \pi(M_0)$.

Property 3. ([19])
Let M_1 and M_2 be two initial markings such that $M_1(\gamma) \le M_2(\gamma), \forall \gamma \in \Gamma$. Then,

(a) $\exists M \in R(M_1), M \le M_2$, i.e. $M(p) \le M_2(p), \forall p \in P$;
(b) $\pi(M_1) \ge \pi(M_2)$.

Property 2 claims that if the initial marking is any marking reachable from M_0 by transition firings instead of M_0, the average cycle time remains the same. Property 3 is a generalization of the monotonicity property of the average cycle time with respect to the initial marking given in [3].

3.2. Stochastic comparison properties

Property 4.([3])

Consider two stochastic timed event graphs with the same net structure and the same initial marking STEG1 = (N, M_0, $\{X_t(k)\}$) and STEG2 = (N, M_0, $\{Y_t(k)\}$). Let $\pi^1(M_0)$ (resp. $\pi^2(M_0)$) be the the average cycle time of STEG1 (resp. STEG2). If

$$X_t \leq_{icx} Y_t, \qquad \forall t \in T,$$

then

$$\pi^1(M_0) \leq \pi^2(M_0).$$

Since the strong ordering $\leq_{st}$ implies convex ordering $\leq_{icx}$, the property still holds if the $\leq_{icx}$ ordering is replaced by the $\leq_{st}$ ordering.

Property 4 claims that the average cycle time is non-decreasing in transition firing times under stochastic comparison relations.

Since $E[X] \leq_{icx} X$ for any random variable X, Property 4 implies that the minimal average cycle time is obtained in the deterministic case as indicated in the following property.

Corollary 1.

$$\pi(M_0) \geq \pi^D(M_0)$$

where $\pi^D(M_0)$ denotes the average cycle time of the deterministic timed event graph (N, M_0, $\{Y_t(k)\}$) with $Y_t(k) = m_t$, i.e.

$$\pi^D(M_0) = \max_{\gamma\in\Gamma} \frac{\sum_{t\in\gamma} m_t}{M_0(\gamma)} \tag{3}$$

3.3. Superposition properties and upper bounds

The superposition properties in [22] are particularly useful in deriving upper bounds of the average cycle time. In particular, the main superposition property claims that the average cycle time is sub-additive when the transition firing times are generated by the superposition of two sets of sequences of random variables.

Main superposition property.

Assuming that $X_t = X_t^1 + X_t^2$, $\forall t \in T$ where X_t^1 and X_t^2 are non-negative random variables, let $\pi^1(M_0)$ (resp. $\pi^2(M_0)$) be the average cycle time of the stochastic timed event graph (N, M_0, $\{X_t^1(k)\}$) (resp. (N, M_0, $\{X_t^2(k)\}$)). Then it holds that :

$$\pi(M_0) \leq \pi^1(M_0) + \pi^2(M_0)$$

Using this property, the following upper bound can be easily derived.

Property 6.

$$\pi(M_0) \le \inf_{Z\ge 0}\left\{\underset{\gamma\in\Gamma}{\mathrm{Max}}\frac{\sum_{t\in\gamma} z_t}{M_0(\gamma)} + \pi^2(M_0, Z)\right\} \tag{4}$$

where Z denotes the vector of constants $(z_1, z_2, \ldots, z_{|T|})^T$ and $\pi^2(M_0, Z)$ denotes the average cycle time of the stochastic timed event graph $(N, M_0, \{X_t^2(k)\})$ with $X_t^2(k) = (X_t(k) - z_t)^+$.

Combining this upper bound and the upper bound presented in [7], the following property can be shown.

Property 7.

$$\pi(M_0) \le \pi^D(M_0) + \underset{Z\in\mathcal{E}}{\mathrm{Min}}\left\{\sum_{t\in T} E\left[(X_t - z_t)^+\right]\right\} \tag{5}$$

where

$$\mathcal{E} = \left\{Z/z_t \ge m_t, \forall t \in T \text{ and } \textstyle\sum_{t\in\gamma} z_t \le \pi^D(M_0)M(\gamma), \forall\gamma\in\Gamma\right\}$$

This bound shows that the firing time randomness of transitions belonging to non-critical elementary circuits has little effect on the average cycle time of the whole system. The effect of their randomness can almost be completely canceled by taking large values for z_t.

By taking $z_t = m_t$, $\forall t \in T$ and by using arguments similar to those in [22], the following upper bound can be derived.

Corollary 2.

$$\pi(M_0) \le \pi^D(M_0) + \sum_{t\in T}\sigma_t \tag{6}$$

We also consider a special case in which the initial marking is a multiple of another live marking. By using the main superposition property, the following tighter upper bound can be derived (see [19]).

Property 8.

$$\pi(nM_0) \le \pi^D(M_0) + \inf_{Z\in\mathcal{E}}\left\{\sum_{t\in T} E\left[\left(\frac{1}{n}\sum_{k=1}^{n} X_t(k) - z_t\right)^+\right]\right\} \tag{7}$$

where $\mathcal{E}$ is the set of vectors defined in Property 6.

From this property, we can prove the following corollary which shows that multiplying the number of tokens can reduce the variance of the transition firing times.

Corollary 3.

$$\pi(nM_0) \le \pi^D(M_0) + \sum_{t \in T} \frac{\sigma_t}{\sqrt{n}} \tag{8}$$

3.4. Upper bounds based on large deviation theory

Without loss of generality, let us assume that there is at most one token in each place, i.e. $M_0(p) \le 1$, $\forall\, p \in P$. Whenever this assumption is violated, an equivalent net in which there is at most one token in each place can be obtained by adding extra places and immediate transitions, as shown in [1].

To present the upper bounds given in [2], we need introduce new notations and rewrite the evolution equation (1).

Consider the set of elementary paths connecting any two transitions $(t_0 p_1 t_1 p_2 \cdots p_u t_u)$ such that only the first place p_1 contains one token and the other places are empty, i.e. $M_0(p_1) = 1$ and $M_0(p_i) = 0$ for $i = 2, \ldots, u$. Let Ψ denote this set of elementary paths.

We also need the following notations :

ρ^- : the first transition t_0 of a path $\rho \in \Psi$

ρ^+ : the last transition t_u of a path $\rho \in \Psi$

ρ^* : the set of internal transitions of a path $\rho \in \Psi$, i.e.

$$\rho^* = \{\, t \in \rho \;/\; t \ne \rho^- \text{ and } t \ne \rho^+ \}$$

$\vartheta^-(t)$: the set of transitions which can be connected to t via a path in Ψ, i.e.

$$\vartheta^-(t) = \{s \in T \;/\; \exists\, \rho \in \Psi,\ \rho^- = s \text{ and } \rho^+ = t\}$$

$\vartheta^+(t)$: the set of transitions which can be reached from t via a path in Ψ, i.e.

$$\vartheta^+(t) = \{s \in T \;/\; \exists\, \rho \in \Psi,\ \rho^+ = s \text{ and } \rho^- = t\}$$

$\Psi(t, s)$: the set of paths in Ψ connecting transition t to transition s, i.e.

$$\Psi(t, s) = \{\rho \in \Psi \;/\; \rho^- = t \text{ and } \rho^+ = s\}$$

Since all the transitions are recycled, the self loop (tpt) with $p^\bullet = {}^\bullet p = t$ belongs to the set $\Psi(t, t)$. Thus, $t \in \vartheta^+(t)$ and $t \in \vartheta^-(t)$ for all transition t.

As shown in [3], the evolution equation (1) can be rewritten as follows :

$$S_t(k) = \max_{\tau \in \vartheta^-(t)} \left\{ S_\tau(k-1) + A_{\tau,t}(k) \right\} \tag{9}$$

where

$$A_{\tau,t}(k) = \underset{\rho \in \Psi(\tau,t)}{\text{Max}} \{\mu(\rho,k)\} \tag{10}$$

and

$$\mu(\rho,k) = X_{\rho^-}(k-1) + \sum_{s \in \rho^*} X_s(k) \tag{11}$$

By convention, $A_{\tau,t}(k) = -\infty$ if $\tau \notin \vartheta^-(t)$. Thus,

$$A_{\tau,t}(k) = \begin{cases} X_\tau(k-1) + \underset{\rho \in \Psi(\tau,t)}{\text{Max}} \left\{\sum_{s \in \rho^*} X_s(k)\right\}, & \text{if } \tau \in \vartheta^-(t); \\ -\infty, & \text{otherwise.} \end{cases} \tag{12}$$

By using the large deviation theory, the following upper bound has been established in [2].

Property 9.

Assume that there exists a random variable χ such that

$$A_{\tau,t}(1) \leq_{st} \chi, \ \forall \tau, t \in T \tag{13}$$

Then,

$$\pi(M_0) \leq \gamma$$

where

$$\gamma = \inf\{x \geq E[\chi] \ / \ h(x) > \log(W)\} \tag{14}$$

$$W = \underset{t \in T}{\text{Max}} \left\{\text{card}\left(\vartheta^+(t)\right)\right\} \tag{15}$$

$$h(x) = \sup_{\theta \geq 0} \left\{x\theta - \log\left(E\left[\exp(\chi\theta)\right]\right)\right\} \tag{16}$$

$h(x)$ is commonly called the Cramér-Legendre transform of χ. It is well known that $h(x)$ is a convex and nonnegative function, and it reaches its minimum at $E[\chi]$ with $h(E[\chi]) = 0$.

4. NEW UPPER BOUNDS OF THE AVERAGE CYCLE TIME

The purpose of this section is to derive upper bounds tighter than the one proposed in [2] by using large deviation theory. More precisely, we intend to establish the following result :

Property 10.

Assume that there exists a random variable χ such that

$$A_{\tau,t}(1) \leq_{icx} \chi, \ \forall \tau, t \in T \tag{17}$$

Then,

$$\pi(M_0) \leq \gamma \tag{18}$$

where γ is defined by equation (14).

Since the strong ordering $\leq_{st}$ implies the convex ordering $\leq_{icx}$, this property allows to derive upper bounds tighter than the upper bound proposed in [2]. In particular, constant transition firing times can be easily taken into account in this new upper bound.

The proof of this property is based on the following four preliminary results, i.e. Lemmas 1 - 4.

To establish this result, we need introduce the following notations :

- $\{a_{\tau,t}(k)\}_{k=0}^{\infty}$ for all τ, $t \in T$ are mutually independent sequences of independent identically distributed random variables with $a_{\tau,t}(1) =_{st} A_{\tau,t}(1)$
- $V_t(k)$ with $t \in T$ and $k > 0$ are random variables defined as follows :

$$V_t(k) = \max_{\tau \in \vartheta^-(t)} \{V_\tau(k-1) + a_{\tau,t}(k)\} \tag{19}$$

Lemma 1.
$S_t(k) \leq_{st} V_t(k)$, $\forall t \in T$ and $\forall\, k > 0$

This property was established in [2]. Its proof is based on two facts : (i) the random variables { $A_{\tau,t}(k)$, $S_t(k)$ for all τ, $t \in T$ and $k \geq 0$} are associated; (ii) For any sequence of transitions $(t_0 t_1 t_2 \ldots)$, $\{A_{t_n, t_{n+1}}(n)\}$ is a sequence of mutually independent random variables (see equation (12)).

We also consider the following random variables :

- $\{b_{\tau,t}(k)\}_{k=0}^{\infty}$ for all τ, $t \in T$ are mutually independent sequences of independent identically distributed random variables with $b_{\tau,t}(1) =_{st} \chi$
- $H_t(k)$ with $t \in T$ and $k > 0$ are random variables defined as follows :

$$H_t(k) = \max_{\tau \in \vartheta^-(t)} \{H_\tau(k-1) + b_{\tau,t}(k)\} \tag{20}$$

Lemma 2.
$V_t(k) \leq_{icx} H_t(k)$, $\forall t \in T$ and $\forall\, k > 0$

Proof :

Let us prove this property by induction. First, since $a_{\tau,t}(1)$ and $b_{\tau,t}(1)$ for all τ, $t \in T$ are mutually independent random variables and since

$a_{\tau,t}(1) =_{st} A_{\tau,t}(1) \leq_{icx} \chi =_{st} b_{\tau,t}(1)$,

from Proposition 3.3.i in [4],

$$V_t(1) = \max_{\tau\in\vartheta^-(t)} \{a_{\tau,t}(1)\} \leq_{icx} \max_{\tau\in\vartheta^-(t)} \{b_{\tau,t}(1)\} = H_t(1)$$

Assume that the property is true for all $k \leq K$ which implies that :

$$V_t(K) \leq_{icx} H_t(K), \forall t \in T$$

Consider that non-decreasing convex mapping $f_t(X, Y)$ defined as follows

$$f_t(X,Y) = \max_{\tau\in\vartheta^-(t)} \{X_\tau + Y_{\tau,t}\}$$

Since the random variables $V_t(K)$, $a_{\tau,t}(K+1)$, $H_t(K)$ and $b_{\tau,t}(K+1)$ for all $\tau, t \in T$ are mutually independent, Proposition 3.3.i in [4] and the definitions of $a_{\tau,t}(K+1)$ and $b_{\tau,t}(K+1)$ imply that :

$$V_t(K+1) = f_t(V(K), a(K+1)) \leq_{icx} f_t(H(K), b(K+1)) = H_t(K+1)$$

Q.E.D.

Let us observe that $H_t(k)$ is the k-th firing time of transition t of a stochastic FIFO event graph which can be defined as follows. The set of transitions is T. There exists a place connecting transition τ to t iff $\tau \in \vartheta^-(t)$. Initially, each place contains exactly one token. A token arrived in any place (τ,t) remains unavailable for a random amount of time $b_{\tau,t}(k)$. The firing times of the transitions are null. According to [1], the following property holds :

Lemma 3.
There exists a positive constant $\pi^*(M_0)$ such that:

$$\lim_{k\to\infty} H_t(k)/k = \lim_{k\to\infty} E[H_t(k)]/k = \pi^*(M_0), \quad \text{a.s.} \ \forall t \in T \tag{21}$$

Lemma 4.
$\pi^*(M_0) \leq \gamma$.

Proof :
Consider the following probability $P\{H_t(k) \leq xk\}$ for any real number x. From Lemma 3, this property is equivalent to :

$$\lim_{k\to\infty} P\{H_t(k) \leq xk\} = 1, \qquad \forall x > \gamma \tag{22}$$

Let us prove relation (22) in the following.

Consider the set of sequences of transitions $(t_0 t_1 \ldots t_k)$ which satisfy the following relations :

$t_k = t$, and $t_{i-1} \in \vartheta^-(t_i)$, $\forall i = 1, 2, \ldots, k$

Let us denote this set as $C_t(k)$. From the recursive equation (20), we have :

$$H_t(k) = \max_{(t_0 t_1 \ldots t_k) \in C_t(k)} \left\{ B(t_0 t_1 \ldots t_k) \right\} \tag{23}$$

where

$$B(t_0 t_1 \ldots t_k) = \sum_{i=1}^{k} b_{t_{i-1} t_i}(i) \tag{24}$$

Since $b_{\tau,t}(k)$ are mutually independent, Lemma 3.1. in [4] implies that the random variables $B(t_0 t_1 \ldots t_k)$ for all sequences of transitions $(t_0 t_1 \ldots t_k) \in C_t(k)$ are associated. From Proposition 3.5. in [4],

$$H_t(k) \leq_{st} \max_{(t_0 t_1 \ldots t_k) \in C_t(k)} \left\{ \overline{B}(t_0 t_1 \ldots t_k) \right\}$$

where $\overline{B}(t_0 t_1 \ldots t_k)$ for all $(t_0 t_1 \ldots t_k) \in C_t(k)$ are the independent version of the random variables $B(t_0 t_1 \ldots t_k)$, i.e. the random variables $\overline{B}(t_0 t_1 \ldots t_k)$ for all $(t_0 t_1 \ldots t_k) \in C_t(k)$ are mutually independent and $\overline{B}(t_0 t_1 \ldots t_k) =_{st} B(t_0 t_1 \ldots t_k)$.

From the property of strong ordering,

$$\begin{aligned} P\left\{ H_t(k) \leq xk \right\} &\geq P\left\{ \max_{(t_0 t_1 \ldots t_k) \in C_t(k)} \left\{ \overline{B}(t_0 t_1 \ldots t_k) \right\} \leq xk \right\} \\ &= \prod_{(t_0 t_1 \ldots t_k) \in C_t(k)} P\left\{ \overline{B}(t_0 t_1 \ldots t_k) \leq xk \right\} \end{aligned}$$

Since $\overline{B}(t_0 t_1 \ldots t_k) =_{st} B(t_0 t_1 \ldots t_k)$,

$$P\left\{ H_t(k) \leq xk \right\} \geq \prod_{(t_0 t_1 \ldots t_k) \in C_t(k)} P\left\{ B(t_0 t_1 \ldots t_k) \leq xk \right\} \tag{25}$$

From equation (24) and since $b_{\tau,t}(k)$ are i.i.d. random variables, Chernoff's theorem implies that :

$$P\left\{ B(t_0 t_1 \ldots t_k) \leq xk \right\} = 1 - P\left\{ B(t_0 t_1 \ldots t_k) \geq xk \right\} = 1 - \exp(-h(x)k + o(k))$$

Combining relations (25),

$$P\{H_t(k) \le xk\} \ge e^{\text{card}(C_t(k))\log(1-\exp(-h(x)k+o(k)))}$$
$$= e^{-\text{card}(C_t(k))\exp(-h(x)k)(1+o(1))}$$

From this relation, if

$$\text{card}(C_t(k))\exp(-h(x)k) \to 0, \text{as } k \to \infty,$$

then

$$\lim_{k\to\infty} P\{H_t(k) \le xk\} = 1, \qquad \forall x > \gamma$$

Since it is obvious that

$$\text{card}(C_t(k)) \le (\text{card}(T))\, W^k,$$

we have

$$\text{card}(C_t(k))\exp(-h(x)k) \le \text{card}(T)\exp\big(k(-h(x) + \log(W))\big)$$

Since h(x) is non-decreasing function, from the definition of γ, then

$$-h(x) + \log(W) < 0,\ \forall\, x > \gamma$$

which implies that

$$\text{card}(C_t(k))\exp(-h(x)k) \to 0, \text{as } k \to \infty$$

Q.E.D.

Proof of Property 10 :

From Lemmas 1 and 2, we have

$$E[S_t(k)] \le E[V_t(k)] \le E[H_t(k)],\ \forall t \in T \text{ and } \forall k>0$$

From relations (2) and (21),

$$\pi(M_0) = \lim_{k\to\infty} E[S_t(k)]/k \le \lim_{k\to\infty} E[H_t(k)]/k = \pi^*(M_0)$$

Combining the above relation with Lemma 4,

$$\pi(M_0) \le \gamma$$

Q.E.D.

Similarly, the following upper bounds can be established.

Property 11.

$$\pi(M_0) \leq \inf\{x \geq E[\chi] \ / \ h(x) > \log(W')\}$$

where

$$W' = \underset{t \in T}{\mathrm{Max}}\left\{\mathrm{card}\left(\vartheta^-(t)\right)\right\}$$

Property 12.

$$\pi(M_0) \leq \inf\{x \geq E[\chi] \ / \ h(x) > \log(W'')\}$$

where W" is the Perron-Frobenius eigen-value of the 0-1 matrix $[L_{\tau,t}]$ with $L_{\tau,t} = 1$ if $\tau \in \vartheta^-(t)$; otherwise $L_{\tau,t} = 0$.

The proof of these properties is similar to that of Property 10 by using the following relations :

$$C_t(k) \leq (W')^k$$

$$\log(C_t(k)) = W'' \, k + o(k)$$

By using slightly different arguments, the following property can be proven.

Property 13.

Assume that there exists a random variable χ such that

$$\mu(\rho, 1) \leq_{icx} \chi, \ \forall \rho \in \Psi$$

Then,

$$\pi(M_0) \leq \gamma$$

where

$\mu(\rho, 1)$ is defined in equation (11)

$$\gamma = \inf\{x \geq E[\chi] \ / \ h(x) > \log(W)\}$$

$$h(x) = \sup_{\theta \geq 0}\left\{x\theta - \log\left(E\left[\exp(\chi\theta)\right]\right)\right\}$$

$$W = \underset{t \in T}{\mathrm{Max}}\left\{\sum_{\tau \in T} \mathrm{card}\left(\Psi(\tau, t)\right)\right\}$$

5. ASYMPTOTIC PROPERTIES WITH RESPECT TO THE NET STRUCTURE

The purpose of this section is to investigate the asymptotic behaviours of stochastic timed event graph as the number of transitions tends to infinity. More precisely, we present sufficient conditions under which the average cycle time tends to a finite positive value as the number of transitions tends to infinity.

First of all, let us consider a sequences of strongly connected stochastic timed event graphs {STEG(n) for n > 0} defined as follows :

$$\mathrm{STEG}(n) = \left(N^n, M_0^n, \left\{X_t^n(k)\right\}\right)$$

where N^n is the net structure of STEG(n) defined by the set of transitions T^n and the set of places P^n (not including the self loop places).

We make the following assumptions :

A1 : The event graph N^n is a sub-graph of N^{n+1} for all $n > 0$, i.e.
$T^n \subseteq T^{n+1}$ and $P^n \subseteq P^{n+1}$

A2 : The marking is identical for any commun places of two STEGs, i.e.
$M_0^n(p) = M_0^{n+1}(p), \quad \forall p \in P^n, \forall n > 0$

A3 : The firing times are identical for any commun transition of two STEGs, i.e.
$X_t^n(k) = X_t^{n+1}(k), \qquad \forall t \in T^n, \forall n, k > 0$

Thanks to assumption A2 and A3, the super indices for the initial marking and the transition firing times can be omitted. We denote by M_0 the initial marking and by $X_t(k)$ the transition firing times. Of course, M_0^n is the restriction of M_0 on P^n and $X_t^n(k)$ is the restriction of $X_t(k)$ on T^n.

Let $\pi^n(M_0)$ be the average cycle time of STEG(n). From the monotonicity property with respect to the net structure presented in [3],

$$\pi^1(M_0) \leq \pi^2(M_0) \leq \pi^3(M_0) \leq \ldots$$

As a result, the average cycle time converges to either a finite positive value or infinity. In the view of Property 13, the following sufficient condition under which the average cycle time converges to a finite positive value can be established.

Property 14.

Assume that the following three assumptions A4 and A5 hold :

A4. There exists a finite random variable χ such that
$\mu(\rho, 1) \leq_{icx} \chi, \forall \rho \in \Psi^n$ and $\forall n > 0$

A5. There exists a positive integer number W such that

$$W \geq \sum_{\tau \in T^n} \mathrm{card}\left(\Psi^n(\tau, t)\right), \qquad \forall t \in T^n, \forall n > 0$$

Then,

$$\pi^n(M_0) \leq \gamma, \forall n > 0$$

where γ is defined by equation (14), Ψ^n and $\Psi^n(\tau, t)$ are the same as Ψ and $\Psi(\tau, t)$ defined in section 3.4. with the stochastic timed event graph $(N, M_0, \{X_t(k)\})$ replaced by STEG(n).

Unfortunately, it is generally difficult to check the assumptions A4 and A5. In the following, we establish more restrictive but easily checkable sufficient conditions.

Let us assume that all the transition firing times are bounded by a finite random variable α in the sense of convex ordering, i.e.

$$X_t(1) \leq_{icx} \alpha, \forall t \in \cup T^n$$

A sufficient condition for A4 to hold is that all the paths in Ψ^n are bounded in length. Furthermore, if each transition is connected to a finite number of places, then the assumption A5 holds.

Property 15.

Assume that the following three assumptions A6, A7 and A8 hold :

A6. Each transition has at most K_1 input places, i.e.

$$\text{card}(\text{in}(t)) \leq K_1, \forall t \in T^n \text{ and } \forall n > 0$$

A7. There exists a finite random variable α such that

$$X_t(1) \leq_{icx} \alpha, \forall t \in \cup T^n$$

A8. There exists a positive integer number K_2 such that

$$L(\rho) \leq K_2 + 1, \forall \rho \in \Psi^n \text{ and } \forall n > 0$$

Then,

$$\pi^n(M_0) \leq \inf\{x \geq K_2 E[\alpha] \;/\; h(x) > K_2 \log(K_1)\}, \forall n > 0$$

where

$L(\rho)$: the number of transitions in path ρ;

$$h(x) = \sup_{\theta \geq 0}\left\{x\theta - K_2 \log\left(E\left[\exp(\alpha\theta)\right]\right)\right\}$$

Proof :

Let

$$\chi = \alpha_1 + \alpha_2 + \dots \alpha_{K_2} \tag{26}$$

where $\alpha_1, \alpha_2, \dots, \alpha_{K_2}$ are independent and identically distributed (i.i.d.) random variables with $\alpha_1 =_{st} \alpha$. From equation (11), for any path $\rho \in \Psi^n$,

$$\mu(\rho,1) = X_{\rho^-}(0) + \sum_{s \in \rho^\bullet} X_s(1) \tag{27}$$

From assumptions A7 and A8 and the mutual independence of random variables appeared on the right-hand side of relation (27),

$$\mu(\rho,1) \leq_{icx} \chi$$

From assumptions A6 and A8,

$$\sum_{\tau \in T^n} \text{card}\left(\Psi^n(\tau,t)\right) \leq (K_1)^{K_2}, \quad \forall t \in T^n, \forall n > 0$$

Finally, Property 14 implies that

$$\pi^n(M_0) \leq \inf\{x \geq E[\chi] \;/\; h(x) > K_2 \log(K_1)\}$$

where

$$E[\chi] = K_2 E[\alpha]$$

$$h(x) = \sup_{\theta \geq 0}\left\{x\theta - \log\left(E\left[\exp(\chi\theta)\right]\right)\right\} = \sup_{\theta \geq 0}\left\{x\theta - K_2 \log\left(E\left[\exp(\alpha\theta)\right]\right)\right\}$$

Q.E.D.

As can be noticed, assumption A8 depends strongly on the initial marking M_0 and the value of K_2 may be reduced if the initial marking is replaced by M which is reachable from M_0. Based on this idea, the following property generalizes Property 15 and presents some easily checkable sufficient conditions under which the average cycle time remains bounded as the number of transitions tends to infinity.

Property 16.

Assume that the following assumption A8' holds as well as A6 and A7.

A8'. There exists a positive integer number K_2 such that

$$L(\gamma) / M_0(\gamma) \le K_2, \forall \gamma \in \Gamma^n \text{ and } \forall n > 0$$

Then

$$\pi^n(M_0) \le \inf\{x \ge K_2 E[\alpha] / h(x) > K_2 \log(K_1)\}, \forall n > 0$$

where Γ^n denotes the set of elementary circuits of STEG(n) and $L(\gamma)$ denotes the number of transitions in γ.

Proof :

Let us consider STEG(n). From assumption A8' and Lemma 5 hereafter, there exists a marking M' reachable from M_0 such that

$$L(\rho) \le K_2 + 1, \forall \rho \in \Psi(M')$$

Let us consider another marking M" with $M''(p) = \text{Min}\{M'(p), 1\}$. Obviously, M" is a 0-1 marking and all the three assumptions A6, A7 and A8 hold in the stochastic timed event graph $(N^n, M'', \{X_t(k)\})$. Let $\pi'(M')$ and $\pi''(M'')$ be the average cycle times of the stochastic timed event graphs $(N^n, M', \{X_t(k)\})$ and $(N^n, M'', \{X_t(k)\})$ respectively.

From Property 15,

$$\pi''(M'') \le \inf\{x \ge K_2 E[\alpha] / h(x) > K_2 \log(K_1)\}$$

From Property 3,

$$\pi'(M') \le \pi''(M'')$$

Finally, from Property 2,

$$\pi^n(M_0) = \pi'(M')$$

which implies that

$$\pi^n(M_0) \le \inf\{x \ge K_2 E[\alpha] / h(x) > K_2 \log(K_1)\}, \forall n > 0$$

Q.E.D.

Lemma 5.

For any strongly connected event graph (N, M_0), the following two statements are equivalent.

(i) There exists a marking $M \in R(M_0)$ and a positive integer number K such that

$$L(\rho) \leq K+1, \forall \rho \in \Psi(M)$$

(ii) There exists a positive integer number K such that

$$L(\gamma) / M_0(\gamma) \leq K, \forall \gamma \in \Gamma$$

where $\Psi(M)$ denotes the set of elementary paths connecting any two transitions $(t_0 p_1 t_1 p_2 \ldots p_u t_u)$ such that only the first place p_1 contains tokens and the other places are empty, i.e. $M(p_1) \geq 1$ and $M(p_i) = 0$ for $i = 2, \ldots, u$.

The proof of Lemma 5 is given in appendix A.

In the view of assumption A8', the question naturally arises as to whether it can be relaxed and replaced by the following assumption :

$$\Sigma_{t \in \gamma} m_t / M_0(\gamma) \leq K_2, \forall \gamma \in \Gamma \quad (28)$$

The answer is negative. A counter-example is given in figure 1. In the example, there are n immediate transitions $\{t_1, t_2, \ldots, t_n\}$ and n timed transitions $\{s_1, s_2, \ldots, s_n\}$.

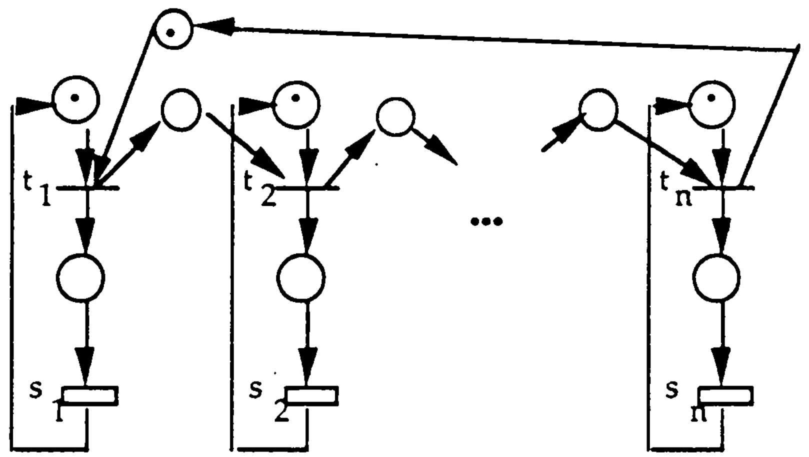

Figure 1: A counter-example

Let $X_i(k)$ be the time needed for the k-th firing of transition s_i and $S_i(k)$ the instant of the k-th firing initiation of transition t_i. Assume that $X_i(k)$ for all i and k are i.i.d. random variables and that they are unbounded.

The assumptions A6, A7 and (28) hold. However, assumption A8 does not hold and we show in the following that the average cycle time tends to infinity as n increases.

First, the following evolution equations can be easily established :

$$S_1(k) = \mathrm{Max}\{S_1(k-1) + X_1(k-1), S_n(k-1)\} \quad (29)$$

$$S_i(k) = \mathrm{Max}\{S_i(k-1) + X_i(k-1), S_{i-1}(k)\}, \forall i > 1 \quad (30)$$

From relation (30), it can be shown that

$$S_n(k) \geq S_{n-1}(k) \geq \ldots \geq S_1(k) \quad (31)$$

Again from relation (30),

$$S_i(k) \geq S_i(k-1) + X_i(k-1) \geq S_1(k-1) + X_i(k-1) \quad (32)$$

From relations (31) and (33),

$$S_n(k) \geq S_i(k) \geq S_1(k-1) + X_i(k-1)$$

which implies that

$$S_n(k) \geq S_1(k-1) + \mathrm{Max}_{1\leq i\leq n} X_i(k-1) \quad (33)$$

From relations (29) and (33),

$$S_1(2k+2) \geq S_n(2k+1) \geq S_1(2k) + \mathrm{Max}_{1\leq i\leq n} X_i(2k)$$

or :

$$S_1(2k+2) \geq \sum_{j=1}^{k} \underset{1\leq i\leq n}{\mathrm{Max}}\{X_i(2j)\}$$

By letting $k \to \infty$,

$$\pi^n(M_0) = \lim_{k\to\infty} \frac{S_1(2k+2)}{2k} \geq \lim_{k\to\infty} \frac{1}{2k} \sum_{j=1}^{k} \underset{1\leq i\leq n}{\mathrm{Max}}\{X_i(2j)\} = 0.5E\left[\underset{1\leq i\leq n}{\mathrm{Max}}\{X_i\}\right]$$

Since the random variables X_i are unbounded random variables,

$$\lim_{n\to\infty} \pi^n(M_0) = \infty$$

6. ASYMPTOTIC PROPERTIES WITH RESPECT TO FIRING TIMES

As shown in Corollary 1, the average cycle time reaches its minimum value when the transition firing times become deterministic, i.e. $\pi^D(M_0)$ (see relation (3)). This section addresses three issues related to this property : (i) the convergence of the average cycle time to its minimum value $\pi^D(M_0)$; (ii) the convergence of the average cycle time to the one of another stochastic timed event graph; (iii) the necessary and sufficient condition under which the average cycle time remains minimal, i.e. $\pi^D(M_0)$.

Now consider the convergence of the average cycle time when the firing times tend to being deterministic. The following property shows that the average cycle time converges to a minimal value as the firing times converges to constants in first moment.

Property 17.

$$\lim_{y\to 0} \pi(M_0) = \pi^D(M_0)$$

where

$$y = \sum_{t \in T} E\left[(X_t - m_t)^+\right]$$

Proof :

From Property 7 with $z_t = m_t$,

$$\pi(M_0) \le \pi^D(M_0) + \sum_{t \in T} E\left[(X_t - m_t)^+\right] = \pi^D(M_0) + y$$

From Corollary 1,

$$\pi(M_0) \ge \pi^D(M_0)$$

Combining the above two relations, we obtain :

$$\lim_{y \to 0} \pi(M_0) = \pi^D(M_0)$$

Q.E.D.

Similar to the proof of Property 17, by using Corollary 2, it can easily be shown that the average cycle time tends to its minimum as the standard deviations of the firing times decrease.

Corollary 4.

$$\lim_{z \to 0} \pi(M_0) = \pi^D(M_0)$$

where $z = \sum_{t \in T} \sigma_t$

Consider also two different stochastic timed event graphs. It was shown in [22] that their average cycle times tend to being identical as their transition firing times converge one to each other in first moment.

Property 18.

Consider two stochastic timed event graphs (N, M_0, $\{X_t^1(k)\}$) and (N, M_0, $\{X_t^2(k)\}$). Let $\pi^1(M_0)$ and $\pi^2(M_0)$ be their respective average cycle times. Then it holds that

$$\lim_{y \to 0} \left(\pi^1(M_0) - \pi^2(M_0)\right) = 0$$

where

$$y = \sum_{t \in T} E\left[\left|X_t^1 - X_t^2\right|\right]$$

In the remainder of this Section, we establish the following property which provides necessary and sufficient conditions under which the average cycle time remains minimal, i.e. is equal to $\pi^D(M_0)$.

Property 19.

The average cycle time remains minimal, i.e. $\pi(M_0) = \pi^D(M_0)$, iff there exists an elementary circuit $\gamma^* \in \Gamma$ for which the following two assumptions hold :

B1. $P\left\{\frac{\sum_{t\in\gamma^*} X_t}{M_0(\gamma^*)} \geq \frac{\sum_{t\in\gamma} X_t}{M_0(\gamma)}\right\} = 1, \quad \forall \gamma \in \Gamma$

B2. The firing times of the transitions belonging to γ^* are deterministic if γ^* contains more than one token, i.e.

$M_0(\gamma^*) \geq 2 \Rightarrow X_t =_{st} m_t, \forall t \in \gamma^*$

The proof of this property is given in Appendix B.

As can be noticed, assumption B1 implies that the firing times of any transition which does not belong to γ^* should be upper bounded and

$$\pi^D(M_0) = \frac{\sum_{t\in\gamma^*} m_t}{M_0(\gamma^*)}$$

Assumption B2 implies that in order to keep the average cycle time minimal, random firing times are not allowed in any critical elementary circuit containing more than one token. Furthermore, we prove the following result which claims that the average cycle times of the critical circuits should always be identical in the case of multiple critical circuits.

Corollary 5.

Assume that the average cycle time remains minimal, i.e. $\pi(M_0) = \pi^D(M_0)$. Then, for any two elementary circuits γ and γ' such that

$$\frac{\sum_{t\in\gamma'} m_t}{M_0(\gamma')} = \frac{\sum_{t\in\gamma''} m_t}{M_0(\gamma'')} = \pi^D(M_0) \tag{34}$$

it holds that :

$$P\left\{\frac{\sum_{t\in\gamma'} X_t}{M_0(\gamma')} = \frac{\sum_{t\in\gamma''} X_t}{M_0(\gamma'')}\right\} = 1 \tag{35}$$

Proof :

Since $\pi(M_0) = \pi^D(M_0)$, from Property 19, there exists an elementary circuit $\gamma^* \in \Gamma$ such that :

$$P\left\{\frac{\sum_{t\in\gamma^*} X_t}{M_0(\gamma^*)} \geq \frac{\sum_{t\in\gamma'} X_t}{M_0(\gamma')}\right\} = 1 \tag{36}$$

To prove relation (35), it is sufficient to prove that it holds for γ^* and γ'. From assumption (34),

$$E\left[\frac{\sum_{t\in\gamma^*} X_t}{M_0(\gamma^*)}\right] = E\left[\frac{\sum_{t\in\gamma} X_t}{M_0(\gamma)}\right] + E\left[\frac{\sum_{t\in\gamma^*} X_t}{M_0(\gamma^*)} - \frac{\sum_{t\in\gamma} X_t}{M_0(\gamma)}\right]$$

or :

$$E\left[\frac{\sum_{t\in\gamma^*}X_t}{M_0(\gamma^*)}\right]=\pi^D(M_0)+E\left[\frac{\sum_{t\in\gamma^*}X_t}{M_0(\gamma^*)}-\frac{\sum_{t\in\gamma}X_t}{M_0(\gamma)}\right] \tag{37}$$

Assume now that

$$P\left\{\frac{\sum_{t\in\gamma^*}X_t}{M_0(\gamma^*)}>\frac{\sum_{t\in\gamma'}X_t}{M_0(\gamma')}\right\}>0 \tag{38}$$

which implies that there exists $\Delta > 0$ and $\varepsilon \geq 0$ such that

$$P\left\{\frac{\sum_{t\in\gamma^*}X_t}{M_0(\gamma^*)}\geq\frac{\sum_{t\in\gamma'}X_t}{M_0(\gamma')}+\Delta\right\}=\varepsilon \tag{39}$$

From relations (36), (37) and (39),

$$E\left[\frac{\sum_{t\in\gamma^*}X_t}{M_0(\gamma^*)}\right]\geq\pi^D(M_0)+\Delta\varepsilon$$

This is in contradiction with the definition of $\pi^D(M_0)$ which concludes that assumption (38) does not hold. Then,

$$P\left\{\frac{\sum_{t\in\gamma^*}X_t}{M_0(\gamma^*)}=\frac{\sum_{t\in\gamma'}X_t}{M_0(\gamma')}\right\}=1$$

Q.E.D.

It was also proven in [7] that the lower bound provided by Corollary 1 cannot be improved by using only the first and second moments of the transition firing times, i.e. m_t and σ_t. We present this property and provide a simple proof below.

Property 20.

Consider an event graph N with initial marking M_0. For any $m_t \geq 0$ and $\sigma_t \geq 0$ for $t \in T$ and for any $\varepsilon > 0$, there exists a stochastic time event graph $(N, M_0, \{X_t(k)\})$ with $E[X_t] = m_t$ and $Var(X_t) = (\sigma_t)^2$ whose average cycle time is smaller than $\pi^D(M_0) + \varepsilon$, i.e. $\pi(M_0) \leq \pi^D(M_0) + \varepsilon$.

Proof :

Consider the following random variables,

$$X_t=\begin{cases}(1-\alpha)m_t, & \text{with probability } p_t;\\ m_t+(\sigma_t)^2/(\alpha m_t), & \text{with probability } 1-p_t;\end{cases}$$

with

$$p_t=\frac{(\sigma_t)^2}{(\sigma_t)^2+(\alpha m_t)^2}$$

It is easy to check that $E[X_t] = m_t$ and $Var(X_t) = (\sigma_t)^2$. From Property 7 with $z_t = m_t$, we have :

$$\pi(M_0) \le \pi^D(M_0) + \sum_{t\in T} E\left[(X_t - m_t)^+\right]$$

$$= \pi^D(M_0) + \alpha \sum_{t\in T} \frac{m_t(\sigma_t)^2}{(\sigma_t)^2 + (\alpha m_t)^2}$$

$$\le \pi^D(M_0) + \alpha \sum_{t\in T} m_t$$

Hence, it holds that for any $\alpha \le \varepsilon / \sum_{t\in T} m_t$,

$$\pi(M_0) \le \pi^D(M_0) + \varepsilon$$

Q.E.D.

7. ASYMPTOTIC PROPERTIES WITH RESPECT TO THE INITIAL MARKING

This subsection is devoted to the asymptotic properties of the average cycle time with respect to the initial marking. We restrict ourselves to two issues including : (i) the reachability of a given cycle time by starting from finite initial marking, and (ii) the convergence of the average cycle time with respect to the number of tokens in a given place. We prove that any cycle time C which is strictly greater than the maximal average transition firing time can be reached (see Property 22), while any C which is strictly smaller than the maximal transition firing time cannot be reached (see Property 21). A necessary and sufficient condition for the reachability of a given cycle time C equal to the maximal average transition firing time is also given (see Property 23). We also prove that when the number of tokens in place p^* increases, the average cycle time converges to the maximum of the average cycle times of the strongly connected components obtained by removing p^*.

Let C^* be the maximal average transition firing time, i.e.

$$C^* = \max_{t\in T}\{m_t\} \tag{40}$$

Property 21.

$$\pi(M_0) \ge C^*, \forall M_0$$

Proof :

From Corollary 1,

$$\pi(M_0) \ge \pi^D(M_0) = \max_{\gamma\in\Gamma} \frac{\sum_{t\in\gamma} m_t}{M_0(\gamma)}$$

where Γ is the set of elementary circuits. Since each transition is involved in a self loop (tpt) with one token, then

$$\pi^D(M_0) \ge \max_{\gamma\in\Gamma} \frac{\sum_{t\in\gamma} m_t}{M_0(\gamma)} \ge \max_{t\in T} m_t = C^*$$

Q.E.D.

Property 22.
Let $M_0 = n\,\mathbf{1}_{|P|}$ where $\mathbf{1}_{|P|}$ is a vector with all its components equal to 1. Then,

$$\pi(M_0) \le C^* + \sum_{t\in T} \frac{\sigma_t}{\sqrt{n}} \tag{41}$$

Proof :
From Corollary 3,

$$\pi(M_0) \le \pi^D(1_{|P|}) + \sum_{t\in T} \frac{\sigma_t}{\sqrt{n}} = C^* + \sum_{t\in T} \frac{\sigma_t}{\sqrt{n}}$$

Q.E.D.

We remind that a similar property has also been shown in [16] by means of a so-called N-POM constrained operating mode.

The following property presents the necessary and sufficient condition for the reachability of C^* proposed in [15]. A simple proof is provided hereafter.

Property 23.
C^* is reachable by using finite initial marking iff the following assumption C1 holds :

C1. There exists $t^* \in T$ such that

$$P\left[X_{t^*} = \operatorname*{Max}_{t\in T} X_t\right] = 1$$

Furthermore,

(a) If this condition holds, $\pi(n\,\mathbf{1}_{|P|}) = C^*$, $\forall n \ge 1$.

(b) If it does not hold, $\pi(M_0) > C^*$, $\forall M_0$

Proof :
First, let us assume that C1 holds. In this case, for $M_0 = n\,\mathbf{1}_{|P|}$, the number of tokens in any elementary circuit is greater than the number of transitions in it, i.e. $M_0(\gamma) \ge L(\gamma)$. This implies that

$$P\left[X_{t^*} \ge \operatorname*{Max}_{\gamma\in\Gamma} \frac{\sum_{t\in\gamma} X_t}{M_0(\gamma)}\right] \ge P\left[X_{t^*} \ge \operatorname*{Max}_{\gamma\in\Gamma} \frac{\sum_{t\in\gamma} X_t}{L(\gamma)}\right] \ge P\left[X_{t^*} = \operatorname*{Max}_{t\in T} X_t\right] = 1$$

From Property 19,

$$\pi(n\,\mathbf{1}_{|P|}) = \pi^D(n\,\mathbf{1}_{|P|}) = C^*$$

We now prove that $\pi(M_0) = C^*$ implies that C1 holds. From the definition of C^*, there exists a transition t' such that

$$E[X_{t'}] = C^*$$

From Property 21 and Corollary 1, $\pi(M_0) = C^*$ implies that

$$\pi(M_0) = \pi^D(M_0) = C^*$$

Combining this relation with Corollary 5 and Property 19, it holds that for the self loop containing t' :

$$P\left[X_{t'} \geq \operatorname*{Max}_{\gamma \in \Gamma} \frac{\sum_{t \in \gamma} X_t}{M_0(\gamma)}\right] = 1$$

which implies that

$$P\left[X_{t'} = \operatorname*{Max}_{t \in T} X_t\right] = 1$$

Q.E.D.

We now consider the asymptotic behaviours with respect to the number of tokens in a give place. Consider a given place p^* and a sequence of initial markings M_n defined as follows :

$$M_n(p^*) = M_0(p^*) + n \text{ and } M_n(p) = M_0(p), \forall p \neq p^*$$

Consider the sequence of stochastic timed event graphs STEG(n) = (N, M_n, $\{X_t(k)\}$). Let $\pi(M_n)$ be the average cycle time of STEG(n). From Property 3,

$$\pi(M_0) \geq \pi(M_1) \geq \pi(M_2) \geq \ldots$$

The following property provides the limiting value of these average cycle times.

Property 24.

$$\lim_{n \to \infty} \pi(M_n) = \operatorname*{Max}_{1 \leq i \leq \upsilon} \lambda_i(M_0)$$

where $\lambda_i(M_0)$ is the isolated average cycle time of the i-th strongly connected component obtained by removing p^*.

The proof of this property is given in Appendix C. From this property, the following decomposition property given in [3] can easily be proved.

Property 25.

Consider a stochastic timed event graph (N, M_0, $\{X_t(k)\}$) where N is connected and is composed of v strongly connected components. Let $\lambda_i(M_0)$ be the isolated average cycle time of the i-th strongly connected component. Then,

$$\operatorname*{Max}_{t \in T}\left\{\lim_{k \to \infty} S_t(k)/k\right\} = \operatorname*{Max}_{1 \leq i \leq \upsilon} \lambda_i(M_0)$$

Proof :

First, according to [1], there exists a positive constant $\mu_i(M_0)$ for each strongly connected component N_i such that

$$\lim_{k \to \infty} S_t(k)/k = \lim_{k \to \infty} E\left[S_t(k)\right]/k = \mu_i(M_0), \quad \text{a.s. } \forall \text{ t of } N_i$$

According to the monotonicity property of the average cycle time with respect to the net structure,

$$\operatorname*{Max}_{t\in T}\left\{\lim_{k\to\infty} S_t(k)/k\right\} = \operatorname*{Max}_{1\le i\le \upsilon} \mu_i(M_0) \ge \operatorname*{Max}_{1\le i\le \upsilon} \lambda_i(M_0)$$

Let us consider a strongly connected event graph N^* obtained from N by adding additional places. Consider the stochastic timed event graph (N^*, M^*, $\{X_t(k)\}$) where $M^*(p) = M_0(p)$ if p belongs to both N and N^*; otherwise $M^*(p) = n$. Let $\pi^*(n)$ be its average cycle time.

From the monotonicity property of the average cycle time with respect to the net structure,

$$\pi^*(n) \ge \operatorname*{Max}_{1\le i\upsilon} \mu_i(M_0), \qquad \forall n$$

By letting $n \to \infty$, Property 24 implies that :

$$\operatorname*{Max}_{1\le i\upsilon} \mu_i(M_0) \le \lim_{n\to\infty} \pi^*(n) = \operatorname*{Max}_{1\le i\upsilon} \lambda_i(M_0)$$

which yields that :

$$\operatorname*{Max}_{t\in T}\left\{\lim_{k\to\infty} S_t(k)/k\right\} = \operatorname*{Max}_{1\le i\le \upsilon} \mu_i(M_0) = \operatorname*{Max}_{1\le i\le \upsilon} \lambda_i(M_0)$$

Q.E.D.

8. STOCHASTIC TRANSFER LINES

Consider a transfer line consisting of a series of K machines ($\mathcal{M}_1$, $\mathcal{M}_2$, ..., $\mathcal{M}_K$) separated by K-1 buffers (B_1, B_2, ..., B_{K-1}). The buffer capacities are respectively C_1, C_2, ..., C_{K-1}. The processing times are generated by a set of random variable sequences $\{X_i(k)\}$ where $X_i(k)$ denotes the processing time of the k-th part on machine $\mathcal{M}_i$.

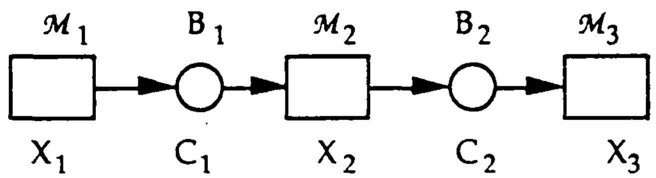

Figure 2: A three-machine transfer line

The behaviour of the system can be described as follows. A part is loaded from outside the system on the first machine $\mathcal{M}_1$. After being processed by $\mathcal{M}_1$, it is unloaded from $\mathcal{M}_1$ and queues in the first buffer B_1 if B_1 is not full, i.e. contains at most C_1 - 1 parts. Otherwise, it stays on $\mathcal{M}_1$ and $\mathcal{M}_1$ is blocked. This blocking situation persists until a part has been removed from the buffer B_1. Then, the part is unloaded from $\mathcal{M}_1$ and queues in B_1. It goes on in this manner through all machines and leaves the system after being processed by the last machine.

The Petri net model of the transfer line is illustrated in Figure 3 (see [23] for more detail).

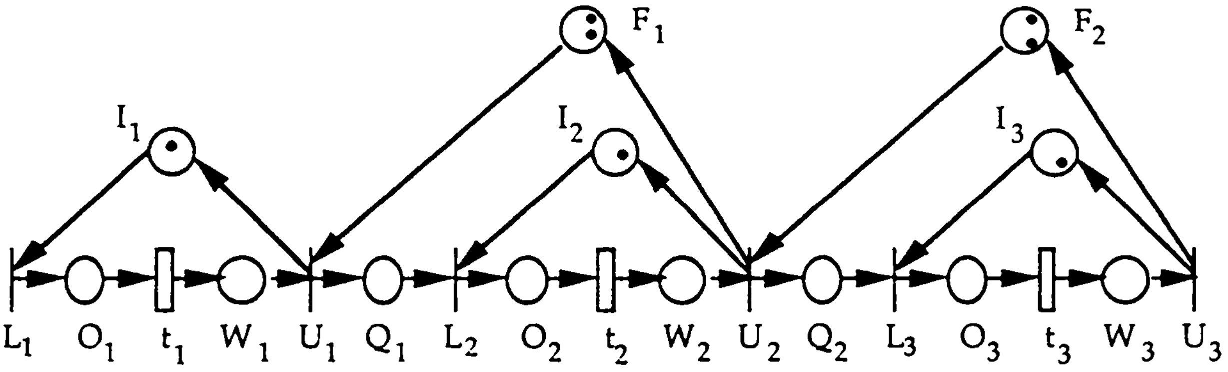

Figure 3: The Petri net model of a 3-machine transfer line

In this Petri net model, only the K transitions t_i are timed transitions and the transition firing times $\{X_t(k)\}_{k=1}^{\infty}$ for all $t \in T$ are defined as follows :

$$X_{L_i}(k) = X_{U_i}(k) = 0 \qquad \forall i = 1, 2, \ldots, K \text{ and } \forall k$$
$$X_{t_i}(k) = X_i(k) \qquad \forall i = 1, 2, \ldots, K \text{ and } \forall k$$

Finally, we denote such a Petri net model by the triplet $(N, M_0, \{X_i(k)\})$. It is a strongly connected event graph (also called marked graph). There are 2K-1 elementary circuits. We denote by $\gamma(\mathcal{M}_i)$ the elementary circuit modeling the state evolution of machine $\mathcal{M}_i$ and by $\gamma(B_{i-1})$ the elementary circuit modeling the storage space constraint of machine $\mathcal{M}$i and buffer Bi-1, i.e. $\gamma(\mathcal{M}_i) = (I_iL_iO_it_iW_iU_iI_i)$ and $\gamma(B_{i-1}) = (F_{i-1}U_{i-1}Q_{i-1}L_iO_it_iW_iU_iF_{i-1})$. Of course, $\gamma(\mathcal{M}_i)$ contains one token and $\gamma(B_{i-1})$ contains $C_{i-1} + 1$ tokens. In the following, the circuits $\gamma(\mathcal{M}_i)$ are called machine circuits and the circuits $\gamma(B_{i-1})$ storage circuits.

From Property 24, it holds that :

Property 26.

$$\lim_{C_{i-1}\to\infty} \pi(M_0) = \mathrm{Max}\{\lambda_1(M_0), \lambda_2(M_0)\}$$

where $\lambda_1(M_0)$ is the average cycle time of the transfer line composed of $(\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_{i-1})$ and $\lambda_2(M_0)$ that of the transfer line composed of $(\mathcal{M}_i, \mathcal{M}_{i+1}, \ldots, \mathcal{M}_K)$.

Property 27.

Let us assume there exists a finite random variable α such that

$$X_i(1) \leq_{icx} \alpha, \forall i$$

Then,

$$\pi(M_0) \leq \inf\{x \geq 2E[\alpha] \ / \ h(x) > \log(3)\}$$

where

$$h(x) = \sup_{\theta \geq 0} \{ x\theta - 2\log(E[\exp(\alpha\theta)]) \}.$$

Proof :

By the monotonicity property with respect to the buffer capacity in [23], we only need to consider the zero buffer case.

Remark that when the buffer capacities are reduced to zero, an equivalent Petri net model can be obtained as shown in Figure 4.

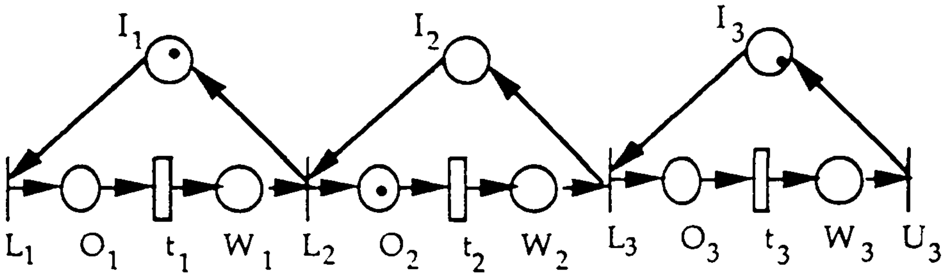

Figure 4: The Petri net model of a 3-machine transfer line without buffers

Consider the following initial marking :

$M_0(I_{2n-1}) = 1$, $M_0(O_{2n}) = 1$ and $M_0(p) = 0$ for all other place p

For this marking,

$A_{\tau,t}(1) \leq_{icx} \alpha_1 + \alpha_2, \forall \tau, t \in T$

where α_1, α_2 are independent and $\alpha_1 =_{st} \alpha_2 =_{st} \alpha$. We can notice that in this timed event graph, all the self loops related to the transitions can be removed since each elementary circuit contains exactly one token. Thus,

$$W' = \operatorname*{Max}_{t \in T} \{ \operatorname{card}(\vartheta^-(t)) \} = 3$$

From Property 11,

$\pi(M_0) \leq \inf\{x \geq 2E[\alpha] \ / \ h(x) > \log(3)\}$

Q.E.D.

Remark that this property proves the conjectures presented in [10] which claims that the productivity of a production line tends to a positive value as the number of machines increases.

Remark that similar results can be obtained in the case of assembly/disassembly systems and kanban systems.

9. CONCLUSION

This paper addresses the performance evaluation and the analysis of asymptotic properties of stochastic timed event graphs. First, upper bounds of the average cycle time are derived by applying large deviation theory. These bounds are similar to but tighter than the bounds proposed in [2].

Based on these new bounds, some asymptotic properties with respect to the net structure are established. We propose in particular sufficient conditions under which the average cycle time tends to a finite positive value as the number of transitions tends to infinity.

We also address the asymptotic behaviour of the average cycle time with respect to the transition firing times. It was proven in [3, 7] that the average cycle time reaches its minimum when the firing times become deterministic. We establish the convergence of the average cycle time to this minimum, the necessary and sufficient conditions under which this minimum is obtained, and the mutual convergence of the average cycle times of any two stochastic timed event graphs.

Another issue addressed is the asymptotic behaviour with respect to the initial marking. We prove that, by putting enough tokens in each place, it is always possible to approach as close as possible to the minimum average cycle time equal to the maximum value of the average transition firing times. A necessary and sufficient condition under which this minimum is reachable is proposed. We also provide the limiting value of the average cycle time when the number of tokens in a given place increases.

Applications to manufacturing systems have been presented. We prove in particular that when the capacity of a buffer in a transfer line increases, the throughput rate tends to the minimum value of the throughput rates of two shorter transfer lines obtained by removing this buffer. We also prove the conjecture presented in [10] which claims that the throughput rate of a transfer line decreases to a positive value when the number of machines increases.

APPENDIX A. Proof of Lemma 5

Lemma 5.

For any strongly connected event graph (N, M_0), the following two statements are equivalent.

(i) There exists a marking $M \in R(M_0)$ and a positive integer number K such that

$$L(\rho) \leq K + 1, \forall \rho \in \Psi(M)$$

(ii) There exists a positive integer number K such that

$$L(\gamma) / M_0(\gamma) \leq K, \forall \gamma \in \Gamma$$

where $\Psi(M)$ denotes the set of elementary paths connecting any two transitions $(t_0 p_1 t_1 p_2 \ldots p_u t_u)$ such that only the first place p_1 contains tokens and the other places are empty, i.e. $M(p_1) \geq 1$ and $M(p_i) = 0$ for $i = 2, \ldots, u$.

In the proof of this Lemma, we need the periodic operating mode of deterministic timed event graphs (see [13]). This operating mode is defined by the first firing initiation instant of each transition $S_t(1)$ and a cycle time α. The other firing initiation instants are determined as follows :

$$S_t(k) = S_t(1) + \alpha k, \forall t \in T \text{ and } \forall k \geq 0$$

Proof :

$i \rightarrow ii$.

Since there exists a marking $M \in R(M_0)$ and a positive integer number K such that

$$L(\rho) \leq K + 1, \forall \rho \in \Psi(M)$$

As a result, there are at least $\lceil L(\gamma)/K \rceil$ marked places in any elementary circuit γ in marking M which implies that

$$M_0(\gamma) = M(\gamma) \geq \lceil L(\gamma)/K \rceil \geq L(\gamma)/K$$

or :

$$L(\gamma) / M_0(\gamma) \leq K$$

$ii \rightarrow i$.

Let us consider a deterministic timed event graph $(N, M_0, X_t(k))$ with $X_t(k) = 1$. Since

$$K_2 \geq \operatorname*{Max}_{\gamma \in \Gamma} \left\{ \frac{L(\gamma)}{M_0(\gamma)} \right\} = \operatorname*{Max}_{\gamma \in \Gamma} \left\{ \frac{\sum_{t \in \gamma} 1}{M_0(\gamma)} \right\},$$

Lemma A presented hereafter implies that there exists a periodic operating mode $\{S_t(1)\}$ with cycle time K where $S_t(1)$ for all $t \in T$ are integers.

Since the transition firing times are equal to 1, this periodic operating mode can be described by a discrete time model. Each transition starts to fire at the beginning of some elementary time periods τ and ends the firing by the end of the same elementary periods τ.

The state of the system (or marking) becomes periodic after the last transition has started its firings, i.e. $\tau \geq \tau_0 = \text{Max}\{S_t(1), \forall t \in T\}$. Thus,

$$M_\tau = M_{\tau+K}, \forall \tau \geq \tau_0$$

where M_τ is the marking at the beginning of period τ. From this instant on, each transition fires exactly once in any time interval of K periods, i.e. $[\tau, \tau + K - 1]$.

In the following, we prove by contradiction that

$$L(\rho) \leq K + 1, \forall \rho \in \Psi(M_{\tau_0})$$

Assume that there exists a path $(t_1 p_1 t_2 p_2 \ldots t_n)$ with $n > K + 1$ such that only the first place p_1 contains tokens and the other places are empty, i.e. $M_{\tau_0}(p_1) \geq 1$ and $M_{\tau_0}(p_i) = 0$ for $i = 2, \ldots, n - 1$.

Since the places $p_2 \ldots p_{n-1}$ are empty at instant τ_0, the time needed to fire the transitions $t_2 \ldots t_n$ is equal to the sum of their transition firing times, i.e. $(n - 1) > K$. This implies that at least one of the transitions $t_2 \ldots t_n$ cannot fire in the time interval $[\tau_0, \tau_0 + K - 1]$ which is in contradiction with the fact that each transition fires exactly once in the time interval $[\tau_0, \tau_0 + K - 1]$.

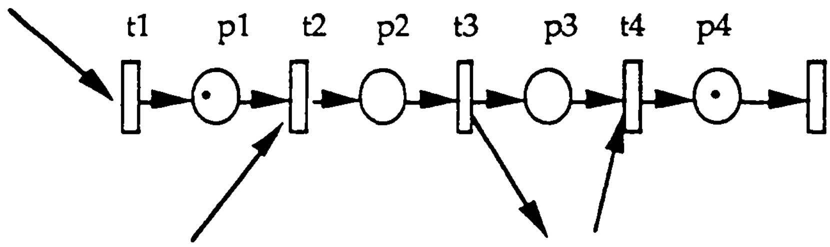

Figure 5: An elementary path

Q.E.D.

Lemma A.

Consider a deterministic timed event graph $(N, M_0, X_t(k))$ with $X_t(k) = m_t$. Assume that the transition firing times are integer number. For any integer number α such that

$$\alpha \geq \underset{\gamma \in \Gamma}{\text{Max}} \left\{ \frac{\sum_{t \in \gamma} m_t}{M_0(\gamma)} \right\} \qquad \text{(a1)}$$

there exists a feasible periodic operating mode $\{S_t(1)\}$ with cycle time α where $S_t(1)$ for all $t \in T$ are integers.

In the proof of this Lemma, we use a generalization of Farkas-Minkowski theorem on the compatibility of linear systems given in [6].

Theorem A1.

One and only one of the two following systems has a solution :

$$S_1 : Ax \geq b \qquad S_2 : \begin{cases} u \geq 0 \\ u^T A = 0 \\ u^T b > 0 \end{cases}$$

where the elements of A, u, b and x are integers.

Proof of Lemma A :

As shown in [17], $\{S_t(1)\}$ is a feasible periodic operating mode with cycle time α iff the following relations hold :

$$S_{\bullet p}(1) + m_{\bullet p} \leq S_{p\bullet}(1) + \alpha\, M_0(p), \forall p \in P$$

or :

$$S_{p\bullet}(1) - S_{\bullet p}(1) \geq m_{\bullet p} - \alpha\, M_0(p), \forall p \in P \qquad \text{(a2)}$$

This linear system can be rewritten in the following matrix form :

$$-C\, x \geq b \qquad \text{(a3)}$$

In relation (a3), x is a integer vector with $x_t = S_t(1)$ for $t \in T$, b is also an integer vector with $bp = m_{\bullet p} - \alpha\, M_0(p)$ for $p \in P$, C is the incidence matrix of the event graph defined as follows :

$$C_{p,t} = \begin{cases} -1, & \text{if } t \in p^\bullet \text{ and } t \notin {}^\bullet p \\ 1, & \text{if } t \in {}^\bullet p \text{ and } t \notin p^\bullet \\ 0, & \text{otherwise} \end{cases}$$

Consider also the following linear system :

$$\left.\begin{array}{l} u \geq 0 \\ -u^T C = 0 \\ u^T b > 0 \end{array}\right\} \qquad \text{(a4)}$$

As can be noticed, any solution u which satisfies the first two relations of system A4 is a p-invariant. In the case of event graphs, it is well known that the set of minimal p-invariants corresponds to the set of elementary circuits.

Consider the minimal p-invariant related to an elementary circuit $\gamma \in \Gamma$, i.e. $u_p = 1$ if $p \in \gamma$ and $u_p = 0$ otherwise. For this p-invariant,

$$u^T b = \sum_{p \in \gamma} \left(m_{\bullet p} - \alpha M_0(p) \right) = \sum_{t \in \gamma} m_t - \alpha M_0(\gamma) \leq 0$$

Since any p-invariant is a linear combination of the minimal p-invariants, the system (a4) does not have any solution.

Thus, theorem A1 implies that the linear system (a3) has at least one integer solution.

Q.E.D.

APPENDIX B. Proof of Property 19

Property 19.

The average cycle time remains minimal, i.e. $\pi(M_0) = \pi^D(M_0)$, iff there exists an elementary circuit $\gamma^* \in \Gamma$ for which the following two assumptions hold :

B1. $P\left\{\frac{\sum_{t\in\gamma^*} X_t}{M_0(\gamma^*)} \geq \frac{\sum_{t\in\gamma} X_t}{M_0(\gamma)}\right\} = 1, \quad \forall \gamma \in \Gamma$

B2. The firing times of the transitions belonging to γ^* are deterministic if γ^* contains more than one token, i.e.

$M_0(\gamma^*) \geq 2 \Rightarrow X_t =_{st} m_t, \forall t \in \gamma^*$

Proof :

In the following, we prove this property by showing the following four claims :

Claim 1. If there exists $\gamma^* \in \Gamma$ such that assumption B1 and the following assumption hold :

(i) $X_t =_{st} m_t, \forall t \in \gamma^*$

then $\pi(M_0) = \pi^D(M_0)$.

Claim 2. If there exists $\gamma^* \in \Gamma$ such that assumption B1 holds and $M(\gamma^*) = 1$, then $\pi(M_0) = \pi^D(M_0)$.

Claim 3. If there exists $\gamma^* \in \Gamma$ such that the following assumptions hold :

(i) $\pi^D(M_0) = \sum_{t\in\gamma^*} m_t / M(\gamma^*)$

(ii) $M(\gamma^*) \geq 2$

(iii) Assumption B2 does not hold

then $\pi(M_0) > \pi^D(M_0)$.

Claim 4. If there exists $\gamma^* \in \Gamma$ such that the following assumptions hold :

(i) $\pi^D(M_0) = \sum_{t\in\gamma^*} m_t / M(\gamma^*)$

(ii) $M(\gamma^*) = 1$

(iii) Assumption B1 does not hold

then $\pi(M_0) > \pi^D(M_0)$.

Q.E.D.

The following notations are needed in the proofs :

$\gamma^* = (p[1]t[1]p[2]t[2] \dots p[u]t[u])$

$\mu(\rho) = \Sigma_{t\in\rho} X_t$ where ρ is an elementary path

$\mu(\rho, k) = \Sigma_{t\in\rho} X_t(k)$

Proving the claims requires the following result.

Lemma B.

For two non-negative random variables X and Y (independent or not) such that $P\{X > Y\} > 0$, there exists $\Delta > 0$, $\alpha > 0$, $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that :

(i) $P\{X \geq Y + \Delta\} = \varepsilon_1$

(ii) $P\{Y < \alpha \text{ and } X \geq \alpha + \Delta\} = \varepsilon_2$

Proof of Lemma B :

Let us first prove (i). Let us observe that the event $\{X > Y\}$ can be expressed as follows :

$$\{X > Y\} = \bigcup_{n=1}^{\infty}\{X \geq Y + 1/n\}$$

Since the right-hand side term is the union of monotone events, we have

$$\lim_{n\to\infty} P\{X \geq Y + 1/n\} = P\{X > Y\} > 0$$

which implies that $\Delta > 0$, $\varepsilon_1 > 0$ such that :

$$P\{X \geq Y + \Delta\} = \varepsilon_1$$

We now prove (ii) by contradiction. For this purpose, assume that for any real value $\alpha > 0$, the following relation holds :

$$P\{Y < \alpha \text{ and } X \geq \alpha + \Delta/2\} = 0 \qquad \text{(b1)}$$

Notice that the probability $P\{X \geq Y + \Delta\}$ can be expressed as follows :

$$P\{X \geq Y + \Delta\} = \sum_{n=0}^{\infty} P\left\{X \geq Y + \Delta \quad / \, n\frac{\Delta}{2} \leq Y < (n+1)\frac{\Delta}{2}\right\}$$

$$\leq \sum_{n=0}^{\infty} P\left\{X \geq (n+1)\frac{\Delta}{2} + \frac{\Delta}{2} \quad / \, n\frac{\Delta}{2} \leq Y < (n+1)\frac{\Delta}{2}\right\}$$

From assumption (b1), we have:

$$P\left\{X \geq (n+1)\frac{\Delta}{2} + \frac{\Delta}{2} \quad / \, n\frac{\Delta}{2} \leq Y < (n+1)\frac{\Delta}{2}\right\} = 0, \quad \forall n$$

which implies that

$$P\{X \geq Y + \Delta\} = 0$$

This contradicts the fact that $P\{X \geq Y + \Delta\} = \varepsilon_1 > 0$. As a result,

$$P\{Y < \alpha \text{ and } X \geq \alpha + \Delta/2\} > 0$$

Q.E.D.

Proof of Claim 1 :

From assumption B1,

$$\pi^D(M_0) = \Sigma_{t\in\gamma^*} m_t / M(\gamma^*)$$

Combining with assumptions B1 and (i),

$$P\left\{\pi^D(M_0) \geq \frac{\sum_{t\in\gamma} X_t}{M_0(\gamma)}\right\} = 1, \ \forall \gamma \in \Gamma \tag{b2}$$

Since the random variables X_t for all $t \in T$ are mutually independent, relation (b2) implies that all the random variables X_t are upper bounded, i.e. $X_t \leq b_t$ with $b_t = m_t$, $\forall t \in \gamma^*$. Furthermore,

$$\pi^D(M_0) \geq \frac{\sum_{t\in\gamma} b_t}{M_0(\gamma)}, \ \forall \gamma \in \Gamma$$

As a result, $\pi^D(M_0)$ is equal to the average cycle time of the deterministic timed event graph $(N, M_0, \{Y_t(k)\})$ with $Y_t(k) = b_t$. From Property 4, we have:

$$\pi(M_0) \leq \pi^D(M_0)$$

which together with Corollary 1 implies that

$$\pi(M_0) = \pi^D(M_0)$$

Q.E.D.

Proof of Claim 2 :

From assumption B1, we have:

$$\pi^D(M_0) = \sum_{t\in\gamma^*} m_t$$

Since the random variables X_t for all $t \in T$ are mutually independent, assumption B1 implies that the random variables X_t for all $t \notin \gamma^*$ are upper bounded, i.e. $X_t \leq b_t$ and the random variables X_t for all $t \in \gamma^*$ are lower bounded X_t, i.e. $X_t \geq a_t$. Furthermore,

$$\pi^* = \sum_{t\in\gamma^*} a_t \geq \frac{\sum_{t\in\gamma\setminus\gamma^*} b_t + \sum_{t\in\gamma\cap\gamma^*} a_t}{M_0(\gamma)}, \ \forall \gamma \in \Gamma \text{ and } \gamma \neq \gamma^* \tag{b3}$$

Consider a second stochastic timed event graph STEG' = $(N, M_0, \{Y_t(k)\})$ with $Y_t(k) = b_t$, $\forall t \notin \gamma^*$ and $Y_t(k) = X_t(k)$, $\forall t \in \gamma^*$. Let $\pi'(M_0)$ be its average cycle time. From Property 4, we have:

$$\pi(M_0) \leq \pi'(M_0) \tag{b4}$$

In the following, we prove that $\pi'(M_0) = \pi^D(M_0)$ which together with Corollary 1 and relation (b4) implies that $\pi(M_0) = \pi^D(M_0)$.

Consider the deterministic timed event graph DTEG = $(N, M_0, \{Z_t(k)\})$ with $Z_t(k) = b_t$, $\forall t \notin \gamma^*$ and $Z_t(k) = a_t$, $\forall t \in \gamma^*$. Its average cycle time is equal to π^*.

As in the proof of Lemma 5 in appendix A, we make use of a periodic operating mode of DTEG $\{S_t(1)\}$ with cycle time π^*, i.e.

$$S_t(k) = S_t(1) + (k-1)\,\pi^*, \ \forall t \in T$$

Consider the first transition of γ^*, i.e. t[1] and consider a firing initiation instant in steady state, i.e. instant $s = S_{t[1]}(k)$ such that $s \geq \mathrm{Max}_{t\in T} S_t(1)$. Let M" be the marking at this instant. Clearly,

$$M''(p[1]) = 1.$$

Let T" be the set of transitions (not including t[1]) that are enabled by marking M" and r(t) the remaining firing time of transition t for all $t \in T''$.

Let us consider the set Ψ of elementary paths $(t_1 p_1 t_2 p_2 \ldots t_{\upsilon-1} p_\upsilon t_\upsilon)$ with $M(p_i) = 0$ for $1 \leq i \leq \upsilon-1$. For any path $\rho \in \Psi$, let $\rho^- = t_1$, $\rho^+ = t_\upsilon$, $\rho^* = \{t_2, \ldots, t_{\upsilon-1}\}$, $n(\rho) = M''(p_\upsilon)$ and :

$$\lambda(\rho) = \begin{cases} \sum_{s\in\rho^*} Z_s, & \text{if } \rho^- \notin T'' \\ r(\rho^-) + \sum_{s\in\rho^*} Z_s, & \text{if } \rho^- \in T'' \end{cases}$$

Let us consider the following two sets of elementary paths :

$$\Psi'(t) = \{\rho\in\Psi \;/\; \rho^- \in T'', \rho^+ = t \text{ and } \rho^*\cap\gamma^* = \varnothing\}$$

$$\Psi''(t) = \{\rho\in\Psi \;/\; \rho^- \in \gamma^*, \rho^+ = t \text{ and } \rho^*\cap\gamma^* = \varnothing\}$$

From the definition of the periodic operating mode, in the time interval $[s, s + \pi^*]$, any transition $t \notin T''$ should fire once, and any transition $t \in T''$ should starts a new firing and its remaining firing time reaches r(t) at instant $s + \pi^*$. As a result, for any $t \in T$ and for any path $\rho \in \Psi'(t)$,

$$\lambda(\rho) + Z_{\rho^+} - r(\rho^+) \leq \pi^*, \quad \text{if } \rho^+ \in T'' \text{ and } n(\rho) = 1 \tag{b5}$$

$$\lambda(\rho) + Z_{\rho^+} \leq \pi^*, \quad \text{if } \rho^+ \notin T'' \tag{b6}$$

Furthermore, let us observe that transition t[i] starts to fire at instant $\sum_{j=1}^{i-1} a_{t[j]}$. Thus, for any $t \in T$ and for any path $\rho \in \Psi''(t)$,

$$\sum_{j=1}^{i} a_{t[j]} + \lambda(\rho) + Z_{\rho^+} - r(\rho^+) \leq \pi^*, \quad \text{if } \rho^+ \in T'' \text{ and } n(\rho) = 1 \tag{b7}$$

$$\sum_{j=1}^{i} a_{t[j]} + \lambda(\rho) + Z_{\rho^+} \leq \pi^*, \quad \text{if } \rho^+ \notin T'' \tag{b8}$$

For any $t[i] \in \gamma^*$,

$$\lambda(\rho) \leq \sum_{j=k+1}^{i-1} a_{t[j]}, \quad \forall \rho \in \Psi''(t[i]) \text{ with } \rho^- = t[k] \tag{b9}$$

$$\lambda(\rho) \leq \sum_{j=1}^{i-1} a_{t[j]}, \quad \forall \rho \in \Psi'(t[i]) \tag{b10}$$

Notice that for any $\rho \in \Psi''(t[i])$ with $\rho^- = t[k]$, it holds that $k < i$. Otherwise, $t[k]p[k+1]t[k+1]p[k+2] \ldots p[i]t[i]\rho$ forms an elementary circuit without token.

Let us consider now a third stochastic timed event graph STEG" = (N, M", $\{Y_t(k)\}$). Initially, the transitions of T" are under firing with the remaining firing time equal to r(t). Let $\pi''(M'')$ be the average cycle time of STEG". From Property 2,

$\pi'(M_0) = \pi''(M'')$ (b11)

In the following, we prove that $\pi''(M'') = \pi^D(M_0)$ which concludes that $\pi'(M_0)=\pi^D(M_0)$. To this end, we consider the following constrained cyclic operating mode. First, the transitions of T'' continue their firing initiated previously and the other transitions start to fire as soon as they are enabled. When a transition t of T'' with $t \neq t[1]$ is enabled again, it starts to fire until the remaining firing time is equal to r(t). At this moment, this transition is frozen until the instant

$$V(1) = \sum_{t\in\gamma^*} X_t(1)$$

In the following, we prove that transition t[i] of γ^* starts to fire at instant

$$H_{t[i]}(1) = \sum_{j=1}^{i-1} X_{t[j]}(1) \tag{b12}$$

and that the system returns to the initial state. As a result, the second cycle can begins at instant V(1). This constrained operating mode is a renewal process and its average cycle time $\pi^c(M'')$ satisfies the following two relations :

$$\pi^c(M'') \geq \pi''(M'') \tag{b13}$$

and

$$\pi^c(M'') = E[V(1)] = \pi^D(M_0) \tag{b14}$$

Relations (b13) and (b14) together with Corollary 1 imply that $\pi''(M'') = \pi^D(M_0)$.

We now need to prove that transition t[i] of γ^* starts to fire at instant $H_{t[i]}(1)$ and

$$H_{t[i]}(1) = \sum_{j=1}^{i-1} X_{t[j]}(1) \tag{b15}$$

and that the system returns to the initial state at instant V(1).

We now prove relation (b15) by induction. Since $H_{t[1]}(1) = 0$ from the definition of the constrained operating mode, the relation (b15) is true for i = 1.

Assume that relation (b15) is true up to i-1. Clearly,

$$H_{t[i]}(1) = \text{Max}\left\{ \underset{\rho\in\Psi'(t[i])}{\text{Max}} \mu(\rho), \underset{\rho\in\Psi''(t[i])}{\text{Max}} \left\{ H_{\rho^-}(1) + X_{\rho^-}(1) + \mu(\rho) \right\} \right\}$$

From relation (b10),

$$\mu(\rho) \leq \sum_{j=1}^{i-1} a_{t[j]} \leq \sum_{j=1}^{i-1} X_{tj}(1), \qquad \forall \rho \in \Psi'(t[i])$$

From relation (b9), for all $\rho \in \Psi''(t[i])$ such that $\rho^- = t[k]$, since $k < i$,

$$H_{\rho^-}(1) + X_{\rho^-}(1) + \mu(\rho) \leq \sum_{j=1}^{k-1} X_{tj}(1) + X_k(1) + \sum_{j=k+1}^{i-1} a_{tj} \leq \sum_{j=1}^{i-1} X_{tj}(1)$$

Combining the above three relations,

$$H_{t[i]}(1) \le \sum_{j=1}^{i-1} X_{t[j]}(1)$$

Since there is only one token in γ^*,

$$H_{t[i]}(1) \ge \sum_{j=1}^{i-1} X_{t[j]}(1)$$

which implies that relation (b15) is true for i. Hence it is for all $1 \le i \le u$ by induction.

Let $H_t(1) > 0$ be the instant a firing of transition t can be initiated. To prove that the system returns to the initial state at instant V(1), we need to show that : (i) any transition $t \notin T''\backslash\rho^*$ has completed one firing by V(1), i.e. $H_t(1) \le V(1) - b_t$; (ii) any transition $t \in T''$ can start a new firing by instant V(1) - r(t), i.e. $H_t(1) \le V(1) - r(t)$. These claims are proven in the following.

For transition $t \notin T''$, it holds that

$$H_t(1) = \mathrm{Max}\left\{ \mathop{\mathrm{Max}}_{\rho \in \Psi'(t)} \mu(\rho), \mathop{\mathrm{Max}}_{\rho \in \Psi''(t)} \left\{ H_{\rho^-}(1) + X_{\rho^-}(1) + \mu(\rho) \right\} \right\}$$

From relation (b6),

$$\mu(\rho) \le \pi^* - b_t \le V(1) - b_t, \quad \forall \rho \in \Psi'(t)$$

and from relation (b8), for all $\rho \in \Psi''(t)$ such that $\rho^- = t[k]$,

$$H_{\rho^-}(1) + X_{\rho^-}(1) + \mu(\rho) \le \sum_{j=1}^{k-1} X_{t[j]}(1) + X_{t[k]}(1) + \sum_{j=k+1}^{u} a_{t[j]} - b_t \le V(1) - b_t$$

These three relations imply that :

$$H_t(1) \le V(1) - b_t, \qquad \forall t \notin T'' \cup \gamma^*$$

For transitions $t \in T''$,

$$H_t(1) = \mathrm{Max}\left\{ r(t), \mathop{\mathrm{Max}}_{\rho \in \Psi'(t)/n(\rho) \le 1} \mu(\rho), \mathop{\mathrm{Max}}_{\rho \in \Psi''(t)/n(\rho) \le 1} \left\{ H_{\rho^-}(1) + X_{\rho^-}(1) + \mu(\rho) \right\} \right\}$$

From relation (b5), for all $\rho \in \Psi'(t)$ such that $n(\rho) \le 1$,

$$\mu(\rho) \le \pi^* - (b_t - r(t)) \le V(1) - (b_t - r(t))$$

From relation (b7), for all $\rho \in \Psi''(t)$ such that $n(\rho) \le 1$ and $\rho^- = t[k]$,

$$H_{\rho^-}(1) + X_{\rho^-}(1) + \mu(\rho) \le \sum_{j=1}^{k-1} X_{t[j]}(1) + X_{t[k]}(1) + \sum_{j=k+1}^{u} a_{t[j]} - (b_t - r(t)) \le V(1) - (b_t - r(t))$$

These three relations imply that :

$$H_t(1) \le V(1) - (b_t - r(t))$$

Q.E.D.

Proof of Claim 3 :

In this proof, we first consider the case where the event graph is an elementary circuit, i.e. $N = \gamma^*$. The general case is then considered.

Case 1 : $N = \gamma^*$.

First, we remind that any marking M such that $M(\gamma^*) = n$ is reachable from the following initial marking :

$M_0(p[1]) = n$, $M_0(p[i]) = 0$ for $2 \le i \le u$

Thanks to Property 2, we only need to consider this initial marking. From the evolution equation (1), we have :

$$S_{t[1]}(k) = \mathrm{Max}\{S_{t[1]}(k-1) + X_{t[1]}(k-1), S_{t[u]}(k-n) + X_{t[u]}(k-n)\} \qquad \text{(b16)}$$

and

$$S_{t[i]}(k) = \mathrm{Max}\{S_{t[i]}(k-1) + X_{t[i]}(k-1), S_{t[i-1]}(k) + X_{t[i-1]}(k)\}, \quad \forall i \ge 2 \qquad \text{(b17)}$$

Combining these two relations, it can be shown that :

$$S_{t[1]}(k) \ge S_{t[1]}(k-i), \qquad \forall i \ge 0 \qquad \text{(b18)}$$

$$S_{t[1]}(k+i\,n) \ge S_{t[1]}(k) + \sum_{j=0}^{i-1} \mu(\gamma^*, k+j\,n), \quad \forall i \ge 0 \qquad \text{(b19)}$$

Combining the above two relations,

$$S_{t[1]}(k+i\,n) \ge S_{t[1]}(k+(i-1)\,n+1) \ge S_{t[1]}(k) + \sum_{j=0}^{i-2} \mu(\gamma^*, k+j\,n+1) \qquad \text{(b20)}$$

From relations (b20) and (b19),

$$S_{t[1]}(k+i\,n) - S_{t[1]}(k) \ge \sum_{j=0}^{i-1} \mu(\gamma^*, k+j\,n) + \mathrm{Max}\left\{0, \sum_{j=0}^{i-2} \mu(\gamma^*, k+j\,n+1) - \sum_{j=0}^{i-1} \mu(\gamma^*, k+j\,n)\right\}$$

By taking expectation,

$$E[S_{t[1]}(k+i\,n) - S_{t[1]}(k)] \ge in\pi^D(M_0) + W(i)$$

with

$$W(i) = E\left[\mathrm{Max}\left\{0, \sum_{j=0}^{i-2} \mu(\gamma^*, k+j\,n+1) - \sum_{j=0}^{i-1} \mu(\gamma^*, k+j\,n)\right\}\right]$$

From this relation, we have

$$E[S_{t[1]}(k\,i\,n+1)] \ge E[S_{t[1]}((k-1)i\,n+1)] + i\,n\pi^D(M_0) + W(i)$$
$$\ge \ldots \ge k\left(i\,n\pi^D(M_0) + W(i)\right)$$

which implies that

$$\pi(M_0) = \lim_{k\to\infty} \frac{E[S_{t[1]}(k\,i\,n)]}{k\,i\,n} \ge \pi^D(M_0) + W/(i\,n)$$

In the following, we prove that there exists i such that W(i) is strictly positive which concludes that $\pi(M_0) > \pi^D(M_0)$.

Since assumption B2 does not hold, $\mu(\gamma^*, k) \ne_{st} E[\mu(\gamma^*, k)] = n\pi^D(M_0)$ which implies that there exists $\Delta > 0$, $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that

$P\{\mu(\gamma^*, j\,n + k + 1) \ge n\pi^D(M_0) + \Delta\} = \varepsilon_1$

and

$P\{\mu(\gamma^*, j\,n + k) \le n\pi^D(M_0)\} = \varepsilon_2$

Let V be the following event :

$\mu(\gamma^*, jn+k+1) \ge n\pi^D(M_0) + \Delta, \forall 0\le j\le i\text{-}2$ and $\mu(\gamma^*, jn + k) \le n\pi^D(M_0), \forall 0 \le j \le i\text{-}1$

Since the random variables $\mu(\gamma^*, jn+k+1)$ $\forall 0\le j\le i\text{-}2$ and $\mu(\gamma^*, jn + k)$ $\forall 0 \le j \le i\text{-}1$ are mutually independent,

$P\{V\} = \varepsilon_1^{i-1}\varepsilon_2^{i}$

and

$$W \ge P\{V\}E\left[\mathrm{Max}\left\{0, \sum_{j=0}^{i-2}\mu(\gamma^*, k+j\,n+1) - \sum_{j=0}^{i-1}\mu(\gamma^*, k+j\,n)\right\} \;/\, V\right]$$

$$\ge P\{V\}\mathrm{Max}\left\{0, (i-1)\left(n\pi^D(M_0)+\Delta\right) - i\,n\pi^D(M_0)\right\}$$

$$\ge \left(i\,\Delta - \left(\Delta + n\pi^D(M_0)\right)\right)(\varepsilon_1)^{i-1}(\varepsilon_2)^{i}$$

which yields that $W > 0, \forall i > 1 + n\pi^D(M_0)/\Delta$.

Case 2 : General case .

Consider a new stochastic timed event graph $(N^*, M_0, \{X_t(k)\})$ uniquely composed of places and transitions of γ^* and let $\pi'(M_0)$ be its average cycle time. From the monotonicity property with respect to the net structure in [3],

$\pi(M_0) \ge \pi'(M_0)$

From Case 1,

$\pi(M_0) \ge \pi'(M_0) > \pi^D(M_0)$

Q.E.D.

Proof of Claim 4 :

In the following, we first consider two special cases in which assumption B1 does not hold. Then, we conclude the proof by showing that the general case can be reduced to one of these special cases.

Case 1. The event graph is composed of the elementary circuit γ^* and an additional elementary path (t[k]p[u+1]t[u+1]...p[u+v]t[u+v]p[u+v+1]t[1]). There is a second elementary circuit γ = (t[1]p[2] ... t[k]p[u+1]t[u+1] ... p[u+v]t[u+v]p[u+v+1]t[1]). We assume that $M_0(\gamma) = \upsilon$ and

$$P\{\mu(\gamma^*) < \mu(\gamma)/\upsilon\} > 0 \tag{b21}$$

Without loss of generality, let us assume that $M_0(p[1]) = 1$ and $M_0(p[u+v+1]) = \upsilon$. By similar arguments as in the proof of Claim 3,

$$S_{t[1]}(k+\upsilon) - S_{t[1]}(k) \ge \mathrm{Max}\left\{\sum_{i=0}^{\upsilon-1}\mu(\gamma^*, k+i), \mu(\gamma, k)\right\}$$

By taking expectation, we have :

$$E\left[S_{t[1]}(k+\upsilon) - S_{t[1]}(k)\right] \ge \upsilon\pi^D(M_0) + W$$

where

$$W = E\left[\operatorname{Max}\left\{0, \mu(\gamma,k) - \sum_{i=0}^{\upsilon-1}\mu(\gamma^*,k+i)\right\}\right]$$

Using similar arguments as in Claim 3, we only need to show that W > 0. From Lemma B and assumption (b21), there exists $\alpha > 0$, $\varepsilon > 0$ such that

$$P\{\mu(\gamma,k)/\upsilon \geq \alpha + \Delta \text{ and } \mu(\gamma^*,k) < \alpha\} = \varepsilon$$

Consider the event V defined as follows :

$$\frac{\mu(\gamma,k)}{\upsilon} \geq \alpha + \Delta, \mu(\gamma^*,k) < \alpha \text{ and } \mu(\gamma^*,k+i) < \alpha \ \forall 0 < i < \upsilon$$

The probability of V can be determined as follows :

$$P[V] = P\left\{\frac{\mu(\gamma,k)}{\upsilon} \geq \alpha + \Delta \text{ and } \mu(\gamma^*,k) < \alpha\right\}\prod_{i=1}^{\upsilon-1}P\{\mu(\gamma^*,k+i) < \alpha\}$$

$$= \varepsilon\prod_{i=1}^{\upsilon-1}P\{\mu(\gamma^*,k+i) < \alpha\}$$

Since $\mu(\gamma^*, k+i)$ for all $i \geq 0$ are i.i.d. random variables, we have:

$$P\{\mu(\gamma^*,k+i) < \alpha\} = P\{\mu(\gamma^*,k) < \alpha\} \geq P\left\{\frac{\mu(\gamma,k)}{\upsilon} \geq \alpha + \Delta \text{ and } \mu(\gamma^*,k) < \alpha\right\} = \varepsilon$$

which yields that

$$P\{V\} \geq \varepsilon^{\upsilon}$$

Finally,

$$W \geq E\left[\operatorname{Max}\left\{0, \mu(\gamma,k) - \sum_{i=0}^{\upsilon-1}\mu(\gamma^*,k+i)\right\}/V\right]P\{V\} \geq \varepsilon^{\upsilon}\upsilon\Delta > 0$$

Case 2. The event graph is composed of two separate elementary circuits γ^* and γ = (p[u+1]t[u+1] ... p[u+u']t[u+u']p[u+1]) connected together by means of two additional places p' and p" where p' (resp. p") connects transition t[u+1] to t[1] (resp. t[1] to t[u+1]). We assume that :

$$P\{\mu(\gamma^*) < \mu(\gamma)/\upsilon\} > 0 \tag{b22}$$

with $\upsilon = M_0(\gamma)$.

Without loss of generality, let us assume that $M_0(p[1]) = 1$, $M_0(p[u+2]) = \upsilon$, $M_0(p") = \upsilon"$ and $M_0(p) = 0$ for all other places.

By similar arguments as in the proof of Claim 3, for $n \geq n_0$ and $n_0\upsilon \geq \upsilon'$,

$$S_{t[1]}(k+n\upsilon) \geq S_{t[1]}(k) + \sum_{i=0}^{n\upsilon-1}\mu(\gamma^*,k+i) \tag{b23}$$

$$S_{t[1]}(k+n\upsilon) \geq S_{t[u+1]}(k+n\upsilon) \geq S_{t[u+1]}(k+n_0\upsilon) + \sum_{i=n_0}^{n-1}\mu(\gamma,k+i\upsilon) \tag{b24}$$

$$S_{t[u+1]}(k+n_0\upsilon) \geq S_{t[u+1]}(k+\upsilon') \geq S_{t[1]}(k) \tag{b25}$$

Combining the above relations,

$$S_{t[1]}(k+n\upsilon)-S_{t[1]}(k)\geq \mathrm{Max}\left\{\sum_{i=0}^{n\upsilon-1}\mu(\gamma^*,k+i),\ \sum_{i=n_0}^{n-1}\mu(\gamma,k+i\upsilon)\right\}$$

By taking expectation, we have :

$$E\left[S_{t[1]}(k+n\upsilon)-S_{t[1]}(k)\right]\geq n\upsilon\pi^D(M_0)+W(n)$$

where

$$W(n)=E\left[\mathrm{Max}\left\{0,\ \sum_{i=n_0}^{n-1}\mu(\gamma,k+i\upsilon)-\sum_{i=0}^{n\upsilon-1}\mu(\gamma^*,k+i)\right\}\right]$$

By similar arguments as in Claim 2, we only need to show that W(n) > 0 for some n. From Lemma B and assumption (b22), there exists $\alpha > 0$, $\varepsilon > 0$ such that

$$P\{\mu(\gamma,k)\geq\alpha+\Delta\}=\varepsilon_1,\qquad \forall k$$
$$P\{\mu(\gamma^*,k)\leq\alpha\}=\varepsilon_2,\qquad \forall k$$

Consider the event V defined as follows :

$$\frac{\mu(\gamma,k+i\upsilon)}{\upsilon}\geq\alpha+\Delta\ \ \forall n_0\leq i<n \text{ and } \mu(\gamma^*,k+i)\leq\alpha\ \forall 0\leq i<n\upsilon$$

The probability of V can be determined as follows :

$$P[V]=(\varepsilon_1)^{n-n_0}(\varepsilon_n)^{n\upsilon}$$

Finally,

$$W(n)\geq P[V]E\left[\mathrm{Max}\left\{0,\ \sum_{i=n_0}^{n-1}\mu(\gamma,k+i\upsilon)-\sum_{i=0}^{n\upsilon-1}\mu\left(\gamma^*,k+i\right)\right\}\ /V\right]$$
$$\geq(\varepsilon_1)^{n-n_0}(\varepsilon_n)^{n\upsilon}(n\upsilon\Delta-n_0\upsilon(\alpha+\Delta))$$

which implies that:

$$W(n)>0,\qquad \forall n>\frac{n\upsilon(\alpha+\Delta)}{\upsilon\Delta}$$

Case 3. General case.

Since assumption B1 does not hold, there exists $\gamma\in\Gamma$ such that

$$P(\mu(\gamma^*)<\mu(\gamma)/M_0(\gamma))>0 \qquad \text{(b26)}$$

In the following, we distinguish two cases according to whether γ^* and γ are connected.

Case 3.1. γ and γ^* are not connected.

In this case, let us express γ as (p[u+1]t[u+1] ... p[u+v]t[u+v]p[u+1]). Let us introduce two implicit places p' and p" connecting t[u+1] to t[1] and t[1] to t[u+1] respectively. Since N is an event graph, we can take M_0(p') (resp. M_0(p")) as being equal to the token distance from t[u+1] to t[1] (resp. t[1] to t[u+1]). Let (N', M_0, $\{X_t(k)\}$) be this new timed even graph and let $\pi'(M_0)$ be its average cycle time. Clearly,

$$\pi'(M_0)=\pi(M_0) \qquad \text{(b27)}$$

Let us consider a third timed event graph (N", M_0, $\{X_t(k)\}$) obtained from (N', M_0, $\{X_t(k)\}$) by removing all places and transitions except γ, γ^* and {p', p"}. Let $\pi''(M_0)$ be its average cycle time. From the monotonicity properties in [3], we have:

$\pi''(M_0) \le \pi'(M_0)$ (b28)

According to Case 2,

$\pi''(M_0) > \pi^D(M_0)$

Case 3.2. γ and γ^* are connected.

Without loss of generality, let us express γ as $\sigma_1\rho_1\sigma_2\rho_2\ldots\sigma_n\rho_n$ where $\sigma_1, \sigma_2, \ldots, \sigma_n$ are subpaths of γ^* whose extremities are transitions, and $\rho_1, \rho_2, \ldots, \rho_n$ are paths whose extremities are places, and

$\rho_i \cap \gamma^* = \varnothing$

Let t[b(i)] and t[e(i)] be the first and last transitions of σ_i respectively. Without loss of generality, assume that

t[b(1)] = t[1] and $M_0(p[1]) = 1$

We can observe that each path ρ_i corresponds to an elementary circuit $\gamma_i = \overline{\sigma}_i\rho_i$ where $\overline{\sigma}_i$ is the subpath of γ^* connecting transition t[b(i+1)] to t[e(i)]. In the following, we prove that at least one of these circuits does not satisfy assumption B1. To this end, we only need to prove that whenever

$\mu(\gamma_i)/M_0(\gamma_i) \le \mu(\gamma^*), \forall 1 \le i \le n$ (b29)

it holds that

$\mu(\gamma)/M_0(\gamma) \le \mu(\gamma^*)$

First, for all γ_i such that b(i+1) > e(i),

$\mu(\gamma_i) = \mu(\rho_i) + \mu(\overline{\sigma}_i)$ and $M_0(\gamma_i) = M_0(\rho_i) + 1$

Meanwhile, for all γ_i such that b(i+1) < b(i) ≤ e(i),

$\mu(\gamma_i) = \mu(\rho_i) + \mu(\overline{\sigma}_i)$ and $M_0(\gamma_i) = M_0(\rho_i)$

Relation (b29) thus becomes :

$\mu(\rho_i) + \mu(\overline{\sigma}_i) - \mu(\gamma^*) \le M_0(\rho_i)\,\mu(\gamma^*)$, if b(i+1) > e(i) (b30)

$\mu(\rho_i) + \mu(\overline{\sigma}_i) \le M_0(\rho_i)\,\mu(\gamma^*)$, if b(i+1) < e(i) (b31)

By summing all relations of (b30) and (b31), we obtain

$$\sum_{i=1}^{n}\mu(\rho_i) + \sum_{i=1}^{n}\mu(\sigma_i) \le \mu(\gamma^*)\sum_{i=1}^{n}M_0(\rho_i)$$

which implies that

$\mu(\gamma)/M_0(\gamma) \le \mu(\gamma^*)$

We still need to prove that the sum of the second terms on the left-hand side of relations (b30) and (b31) is equal to $\sum\mu(\sigma_i)$. For this purpose, let us consider the counting process defined by the following algorithm.

Algorithm.

Step1. Set N0(t) = N1(t) = N2(t) = 0 for all $t \in T$

Step 2. For i = 1 to n

2.1. For t^* = t[b(i)] to t[e(i)], set N0(t^*) := N0(t^*) +1;

2.2. if b(i+1) > b(i),

For $t^* = t[e(i)+1]$ to $t[b(i+1)-1]$, set $N1(t^*) := N1(t^*) + 1$;

else

For $t^* = t[e(i)]$ to $t[b(i+1)]$, set $N2(t^*) := N2(t^*) + 1$;

Clearly, the sum of the left-hand side terms of relations (b30) and (b31) is equal to :

$$\sum_{i=1}^{n} \mu(\rho_i) + \sum_{t \in T} (N2(t) - N1(t)) X_t$$

We notice that in this counting process, the pointer t^* draws a circuit along γ^* which implies that

$$N2(t) = N1(t) + N0(t), \forall t \in \gamma^*$$

Obviously, $N0(t) = 1$ for all t belonging to some σ_i; otherwise $N0(t) = 0$. That is :

$$N2(t) - N1(t) = \begin{cases} 1, & \forall t \in \bigcup_{i=1}^{n} \sigma_i \\ 0, & \forall t \notin \bigcup_{i=1}^{n} \sigma_i \end{cases}$$

and the proof is thus completed.

Q.E.D.

APPENDIX C. Proof of Property 24

Consider the sequence of stochastic timed event graphs STEG(n) = $(N, M_n, \{X_t(k)\})$ where the initial marking is defined as follows :

$$M_n(p^*) = M_0(p^*) + n \text{ and } M_n(p) = M_0(p), \forall p \neq p^*$$

Let $\pi(M_n)$ be the average cycle time of STEG(n). From Property 2,

$$\pi(M_0) \geq \pi(M_1) \geq \pi(M_2) \geq \ldots$$

Property 24.

$$\lim_{n \to \infty} \pi(M_n) = \max_{1 \leq i \leq \upsilon} \lambda_i(M_0)$$

where $\lambda_i(M_0)$ is the isolated average cycle time of the i-th strongly connected component obtained by removing p^*.

Proof :

Consider the strongly connected stochastic timed event graph related to the i-th strongly connected component obtained by removing p^* and let us denote it as SCSTEG(i) = $(N_i, M_0, \{X_t(k)\})$.

Let

$$\pi^* = \max_{1\le i\le \upsilon} \lambda_i(M_0) \tag{c1}$$

From the monotonicity property of the average cycle time with respect to the net structure in [3],

$$\pi(M_n) \ge \lambda_i(M_0), \qquad \forall i, \forall n$$

which implies that

$$\lim_{n\to\infty} \pi(M_n) \ge \pi^* \tag{c2}$$

In the following, we intend to show that :

$$\lim_{n\to\infty} \pi(M_n) \le \pi^* + \varepsilon \tag{c3}$$

for all $\varepsilon > 0$. Thus, relations (c2) and (c3) imply that :

$$\lim_{n\to\infty} \pi(M_n) = \pi^*$$

To prove relation (c2), let us notice that the strongly connected components can be denoted in such a way that any component N_i has not input places from component N_j with $j > i$, i.e. there do not exist any place belonging to $P - \{p^*\}$ which connects N_j to N_i.

It can be proven that there exists a marking $M'_{n\upsilon}$ reachable from the initial marking $M_{n\upsilon}$ such that :

$$M'_{n\upsilon}(p) = M_{n\upsilon}(p), \ \forall p \in \bigcup_{i=1}^{\upsilon} N_i$$

$$M'_{n\upsilon}(p) \ge n, \ \forall p \notin \bigcup_{i=1}^{\upsilon} N_i$$

Let us consider a new stochastic timed event graph $(N, M'_{n\upsilon}, \{X_t(k)\})$ and let its average cycle time be $\pi(M'_{n\upsilon})$. From Property 2,

$$\pi(M'_{n\upsilon}) = \pi(M_{n\upsilon}) \tag{c4}$$

Consider also the following constrained operating mode of $(N, M'_{n\upsilon}, \{X_t(k)\})$. From the definition of $M'_{n\upsilon}$, each transition of any strongly connected component SCSTEG(i) can fire n times, whatever the firing of transitions belonging to other components. Let $H_i(n)$ be the instant at which the last transition of component N_i finishes its n-th firing. Thus, at instant

$$H(n) = \max_{1\le i\le \upsilon} H_i(n),$$

each transition of the net has completed its n-th firing. We assume that the transitions completes its n-th firing before instant H(n) are frozen until H(n). As a result, the initial marking $M'_{n\upsilon}$ is again reached at instant H(n) and we restart the system in exactly the same way.

Obviously, the constrained operating mode is a renewal process and its average transition firing cycle time $\pi^c(M'_{n\upsilon})$ can be determined as follows :

$$\pi^c(M'_{n\upsilon}) = \frac{E[H(n)]}{n} \tag{c5}$$

Furthermore, we have :

$$\pi(M'_{n\upsilon}) \le \pi^c(M'_{n\upsilon}) \tag{c6}$$

From the definition of $\lambda_i(M_0)$ and relation (2), it can easily be shown that :

$$\lim_{n\to\infty} \frac{H_i(n)}{n} = \lim_{n\to\infty} \frac{E[H_i(n)]}{n} = \lambda_i(M_0), \quad \text{a.s.} \tag{c7}$$

As a result, for all $\varepsilon_1 > 0$,

$$\lim_{n\to\infty} E\left[\text{Max}\left\{\frac{H_i(n)}{n}, \lambda_i(M_0) + \varepsilon_1\right\}\right] = \lambda_i(M_0)$$

Since

$$\frac{H_i(n)}{n} = \text{Max}\left\{\frac{H_i(n)}{n}, \lambda_i(M_0) + \varepsilon_1\right\} + \left(\frac{H_i(n)}{n} - \lambda_i(M_0) - \varepsilon_1\right)^+,$$

then

$$\lim_{n\to\infty} E\left[\left(\frac{H_i(n)}{n} - \lambda_i(M_0) - \varepsilon_1\right)^+\right] = 0$$

which implies that for all $\varepsilon_2 > 0$, there exists $n_i > 0$ such that for all $n \ge n_i$,

$$E\left[\left(\frac{H_i(n)}{n} - \lambda_i(M_0) - \varepsilon_1\right)^+\right] \le \varepsilon_2 \tag{c8}$$

From the definition of $H_i(n)$,

$$\begin{aligned}
\pi^c(M'_{n\upsilon}) &= E\left[\underset{1\le i\le \upsilon}{\text{Max}}\left\{\frac{H_i(n)}{n}\right\}\right] \\
&\le \pi^* + \varepsilon_1 + E\left[\underset{1\le i\le \upsilon}{\text{Max}}\left\{\left(\frac{H_i(n)}{n} - \pi^* - \varepsilon_1\right)^+\right\}\right] \\
&\le \pi^* + \varepsilon_1 + E\left[\underset{1\le i\le \upsilon}{\text{Max}}\left\{\left(\frac{H_i(n)}{n} - \lambda_i(M_0) - \varepsilon_1\right)^+\right\}\right] \\
&\le \pi^* + \varepsilon_1 + \sum_{i=1}^{\upsilon}\left[\left(\frac{H_i(n)}{n} - \lambda_i(M_0) - \varepsilon_1\right)^+\right]
\end{aligned}$$

From relation (c8), for all $n \ge \text{Max}\{n_1, n_2, \ldots, n_\upsilon\}$,

$$\pi^c(M'_{n\upsilon}) \le \pi^* + \varepsilon_1 + \upsilon\varepsilon_2$$

which implies that

$$\lim_{n\to\infty} \pi^c(M'_{n\upsilon}) \le \pi^* + \varepsilon_1 + \upsilon\varepsilon_2, \qquad \forall\varepsilon_1 > 0, \forall\varepsilon_2 > 0$$

By choosing $\varepsilon_1 = \varepsilon_2 = \varepsilon/(1+n)$,

$$\lim_{n\to\infty} \pi^c(M'_{n\upsilon}) \le \pi^* + \varepsilon, \qquad \forall\varepsilon > 0$$

Q.E.

D.

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