C1-curved finite elements with numerical integration for thin plate and thin shell problems : part 2 : approximation of thin plate and thin shell problems

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Part 2 : Approximation of thin plate and thin shell problems

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C^1-CURVED FINITE ELEMENTS WITH NUMERICAL INTEGRATION FOR THIN PLATE AND THIN SHELL PROBLEMS (*)(**)

Part 2 : Approximation of thin plate and thin shell problems

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Abstract

In the first part of this work we have studied a family of curved finite elements of class C^1 which are compatible with ARGYRIS or BELL triangles.

In this second part, we use these results to approximate the solutions of linear thin shell problems formulated on plane reference domains with curved boundary, according to the model of W.T. KOITER. Sufficient conditions are given which preserve the order of convergence of the method. These conditions involve both the approximation of the components of the displacement with straight and curved finite element families of class C^1, and the degree of accuracy of the numerical quadrature schemes. We conclude by examining various examples. Of course, as a particular case, we can apply the results to plate bending problems.

In a subsequent work (in collaboration with J.M. BOISSERIE) we will illustrate the effectiveness of these methods by some numerical experiments.

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ELEMENTS FINIS COURBES DE CLASSE $\mathcal{C}^1$ AVEC INTEGRATION NUMERIQUE POUR DES PROBLEMES DE PLAQUES ET DE COQUES MINCES

Partie 2 : Approximation de problèmes de plaques et de coques minces

Résumé

Dans la première de ce travail nous avions étudié une famille d'éléments finis courbes de classe $\mathcal{C}^1$ compatibles avec les triangles d'ARGYRIS et de BELL.

Dans cette seconde partie, nous utilisons ces résultats pour approcher les solutions de problèmes linéaires de coques minces formulés sur des domaines de références plans à frontière curviligne, en utilisant la modélisation de W.T. KOITER. Des conditions suffisantes sont données qui préservent l'ordre de convergence de la méthode. Ces conditions prennent en compte l'approximation des composantes du déplacement par des familles d'éléments finis de classe $\mathcal{C}^1$, droits ou courbes, et le degré de précision des schémas d'intégration numérique. On termine par l'examen de quelques exemples. Naturellement, comme cas particulier, on peut appliquer ces résultats aux problèmes de flexion de plaques.

Dans un travail ultérieur, en collaboration avec J.M. BOISSERIE, nous illustrerons l'efficacité de ces méthodes par quelques exemples numériques.

1 INTRODUCTION

In [1], we have studied the approximation of the linear thin shell equations of [2] by using conforming finite element methods and numerical integration techniques, we have proved the convergence and we have obtained an explicit estimate of the asymptotic error. This estimate takes into account the approximation of the displacement components and the degree of accuracy of the numerical integration scheme. In this way, we have assumed that the middle surface of the shell is the image of a polygonal domain through a regular mapping $\vec{\varphi}$.

The aim of this work is to extend the previous results to the case of shells whose middle surface is defined as the image of a plane reference domain with curved boundary through a regular mapping $\vec{\varphi}$. In this way, we will use finite element spaces constructed from curved finite elements of class $\mathcal{C}^1$ as those developed in Part 1 (see [3]) which are $\mathcal{C}^1$-compatible with classical Argyris (see [4]) or Bell [5] triangles.

For clarity we have subdivided this paper in nine paragraphs. Firstly, paragraph 2 briefly records the statement of the continuous problem. Two discrete problems are formulated in paragraph 3, without or with use of numerical integration techniques. An "abstract" error estimate is obtained in paragraph 4 while local error estimates are derived in paragraph 5. Next, in order to have the uniform $\vec{V}_h$-ellipticity condition satisfied, sufficient conditions are formulated in paragraph 6.

From the results of paragraphs 4 to 6, we obtain the asymptotic error estimate theorem. The criteria to be observed by the numerical integration schemes are the same than those previously obtained in [1] when the finite elements into consideration have straight sides. For curved finite elements, we prove that the degree of accuracy of the numerical integration schemes is obtained by adding $2n-2$ to the required degree of accuracy of the numerical integration schemes used for corresponding straight finite elements. Here, n means the degree of the polynomials used in the definition of the application F_K which maps the reference triangle $\hat{K}$ onto the curved triangle K.

We conclude this study (paragraph 8) by some examples of convenient finite element spaces which can be used to approximate plate bending problems (paragraph 8) and general thin shell problems (paragraph 9). In each case, we recall the sufficient conditions which preserve the order of convergence of the method and we give some references of appropriate numerical integration schemes.

In conclusion, as in the case of polygonal domains, we emphasize that this study
(i) is valid for general thin shell equations, and,
(ii) provides precise criteria for choosing optimal numerical integration schemes.

A description of the implementation and some numerical experiments are given in [6].

Contents

Notations

This paper is a continuation of [3] and it uses the same notations. References to this paper will be made by Part I.

2 THE CONTINUOUS PROBLEM

For a detailed formulation of the continuous thin shell problem we refer to [2,7]. In [8], we have briefly recorded this formulation and we have given expressions of the bilinear and linear forms which are well adapted to the approximation by finite element methods. We recall these expressions hereunder :

The bilinear form of the thin shell problem, associated to the strain energy of the shell can be written (see [9, prop. 2.1], [8, theorem 1.5.1])

$$a(\vec{u},\vec{v}) = \int_\Omega {}^tU[A_{IJ}]V d\xi^1 d\xi^2, \tag{2.1}$$

where the column matrix V (and similarly for the matrix U) is given by

$${}^tV = [v_1 \;\; v_{1,1} \;\; v_{1,2} \;\; v_2 \;\; v_{2,1} \;\; v_{2,2} \;\; v_3 \;\; v_{3,1} \;\; v_{3,2} \;\; v_{3,11} \;\; v_{3,12} \;\; v_{3,22}] \tag{2.2}$$

and where the 12-square matrix $[A_{IJ}]$, is only dependent on the first, second and third partial derivatives of the application $\vec{\varphi} : \bar{\Omega} \longrightarrow \bar{S}$ which maps the plane reference domain Ω onto the middle surface S of the thin shell. Subsequently, it is convenient to assume that this mapping $\vec{\varphi}$ verifies the following hypothesis :

Hypothesis 2.1 : the mapping $\vec{\varphi}$ is defined on a domain $\tilde{\Omega} \supset \bar{\Omega}, \vec{\varphi} \in (C^3(\bar{\tilde{\Omega}}))^3$ and all the points of the surface $\bar{\tilde{S}} = \vec{\varphi}(\bar{\tilde{\Omega}})$ are regular in the sense that $\vec{\varphi}_{,1}(\xi^1,\xi^2) \times \vec{\varphi}_{,2}(\xi^1,\xi^2) \neq \vec{O}$ for any $(\xi^1,\xi^2) \in \bar{\tilde{\Omega}}$. ■

The linear form of the thin shell problem : For simplicity, we assume that the shell is clamped along its boundary and submitted to a distribution of external loads whose resultant is noted $\vec{p}$ and whose moment is $\vec{0}$. Then the work of the external loads associated with a displacement $\vec{v}$ of the middle surface is given by (see [8, theorem 1.5.2]) :

$$(2.3) \qquad f(\vec{v}) = \int_{\Omega} {}^tFV d\xi^1 d\xi^2$$

where the column matrix V is given by (2.2) and where the column matrix F is only dependent on the first partial derivatives of the mapping $\vec{\varphi} : \bar{\Omega} \longrightarrow \bar{S}$ and on the components of $\vec{p}$.

Then, we can formulate the thin shell problem as follows :

Problem 2.1 : For $\vec{\varphi} \in (\mathcal{C}^3(\bar{\bar{\Omega}}))^3$, *for* $\vec{p} \in (\mathcal{C}^3(\bar{\Omega}))^3$, *find* $\vec{u} \in \vec{V}$ such that

$$(2.4) \qquad a(\vec{u}, \vec{v}) = f(\vec{v}), \ \ \forall \vec{v} \in \vec{V}$$

where the space $\vec{V}$ *of admissible displacements is given by*

$$(2.5) \qquad \vec{V} = H_0^1(\Omega) \ \times \ H_0^1(\Omega) \ \times \ H_0^2(\Omega).$$

■

Theorem 2.1 ([7, Theorem 6.4.1]) : Problem 2.1 has one and only one solution.

Remark 2.1 : The hypotheses $\vec{\varphi} \in (\mathcal{C}^3(\bar{\bar{\Omega}}))^3$ and $\vec{p} \in (\mathcal{C}^0(\bar{\Omega}))^3$ are in particular linked to the use of numerical integration schemes. Indeed, they involve $[A_{IJ}] \in (\mathcal{C}^0(\bar{\bar{\Omega}}))^{144}$ and ${}^tF \in (\mathcal{C}^0(\bar{\Omega}))^{12}$. ■

Remark 2.2 : The definition (2.5) of space $\vec{V}$ corresponds to the study of deformations of *clamped* shells along their boundaries. For more general boundary conditions, see [7]. ■

3 THE DISCRETE PROBLEMS

According to the method developed in Part 1, paragraph 2, we associate to the domain Ω, an approximate domain Ω_h. We have

$$(3.1) \qquad \bar{\Omega}_h = \bigcup_{K \in \mathcal{T}_h} K \quad \text{where } \mathcal{T}_h = \mathcal{T}_h^1 \ \cup \ \mathcal{T}_h^2.$$

The triangulation $\mathcal{T}_h^1$ is composed of straight triangles which are in affine correspondence with a reference triangle $\hat{K}$. The triangulation $\mathcal{T}_h^2$ is composed of triangles which have two straight sides while the third one approximates an arc of the boundary (Γ). These curved triangles are the images of a reference triangle $\hat{K}$ by a non affine mapping F_K.

With each triangulation $\mathcal{T}_h$ of the domain Ω_h, we associate a product of finite element spaces $\vec{X}_h = X_{h1} \times X_{h1} \times X_{h2}$ and then we define a suitable subspace $\vec{V}_h = V_{h1} \times V_{h1} \times V_{h2}$ of $\vec{X}_h$ which takes into account the boundary conditions. More precisely these spaces are defined as follows $(\alpha = 1, 2)$:

$$
(3.2)\quad \begin{cases} \textit{Spaces } X_{h\alpha} : \text{the functions of spaces } X_{h\alpha}, \alpha = 1,2, \text{ are such that} \\ \quad (i) \text{ upon each } K \in \mathcal{T}_h, \text{ they belong to a finite dimensional} \\ \quad\quad \text{space which is contained in } H^\alpha(K); \\ \quad (ii) \text{ upon each } K \in \mathcal{T}_h, \text{ they are determined by their values} \\ \quad\quad \text{and by the values of their derivatives upon the set of the} \\ \quad\quad \text{degrees of freedom of the element } K; \\ \quad (iii)\ X_{h\alpha} \subset \mathcal{C}^{\alpha-1}(\bar{\Omega}_h). \end{cases}
$$

$$
(3.3)\quad \begin{cases} \textit{Spaces } V_{h\alpha}, \alpha = 1,2 : \\ \quad V_{h1} = \{v_h \in X_{h1}\ ;\ v_h = 0 \text{ over } \Gamma_h = \partial\Omega_h\}, \\ \quad V_{h2} = \{v_h \in X_{h2}\ ;\ v_h = 0,\ \partial_\nu v_h = 0 \text{ over } \Gamma_h\}, \text{ where } \partial_\nu \text{ means} \\ \quad\quad \text{the unit external normal derivative to } \Gamma_h. \end{cases}
$$

Thus, the space $\vec{V}_h = V_{h1} \times V_{h1} \times V_{h2}$ satisfies the inclusion

$$
(3.4)\quad \vec{V}_h \subset \vec{V}(\Omega_h) = H_0^1(\Omega_h) \times H_0^1(\Omega_h) \times H_0^2(\Omega_h).
$$

Since the parameter h is supposed to decrease to zero, we can assume the existence of a parameter $h_0 > 0$ such that

$$
(3.5)\quad \Omega_h \subset \tilde{\Omega},\ \ \forall h \in]0, h_0[\,,
$$

where $\tilde{\Omega}$ is the domain introduced in the Hypothesis 2.1.

Construction of spaces X_{h1} and X_{h2} in practice

Concerning the approximation of the displacement field, two solutions are possible :

(i) the tangential and normal components of the displacement are approximated with the help of the same finite element family which is then of class $\mathcal{C}^1$: to the straight triangles $K \in \mathcal{T}_h^1$, we associate a finite element family of class $\mathcal{C}^1$ while to the curved triangles $K \in \mathcal{T}_h^2$, we associate a family of curved finite elements which are $\mathcal{C}^1$-compatible with the straight finite elements into consideration ;

(ii)the tangential and normal components of the displacement are approximated with the help of distinct finite element families. For the normal component we proceed like in (i). For the tangential components, we associate to the straight triangles $K \in \mathcal{T}_h^1$, a family of straight finite elements of class $\mathcal{C}^0$, while to the curved triangles $K \in \mathcal{T}_h^2$ (constructed with the help of the application F_K used to define the curved finite element of class $\mathcal{C}^1$) we associate a family of curved finite elements, $\mathcal{C}^0$-compatible with the considered straight finite elements.

Subsequently, we construct spaces X_{h2} which satisfy conditions (3.2), with the help of curved finite elements which are C^1-compatible with the ARGYRIS or BELL triangles (see Part 1, section 3.2 when $F_K \in (P_5)^2$, and section 3.3 when $F_K \in (P_3)^2$). With the help of the same finite elements or of curved finite elements which are C_0-compatible with the Hermite finite element of type (3) described in Part 1, section 3.4, we construct spaces X_{h1} that satisfy conditions (3.2).

From spaces $X_{h\alpha}$ to spaces $V_{h\alpha}, \alpha = 1, 2$:

We define the subspaces $V_{h\alpha}$ from the spaces $X_{h\alpha}$ by taking into account the boundary conditions. When the triangles are straight, the treatment of homogeneous boundary conditions is detailed in [8] for several kinds of conforming finite elements. Subsequently, we will concentrate on the construction of the space V_{h2} when using curved finite elements of class C^1 which are extensively described in Part 1, sections 3.2 and 3.3.

Case $F_K \in (P_3)^2$: Let us thoroughly examine the case of a curved triangle $K = F_K(\hat{K})$ constructed from the application $F_K \in (P_3)^2$ introduced in Part 1, Example 2.2. Let $v \in C^2(\tilde{\Omega})$ be a function such that

$$v|_\Gamma = 0, \quad \frac{\partial v}{\partial n}|_\Gamma = 0 \tag{3.6}$$

and let us show that the corresponding interpolate function $\pi_K v$ of function v, associated to the curved finite element C^1-compatible with the ARGYRIS triangle, verifies the conditions

$$(\pi_K v)|_{\gamma_h} = 0, \quad \left.\frac{\partial \pi_K v}{\partial n}\right|_{\gamma_h} = 0, \tag{3.7}$$

where γ_h denotes the approximate arc $a_1 a_2$. Then, such elements permit to construct space V_{h2}.

Let a be any point of γ_h (Fig. 3.1) and let $\hat{a} = F_K^{-1}(a) \in \hat{a}_1\hat{a}_2$. We set

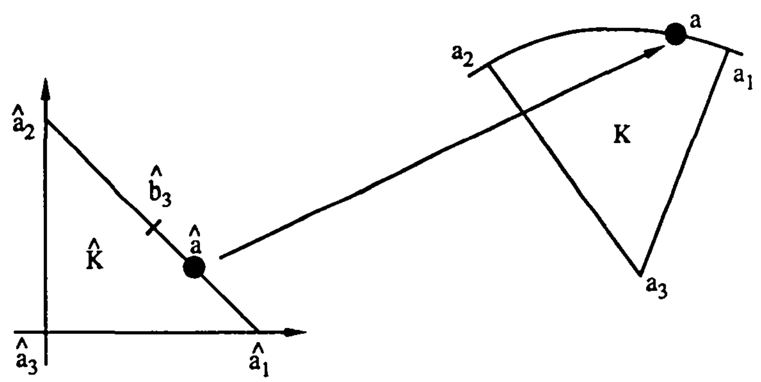

Figure 3.1 : Taking into account boundary conditions

$$\widehat{\pi_K v} = \pi_K v \circ F_K$$

so that conditions (3.7) are equivalent to

$$\widehat{\pi_K v}(\hat{a}) = 0, \quad D\widehat{\pi_K v}(\hat{a})(\hat{a}_3 - \hat{b}_3) = 0. \tag{3.8}$$

But, by construction, $\widehat{\pi_K v}|_{[\hat{a}_1,\hat{a}_2]}$ coincides with the polynomial $\hat{f}_3$, of degree five, which is entirely determined by the data of the degrees of freedom

$$\begin{cases} \{\hat{v}(\hat{a}_1), \hat{v}(\hat{a}_2), D\hat{v}(\hat{a}_1)(\hat{a}_2 - \hat{a}_1),\ D\hat{v}(\hat{a}_2)(\hat{a}_1 - \hat{a}_2), \\ \qquad D^2\hat{v}(\hat{a}_1)(\hat{a}_2 - \hat{a}_1)^2,\ D^2\hat{v}(\hat{a}_2)(\hat{a}_1 - \hat{a}_2)^2\} \end{cases} \tag{3.9}$$

while $D\widehat{\pi_K v}(.)(\hat{a}_3 - \hat{b}_3)|_{[\hat{a}_1,\hat{a}_2]}$ agrees with the fourth degree polynomial $\hat{g}_3$, which is entirely determined by the data of the degrees of freedom

$$\begin{cases} \{D\hat{v}(\hat{a}_\alpha)(\hat{a}_3 - \hat{b}_3),\ \alpha = 1, 2\ ;\ D^2\hat{v}(\hat{a}_1)(\hat{a}_3 - \hat{b}_3, \hat{a}_2 - \hat{a}_1); \\ \qquad D^2\hat{v}(\hat{a}_2)(\hat{a}_3 - \hat{b}_3, \hat{a}_1 - \hat{a}_2)\ ;\ D\hat{v}(\hat{b}_3)(\hat{a}_3 - \hat{b}_3)\}. \end{cases} \tag{3.10}$$

The boundary conditions (3.6) immediately involve

$$v|_\Gamma = 0, \ \ Dv|_\Gamma = 0, \ \ \text{and} \ \ D^2v(\chi_1(s), \chi_2(s))(\vec{\chi}'(s), \vec{\tau}) = 0,$$

where $\vec{\tau}$ is any vector of $\mathbb{R}^2$ and $\vec{\chi}(s) = (\chi_1(s), \chi_2(s))$ is any parameterization of the arc Γ (see Part 1, (2.1)). With the relation $\hat{v} = v \circ F_K$ and the expression [Part 1, (2.33)] of F_K, we obtain the following relations, satisfied at points $\hat{a}_1$ and $\hat{a}_2$:

$$\begin{cases} \hat{v}(\hat{a}_1) = \hat{v}(\hat{a}_2) = 0, \\ D\hat{v}(\hat{a}_1) = D\hat{v}(\hat{a}_2) = 0, \\ D^2\hat{v}(\hat{a}_1)(\hat{a}_2 - \hat{a}_1, \vec{\hat{\tau}}) = D^2\hat{v}(\hat{a}_2)(\hat{a}_1 - \hat{a}_2, \vec{\hat{\tau}}) = 0, \end{cases} \tag{3.11}$$

where $\vec{\hat{\tau}}$ is any vector of plane $\mathbb{R}^2$. Then, relations (3.11) involve that the degrees of freedom included in the sets (3.9) and (3.10) are zero, so that we get (3.8).

Case $F_K \in (P_5)^2$: the above proof can be immediately adapted to this case.

Case of curved triangles which are C^1-compatible with BELL triangle ($F_K \in (P_3)^2$ *or* $F_K \in (P_5)^2$) : a same kind of proof can be applied to this case.

Thus, for all these cases, conditions (3.8) involve that conditions (3.7) are satisfied and then we obtain space V_{h2}.

Analogously, one can show that the space V_{h1} can be constructed from curved finite elements which are $\mathcal{C}_0$-compatible with the Hermite finite element of type 3. Of course, as an alternative, we can take V_{h2} for V_{h1}.

Remark 3.1 : Thus, the use of an interpolation of the boundary Γ with the help of polynomials of degree 3 for the construction of curved finite elements which are C^1-compatible with ARGYRIS or BELL triangles, is possible when the boundary conditions are homogeneous Dirichlet type.

The case of general boundary conditions is more complicated and beyond the scope of this paper. In this direction, let us mention [10,11] which contains in particular i) interesting results for the approximation of elliptic equations of order $2m+2, m \in \mathbb{N}$, for Dirichlet nonhomogeneous boundary conditions by means of straight and curved compatible Bell triangles, and ii) an interesting discussion on the degree of polynomials ($n = 3$ or $n = 5$) that we need to use to interpolate the boundary, depending on the kind of nonhomogeneous boundary conditions we have to approximate. ■

Now we are in position to define the discrete problems associated to the continuous problem 2.1 (see (2.4)) :

First discrete problem : For $h < h_0$ (see (3.5)), *find* $\vec{\tilde{u}}_h \in \vec{V}_h$ *such that*

$$\int_{\Omega_h} {}^t\tilde{U}_h[A_{IJ}]V_h dx = \int_{\Omega_h} {}^t\tilde{F}V_h dx, \ \ \forall \vec{v}_h \in \vec{V}_h, \tag{3.12}$$

where the elements of the matrix ${}^t\tilde{F}$ are some extensions of the elements of the matrix ${}^tF$ to the domain $\tilde{\Omega}$, i.e.,

$$ {}^t\tilde{F}|_{\bar{\Omega}} = {}^tF. \tag{3.13}$$

The elements of ${}^tF$ are assumed to be continuous (cf. statement of problem 2.1) so that the existence of continuous extensions is ensured by Tietze-Urysohn theorem (see [12]). Let us note that the coefficients A_{IJ} are defined and continuous upon $\bar{\Omega}_h$ thanks to Hypothesis 2.1 and to the relation (3.5). ■

Set $\vec{X}(\Omega_h) = (H^1(\Omega_h))^2 \times H^2(\Omega_h)$ and consider the following bilinear and linear forms :

$$\vec{u}, \vec{v} \in \vec{X}(\Omega_h) \longrightarrow \tilde{a}_h(\vec{u}, \vec{v}) = \int_{\Omega_h} {}^tU[A_{IJ}]V dx, \tag{3.14}$$

$$\vec{v} \in \vec{X}(\Omega_h) \longrightarrow \tilde{f}_h(\vec{v}) = \int_{\Omega_h} {}^t\tilde{F}V dx. \tag{3.15}$$

The inclusions (3.5) involve the existence of a constant $\tilde{M} > 0$, independent of h, such that for any $\vec{u}, \vec{v} \in \vec{X}(\Omega_h)$ we obtain

$$\left|\int_{\Omega_h} {}^tU[A_{IJ}]V dx\right| \leq \tilde{M} \|\vec{u}\|_{\vec{X}(\Omega_h)} \|\vec{v}\|_{\vec{X}(\Omega_h)}, \tag{3.16}$$

$$\left| \int_{\Omega_h} {}^t\tilde{F} V dx \right| \leq \tilde{M} \|\vec{v}\|_{\vec{X}(\Omega_h)}. \tag{3.17}$$

In paragraph 6, we will check that the bilinear form $\tilde{a}_h(.,.)$ is uniformly $\vec{V}_h$-elliptic. Here and subsequently, we note

$$\begin{cases} \|\vec{v}\|_{\vec{X}(\Omega_h)} = \left[\|v_1\|^2_{1,\Omega_h} + \|v_2\|^2_{1,\Omega_h} + \|v_3\|^2_{2,\Omega_h}\right]^{1/2} \\ |\vec{v}|_{\vec{X}(\Omega_h)} = \left[|v_1|^2_{1,\Omega_h} + |v_2|^2_{1,\Omega_h} + |v_3|^2_{2,\Omega_h}\right]^{1/2}. \end{cases} \tag{3.18}$$

Since it is expensive, or even impossible, to exactly compute integrals like those appearing in (3.12), we are led to use numerical integration techniques. Thus consider the following numerical integration scheme over the reference triangle $\hat{K}$:

$$\int_{\hat{K}} \hat{\phi}(\hat{x}) d\hat{x} \sim \sum_{\ell=1}^{L} \hat{\omega}_\ell \hat{\phi}(\hat{b}_\ell). \tag{3.19}$$

Given two functions $\hat{\phi} : \hat{K} \longrightarrow \mathbb{R}$ and $\phi : K = F_K(\hat{K}) \longrightarrow \mathbb{R}$ in the usual correspondences $\phi = \hat{\phi} \circ F_K^{-1}$, $\hat{\phi} = \phi \circ F_K$ we have

$$\int_K \phi(x) dx = \int_{\hat{K}} \hat{\phi}(\hat{x}) J_{F_K}(\hat{x}) d\hat{x}, \tag{3.20}$$

where the Jacobian J_{F_K} of the application F_K can be assumed to be strictly positive without lost of generality. Thus the numerical integration scheme (3.19) upon the reference triangle $\hat{K}$ induces the following numerical integration scheme upon the triangle K :

$$\int_K \phi(x) dx \sim \sum_{\ell=1}^{L} \omega_{\ell,K} \phi(b_{\ell,K}), \tag{3.21}$$

with

$$\omega_{\ell,K} = \hat{\omega}_\ell J_{F_K}(\hat{b}_\ell) \text{ and } b_{\ell,K} = F_K(\hat{b}_\ell), 1 \leq \ell \leq L. \tag{3.22}$$

Parallely, we define the error functionals

$$E_K(\phi) = \int_K \phi(x) dx - \sum_{\ell=1}^{L} \omega_{\ell,K} \phi(b_{\ell,K}) \tag{3.23}$$

$$\hat{E}(\hat{\phi}) = \int_{\hat{K}} \hat{\phi}(\hat{x}) d\hat{x} - \sum_{\ell=1}^{L} \hat{\omega}_\ell \hat{\phi}(\hat{b}_\ell), \tag{3.24}$$

so that

$$E_K(\phi) = \hat{E}(\hat{\phi} J_{F_K}). \tag{3.25}$$

To the bilinear and linear forms (3.14) and (3.15), the relations (3.1) (3.21) associate the following forms

$$\vec{u}_h, \vec{v}_h \in \vec{V}_h \longrightarrow a_h(\vec{u}_h, \vec{v}_h) = \sum_{K \in \mathcal{T}_h} \sum_{\ell=1}^{L} \omega_{\ell,K} \, {}^tU_h(b_{\ell,K})[A_{IJ}(b_{\ell,K})]V_h(b_{\ell,K}) \tag{3.26}$$

$$\vec{v}_h \in \vec{V}_h \longrightarrow f_h(\vec{v}_h) = \sum_{K \in \mathcal{T}_h} \sum_{\ell=1}^{L} \omega_{\ell,K} \, {}^tF(b_{\ell,K})V_h(b_{\ell,K}). \tag{3.27}$$

In the last relation we have written ${}^tF(b_{\ell,K})$ which requires that all the integration nodes are inside of $\bar{\Omega}$. This is a consequence of the following hypothesis that we subsequently assume to be verified.

Hypothesis 3.1 : Upon the reference triangle $\hat{K}$, all the numerical integration nodes are internal or they coincide with vertices $\hat{a}_1, \hat{a}_2, \hat{a}_3$. ■

Upon relations (3.26) (3.27) it only appears a single numerical integration scheme. In practice, we will use *different schemes* depending on the kind of triangle into consideration : we will use very accurate schemes upon curved triangles and most simple schemes upon straight triangles.

Thus we obtain :

Second discrete problem : Find $\vec{u}_h \in \vec{V}_h$ *such that*

$$a_h(\vec{u}_h, \vec{v}_h) = f_h(\vec{v}_h), \quad \forall \vec{v}_h \in \vec{V}_h. \tag{3.28}$$

■

4 "ABSTRACT" ERROR ESTIMATE

To prepare the obtention of explicit error estimates, we start by giving an "abstract" error estimate. In this way, we have to assume the uniform $\vec{V}_h$-ellipticity property of the bilinear form $a_h(.,.)$. We will come back in paragraph 6 on this property.

Theorem 4.1 : Let us consider a family of discrete problems (3.28) *satisfying assumptions* (3.4) (3.5) *and such that there exists a constant* $\beta > 0$, *independent of* h, *for which*

$$\beta \|\vec{v}_h\|^2_{\vec{X}(\Omega_h)} \leq a_h(\vec{v}_h, \vec{v}_h), \quad \forall \vec{v}_h \in \vec{V}_h, \quad \forall h \text{ sufficiently small.} \tag{4.1}$$

For any $\vec{v}, \vec{w} \in \vec{X}(\Omega_h)$, *the bilinear form* $\tilde{a}_h(\vec{v}, \vec{w})$ *is defined by relation* (3.14), *i.e.,*

$$\tilde{a}_h(\vec{v}, \vec{w}) = \int_{\Omega_h} {}^tV[A_{IJ}]W dx. \tag{4.2}$$

Then, there exists a constant c, *independent of* h, *such that*

$$
(4.3) \quad \left\{ \begin{aligned} & \|\vec{\tilde{u}} - \vec{u}_h\|_{\vec{X}(\Omega_h)} \\ & \quad \leq c \left[\inf_{\vec{v}_h \in \vec{V}_h} \left\{ \|\vec{\tilde{u}} - \vec{v}_h\|_{\vec{X}(\Omega_h)} + \sup_{\vec{w}_h \in \vec{V}_h} \frac{|\tilde{a}_h(\vec{v}_h, \vec{w}_h) - a_h(\vec{v}_h, \vec{w}_h)|}{\|\vec{w}_h\|_{\vec{X}(\Omega_h)}} \right\} \right. \\ & \quad \left. + \sup_{\vec{w}_h \in \vec{V}_h} \frac{|\tilde{a}_h(\vec{\tilde{u}}, \vec{w}_h) - f_h(\vec{w}_h)|}{\|\vec{w}_h\|_{\vec{X}(\Omega_h)}} \right], \end{aligned} \right.
$$

where $\vec{\tilde{u}}$ *is any function of* $\vec{V}(\tilde{\Omega})$.

Proof : From (4.1) we obtain the existence of a unique solution $\vec{u}_h$ for every discrete problem (3.28). Then, for any $\vec{v}_h \in \vec{V}_h$, the inequality (4.1) implies

$$
\begin{aligned} \beta\|\vec{u}_h - \vec{v}_h\|^2_{\vec{X}(\Omega_h)} & \leq a_h(\vec{u}_h - \vec{v}_h, \vec{u}_h - \vec{v}_h) = \tilde{a}_h(\vec{\tilde{u}} - \vec{v}_h, \vec{u}_h - \vec{v}_h) \\ & + \{\tilde{a}_h(\vec{v}_h, \vec{u}_h - \vec{v}_h) - a_h(\vec{v}_h, \vec{u}_h - \vec{v}_h)\} + \{f_h(\vec{u}_h - \vec{v}_h) - \tilde{a}_h(\vec{\tilde{u}}, \vec{u}_h - \vec{v}_h)\}, \end{aligned}
$$

and hence, with inequality (3.16),

$$
\left\{ \begin{aligned} & \beta\|\vec{u}_h - \vec{v}_h\|_{\vec{X}(\Omega_h)} \leq \tilde{M}\|\vec{\tilde{u}} - \vec{v}_h\|_{\vec{X}(\Omega_h)} \\ & \quad + \sup_{\vec{w}_h \in \vec{V}_h} \frac{|\tilde{a}_h(\vec{v}_h, \vec{w}_h) - a_h(\vec{v}_h, \vec{w}_h)|}{\|\vec{w}_h\|_{\vec{X}(\Omega_h)}} + \sup_{\vec{w}_h \in \vec{V}_h} \frac{|\tilde{a}_h(\vec{\tilde{u}}, \vec{w}_h) - f_h(\vec{w}_h)|}{\|\vec{w}_h\|_{\vec{X}(\Omega_h)}}. \end{aligned} \right.
$$

Then, it remains to combine this inequality with

$$
\|\vec{\tilde{u}} - \vec{u}_h\|_{\vec{X}(\Omega_h)} \leq \|\vec{\tilde{u}} - \vec{v}_h\|_{\vec{X}(\Omega_h)} + \|\vec{u}_h - \vec{v}_h\|_{\vec{X}(\Omega_h)},
$$

and next to take the infinimum with respect to $\vec{v}_h \in \vec{V}_h$. ∎

Remark 4.1 : In the above theorem, $\vec{\tilde{u}}$ is any function of space $\vec{V}(\tilde{\Omega})$. Subsequently, as function $\vec{\tilde{u}}$, we will use any extension of the solution $\vec{u}$ to the domain $\tilde{\Omega}$. ∎

Remark 4.2 : In the second member of inequality (4.3), it appears the generalization of the usual term of the interpolation theory, i.e., $\inf_{\vec{v}_h \in \vec{V}_h} \|\vec{\tilde{u}} - \vec{v}_h\|_{\vec{X}(\Omega_h)}$, and next two additional terms which evaluate the consistency of the integration scheme. ∎

In order to find an explicit error estimate, we have to :

(i) check the property (4.1) of uniform $\vec{V}_h$-ellipticity ;
(ii) evaluate the term of interpolation error ;

(iii) evaluate the consistency terms of the inequality (4.3).

In case of a polygonal boundary, we have proved in [1] that these properties (i) (ii) (iii) are obtained through a suitable choice of numerical integration schemes formulated upon the reference triangle $\hat{K}$. When the boundary is curved we will check subsequently that we get similar results.

5 LOCAL ERROR ESTIMATES

In this paragraph we give two theorems which allow us to prove the property (4.1) of uniform $\vec{V}_h$-ellipticity and to evaluate the consistency terms of the inequality (4.3). Subsequently, we assume that :

(i) the current triangle K is the image of the reference triangle $\hat{K}$ by an application F_K which satisfies the properties stated in Part 1, theorem 2.1, in particular

$$F_K \in [P_n(\hat{K})]^2, \;\; n = 1 \text{ if } K \in \mathcal{T}_h^1, \;\; n \geq 1 \text{ if } K \in \mathcal{T}_h^2 \;; \tag{5.1}$$

$$\begin{cases} |F_K|_{\ell,\infty,\hat{K}} \leq c h_K^\ell, & \ell = 0, 1, \ldots \\ |F_K^{-1}|_{\ell,\infty,K} \leq c h_K^{-1}, & \ell = 1, 2, \ldots \\ |J_{F_K}|_{\ell,\infty,\hat{K}} \leq c h_K^{2+\ell}, & \ell = 0, \ldots, n \\ \dfrac{c_1}{h_K^2} \leq |J_{F_K^{-1}}|_{0,\infty,K} \leq \dfrac{c_2}{h_K^2} \;; & \end{cases} \tag{5.2}$$

(ii) the components u_1, u_2 of the displacement are approximated in a discrete space $V_{h1} \subset H_0^1(\Omega_h)$ associated to a straight or curved finite element $(K, P_{K1}, \Sigma_{K1}), \forall K \in \mathcal{T}_h$. Let

$$\hat{P}_{K1} = \{\hat{p}_K : \hat{K} \longrightarrow \mathbb{R} \;;\; \hat{p}_K = p \circ F_K, \;\; p \in P_{K1}\}. \tag{5.3}$$

Then we assume that there exists integers m_1, n_1 such that

$$\begin{cases} 1 \leq m_1 \leq n_1 \\ P_{M_1}(\hat{K}) \subset \hat{P}_{K1} \subset P_{N_1}(\hat{K}), \\ M_1 = m_1, \;\; N_1 = n_1 \text{ for a straight or a curved } \mathcal{C}^0\text{-compatible element,} \\ M_1 = m_1 + n - 1, \;\; N_1 = n_1 + n - 1 \text{ for a curved } \mathcal{C}^1\text{-compatible element.} \end{cases} \tag{5.4}$$

This definition of M_1, N_1 takes into account the increase of the degree of the polynomials used upon the reference triangle when we construct curved finite elements of class $\mathcal{C}^1$ (cf. Part 1, (3.9), for the curved finite elements which are $\mathcal{C}^1$-compatible with the ARGYRIS or BELL triangles). On the other hand, the degrees of the polynomials used upon the reference triangle in order to define curved finite elements which are $\mathcal{C}^0$-compatible with given straight finite elements are the same.

(iii) the component u_3 of the displacement is approximated in a discrete finite element $V_{h2} \subset H_0^2(\Omega_h)$ associated to a straight or curved finite element (K, P_{K2}, Σ_{K2}), of class $\mathcal{C}^1, \forall K \in \mathcal{T}_h$. Let

$$\hat{P}_{K2} = \{\hat{p}_K : \hat{K} \longrightarrow \mathbf{R}\ ;\ \hat{p}_K = p \circ F_K,\ \ p \in P_{K2}\}. \tag{5.5}$$

Then we assume that there exists integers m_2, n_2 such that

$$\begin{cases} 2 \le m_2 \le n_2 \\ P_{M_2}(\hat{K}) \subset \hat{P}_{K2} \subset P_{N_2}(\hat{K}), \\ M_2 = m_2,\ \ N_2 = n_2 \text{ for a straight element } (n=1) \\ M_2 = m_2 + n - 1,\ \ N_2 = n_2 + n - 1 \text{ for a curved } \mathcal{C}^1\text{-compatible element.} \end{cases} \tag{5.6}$$

Remark 5.1 : Once more it is worth to note that this construction of curved finite elements uses polynomial spaces upon the reference triangle, i.e., the spaces $\hat{P}_{K\alpha},\ \alpha = 1, 2$, are polynomial spaces. In general, when F_K is non affine, the corresponding spaces $P_{K\alpha}$ are not polynomial spaces except their traces (and the traces of their normal derivatives for $\mathcal{C}^1$-curved elements) along the straight sides $a_3a_1,\ a_3a_2$ of the triangle K. Indeed $p = \hat{p}_K \circ F_K^{-1}$ and, except along the sides a_3a_1, a_3a_2, there are no reasons to obtain a polynomial space for functions p. ■

Local error estimate in view of the evaluation of the first consistency term

We have to evaluate the difference $\tilde{a}_h(\vec{v}_h, \vec{w}_h) - a_h(\vec{v}_h, \vec{w}_h)$. The relations (3.14) (3.26) and (3.23) involve

$$\tilde{a}_h(\vec{v}_h, \vec{w}_h) - a_h(\vec{v}_h, \vec{w}_h) = \sum_{K \in \mathcal{T}_h} E_K[{}^tV_h[A_{IJ}]W_h]. \tag{5.7}$$

The second member of the relation (5.7) is a sum of terms like

$$E_K(a\partial^\nu v \partial^\mu w),\ \ v \in P_{K\alpha},\ \ w \in P_{K\beta},\ \text{ with }\ \alpha, \beta = 1, 2 \text{ and } |\nu| \le \alpha,\ \ |\mu| \le \beta. \tag{5.8}$$

But, to the function $v \in P_{K\alpha}$, the application F_K associates the function $\hat{v} \in \hat{P}_{K\alpha}$ (cf. relations (5.3) and (5.5)). More generally, we obtain

(5.9) $$\hat{v}(\hat{x}) = v(x), \quad x = F_K(\hat{x}),$$

(5.10) $$\widehat{\frac{\partial v}{\partial x_\alpha}}(\hat{x}) = \sum_{\varepsilon=1}^{2} \frac{A_{\varepsilon\alpha}(\hat{x})}{J_{F_K}(\hat{x})} \frac{\partial \hat{v}}{\partial \hat{x}_\varepsilon}(\hat{x})$$

(5.11) $$\widehat{\frac{\partial^2 v}{\partial x_\alpha \partial x_\beta}}(\hat{x}) = \sum_{\rho=1}^{2} \frac{A_{\rho\beta}(\hat{x})}{J_{F_K}(\hat{x})} \sum_{\varepsilon=1}^{2} \left[\frac{A_{\varepsilon\alpha}(\hat{x})}{J_{F_K}(\hat{x})} \frac{\partial^2 \hat{v}}{\partial \hat{x}_\rho \partial \hat{x}_\varepsilon}(\hat{x}) + \frac{\partial \hat{v}}{\partial \hat{x}_\varepsilon}(\hat{x}) \frac{\partial}{\partial \hat{x}_\rho} \left(\frac{A_{\varepsilon\alpha}(\hat{x})}{J_{F_K}(\hat{x})} \right) \right]$$

with

(5.12) $$A_{1\alpha}(\hat{x}) = det \left[\vec{e}_\alpha, \frac{\partial F_K}{\partial \hat{x}_2} \right], \; A_{2\alpha}(\hat{x}) = det \left[\frac{\partial F_K}{\partial \hat{x}_1}, \vec{e}_\alpha \right].$$

Proof of relation (5.10) :

$$\widehat{\frac{\partial v}{\partial x_\alpha}}(\hat{x}) = \frac{\partial v}{\partial x_\alpha}(x) = Dv(x)\vec{e}_\alpha = D\hat{v}(\hat{x})DF_K^{-1}(x)\vec{e}_\alpha = D\hat{v}(\hat{x})[DF_K(\hat{x})]^{-1}\vec{e}_\alpha.$$

The vector $\vec{f}_\alpha = [DF_K(\hat{x})]^{-1}\vec{e}_\alpha$ is solution of the linear system $DF_K(\hat{x})\vec{f}_\alpha = \vec{e}_\alpha$ whose determinent is precisely $J_{F_K}(\hat{x})$. Then Cramer's rule and relations (5.12) gives

$$\vec{f}_\alpha = \sum_{\varepsilon=1}^{2} \frac{A_{\varepsilon\alpha}(\hat{x})}{J_{F_K}(\hat{x})} \vec{e}_\varepsilon,$$

and hence,

$$\widehat{\frac{\partial v}{\partial x_\alpha}}(\hat{x}) = D\hat{v}(\hat{x})\vec{f}_\alpha = \sum_{\varepsilon=1}^{2} \frac{A_{\varepsilon\alpha}(\hat{x})}{J_{F_K}(\hat{x})} \frac{\partial \hat{v}}{\partial \hat{x}_\varepsilon}(\hat{x}).$$

■

Proof of relation (5.11) :

Let us set $w_\alpha(x) = Dv(x)\vec{e}_\alpha$. By definition, we obtain

$$\widehat{\frac{\partial^2 v}{\partial x_\alpha \partial x_\beta}}(\hat{x}) = \frac{\partial^2 v}{\partial x_\alpha \partial x_\beta}(x) = D^2v(x)(\vec{e}_\alpha, \vec{e}_\beta) = Dw_\alpha(x)\vec{e}_\beta.$$

The relation (5.10) involves

$$Dw_\alpha(x)\vec{e}_\beta = \sum_{\rho=1}^{2} \frac{A_{\rho\beta}(\hat{x})}{J_{F_K}(\hat{x})} \frac{\partial \hat{w}_\alpha}{\partial \hat{x}_\rho}(\hat{x}), \; \hat{w}_\alpha(\hat{x}) = \widehat{\frac{\partial v}{\partial x_\alpha}}(\hat{x}) = \sum_{\varepsilon=1}^{2} \frac{A_{\varepsilon\alpha}(\hat{x})}{J_{F_K}(\hat{x})} \frac{\partial \hat{v}}{\partial \hat{x}_\varepsilon}(\hat{x}).$$

Finally,

$$\frac{\widehat{\partial^2 v}}{\partial x_\alpha \partial x_\beta}(\hat{x}) = \sum_{\rho=1}^{2} \frac{A_{\rho\beta}(\hat{x})}{J_{F_K}(\hat{x})} \sum_{\varepsilon=1}^{2} \left[\frac{A_{\varepsilon\alpha}(\hat{x})}{J_{F_K}(\hat{x})} \frac{\partial^2 \hat{v}}{\partial \hat{x}_\rho \partial \hat{x}_\varepsilon}(\hat{x}) + \frac{\partial \hat{v}}{\partial \hat{x}_\varepsilon}(\hat{x}) \frac{\partial}{\partial \hat{x}_\rho} \left(\frac{A_{\varepsilon\alpha}(\hat{x})}{J_{F_K}(\hat{x})} \right) \right].$$

■

For simplicity, we write the relations (5.10) (5.11) as

(5.13) $$\widehat{\partial^\nu v} = \sum_{|\gamma|=1}^{|\nu|} \hat{d}_{\nu\gamma} \partial^\gamma \hat{v} \text{ where } \nu \text{ and } \gamma \text{ are multi-indices.}$$

By substitution of relations (3.25) and (5.13) into the expression (5.8), we get :

(5.14) $$E_K(a\partial^\nu v \partial^\mu w) = \sum_{|\gamma|=1}^{|\nu|} \sum_{|\delta|=1}^{|\mu|} \hat{E}[\hat{a} J_{F_K} \hat{d}_{\nu\gamma} \hat{d}_{\mu\delta} \partial^\gamma \hat{v} \partial^\delta \hat{w}], \text{ where } \hat{a} = a \circ F_K.$$

We collect hereunder some results which are subsequently of constant use :

Lemma 5.1 : the functions $\hat{d}_{\nu\gamma}$ which are defined by relation (5.13) *verify estimates :*

(5.15) $$\partial^\rho[\hat{d}_{\nu\gamma} \hat{d}_{\mu\delta} J_{F_K}] = 0[h_K^{2-|\gamma|-|\delta|+|\rho|}].$$

Proof : this is a direct consequence of relations (5.9) to (5.13) and of estimates (5.2). ■

Lemma 5.2 : let $\phi \in W^{k,q}(K)$, $w \in W^{k,\infty}(K)$. Then the function $\phi w \in W^{k,q}(K)$ and satisfies

(5.16) $$|\phi w|_{k,q,K} \leq c \sum_{j=0}^{k} |\phi|_{k-j,q,K} \, |w|_{j,\infty,K}$$

where the constant c is only dependent of the integers k and q. ■

Lemma 5.3 : let k be a given integer. There exists a constant c, independent of $\hat{v} \in P_k$, such that

(5.17) $$|\hat{v}|_{j,\hat{K}} \leq c|\hat{v}|_{i,\hat{K}}, \quad 0 \leq i \leq j \leq k, \quad \forall \hat{v} \in P_k,$$

(5.18) $$|\hat{v}|_{j,\infty,\hat{K}} \leq c|\hat{v}|_{j,\hat{K}}, \quad 0 \leq j \leq k, \quad \forall \hat{v} \in P_k,$$

P_k being the linear space of all the polynomials whose degree is less or equal to k. ■

Lemma 5.4 : let $p \geq 0$, $q > 0$ be given integers. There exists a constant c, independent of h_K, such that

(5.19) $$|\hat{v}|_{p,q,\hat{K}} \leq c h_K^{p-2/q} \|v\|_{p,q,K}, \quad \forall \hat{v} \in W^{p,q}(\hat{K}).$$

Proof : it suffices to substitute the estimates (5.2) into the expression (4.15) of Part 1. ■

Then, the following theorem gives an estimate of the term $E_K(a\partial^\nu v\partial^\mu w)$:

Theorem 5.1 : let (K, P_{K1}, Σ_{K1}) *and* (K, P_{K2}, Σ_{K2}) *be two families of finite elements, straight or curved, which satisfy the hypotheses (5.1) to (5.6).*

Let k, ℓ, m *be given integers* ≥ 0. *We assume that the numerical integration scheme over the reference triangle* $\hat{K}$ *satisfies the following properties :*

(i) if $m+1 \leq k+\ell-|\nu|-|\mu|$

$$(5.20) \qquad \begin{cases} \text{if } k-|\nu| \leq m, \ \hat{E}(\hat{\phi}) = 0, \ \forall \hat{\phi} \in P_{m-k+N_\alpha}, \\ \text{if } \ell-|\mu| \leq m, \ \hat{E}(\hat{\phi}) = 0, \ \forall \hat{\phi} \in P_{m-\ell+N_\beta}, \\ \hat{E}(\hat{\phi}) = 0, \ \forall \hat{\phi} \in P_m \ ; \end{cases}$$

(ii) if $|\nu| = \alpha = k, \ \ |\mu| = \beta = \ell$ *and* $m = 0$,

$$(5.21) \qquad \hat{E}(\hat{\phi}) = 0, \ \forall \hat{\phi} \in P_{N_\alpha+N_\beta-\alpha-\beta}.$$

Then, there exists a constant $c > 0$, *independent of* K, *such that, for all function* $a \in W^{m+1,\infty}(K)$, *for all* $v \in P_{K\alpha}, \alpha = 1,2$, *for all* $w \in P_{K\beta}, \beta = 1,2$, *for all multi-indices* ν *and* μ *such that* $0 \leq |\nu| \leq \alpha \leq k, 0 \leq |\mu| \leq \beta \leq \ell$, *we have the upper bound :*

$$(5.22) \qquad |E_K(a\partial^\nu v\partial^\mu w)| \ \leq ch_K^{m+1} \ \|a\|_{m+1,\infty,K} \ \|v\|_{k,K} \ \|w\|_{\ell,K}.$$

Proof :

(i) Case $m+1 \leq k+\ell-|\nu|-|\mu|$: *according to (5.14) we have*

$$(5.23) \qquad \begin{cases} E_K(a\partial^\nu v\partial^\mu w) = \displaystyle\sum_{|\gamma|=1}^{|\nu|} \sum_{|\delta|=1}^{|\mu|} \hat{E}[\hat{a}J_{F_K} \ \hat{d}_{\nu\gamma}\hat{d}_{\mu\delta} \ \partial^\gamma\hat{v}\partial^\delta\hat{w}] \\ \text{where } 1 \leq |\nu| \leq \alpha \text{ and } 1 \leq |\mu| \leq \beta. \end{cases}$$

The assumptions (5.3) to (5.6) involve $\hat{v} \in P_{N_\alpha}$, $\hat{w} \in P_{N_\beta}$, so that $\partial^\gamma\hat{v} \in P_{N_\alpha-|\gamma|}$, $\partial^\delta\hat{w} \in P_{N_\beta-|\delta|}$. Let us set

$$(5.24) \qquad \hat{b} = \hat{a}J_{F_K} \ \hat{d}_{\nu\gamma}\hat{d}_{\mu\delta}, \ \ \hat{p} = \partial^\gamma\hat{v}, \ \ \hat{q} = \partial^\delta\hat{w},$$

and let us estimate $\hat{E}(\hat{b}\hat{p}\hat{q})$. For any $|\gamma| \leq k, |\delta| \leq \ell$ one can write

$$
(5.25) \quad \begin{cases} \hat{E}(\hat{b}\hat{p}\hat{q}) = \hat{E}[\hat{b}(\hat{\pi}_{k-|\gamma|}\hat{p})(\hat{\pi}_{\ell-|\delta|}\hat{q})] + \hat{E}[\hat{b}(\hat{\pi}_{k-|\gamma|}\hat{p})(\hat{q} - \hat{\pi}_{\ell-|\delta|}\hat{q})] \\ \qquad + \hat{E}[\hat{b}(\hat{p} - \hat{\pi}_{k-|\gamma|}\hat{p})(\hat{\pi}_{\ell-|\delta|}\hat{q})] + \hat{E}[\hat{b}(\hat{p} - \hat{\pi}_{k-|\gamma|}\hat{p})(\hat{q} - \hat{\pi}_{\ell-|\delta|}\hat{q})], \end{cases}
$$

where $\hat{\pi}_\ell$ denotes the $L^2(\hat{K})$-orthogonal projection operator onto the subspace $P_\ell(\hat{K})$. For clarity, we set

$$
(5.26) \qquad k_1 = k - |\gamma|, \;\; \ell_1 = \ell - |\delta|.
$$

Then the proof takes the four following steps :

Step 1 : Upper bound of $|\hat{E}[\hat{b}(\hat{\pi}_{k_1}\hat{p})(\hat{\pi}_{\ell_1}\hat{q})]|$, $\forall \hat{b} \in W^{m+1,\infty}(\hat{K})$:

Assumptions (5.20) and Bramble-Hilbert lemma [13,14,15] involve :

$$
\forall \hat{\psi} \in W^{m+1,\infty}(\hat{K}), \;\; |\hat{E}(\hat{\psi})| \leq c|(\hat{\psi})|_{m+1,\infty,,\hat{K}},
$$

where, here and subsequently, the letters c denote constants which are independent of the triangle into consideration and which can change from an inequality to the next.

Let us substitute $\hat{\psi}$ by $\hat{b}(\hat{\pi}_{k_1}\hat{p})(\hat{\pi}_{\ell_1}\hat{q})$ and let us use Leibniz formula

$$
|\hat{E}[\hat{b}(\hat{\pi}_{k_1}\hat{p})(\hat{\pi}_{\ell_1}\hat{q})]| \leq c \sum_{\substack{i+j=0 \\ i \leq k_1, j \leq \ell_1}}^{m+1} |\hat{b}|_{m+1-i-j,\infty,,\hat{K}} \; |\hat{\pi}_{k_1}\hat{p}|_{i,\infty,\hat{K}} \; |\hat{\pi}_{\ell_1}\hat{q}|_{j,\infty,\hat{K}},
$$

or equivalently, with inequality (5.18)

$$
|\hat{E}[\hat{b}(\hat{\pi}_{k_1}\hat{p})(\hat{\pi}_{\ell_1}\hat{q})]| \leq c \sum_{\substack{i+j=0 \\ i \leq k_1, j \leq \ell_1}}^{m+1} |\hat{b}|_{m+1-i-j,\infty,,\hat{K}} \; |\hat{\pi}_{k_1}\hat{p}|_{i,\hat{K}} \; |\hat{\pi}_{\ell_1}\hat{q}|_{j,\hat{K}}.
$$

Another application of Bramble-Hilbert lemma gives

$$
(5.27) \quad \begin{cases} |\hat{\pi}_{k_1}\hat{p}|_{i,\hat{K}} \leq c|\hat{p}|_{i,\hat{K}}, \;\; i = 0, ..., min(k_1, m+1), \\ |\hat{\pi}_{\ell_1}\hat{q}|_{j,\hat{K}} \leq c|\hat{q}|_{j,\hat{K}}, \;\; j = 0, ..., min(\ell_1, m+1). \end{cases}
$$

By combining the previous inequalities, we deduce the existence of a constant c such that

$$
|\hat{E}[\hat{b}(\hat{\pi}_{k_1}\hat{p})(\hat{\pi}_{\ell_1}\hat{q})]| \leq c \sum_{\substack{i+j=0 \\ i \leq k_1, j \leq \ell_1}}^{m+1} |\hat{b}|_{m+1-i-j,\infty,,\hat{K}} \; |\hat{p}|_{i,\hat{K}} \; |\hat{q}|_{j,\hat{K}},
$$

Relation (5.24) and lemmas 5.1 and 5.2 imply

$$
(5.28)\qquad \begin{cases} |\hat{b}|_{m+1-i-j,\infty,,\hat{K}} \le c \displaystyle\sum_{r=0}^{m+1-i-j} |\hat{a}|_{m+1-i-j-r,\infty,\hat{K}} \, |J_{F_K} \hat{d}_{\nu\gamma} \hat{d}_{\mu\delta}|_{r,\infty,\hat{K}} \\ \le c h_K^{2-|\gamma|-|\delta|} \displaystyle\sum_{r=0}^{m+1-i-j} h_K^r |\hat{a}|_{m+1-i-j-r,\infty,\hat{K}} . \end{cases}
$$

Thus

$$
(5.29)\qquad \begin{cases} \forall \hat{a} \in W^{m+1,\infty}(\hat{K}), \ \forall \hat{p} \in P_{N_\alpha-|\gamma|}, \ \forall \hat{q} \in P_{N_\beta-|\delta|}, \\ |\hat{E}[\hat{b}(\hat{\pi}_{k_1}\hat{p})(\hat{\pi}_{\ell_1}\hat{q})]| \\ \le c h_K^{2-|\gamma|-|\delta|} \displaystyle\sum_{\substack{i+j=0 \\ i \le k_1, j \le \ell_1}}^{m+1} \sum_{r=0}^{m+1-i-j} h_K^r \, |\hat{a}|_{m+1-i-j-r,\infty,\hat{K}} \, |\hat{p}|_{i,\hat{K}} \, |\hat{q}|_{j,\hat{K}} . \end{cases}
$$

Step 2 : Upper bound of $|\hat{E}[\hat{b}(\hat{\pi}_{k_1}\hat{p})(\hat{q}-\hat{\pi}_{\ell_1}\hat{q})]|$ *and* $|\hat{E}[\hat{b}(\hat{p}-\hat{\pi}_{k_1}\hat{p})(\hat{\pi}_{\ell_1}\hat{q})]|$:

Let us start by the first term.

If $0 \le \ell_1 < m+1$ we have the following inclusions with continuous injections

$$
(5.30)\qquad W^{m+1,\infty}(\hat{K}) \hookrightarrow W^{m+1-\ell_1,r}(\hat{K}) \hookrightarrow C^0(\hat{K}),
$$

with $r \ge 1$ and $r > \frac{2}{m+1-\ell_1}$. In particular, $W^{\ell_1,\infty}(\hat{K}) \hookrightarrow L^r(\hat{K})$, so that

$$
\forall \hat{\phi} \in W^{\ell_1,\infty}(\hat{K}), \ \ |\hat{\phi}|_{0,r,\hat{K}} \le c \sum_{j=0}^{\ell_1} |\hat{\phi}|_{\ell_1-j,\infty,\hat{K}},
$$

hence

$$
(5.31)\qquad \forall \hat{\phi} \in W^{m+1,\infty}(\hat{K}), \ \ |\hat{\phi}|_{m+1-\ell_1,r,\hat{K}} \le c \sum_{j=0}^{\ell_1} |\hat{\phi}|_{m+1-j,\infty,\hat{K}} .
$$

Analogously with the arguments of step 1 and with inequality (5.18), we find for any $\hat{\phi} \in W^{m+1-\ell_1,r}(\hat{K}) \hookrightarrow C^0(\hat{K})$ and for any $\hat{q} \in P_{N_\beta-|\delta|}$

$$
\begin{cases} |\hat{E}[\hat{\phi}(\hat{q}-\hat{\pi}_{\ell_1}\hat{q})]| \le c |\hat{\phi}(\hat{q}-\hat{\pi}_{\ell_1}\hat{q})|_{0,\infty,\hat{K}} \\ \le c |\hat{\phi}|_{0,\infty,\hat{K}} \, |\hat{q}-\hat{\pi}_{\ell_1}\hat{q}|_{0,\infty,\hat{K}} \le c \|\hat{\phi}\|_{m+1-\ell_1,r,\hat{K}} \, |\hat{q}-\hat{\pi}_{\ell_1}\hat{q}|_{0,\hat{K}} . \end{cases}
$$

For a given $\hat{q} \in P_{N_\beta-|\delta|}$, the linear form

$$\hat{\phi} \in W^{m+1-\ell_1,r}(\hat{K}) \longrightarrow \hat{E}[\hat{\phi}(\hat{q} - \hat{\pi}_{\ell_1}\hat{q})]$$

is continuous and zero over the space $P_{m-\ell_1}(\hat{K})$, thanks to assumption (5.20) (note that assumption $\ell_1 \leq m$ and relations (5.23) (5.26) involve $\ell - |\mu| \leq m$). With Bramble-Hilbert lemma, we obtain :

$$(5.32) \qquad \begin{cases} \forall \hat{\phi} \in W^{m+1-\ell_1,r}(\hat{K}), \ \forall \hat{q} \in P_{N_\beta - |\delta|}, \\ |\hat{E}[\hat{\phi}(\hat{q} - \hat{\pi}_{\ell_1}\hat{q})]| \leq c|\hat{\phi}|_{m+1-\ell_1,r,\hat{K}} \ |\hat{q} - \hat{\pi}_{\ell_1}\hat{q}|_{0,\hat{K}}. \end{cases}$$

Since the operator $\hat{\pi}_{\ell_1}$ leaves (in particular) invariant the space $P_{j-1}(\hat{K}), j = 1, ..., \ell_1$, we have

$$(5.33) \qquad \forall \hat{q} \in P_{N_\beta - |\delta|}, \ |\hat{q} - \hat{\pi}_{\ell_1}\hat{q}|_{0,\hat{K}} \leq c|\hat{q}|_{j,\hat{K}}, \ 0 \leq j \leq \ell_1$$

(for $j = 0$ we use the projection property into $L^2(\hat{K})$). Then relation (5.30) and inequalities (5.31) (5.32) and (5.33) involve :

$$\begin{cases} \forall \hat{\phi} \in W^{m+1,\infty}(\hat{K}), \ \forall \hat{q} \in P_{N_\beta - |\delta|}, \\ |\hat{E}[\hat{\phi}(\hat{q} - \hat{\pi}_{\ell_1}\hat{q})]| \leq c\sum_{j=0}^{\ell_1} |\hat{\phi}|_{m+1-j,\infty,\hat{K}} \ |\hat{q}|_{j,\hat{K}}. \end{cases}$$

Next, we set $\hat{\phi} = \hat{b}\hat{\pi}_{k_1}\hat{p}$ and we use Leibniz formula and inequalities (5.27). We find

$$|\hat{E}[\hat{b}(\hat{\pi}_{k_1}\hat{p})(\hat{q} - \hat{\pi}_{\ell_1}\hat{q})]| \ \leq c \sum_{\substack{i+j=0 \\ i \leq k_1, j \leq \ell_1}}^{m+1} |\hat{b}|_{m+1-i-j,\infty,\hat{K}} \ |\hat{p}|_{i,\hat{K}} \ |\hat{q}|_{j,\hat{K}}.$$

Finally, with inequalities (5.28), we obtain

$$(5.34) \qquad \begin{cases} \forall \hat{a} \in W^{m+1,\infty}(\hat{K}), \ \forall \hat{p} \in P_{N_\alpha - |\gamma|}, \ \forall \hat{q} \in P_{N_\beta - |\delta|}, \\ |\hat{E}[\hat{b}(\hat{\pi}_{k_1}\hat{p})(\hat{q} - \hat{\pi}_{\ell_1}\hat{q})]| \\ \quad \leq ch_K^{2-|\gamma|-|\delta|} \displaystyle\sum_{\substack{i+j=0 \\ i \leq k_1, j \leq \ell_1}}^{m+1} \sum_{r=0}^{m+1-i-j} h_K^r \ |\hat{a}|_{m+1-i-j-r,\infty,\hat{K}} \ |\hat{p}|_{i,\hat{K}} \ |\hat{q}|_{j,\hat{K}}. \end{cases}$$

If $m + 1 \leq \ell_1$, then $W^{m+1,\infty}(\hat{K}) \hookrightarrow C^0(\hat{K})$ and inequalities (5.33) involve,

$$\forall \hat{\phi} \in W^{m+1,\infty}(\hat{K}), \ \forall \hat{q} \in P_{N_\beta - |\delta|}, \ |\hat{E}[\hat{\phi}(\hat{q} - \hat{\pi}_{\ell_1}\hat{q})]| \leq c|\hat{\phi}|_{0,\infty,\hat{K}} \ |\hat{q}|_{\ell_1,\hat{K}}.$$

By taking $\hat{\phi} = \hat{b}\hat{\pi}_{k_1}\hat{p}$, we obtain

$$|\hat{E}[\hat{b}(\hat{\pi}_{k_1}\hat{p})(\hat{q} - \hat{\pi}_{\ell_1}\hat{q})]| \leq c|\hat{b}|_{0,\infty,\hat{K}} \ |\hat{p}|_{0,\hat{K}} \ |\hat{q}|_{\ell_1,\hat{K}}.$$

The inequalities (5.28) involve

$$(5.35) \qquad \begin{cases} \forall \hat{a} \in W^{m+1,\infty}(\hat{K}), \ \forall \hat{p} \in P_{N_\alpha - |\gamma|}, \ \forall \hat{q} \in P_{N_\beta - |\delta|}, \\ |\hat{E}[\hat{b}(\hat{\pi}_{k_1}\hat{p})(\hat{q} - \hat{\pi}_{\ell_1}\hat{q})]| \leq c h_K^{2-|\gamma|-|\delta|} \ |\hat{a}|_{0,\infty,\hat{K}} \ |\hat{p}|_{0,\hat{K}} \ |\hat{q}|_{\ell_1,\hat{K}}. \end{cases}$$

With symmetry considerations the second term admits the following upper bounds :

If $0 \leq k_1 < m+1$ *(so that* $k - |\nu| \leq m$*)*

$$(5.36) \qquad \begin{cases} \forall \hat{a} \in W^{m+1,\infty}(\hat{K}), \ \forall \hat{p} \in P_{N_\alpha - |\gamma|}, \ \forall \hat{q} \in P_{N_\beta - |\delta|}, \\ |\hat{E}[\hat{b}(\hat{p} - \hat{\pi}_{k_1}\hat{p})(\hat{\pi}_{\ell_1}\hat{q})]| \\ \leq c h_K^{2-|\gamma|-|\delta|} \displaystyle\sum_{\substack{i+j=0 \\ i \leq k_1, j \leq \ell_1}}^{m+1} \ \sum_{r=0}^{m+1-i-j} h_K^r \ |\hat{a}|_{m+1-i-j-r,\infty,\hat{K}} \ |\hat{p}|_{i,\hat{K}} \ |\hat{q}|_{j,\hat{K}}. \end{cases}$$

If $m + 1 \leq k_1$

$$(5.37) \qquad \begin{cases} \forall \hat{a} \in W^{m+1,\infty}(\hat{K}), \ \forall \hat{p} \in P_{N_\alpha - |\gamma|}, \ \forall \hat{q} \in P_{N_\beta - |\delta|}, \\ |\hat{E}[\hat{b}(\hat{p} - \hat{\pi}_{k_1}\hat{p})(\hat{\pi}_{\ell_1}\hat{q})]| \leq c h_K^{2-|\gamma|-|\delta|} \ |\hat{a}|_{0,\infty,\hat{K}} \ |\hat{p}|_{k_1,\hat{K}} \ |\hat{q}|_{0,\hat{K}}. \end{cases}$$

Step 3 : Upper bound of $|\hat{E}[\hat{b}(\hat{p} - \hat{\pi}_{k_1}\hat{p})(\hat{q} - \hat{\pi}_{\ell_1}\hat{q})]|$

The inequality (5.18) implies

$$|\hat{E}[\hat{b}(\hat{p} - \hat{\pi}_{k_1}\hat{p})(\hat{q} - \hat{\pi}_{\ell_1}\hat{q})]| \leq c|\hat{b}|_{0,\infty,\hat{K}} \ |\hat{p} - \hat{\pi}_{k_1}\hat{p}|_{0,\hat{K}} \ |\hat{q} - \hat{\pi}_{\ell_1}\hat{q}|_{0,\hat{K}}.$$

Since the assumption $k + \ell - |\nu| - |\mu| \geq m+1$ involves $k_1 + \ell_1 \geq m+1$, we can arbitrarily choose the integers i and j with $0 \leq i \leq k_1, 0 \leq j \leq \ell_1$ and $i + j = m+1$ so that the inequalities (5.28) and (5.33) give

$$(5.38) \qquad \begin{cases} \forall \hat{a} \in W^{m+1,\infty}(\hat{K}), \ \forall \hat{p} \in P_{N_\alpha - |\gamma|}, \ \forall \hat{q} \in P_{N_\beta - |\delta|}, \\ |\hat{E}[\hat{b}(\hat{p} - \hat{\pi}_{k_1}\hat{p})(\hat{q} - \hat{\pi}_{\ell_1}\hat{q})]| \leq c h_K^{2-|\gamma|-|\delta|} \ |\hat{a}|_{0,\infty,\hat{K}} \ |\hat{p}|_{i,\hat{K}} \ |\hat{q}|_{j,\hat{K}}. \end{cases}$$

Step 4 : Final estimate (5.22)

To obtain the final estimate (5.22), it remains to collect estimates (5.29) and (5.34) to (5.38). Thanks to relation (5.24) and estimates (5.19), estimate (5.29) can be written

$$\begin{cases} \forall \hat{a} \in W^{m+1,\infty}(\hat{K}), \ \ \forall \hat{v} \in P_{N_\alpha}, \ \ \forall \hat{w} \in P_{N_\beta}, \\ |\hat{E}[\hat{b}\hat{\pi}_{k_1}(\partial^\gamma \hat{v})\hat{\pi}_{\ell_1}(\partial^\delta \hat{w})]| \\ \le c h_K^{2-|\gamma|-|\delta|} \displaystyle\sum_{\substack{i+j=0 \\ i \le k_1, j \le \ell_1}}^{m+1} \sum_{r=0}^{m+1-i-j} h_K^r \, |\hat{a}|_{m+1-i-j-r,\infty,\hat{K}} \, |\hat{v}|_{i+|\gamma|,\hat{K}} \, |\hat{w}|_{i+|\delta|,\hat{K}} \\ \le C h_K^{m+1} \|a\|_{m+1,\infty,K} \|v\|_{k,K} \|w\|_{\ell,K}. \end{cases}$$

We obtain the same kind of result from estimates (5.34) to (5.38) and thus, relations (5.23) to (5.26) permit to complete the proof of theorem 5.1 when $m+1 \le k+\ell-|\nu|-|\mu|$.

(ii) Case $|\nu| = \alpha = k, |\mu| = \beta = \ell$ *and* $m = 0 (\alpha, \beta = 1,2)$:

We have to prove that $\forall a \in W^{1,\infty}(K), \ \ \forall v \in P_{K\alpha}, \ \ \forall w \in P_{K\beta}$

$$|E_K(a\partial^\nu v \partial^\mu w)| \le c h_K \ \|a\|_{1,\infty,K} \ \|v\|_{\alpha,K} \ \|w\|_{\beta,K}.$$

But

$$E_K(a\partial^\nu v \partial^\mu w) = \sum_{|\gamma|=1}^{\alpha} \sum_{|\delta|=1}^{\beta} \hat{E}[\hat{a} J_{F_K} \ \hat{d}_{\nu\gamma} \hat{d}_{\mu\delta} \ \partial^\gamma \hat{v} \partial^\delta \hat{w}] \text{ where } \hat{v} \in P_{N_\alpha}, \ \ \hat{w} \in P_{N_\beta}.$$

Then we have two cases to examine :

* when $|\gamma| + |\delta| \le \alpha + \beta - 1$, we use arguments similar to those of case (i) since condition $k + \ell - |\gamma| - |\delta| \ge 1$ is satisfied (note that here $m = 0$) and since assumption (5.21) involves the verification of hypotheses (5.20) ;

** when $|\gamma| = \alpha, \ \ |\delta| = \beta$, set

$$\begin{cases} \hat{b} = \hat{a} \ J_{F_K} \ \hat{d}_{\nu\gamma} \hat{d}_{\mu\delta}, \ \ \hat{p} = \partial^\gamma \hat{v}, \ \ \hat{q} = \partial^\delta \hat{w}, \\ \hat{b} \in W^{1,\infty}(\hat{K}), \ \ \hat{p} \in P_{N_\alpha - \alpha}, \ \ \hat{q} \in P_{N_\beta - \beta}. \end{cases}$$

We obtain

$$|\hat{E}(\hat{b}\hat{p}\hat{q})| \leq c|\hat{b}|_{0,\infty,\hat{K}}\ |\hat{p}|_{0,\infty,\hat{K}}\ |\hat{q}|_{0,\infty,\hat{K}} \leq c\|\hat{b}\|_{1,\infty,\hat{K}}\ |\hat{p}|_{0,\hat{K}}\ |\hat{q}|_{0,\hat{K}}.$$

Then assumption (5.21) and Bramble-Hilbert lemma imply

$$|\hat{E}(\hat{b}\hat{p}\hat{q})| \leq c|\hat{b}|_{1,\infty,\hat{K}}\ |\hat{p}|_{0,K}\ |\hat{q}|_{0,\hat{K}}.$$

But Lemmas 5.1 and 5.4 involve

$$|\hat{b}|_{1,\infty,\hat{K}} \leq c\sum_{i=0}^{1} |\hat{a}|_{1-i,\infty,\hat{K}}\ |J_{F_K}\ \hat{d}_{\nu\gamma}\hat{d}_{\mu\delta}|_{i,\infty,\hat{K}} \leq ch_K^{3-\alpha-\beta}\ \|a\|_{1,\infty,K}$$

$$|\hat{p}|_{0,\hat{K}} \leq c|\hat{v}|_{\alpha,\hat{K}} \leq ch_K^{\alpha-1}\ \|v\|_{\alpha,K},$$

$$|\hat{q}|_{0,\hat{K}} \leq c|\hat{w}|_{\beta,\hat{K}} \leq ch_K^{\beta-1}\ \|w\|_{\beta,K},$$

so that we get the expected estimate (5.22) when $|\nu| = \alpha = k, |\mu| = \beta = 1$ and $m = 0$. ∎

Local error estimate in view of the evaluation of the second consistency term :

Theorem 5.2 : Let (K, P_{K1}, Σ_{K1}) *and* (K, P_{K2}, Σ_{K2}) *be two straight or curved finite element families verifying the hypotheses* (5.1) *to* (5.6).

Let ℓ, m *be given integers* ≥ 0 *and let* q *be any real number* $> \frac{2}{m+1}$. *We assume that the numerical integration scheme upon the reference triangle* $\hat{K}$ *satisfies the following properties :*

(5.39)
$$\begin{cases} \text{if } \ell \leq m, & \hat{E}(\phi) = 0, \ \forall\hat{\phi} \in P_{m-\ell+N_\beta}(\hat{K}), \\ \hat{E}(\phi) = 0, & \forall\hat{\phi} \in P_m(\hat{K}). \end{cases}$$

Then, there exists a constant $c > 0$, *independent of* h_K, *such that*

(5.40)
$$\begin{cases} |E_K(\phi w)| \leq ch_K^{m+1}\ [measure(K)]^{\frac{1}{2}-\frac{1}{q}} \|\phi\|_{m+1,q,K}\ \|w\|_{\ell,K}, \\ \forall\phi \in W^{m+1,q}(K), \ \forall w \in P_{K\beta}, \ \beta = 1, 2, \end{cases}$$

where ℓ *is an integer* $\geq \beta$.

Proof : For any $\phi \in W^{m+1,q}(K)$ and for any $w \in P_{K\beta}$, the relations (3.25) (5.4) and (5.6) involve :

(5.41)
$$E_K(\phi w) = \hat{E}[\hat{\phi}\hat{w}J_{F_K}] \text{ where } \hat{\phi} \in W^{m+1,q}(\hat{K}), \ \hat{w} \in P_{N_\beta}(\hat{K}).$$

We can write

$$\hat{E}[\hat{\phi}\hat{w}J_{F_K}] = \hat{E}[\hat{\phi}J_{F_K}\ \hat{\pi}_\ell\hat{w}] + \hat{E}[\hat{\phi}J_{F_K}\ (\hat{w} - \hat{\pi}_\ell\hat{w})]$$

where $\hat{\pi}_\ell$ denotes the $L^2(\hat{K})$-orthogonal projection operator onto the finite dimensional subspace $P_\ell(\hat{K})$. Then the proof is similar to that of Theorem 5.1 ; we record the main ideas :

(i) Estimate of $\hat{E}[\hat{\phi} J_{F_K}\ \hat{\pi}_\ell \hat{w}]$:

The assumption (5.39) and $q > \frac{2}{m+1}$ involve

$$\forall \hat{\psi} \in W^{m+1,q}(\hat{K}),\ \ |\hat{E}(\hat{\psi})| \leq c|\hat{\psi}|_{m+1,q,\hat{K}},$$

and hence with (5.16) (5.18) (5.27)

$$(5.42)\qquad \begin{cases} \forall \hat{\phi} \in W^{m+1,q}(\hat{K}),\ \ \forall \hat{w} \in P_{N_\beta}, \\ |\hat{E}[\hat{\phi} J_{F_K}\ \hat{\pi}_\ell \hat{w}]| \leq c \displaystyle\sum_{j=0}^{\min(\ell,m+1)} |\hat{\phi} J_{F_K}|_{m+1-j,q,\hat{K}}\ |\hat{w}|_{j,\hat{K}}. \end{cases}$$

(ii) Estimate of $\hat{E}[\hat{\phi} J_{F_K}\ (\hat{w} - \hat{\pi}_\ell \hat{w})]$:

If $1 \leq \ell \leq m$, we have the inclusions

$$W^{m+1,q}(\hat{K}) \hookrightarrow W^{m+1-\ell,r}(\hat{K}) \hookrightarrow C^0(\hat{K}),$$

where r is given by

$$\begin{cases} \dfrac{1}{r} = \dfrac{1}{q} - \dfrac{\ell}{2} \text{ if } 1 \leq q < \dfrac{2}{\ell}, \\ r \text{ large enough so that } m+1-\ell-\dfrac{2}{r} > 0 \text{ if } q \geq \dfrac{2}{\ell}. \end{cases}$$

Then,

$$\forall \hat{\phi} \in W^{m+1,q}(\hat{K}),\ \ |\hat{\phi}|_{m+1-\ell,r,\hat{K}} \leq c\sum_{j=0}^{\ell} |\hat{\phi}|_{m+1-j,q,\hat{K}}$$

and

$$\begin{cases} \forall \hat{\phi} \in W^{m+1-\ell,r}(\hat{K}),\ \ \forall \hat{w} \in P_{N_\beta}, \\ |\hat{E}[\hat{\phi} J_{F_K}\ (\hat{w} - \hat{\pi}_\ell \hat{w})]| \leq c|\hat{\phi} J_{F_K}|_{m+1-\ell,r,\hat{K}}\ |\hat{w} - \hat{\pi}_\ell \hat{w}|_{0,\hat{K}} \end{cases}$$

so that, we get

$$(5.43)\qquad \begin{cases} \forall \hat{\phi} \in W^{m+1,q}(\hat{K}),\ \ \forall \hat{w} \in P_{N_\beta}, \\ |\hat{E}[\hat{\phi} J_{F_K}\ (\hat{w} - \hat{\pi}_\ell \hat{w})]| \leq c\displaystyle\sum_{j=0}^{\ell} |\hat{\phi} J_{F_K}|_{m+1-j,q,\hat{K}}\ |\hat{w}|_{j,\hat{K}}. \end{cases}$$

If $m+1 \leq \ell$ we have the inclusion $W^{m+1,q}(\hat{K}) \hookrightarrow C^0(\hat{K})$. Then,

$$(5.44) \qquad \begin{cases} \forall \hat{\phi} \in W^{m+1,q}(\hat{K}), \ \forall \hat{w} \in P_{N_\beta}, \\ |\hat{E}[\hat{\phi} J_{F_K}\ (\hat{w} - \hat{\pi}_\ell \hat{w})]| \leq c \|\hat{\phi} J_{F_K}\|_{m+1,q,\hat{K}}\ |\hat{w}|_{\ell,\hat{K}}. \end{cases}$$

To conclude, it remains to combine the relation (5.41), the inequalities (5.42) (5.43) for $1 \leq \ell \leq m$, the inequalities (5.42) (5.44) for $m+1 \leq \ell$, the inequalities (5.16) and the estimates (5.2) and (5.19). ■

6 SUFFICIENT CONDITIONS TO ENSURE THE UNIFORM $\vec{V}_h$-ELLIPTICITY PROPERTY

Let us prove the following theorem :

Theorem 6.1 : Let $\mathcal{T}$ be a regular family of triangulations of the domain Ω verifying the properties of Part 1, (2.2) *and* (2.3). *Let (K, P_{K1}, Σ_{K1}) and (K, P_{K2}, Σ_{K2}) be two families of straight or curved finite elements verifying assumptions* (5.1) *to* (5.6). *We assume that the numerical integration scheme over the reference triangle $\hat{K}$ is such that :*

$$(6.1) \qquad \forall \hat{\phi} \in P_{-2+2max(N_1,N_2-1)}, \ \hat{E}(\hat{\phi}) = 0.$$

Then, if $A_{IJ} \in W^{1,\infty}(\tilde{\Omega}), I, J = 1, ..., 12$, there exists a constant $\beta > 0$, independent of h, and an $h_1 > 0$ such that

$$(6.2) \qquad \beta \|\vec{v}_h\|^2_{\vec{X}(\Omega_h)} \leq a_h(\vec{v}_h, \vec{v}_h), \ \forall \vec{v}_h \in \vec{V}_h, \ \forall h < h_1,$$

where the bilinear form a_h is defined by relation (3.26).

Proof : For any $\vec{v}_h \in \vec{V}_h$, relation (3.14) involves

$$(6.3) \qquad a_h(\vec{v}_h, \vec{v}_h) = \tilde{a}_h(\vec{v}_h, \vec{v}_h) + a_h(\vec{v}_h, \vec{v}_h) - \tilde{a}_h(\vec{v}_h, \vec{v}_h).$$

The proof takes three steps :

Step 1 : There exists a constant $\alpha > 0$, independent of h, such that

$$(6.4) \qquad \forall \vec{v}_h \in \vec{V}_h, \ \alpha \|\vec{v}_h\|^2_{\vec{X}(\Omega_h)} \leq \tilde{a}_h(\vec{v}_h, \vec{v}_h).$$

By consideration of [7, theorem 6.1.3] and from Hypothesis 2.1, we obtain that the bilinear form associated to the mapping $\vec{\varphi}$ defined upon the domain $\tilde{\Omega}$, i.e.,

$$\tilde{a}(\vec{u}, \vec{v}) = \int_{\tilde{\Omega}} {}^tU[A_{IJ}]V dx,$$

is $(H_0^1(\tilde{\Omega}))^2 \times H_0^2(\tilde{\Omega})$-elliptic, i.e., there exists a constant $\tilde{\beta} > 0$ such that

$$\forall \vec{v} \in (H_0^1(\tilde{\Omega}))^2 \times H_0^2(\tilde{\Omega}), \quad \tilde{\beta} \|\vec{v}_h\|^2_{\vec{X}(\tilde{\Omega})} \leq \tilde{a}(\vec{v}, \vec{v}). \tag{6.5}$$

From (3.5) we obtain $\Omega_h \subset \tilde{\Omega}$ for any $h < h_0$. In addition, any function $\vec{v}_h \in \vec{V}_h$ verifies

$$\vec{v}_h \in [(H_0^1(\Omega_h))^2 \times H_0^2(\Omega_h)] \cap [(\mathcal{C}^0(\Omega_h))^2 \times \mathcal{C}^1(\Omega_h)].$$

Hence, any function $\vec{v}_h \in \vec{V}_h$ can be extended by 0 over $\tilde{\Omega} - \Omega_h$ and this new function $\bar{\vec{v}}_h \in (H_0^1(\tilde{\Omega}))^2 \times H_0^2(\tilde{\Omega})$. If we note that $\|\bar{\vec{v}}_h\|_{\vec{X}(\tilde{\Omega})} = \|\vec{v}_h\|_{\vec{X}(\Omega_h)}$ and that $\tilde{a}_h(\bar{\vec{v}}_h, \bar{\vec{v}}_h) = \tilde{a}_h(\vec{v}_h, \vec{v}_h)$, the inequality (6.5) can be written :

$$\forall \vec{v}_h \in \vec{V}_h, \quad \tilde{\beta} \|\vec{v}_h\|^2_{\vec{X}(\Omega_h)} \leq \tilde{a}_h(\vec{v}_h, \vec{v}_h),$$

and hence we get (6.4) with $\alpha = \tilde{\beta}$.

Step 2 : There exists a constant $c > 0$, independent of h, such that

$$\forall \vec{v}_h \in \vec{V}_h, \quad |\tilde{a}_h(\vec{v}_h, \vec{v}_h) - a_h(\vec{v}_h, \vec{v}_h)| \leq ch \|\vec{v}_h\|^2_{\vec{X}(\Omega_h)}. \tag{6.6}$$

Figure 6.1 shows that the hypotheses of Theorem 6.1, in particular the hypothesis (6.1), allow us to apply Theorem 5.1 to the different terms which appear at the right hand side of the relation

$$\tilde{a}_h(\vec{v}_h, \vec{v}_h) - a_h(\vec{v}_h, \vec{v}_h) = \sum_{K \in \mathcal{T}_h} \sum_{I,J=1}^{12} E_K[A_{IJ}(V_h)_I (V_h)_J].$$

Thus,

$$\left\{ \begin{aligned} & \forall \vec{v}_h \in \vec{V}_h, \quad \forall A_{IJ} \in W^{1,\infty}(\tilde{\Omega}) \\ & |\tilde{a}_h(\vec{v}_h, \vec{v}_h) - a_h(\vec{v}_h, \vec{v}_h)| \leq \sum_{K \in \mathcal{T}_h} \sum_{I,J=1}^{12} |E_K[A_{IJ}(V_h)_I (V_h)_J]| \\ & \leq c \sum_{K \in \mathcal{T}_h} h_K \left[\sum_{I,J=1}^{12} \|A_{IJ}\|_{1,\infty,K} \right] \|\vec{v}_h\|^2_{\vec{V}(K)} \leq ch \|\vec{v}_h\|^2_{\vec{X}(\Omega_h)}. \end{aligned} \right. \tag{6.7}$$

Step 3 : Inequality (6.2)

The relations (6.3) (6.4) and (6.6) imply

$$\forall \vec{v}_h \in \vec{V}_h, \quad a_h(\vec{v}_h, \vec{v}_h) \geq (\alpha - ch) \|\vec{v}_h\|^2_{\vec{X}(\Omega_h)}, \tag{6.8}$$

and hence we get the inequality (6.2) with $\beta = \frac{\alpha}{2}$ and $h_1 = min(h_0, \frac{\alpha}{2c})$. ■

7 THEOREM OF ASYMPTOTIC ERROR ESTIMATE

From now on, we are able to evaluate the different terms of the inequality (4.3) and thus, to give an estimate of the asymptotic error $\|\tilde{\vec{u}} - \vec{u}_h\|^2_{\dot{X}(\Omega_h)}$ between the extension $\tilde{\vec{u}}$ of the exact solution $\vec{u}$ and the approximate solution $\vec{u}_h$ of the problem (3.28). Below, Theorem 7.1 gives the error estimate result for some finite element spaces constructed from the curved elements introduced in [Part 1, paragraph 3], and from the corresponding straight finite elements. In particular, it clarifies the criteria to be observed when choosing the numerical integration schemes in order to obtain the same asymptotic error estimate than the one obtained when integrating exactly. These criteria are different according to the kind of finite elements we consider : straight on the one hand ($n = 1$) or curved on the other hand ($n = 3$ or $n = 5$).

Theorem 7.1 : Let $\mathcal{T}$ be a regular family of triangulations of the domain Ω verifying the properties [Part 1, (2.2) *and* (2.3)]. *Let (K, P_{K1}, Σ_{K1}) and (K, P_{K2}, Σ_{K2}) be two families of straight or curved finite elements verifying the assumptions* (5.1) *to* (5.6), *particularly*

$$P_{M_\alpha}(\hat{K}) \subset \hat{P}_{K\alpha} \subset P_{N_\alpha}(\hat{K}), \quad \alpha = 1, 2. \tag{7.1}$$

Let V_{h1}, V_{h2} be the associate finite element spaces verifying conditions (3.2) *and* (3.3). *Set $m = -1 + min(m_1, m_2 - 1)$. We assume that the numerical integration scheme upon the reference triangle $\hat{K}$ satisfies the following properties :*

$$\begin{cases} \textit{all the integration nodes } \hat{b}_\ell \textit{ are inside of } \hat{K} \textit{ or they coincide} \\ \textit{with the vertices of } \hat{K} \textit{ (cf. Hypothesis 3.1) ;} \end{cases} \tag{7.2}$$

$$\hat{E}(\hat{\phi}) = 0, \quad \forall \hat{\phi} \in P_{-2+2max(N_1, N_2-1)}. \tag{7.3}$$

Let $\tilde{A}$ be the system of partial differential operators associated to the bilinear form

$$\tilde{\vec{u}}, \tilde{\vec{v}} \in \vec{V}(\tilde{\Omega}) \longrightarrow \tilde{a}(\tilde{\vec{u}}, \tilde{\vec{v}}) = \int_{\tilde{\Omega}} {}^t\tilde{U}[A_{IJ}]\tilde{V}dx. \tag{7.4}$$

Then, if the extension $\tilde{\vec{u}}$ of the solution $\vec{u}$ of the problem 2.1 belongs to the space $H^{m+2}(\tilde{\Omega}) \times H^{m+2}(\tilde{\Omega}) \times H^{m+3}(\tilde{\Omega})$, if $H^{m+1+\alpha}(\tilde{\Omega}) \hookrightarrow C^{s_\alpha}(\tilde{\Omega}), \alpha = 1, 2,$ where s_α denotes the maximal order of the partial derivatives which appear in the definition of the degrees of freedom Σ_K, if $A_{IJ} \in W^{m+1,\infty}(\tilde{\Omega}), I, J = 1, ..., 12,$ if $\tilde{A}\tilde{\vec{u}} \in (W^{m+1,q}(\tilde{\Omega}))^3$ for a number $q > \dfrac{2}{m+1}$ with $q \geq 2$, if h is small enough, there exists a constant c independent of h such that :

$$\begin{cases} \|\tilde{\vec{u}} - \vec{u}_h\|_{\vec{X}(\Omega_h)} \\ \qquad \leq ch^{m+1} \left\{ \left[\sum_{\alpha=1}^{2} \|\tilde{u}_\alpha\|^2_{m+2,\tilde{\Omega}} + \|\tilde{u}_3\|^2_{m+3,\tilde{\Omega}} \right]^{1/2} + \left[\sum_{i=1}^{3} \|(\tilde{A}\tilde{\vec{u}})_i\|^q_{m+1,q,\tilde{\Omega}} \right]^{1/q} \right\}, \end{cases} \tag{7.5}$$

$w = (V_h)_J$ / $v = (V_h)_I$	$w \in P_{K1}$	$Dw, (w \in P_{K1})$	$w \in P_{K2}$	$Dw, (w \in P_{K2})$	$D^2w, (w \in P_{K2})$
$v \in P_{K1}$	$\alpha = \beta = 1$ $k = \ell = 1 ; m = 0$ $\lvert\nu\rvert = \lvert\mu\rvert = 0$ $\forall\hat{\phi} \in P_0, \ \hat{E}(\hat{\phi}) = 0$	$\alpha = \beta = 1$ $k = \ell = 1 ; m = 0$ $\lvert\nu\rvert = 0 ; \lvert\mu\rvert = 1$ $\forall\hat{\phi} \in P_{N_1-1}, \ \hat{E}(\hat{\phi}) = 0$	$\alpha = 1 ; \beta = 2$ $k = 1 ; \ell = 2 ; m = 0$ $\lvert\nu\rvert = \lvert\mu\rvert = 0$ $\forall\hat{\phi} \in P_0, \ \hat{E}(\hat{\phi}) = 0$	$\alpha = 1 ; \beta = 2$ $k = 1 ; \ell = 2 ; m = 0$ $\lvert\nu\rvert = 0 ; \lvert\mu\rvert = 1$ $\forall\hat{\phi} \in P_0, \ \hat{E}(\hat{\phi}) = 0$	$\alpha = 1 ; \beta = 2$ $k = 1 ; \ell = 2 ; m = 0$ $\lvert\nu\rvert = 0 ; \lvert\mu\rvert = 2$ $\forall\hat{\phi} \in P_{N_2-2}, \ \hat{E}(\hat{\phi}) = 0$
$Dv, (v \in P_{K1})$	$\alpha = \beta = 1$ $k = \ell = 1 ; m = 0$ $\lvert\nu\rvert = 1 ; \lvert\mu\rvert = 0$ $\forall\hat{\phi} \in P_{N_1-1}, \ \hat{E}(\hat{\phi}) = 0$	$\alpha = \beta = 1$ $k = \ell = 1 ; m = 0$ $\lvert\nu\rvert = \lvert\mu\rvert = 1$ $\forall\hat{\phi} \in P_{2N_1-2}, \ \hat{E}(\hat{\phi}) = 0$	$\alpha = 1 ; \beta = 2$ $k = 1 ; \ell = 2 ; m = 0$ $\lvert\nu\rvert = 1 ; \lvert\mu\rvert = 0$ $\forall\hat{\phi} \in P_{N_1-1}, \ \hat{E}(\hat{\phi}) = 0$	$\alpha = 1 ; \beta = 2$ $k = 1 ; \ell = 2 ; m = 0$ $\lvert\nu\rvert = \lvert\mu\rvert = 1$ $\forall\hat{\phi} \in P_{N_1-1}, \ \hat{E}(\hat{\phi}) = 0$	$\alpha = 1 ; \beta = 2$ $k = 1 ; \ell = 2 ; m = 0$ $\lvert\nu\rvert = 1 ; \lvert\mu\rvert = 2$ $\forall\hat{\phi} \in P_{N_1+N_2-3}, \ \hat{E}(\hat{\phi}) = 0$
$v \in P_{K2}$	$\alpha = 2 ; \beta = 1$ $k = 2 ; \ell = 1 ; m = 0$ $\lvert\nu\rvert = \lvert\mu\rvert = 0$ $\forall\hat{\phi} \in P_0, \ \hat{E}(\hat{\phi}) = 0$	$\alpha = 2 ; \beta = 1$ $k = 2 ; \ell = 1 ; m = 0$ $\lvert\nu\rvert = 0 ; \lvert\mu\rvert = 1$ $\forall\hat{\phi} \in P_{N_1-1}, \ \hat{E}(\hat{\phi}) = 0$	$\alpha = \beta = 2$ $k = \ell = 2 ; m = 0$ $\lvert\nu\rvert = \lvert\mu\rvert = 0$ $\forall\hat{\phi} \in P_0, \ \hat{E}(\hat{\phi}) = 0$	$\alpha = \beta = 2$ $k = \ell = 2 ; m = 0$ $\lvert\nu\rvert = 0 ; \lvert\mu\rvert = 1$ $\forall\hat{\phi} \in P_0, \ \hat{E}(\hat{\phi}) = 0$	$\alpha = \beta = 2$ $k = \ell = 2 ; m = 0$ $\lvert\nu\rvert = 0 ; \lvert\mu\rvert = 2$ $\forall\hat{\phi} \in P_{N_2-2}, \ \hat{E}(\hat{\phi}) = 0$
$Dv, (v \in P_{K2})$	$\alpha = 2 ; \beta = 1$ $k = 2 ; \ell = 1 ; m = 0$ $\lvert\nu\rvert = 1 ; \lvert\mu\rvert = 0$ $\forall\hat{\phi} \in P_0, \ \hat{E}(\hat{\phi}) = 0$	$\alpha = 2 ; \beta = 1$ $k = 2 ; \ell = 1 ; m = 0$ $\lvert\nu\rvert = \lvert\mu\rvert = 1$ $\forall\hat{\phi} \in P_{N_1-1}, \ \hat{E}(\hat{\phi}) = 0$	$\alpha = \beta = 2$ $k = \ell = 2 ; m = 0$ $\lvert\nu\rvert = 1 ; \lvert\mu\rvert = 0$ $\forall\hat{\phi} \in P_0, \ \hat{E}(\hat{\phi}) = 0$	$\alpha = \beta = 2$ $k = \ell = 2 ; m = 0$ $\lvert\nu\rvert = \lvert\mu\rvert = 1$ $\forall\hat{\phi} \in P_0, \ \hat{E}(\hat{\phi}) = 0$	$\alpha = \beta = 2$ $k = \ell = 2 ; m = 0$ $\lvert\nu\rvert = 1 ; \lvert\mu\rvert = 2$ $\forall\hat{\phi} \in P_{N_2-2}, \ \hat{E}(\hat{\phi}) = 0$
$D^2v, (v \in P_{K2})$	$\alpha = 2 ; \beta = 1$ $k = 2 ; \ell = 1 ; m = 0$ $\lvert\nu\rvert = 2 ; \lvert\mu\rvert = 0$ $\forall\hat{\phi} \in P_{N_2-2}, \ \hat{E}(\hat{\phi}) = 0$	$\alpha = 2 ; \beta = 1$ $k = 2 ; \ell = 1 ; m = 0$ $\lvert\nu\rvert = 2 ; \lvert\mu\rvert = 1$ $\forall\hat{\phi} \in P_{N_1+N_2-3}, \ \hat{E}(\hat{\phi}) = 0$	$\alpha = \beta = 2$ $k = \ell = 2 ; m = 0$ $\lvert\nu\rvert = 2 ; \lvert\mu\rvert = 0$ $\forall\hat{\phi} \in P_{N_2-2}, \ \hat{E}(\hat{\phi}) = 0$	$\alpha = \beta = 2$ $k = \ell = 2 ; m = 0$ $\lvert\nu\rvert = 2 ; \lvert\mu\rvert = 1$ $\forall\hat{\phi} \in P_{N_2-2}, \ \hat{E}(\hat{\phi}) = 0$	$\alpha = \beta = 2$ $k = \ell = 2 ; m = 0$ $\lvert\nu\rvert = \lvert\mu\rvert = 2$ $\forall\hat{\phi} \in P_{2N_2-4}, \ \hat{E}(\hat{\phi}) = 0$

Figure 6.1 : Estimation of terms $E_K[A_{IJ}(V_h)_I(V_h)_J]$, $I, J = 1, ..., 12$, with help of Theorem 5.1 (in each case we indicate the values of parameters $k, \ell, |\nu|, |\mu|$, defined in the statement of Theorem 5.1, as well as the criterion on the numerical integration scheme).

Then, if the extension $\vec{\tilde{u}}$ of the solution $\vec{u}$ of the problem 2.1 belongs to the space $H^{m+2}(\tilde{\Omega}) \times H^{m+2}(\tilde{\Omega}) \times H^{m+3}(\tilde{\Omega})$, if $H^{m+1+\alpha}(\tilde{\Omega}) \hookrightarrow C^{s_\alpha}(\tilde{\Omega}), \alpha = 1,2$, where s_α denotes the maximal order of the partial derivatives which appear in the definition of the degrees of freedom Σ_K, if $A_{IJ} \in W^{m+1,\infty}(\tilde{\Omega}), I, J = 1, ..., 12$, if $\tilde{A}\vec{\tilde{u}} \in (W^{m+1,q}(\tilde{\Omega}))^3$ for a number $q > \dfrac{2}{m+1}$ with $q \geq 2$, if h is small enough, there exists a constant c independent of h such that :

$$(7.6) \begin{cases} \|\vec{\tilde{u}} - \vec{u}_h\|_{\vec{X}(\Omega_h)} \\ \qquad \leq ch^{m+1} \left\{ \left[\sum_{\alpha=1}^{2} \|\tilde{u}_\alpha\|^2_{m+2,\tilde{\Omega}} + \|\tilde{u}_3\|^2_{m+3,\tilde{\Omega}} \right]^{1/2} + \left[\sum_{i=1}^{3} \|(\tilde{A}\vec{\tilde{u}})_i\|^q_{m+1,q,\tilde{\Omega}} \right]^{1/q} \right\}, \end{cases}$$

where $\vec{u}_h \in \vec{V}_h$ is the solution of the discrete problem (3.28).

Proof : Firstly let us observe that the hypotheses of Theorem 6.1 are satisfied so that the bilinear form $a_h(.,.)$ is uniformly $\vec{V}_h$-elliptic. Then, we can use Theorem 4.1. According to inequality (4.3), we estimate each of the terms of the right hand side of the inequality obtained for $\vec{v}_h = \vec{\pi}_h \vec{\tilde{u}}$.

Step 1 : Estimate of $\|\vec{\tilde{u}} - \vec{\pi}_h \vec{\tilde{u}}\|_{\vec{X}(\Omega_h)}$

The curved finite elements which are used to construct the spaces V_{h1} and V_{h2} and which are C^0 or C^1-compatible, have the same order of interpolation error than the corresponding straight finite elements. Then [Part 1, Theorems 4.1, 4.2 and 4.3] and the interpolation properties of the Hermite finite element of type (3), of the Argyris triangle [4] and of the Bell triangle [5] involve :

$$(7.7) \qquad \|\vec{\tilde{u}} - \vec{\pi}_h \vec{\tilde{u}}\|_{\vec{X}(\Omega_h)} \leq ch^{m+1} \left[\sum_{\alpha=1}^{2} \|\tilde{u}_\alpha\|^2_{m+2,\tilde{\Omega}} + \|\tilde{u}_3\|^2_{m+3,\tilde{\Omega}} \right]^{1/2}.$$

In this inequality $\vec{\pi}_h \vec{\tilde{u}}_h = (\pi_{h1}\tilde{u}_1, \pi_{h1}\tilde{u}_2, \pi_{h2}\tilde{u}_3)$ denotes the $\vec{V}_h$-interpolate function of $\vec{\tilde{u}} \in \vec{V}(\Omega_h)$.

Step 2 : Estimate of $\sup_{\vec{w}_h \in \vec{V}_h} \dfrac{|\tilde{a}_h(\vec{\pi}_h \vec{\tilde{u}}, \vec{w}_h) - a_h(\vec{\pi}_h \vec{\tilde{u}}, \vec{w}_h)|}{\|\vec{w}_h\|_{\vec{X}(\Omega_h)}}$

Relations (3.14) (3.23) and (3.26) imply

$$(7.8) \qquad \tilde{a}_h(\vec{\pi}_h \vec{\tilde{u}}, \vec{w}_h) - a_h(\vec{\pi}_h \vec{\tilde{u}}, \vec{w}_h) = \sum_{K \in \mathcal{T}_h} \sum_{I,J=1}^{12} |E_K\{A_{IJ}(\pi_h \tilde{U})_I (W_h)_J\}|,$$

where $\vec{w}_h$ denotes any element of the space $\vec{V}_h$. Then, Figure 7.1 shows that a suitable application of Theorem 5.1 to the different kinds of terms which appear at the right hand side of inequality (7.7), gives for any $K \in \mathcal{T}_h$:

$$\left\{\begin{array}{l}\displaystyle\sum_{I,J=1}^{12}|E_K\{A_{IJ}(\pi_h\tilde{U})_I(W_h)_J\}|\leq ch_K^{m+1}\left(\sum_{I,J=1}^{12}\|A_{IJ}\|_{m+1,\infty,K}\right)\\[2ex] \qquad\left(\|\pi_{h1}\tilde{u}_1\|^2_{H^{m+2}(K)}+\|\pi_{h1}\tilde{u}_2\|^2_{H^{m+2}(K)}+\|\pi_{h2}\tilde{u}_3\|^2_{H^{m+3}(K)}\right)^{1/2}\|\vec{w}_h\|_{\vec{V}(K)}.\end{array}\right.$$

But with [Part 1, Theorems 4.1, 4.2 and 4.3] we get

$$\left\{\begin{array}{l}\|\pi_{h1}\tilde{u}_\alpha\|_{m+2,K}\leq\|\tilde{u}_\alpha\|_{m+2,K}+\|\tilde{u}_\alpha-\pi_{h1}\tilde{u}_\alpha\|_{m+2,K}\leq c\|\tilde{u}_\alpha\|_{m+2,K},\quad\alpha=1,2,\\[1ex] \|\pi_{h2}\tilde{u}_3\|_{m+3,K}\leq\|\tilde{u}_3\|_{m+3,K}+\|\tilde{u}_3-\pi_{h2}\tilde{u}_3\|_{m+3,K}\leq c\|\tilde{u}_3\|_{m+3,K}.\end{array}\right.$$

Thanks to Cauchy-Schwarz inequality and to [Part 1, hypothesis (2.3)]

$$\left\{\begin{array}{l}|\tilde{a}_h(\vec{\pi}_h\vec{\tilde{u}},\vec{w}_h)-a_h(\vec{\pi}_h\vec{\tilde{u}},\vec{w}_h)|\leq ch^{m+1}\left(\displaystyle\sum_{I,J=1}^{12}\|A_{IJ}\|_{m+1,\infty,\tilde{\Omega}}\right)\\[2ex] \qquad\left[\displaystyle\sum_{\alpha=1}^{2}\|\tilde{u}_\alpha\|^2_{m+2,\tilde{\Omega}}+\|\tilde{u}_3\|^2_{m+3,\tilde{\Omega}}\right]^{1/2}\|\vec{w}_h\|_{\vec{X}(\Omega_h)}\end{array}\right.$$

and thus

$$(7.9)\qquad\sup_{\vec{w}_h\in\vec{V}_h}\frac{|\tilde{a}_h(\vec{\pi}_h\vec{\tilde{u}},\vec{w}_h)-a_h(\vec{\pi}_h\vec{\tilde{u}},\vec{w}_h)|}{\|\vec{w}_h\|_{\vec{X}(\Omega_h)}}\leq ch^{m+1}\left[\sum_{\alpha=1}^{2}\|\tilde{u}_\alpha\|^2_{m+2,\tilde{\Omega}}+\|\tilde{u}_3\|^2_{m+3,\tilde{\Omega}}\right]^{1/2}.$$

Step 3 : Estimate of $\displaystyle\sup_{\vec{w}_h\in\vec{V}_h}\frac{|\tilde{a}_h(\vec{\tilde{u}},\vec{w}_h)-f_h(\vec{w}_h)|}{\|\vec{w}_h\|_{\vec{X}(\Omega_h)}}$.

The relations (3.14) and (3.27) involve for any $\vec{w}_h\in\vec{V}_h$

$$(7.10)\qquad\tilde{a}_h(\vec{\tilde{u}},\vec{w}_h)-f_h(\vec{w}_h)=\int_{\Omega_h}{}^t\tilde{U}[A_{IJ}]W_h dx-\sum_{K\in\mathcal{T}_h}\sum_{\ell=1}^{L}\omega_{\ell,K}\,{}^tF(b_{\ell,K})W_h(b_{\ell,K}).$$

We write this relation in another way. Since $\vec{\tilde{u}}\in(H^2(\tilde{\Omega}))^2\times H^4(\tilde{\Omega})$, Green's formula implies the existence of a system of partial differential operators $\tilde{A}:(H^2(\tilde{\Omega}))^2\times H^4(\tilde{\Omega})\longrightarrow(L^2(\tilde{\Omega}))^3$ such that

$$\left\{\begin{array}{l}\forall\vec{v}\in(H^2(\tilde{\Omega}))^2\times H^4(\tilde{\Omega}),\ \ \forall\vec{w}\in(H^1(\tilde{\Omega}))^2\times H^2(\tilde{\Omega}),\\[1ex] \displaystyle\int_{\tilde{\Omega}}{}^tV[A_{IJ}]W dx=\int_{\tilde{\Omega}}(\tilde{A}\vec{v})\vec{w}dx+\int_{\partial\tilde{\Omega}}(\quad)ds.\end{array}\right.$$

$w_h = (W_h)_J$ $v_h = (\pi_h\tilde{U})_I$	$w_h \in P_{K1}$	$Dw_h, w_h \in P_{K1}$	$w_h \in P_{K2}$	$Dw_h, w_h \in P_{K2}$	$D^2w_h, w_h \in P_{K2}$
$v_h \in P_{K1}$	$\ell = 1\ ;\ \lvert\nu\rvert = \lvert\mu\rvert = 0$ $k = m+2$ $\forall\hat{\phi} \in P_{m-1+N_1},\ \hat{E}(\hat{\phi}) = 0$ if $1 \leq m$ $\forall\hat{\phi} \in P_m,\ \hat{E}(\hat{\phi}) = 0$	$\ell = 1\ ;\ \lvert\nu\rvert = 0\ ;\ \lvert\mu\rvert = 1$ $k = m+2$ $\forall\hat{\phi} \in P_{m-1+N_1},\ \hat{E}(\hat{\phi}) = 0$ $\forall\hat{\phi} \in P_m,\ \hat{E}(\hat{\phi}) = 0$	$\ell = 2\ ;\ \lvert\nu\rvert = \lvert\mu\rvert = 0$ $k = m+2$ $\forall\hat{\phi} \in P_{m-2+N_2},\ \hat{E}(\hat{\phi}) = 0$ if $2 \leq m$ $\forall\hat{\phi} \in P_m,\ \hat{E}(\hat{\phi}) = 0$	$\ell = 2\ ;\ \lvert\nu\rvert = 0\ ;\ \lvert\mu\rvert = 1$ $k = m+2$ $\forall\hat{\phi} \in P_{m-2+N_2},\ \hat{E}(\hat{\phi}) = 0$ if $1 \leq m$ $\forall\hat{\phi} \in P_m,\ \hat{E}(\hat{\phi}) = 0$	$\ell = 2\ ;\ \lvert\nu\rvert = 0\ ;\ \lvert\mu\rvert = 2$ $k = m+2$ $\forall\hat{\phi} \in P_{m-2+N_2},\ \hat{E}(\hat{\phi}) = 0$ $\forall\hat{\phi} \in P_m,\ \hat{E}(\hat{\phi}) = 0$
Dv_h, $v_h \in P_{K1}$	$\ell = 1\ ;\ \lvert\nu\rvert = 1\ ;\ \lvert\mu\rvert = 0$ $k = m+2$ $\forall\hat{\phi} \in P_{m-1+N_1},\ \hat{E}(\hat{\phi}) = 0$ if $1 \leq m$ $\forall\hat{\phi} \in P_m,\ \hat{E}(\hat{\phi}) = 0$	$\ell = 1\ ;\ \lvert\nu\rvert = \lvert\mu\rvert = 1$ $k = m+2$ $\forall\hat{\phi} \in P_{m-1+N_1},\ \hat{E}(\hat{\phi}) = 0$ $\forall\hat{\phi} \in P_m,\ \hat{E}(\hat{\phi}) = 0$	$\ell = 2\ ;\ \lvert\nu\rvert = 1\ ;\ \lvert\mu\rvert = 0$ $k = m+2$ $\forall\hat{\phi} \in P_{m-2+N_2},\ \hat{E}(\hat{\phi}) = 0$ if $2 \leq m$ $\forall\hat{\phi} \in P_m,\ \hat{E}(\hat{\phi}) = 0$	$\ell = 2\ ;\ \lvert\nu\rvert = \lvert\mu\rvert = 1$ $k = m+2$ $\forall\hat{\phi} \in P_{m-2+N_2},\ \hat{E}(\hat{\phi}) = 0$ if $1 \leq m$ $\forall\hat{\phi} \in P_m,\ \hat{E}(\hat{\phi}) = 0$	$\ell = 2\ ;\ \lvert\nu\rvert = 1\ ;\ \lvert\mu\rvert = 2$ $k = m+2$ $\forall\hat{\phi} \in P_{m-2+N_2},\ \hat{E}(\hat{\phi}) = 0$ $\forall\hat{\phi} \in P_m,\ \hat{E}(\hat{\phi}) = 0$
$v_h \in P_{K2}$	$\ell = 1\ ;\ \lvert\nu\rvert = \lvert\mu\rvert = 0$ $k = m+3$ $\forall\hat{\phi} \in P_{m-1+N_1},\ \hat{E}(\hat{\phi}) = 0$ if $1 \leq m$ $\forall\hat{\phi} \in P_m,\ \hat{E}(\hat{\phi}) = 0$	$\ell = 1\ ;\ \lvert\nu\rvert = 0\ ;\ \lvert\mu\rvert = 1$ $k = m+3$ $\forall\hat{\phi} \in P_{m-1+N_1},\ \hat{E}(\hat{\phi}) = 0$ $\forall\hat{\phi} \in P_m,\ \hat{E}(\hat{\phi}) = 0$	$\ell = 2\ ;\ \lvert\nu\rvert = \lvert\mu\rvert = 0$ $k = m+3$ $\forall\hat{\phi} \in P_{m-2+N_2},\ \hat{E}(\hat{\phi}) = 0$ if $2 \leq m$ $\forall\hat{\phi} \in P_m,\ \hat{E}(\hat{\phi}) = 0$	$\ell = 2\ ;\ \lvert\nu\rvert = 0\ ;\ \lvert\mu\rvert = 1$ $k = m+3$ $\forall\hat{\phi} \in P_{m-2+N_2},\ \hat{E}(\hat{\phi}) = 0$ if $1 \leq m$ $\forall\hat{\phi} \in P_m,\ \hat{E}(\hat{\phi}) = 0$	$\ell = 2\ ;\ \lvert\nu\rvert = 0\ ;\ \lvert\mu\rvert = 2$ $k = m+3$ $\forall\hat{\phi} \in P_{m-2+N_2},\ \hat{E}(\hat{\phi}) = 0$ $\forall\hat{\phi} \in P_m,\ \hat{E}(\hat{\phi}) = 0$
Dv_h, $v_h \in P_{K2}$	$\ell = 1\ ;\ \lvert\nu\rvert = 1\ ;\ \lvert\mu\rvert = 0$ $k = m+3$ $\forall\hat{\phi} \in P_{m-1+N_1},\ \hat{E}(\hat{\phi}) = 0$ if $1 \leq m$ $\forall\hat{\phi} \in P_m,\ \hat{E}(\hat{\phi}) = 0$	$\ell = 1\ ;\ \lvert\nu\rvert = \lvert\mu\rvert = 1$ $k = m+3$ $\forall\hat{\phi} \in P_{m-1+N_1},\ \hat{E}(\hat{\phi}) = 0$ $\forall\hat{\phi} \in P_m,\ \hat{E}(\hat{\phi}) = 0$	$\ell = 2\ ;\ \lvert\nu\rvert = 1\ ;\ \lvert\mu\rvert = 0$ $k = m+3$ $\forall\hat{\phi} \in P_{m-2+N_2},\ \hat{E}(\hat{\phi}) = 0$ if $2 \leq m$ $\forall\hat{\phi} \in P_m,\ \hat{E}(\hat{\phi}) = 0$	$\ell = 2\ ;\ \lvert\nu\rvert = \lvert\mu\rvert = 1$ $k = m+3$ $\forall\hat{\phi} \in P_{m-2+N_2},\ \hat{E}(\hat{\phi}) = 0$ if $1 \leq m$ $\forall\hat{\phi} \in P_m,\ \hat{E}(\hat{\phi}) = 0$	$\ell = 2\ ;\ \lvert\nu\rvert = 1\ ;\ \lvert\mu\rvert = 2$ $k = m+3$ $\forall\hat{\phi} \in P_{m-2+N_2},\ \hat{E}(\hat{\phi}) = 0$ $\forall\hat{\phi} \in P_m,\ \hat{E}(\hat{\phi}) = 0$
D^2v_h, $v_h \in P_{K2}$	$\ell = 1\ ;\ \lvert\nu\rvert = 2\ ;\ \lvert\mu\rvert = 0$ $k = m+3$ $\forall\hat{\phi} \in P_{m-1+N_1},\ \hat{E}(\hat{\phi}) = 0$ if $1 \leq m$ $\forall\hat{\phi} \in P_m,\ \hat{E}(\hat{\phi}) = 0$	$\ell = 1\ ;\ \lvert\nu\rvert = 2\ ;\ \lvert\mu\rvert = 1$ $k = m+3$ $\forall\hat{\phi} \in P_{m-1+N_1},\ \hat{E}(\hat{\phi}) = 0$ $\forall\hat{\phi} \in P_m,\ \hat{E}(\hat{\phi}) = 0$	$\ell = 2\ ;\ \lvert\nu\rvert = 2\ ;\ \lvert\mu\rvert = 0$ $k = m+3$ $\forall\hat{\phi} \in P_{m-2+N_2},\ \hat{E}(\hat{\phi}) = 0$ if $2 \leq m$ $\forall\hat{\phi} \in P_m,\ \hat{E}(\hat{\phi}) = 0$	$\ell = 2\ ;\ \lvert\nu\rvert = 2\ ;\ \lvert\mu\rvert = 1$ $k = m+3$ $\forall\hat{\phi} \in P_{m-2+N_2},\ \hat{E}(\hat{\phi}) = 0$ if $1 \leq m$ $\forall\hat{\phi} \in P_m,\ \hat{E}(\hat{\phi}) = 0$	$\ell = 2\ ;\ \lvert\nu\rvert = \lvert\mu\rvert = 2$ $k = m+3$ $\forall\hat{\phi} \in P_{m-2+N_2},\ \hat{E}(\hat{\phi}) = 0$ $\forall\hat{\phi} \in P_m,\ \hat{E}(\hat{\phi}) = 0$

Figure 7.1 : Estimate of terms $E_K\{A_{IJ}(\pi_h\tilde{U})_I(W_h)_J\}$ with the help of Theorem 5.1 ($m = -1 + min(m_1, m_2 - 1)$).

In particular, the restriction of this operator $\tilde{A}$ to the space $(H^2(\Omega_h))^2 \times H^4(\Omega_h)$, restriction which is independent of h and that we continue to note $\tilde{A}$, verifies

$$\begin{cases} \forall \vec{v} \in (H^2(\tilde{\Omega}))^2 \times H^4(\tilde{\Omega}), \ \forall \vec{w}_h \in \vec{V}_h \subset (H_0^1(\Omega_h))^2 \times H_0^2(\Omega_h), \\ \tilde{a}_h(\vec{v}, \vec{w}_h) = \int_{\Omega_h} {}^tV[A_{IJ}]W_h dx = \int_{\Omega_h} (\tilde{A}\vec{v})\vec{w}_h dx. \end{cases}$$

Analogously, we define the system of operators $A : (H^2(\Omega))^2 \times H^4(\Omega) \longrightarrow (L^2(\Omega))^3$ as follows :

$$\begin{cases} \forall \vec{v} \in (H^2(\Omega))^2 \times H^4(\Omega), \ \forall \vec{w} \in \vec{V} = (H_0^1(\Omega))^2 \times H_0^2(\Omega), \\ a(\vec{v}, \vec{w}) = \int_{\Omega} {}^tV[A_{IJ}]W dx = \int_{\Omega} (A\vec{v})\vec{w} dx. \end{cases}$$

Thus, since $\tilde{\vec{u}}$ is an extension of $\vec{u}$ such that $\tilde{\vec{u}} \in (H^2(\tilde{\Omega}))^2 \times H^4(\tilde{\Omega})$, we obtain

$$\tilde{A}\tilde{\vec{u}} = A\vec{u} = \vec{f} \quad \text{a.e. on } \bar{\Omega},$$

where $\vec{p}$ is defined here by $f(\vec{v}) = \int_{\Omega} \vec{p}\vec{v} dx$ (cf. (2.3)). But by hypothesis $\tilde{A}\tilde{\vec{u}} \in (W^{m+1,q}(\tilde{\Omega}))^3$ for a certain number $q > \frac{2}{m+1}$. Thanks to Sobolev's theorem, we have $\tilde{A}\tilde{\vec{u}} \in (\mathcal{C}^0(\tilde{\Omega}))^3$. Moreover, we have assumed $\vec{p} \in (\mathcal{C}^0(\bar{\Omega}))^3$. Since all the integration nodes belong to $\bar{\Omega}$ (see (7.2)), we obtain :

$$\tilde{A}\tilde{\vec{u}}(b_{\ell,K}) = \vec{p}(b_{\ell,K}), \ \forall b_{\ell,K}, \ \forall K \in \mathcal{T}_h.$$

Thus, the relation (7.9) can be written

$$\tilde{a}_h(\tilde{\vec{u}}, \vec{w}_h) - f_h(\vec{w}_h) = \int_{\Omega_h} (\tilde{A}\tilde{\vec{u}})\vec{w}_h dx - \sum_{K \in \mathcal{T}_h} \sum_{\ell=1}^{L} \omega_{\ell,K}((\tilde{A}\tilde{\vec{u}})\vec{w}_h)(b_{\ell,K}) = \sum_{K \in \mathcal{T}_h} E_K((\tilde{A}\tilde{\vec{u}})\vec{w}_h).$$

Let us set

$$(\tilde{A}\tilde{\vec{u}})\vec{w}_h = \sum_{i=1}^{3} (\tilde{A}\tilde{\vec{u}})_i w_{hi}.$$

Figure 7.2 shows that a suitable application of Theorem 5.2 to both kinds of terms, involves :

$$\begin{aligned} |\tilde{a}_h(\tilde{\vec{u}}, \vec{w}_h) - f_h(\vec{w}_h)| &\le \sum_{K \in \mathcal{T}_h} |E_K((\tilde{A}\tilde{\vec{u}})\vec{w}_h| \\ &\le ch^{m+1} \sum_{K \in \mathcal{T}_h} (measure(K))^{1/2-1/q} \left(\sum_{i=1}^{3} \|(\tilde{A}\tilde{\vec{u}})_i\|_{m+1,q,K}^q \right)^{1/q} \|\vec{w}_h\|_{\vec{X}(K)} \\ &\le ch^{m+1} (measure(\tilde{\Omega}))^{1/2-1/q} \left(\sum_{i=1}^{3} \|(\tilde{A}\tilde{\vec{u}})_i\|_{m+1,q,\tilde{\Omega}}^q \right)^{1/q} \|\vec{w}_h\|_{\vec{X}(\Omega_h)}. \end{aligned}$$

1) $E_K((\tilde{A}\vec{\tilde{u}})_\alpha w_{h\alpha})$, $(\tilde{A}\vec{\tilde{u}})_\alpha \in W^{m+1,q}(K)$, $w_{h\alpha} \in P_{K1}$, $\alpha = 1, 2$.

Then $\ell = 1$, and we need

$$\begin{cases} \forall \hat{\phi} \in P_{m-1+N_1}(\hat{K}), \ \hat{E}(\hat{\phi}) = 0, \ \text{if } 1 \leq m, \\ \forall \hat{\phi} \in P_m(\hat{K}), \ \hat{E}(\hat{\phi}) = 0; \end{cases}$$

2) $E_K((\tilde{A}\vec{\tilde{u}})_3 w_{h3})$, $(\tilde{A}\vec{\tilde{u}})_3 \in W^{m+1,q}(K)$, $w_{h3} \in P_{K2}$.

Then $\ell = 2$, and we need

$$\begin{cases} \forall \hat{\phi} \in P_{m-2+N_2}(\hat{K}), \ \hat{E}(\hat{\phi}) = 0, \ \text{if } 2 \leq m, \\ \forall \hat{\phi} \in P_m(\hat{K}), \ \hat{E}(\hat{\phi}) = 0. \end{cases}$$

Figure 7.2 : Estimation of $E_K((\tilde{A}\vec{\tilde{u}})\vec{w}_h)$ with the help of Theorem 5.2 $(m = -1 + min(m_1, m_2 - 1))$

In order to set up the last inequality, we use arguments similar to those of the proof of [1, Theorem 4.5], in the course of which we have assumed $q \geq 2$. Thus, we obtain

$$\text{(7.11)} \qquad \sup_{\vec{w}_h \in \vec{V}_h} \frac{|\tilde{a}_h(\vec{\tilde{u}}, \vec{w}_h) - f_h(\vec{w}_h)|}{\|\vec{w}_h\|_{\vec{X}(\Omega_h)}} \leq ch^{m+1} \left(\sum_{i=1}^{3} \|(\tilde{A}\vec{\tilde{u}})_i\|^q_{m+1,q,\tilde{\Omega}} \right)^{1/q}.$$

Step 4 : Final estimate (7.5)

To obtain the result of asymptotic error estimate (7.5), it suffices to substitute estimates (7.6) (7.8) and (7.10) into the inequality (4.3), written for $\vec{v}_h = \vec{\pi}_h \vec{\tilde{u}}$. ■

8 EXAMPLES (Case of plate bending problems)

When the thin shell is reduced to a plate, i.e., when the middle surface of the shell is plane, there are some important simplifications :

(i) For the linear plate theory, membrane and bending deformations are uncoupled so that we can restrict our attention only to the bending plate problem. The membrane deformations are modelized by two-dimensional elasticity equations. These equations are of the second order and they can be approximated according to [14,15] ;

(ii) Theorem 7.1 remains valid but since we restrict our attention to the bending problem we have just to consider the finite element space V_{h2}. That means instead of $m = -1 + min(m_1, m_2 - 1)$, we take now $m = m_2 - 2$ and condition (7.3) can be simplified as $\forall\hat{\phi} \in P_{2N_2-4}$, $\hat{E}(\hat{\phi}) = 0$.

Thus, when the shell becomes a plate and when we restrict our attention to the bending of this plate, the asymptotic error estimate results are summarized in Figure 8.1.

9 EXAMPLES (Case of general thin shell problems)

Now we illustrate the above results in case of general thin shell problems when the spaces V_{h1} and V_{h2} are constructed from ARGYRIS, BELL or cubic Hermite finite elements combined with their associate curved versions. For each finite element space $\vec{V}_h = V_{h1} \times V_{h1} \times V_{h2}$, there are two different sets of criteria to be observed when choosing the numerical integration schemes :

(i) the first set of criteria concerns the straight finite elements which are associated to triangles $K \in \mathcal{T}_h^1$; then we find back the criteria previously stated in [1] ;

(ii) the second set of criteria concerns the curved finite elements which are associated to curved triangles $K \in \mathcal{T}_h^2$. Then, as a general rule, the degree of the polynomials we have to exactly integrate is higher than in case (i) : we have to add $2n-2$ to the degree obtained in (i), where the integer n comes from $F_K \in (P_n)^2$. Indeed the degree of the polynomials of the functional space $\hat{P}$ is increased by $n-1$ for the curved $\mathcal{C}^1$-compatible finite elements.

The different results are summarized in Figures 9.1 to 9.3 and are obtained by applying Theorem 7.1. In each case, we indicate :
(i) the value of $m = -1 + min(m_1, m_2 - 1)$;
(ii) the result of asymptotic error estimate as $0(h^{m+1})$;
(iii) the criterion on the numerical integration scheme ;
(vi) the reference of a suitable numerical integration scheme ;
(v) the regularity hypotheses on coefficients A_{IJ}, on $\tilde{A}\tilde{\vec{u}}$, and on an extension $\tilde{\vec{u}}$ of the solution $\vec{u}$.

10 CONCLUDING REMARKS

These curved finite element methods lead to approximations with very high degree of accuracy. They can be used to approximate plate or shell problems set on reference domains with curved boundary.

Their implementation, which is technical enough, is thoroughly detailed in [6]. In practice the user will be able to use them as "black box".

Concerning computing time, these methods are a little more expensive than the corresponding ones just using straight finite elements. But it is worth to note

i) when we have curved boundaries, we have to approximate these boundaries by using appropriate curved elements, otherwise the degree of accuracy decreases ;

ii) the computation of the rigidity matrix attached to a $\mathcal{C}^1$-curved element is more expensive since we use higher polynomial spaces and notably more numerical integration nodes. But we have to remember that a mesh including $O(N)$ triangles has $0(\sqrt{N})$ boundary triangles so that the additional computing time is not very important ;

iii) for the approximation of thin shell problems formulated on polygonal reference domains, we show in [20] that the combination of Argyris triangles (for the bending component) and of Ganev triangles [21] (for the membrane components) allow to save about 40% of computing time. These Ganev triangles are $\mathcal{C}^0$-elements and they use complete P_4-polynomials ; when combined with Argyris triangles, they require the use of numerical integration scheme exact for polynomials of degree 6 instead of 8 as reported in Figure 9.1. In this direction, it should be interesting to develop a curved version of such elements in the same spirit than for the $\mathcal{C}^0$-compatible elements introduced in [Part 1, §3.4].

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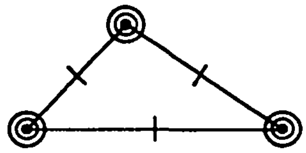
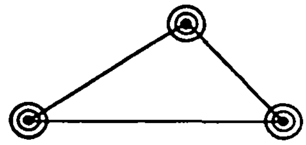
Straight or curved finite element used to approximate u_3	 ARGYRIS triangle ($m_2 = n_2 = 5$; $M_2 = N_2 = n + 4$)	 BELL triangle ($m_2 = 4$; $n_2 = 5$; $M_2 = n + 3$; $N_2 = n + 4$)
Error estimate	$m = 3$; $0(h^4)$	$m = 2$; $0(h^3)$
regularity assumptions	$A_{IJ} \in W^{4,\infty}(\tilde{\Omega})$, $6 \leq I, J \leq 12$ $\tilde{A}\tilde{u}_3 \in W^{4,q}(\tilde{\Omega})$, $q \geq 2$ $\tilde{u}_3 \in H^6(\tilde{\Omega})$.	$A_{IJ} \in W^{3,\infty}(\tilde{\Omega})$, $6 \leq I, J \leq 12$ $\tilde{A}\tilde{u}_3 \in W^{3,q}(\tilde{\Omega})$, $q \geq 2$ $\tilde{u}_3 \in H^5(\tilde{\Omega})$.
a) Case of a straight triangle ($K \in \mathcal{T}_h^1$, $n = 1$)	$M_2 = N_2 = 5$ $\forall \hat{\phi} \in P_6$, $\hat{E}(\hat{\phi}) = 0$ Scheme with 12 nodes [16,17]	$M_2 = 4$; $N_2 = 5$ $\forall \hat{\phi} \in P_6$, $\hat{E}(\hat{\phi}) = 0$ Scheme with 12 nodes [16,17]
b) Case of a curved triangle and clamped conditions ($K \in \mathcal{T}_h^2$, $n = 3$)	$M_2 = N_2 = 7$ $\forall \hat{\phi} \in P_{10}$, $\hat{E}(\hat{\phi}) = 0$ Scheme with 25 nodes [18,19]	$M_2 = 6$; $N_2 = 7$ $\forall \hat{\phi} \in P_{10}$, $\hat{E}(\hat{\phi}) = 0$ Scheme with 25 nodes [18,19]
c) Case of a curved triangle and general boundary conditions ($K \in \mathcal{T}_h^2$, $n = 5$)	$M_2 = N_2 = 9$ $\forall \hat{\phi} \in P_{14}$, $\hat{E}(\hat{\phi}) = 0$ Scheme with 42 nodes [18]	$M_2 = 8$; $N_2 = 9$ $\forall \hat{\phi} \in P_{14}$, $\hat{E}(\hat{\phi}) = 0$ Scheme with 42 nodes [18]

Figure 8.1 : Case of a plate bending ; examples of error estimates and criteria on numerical integration schemes depending on the type of triangle (straight or curved ($n = 3$ or $n = 5$))

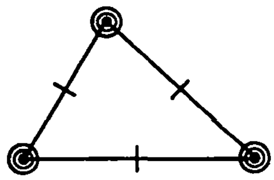
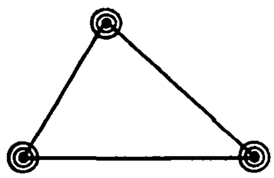
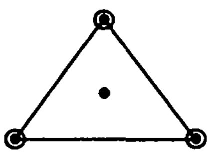
Finite elements used to construct V_{h2} / Finite elements used to construct V_{h1}	ARGYRIS element $n = 1$; $m_2 = n_2 = 5$ $M_2 = N_2 = 5$ 	BELL element $n = 1$; $m_2 = 4$; $n_2 = 5$ $M_2 = 4$; $N_2 = 5$ 
ARGYRIS element $n = 1$ $m_1 = n_1 = 5$ $M_1 = N_1 = 5$	$m = 3$; $0(h^4)$, $\forall \hat{\phi} \in P_8$, $\hat{E}(\hat{\phi}) = 0$, Scheme with 16 nodes [17] $A_{IJ} \in W^{4,\infty}(\tilde{\Omega})$, $\tilde{A}\tilde{\vec{u}} \in (W^{4,q}(\tilde{\Omega}))^3$, $q \geq 2$, $\tilde{\vec{u}} \in (H^5(\tilde{\Omega}))^2 \times H^6(\tilde{\Omega})$.	
BELL element $n = 1$ $m_1 = 4$; $n_1 = 5$ $M_1 = 4$; $N_1 = 5$		$m = 2$; $0(h^3)$, $\forall \hat{\phi} \in P_8$, $\hat{E}(\hat{\phi}) = 0$, Scheme with 16 nodes [17] $A_{IJ} \in W^{3,\infty}(\tilde{\Omega})$, $\tilde{A}\tilde{\vec{u}} \in (W^{3,q}(\tilde{\Omega}))^3$, $q \geq 2$, $\tilde{\vec{u}} \in (H^4(\tilde{\Omega}))^2 \times H^5(\tilde{\Omega})$.
P_3-HERMITE element $n = 1$; $m_1 = n_1 = 3$ $M_1 = N_1 = 3$ 	$m = 2$; $0(h^3)$, $\forall \hat{\phi} \in P_6$, $\hat{E}(\hat{\phi}) = 0$, Scheme with 12 nodes [16,17] $A_{IJ} \in W^{3,\infty}(\tilde{\Omega})$, $\tilde{A}\tilde{\vec{u}} \in (W^{3,q}(\tilde{\Omega}))^3$, $q \geq 2$, $\tilde{\vec{u}} \in (H^4(\tilde{\Omega}))^2 \times H^5(\tilde{\Omega})$.	$m = 2$; $0(h^3)$, $\forall \hat{\phi} \in P_6$, $\hat{E}(\hat{\phi}) = 0$, Scheme with 12 nodes [16,17] $A_{IJ} \in W^{3,\infty}(\tilde{\Omega})$, $\tilde{A}\tilde{\vec{u}} \in (W^{3,q}(\tilde{\Omega}))^3$, $q \geq 2$, $\tilde{\vec{u}} \in (H^4(\tilde{\Omega}))^2 \times H^5(\tilde{\Omega})$.

Figure 9.1 : Case of straight finite elements, i.e., $K \in \mathcal{T}_h^1$, $n = 1$.
(In each case, we successively indicate : (i) $m = -1 + min(m_1, m_2 - 1)$;
(ii) error estimate $0(h^{m+1})$;
(iii) criterion on the numerical integration scheme ;
(iv) references of appropriate numerical integration schemes ;
(v) regularity hypotheses upon $A_{IJ}, \tilde{A}\tilde{\vec{u}}, \tilde{\vec{u}}$).

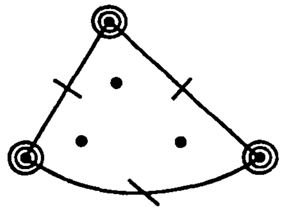
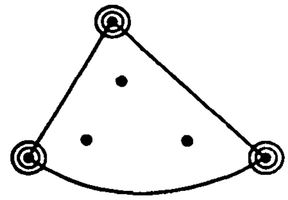
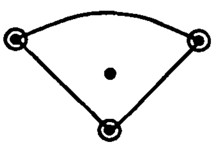
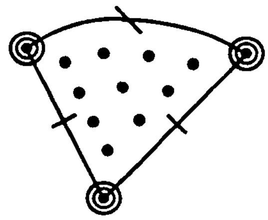
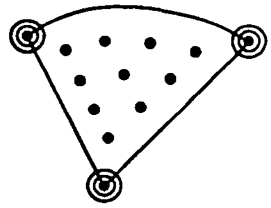
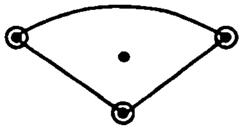
Finite elements used to construct V_{h2} / Finite elements used to construct V_{h1}	Curved C^1-compatible element with ARGYRIS triangle $n = 3$; $m_2 = n_2 = 5$ $M_2 = N_2 = 7$ 	Curved C^1-compatible element with BELL triangle $n = 3$; $m_2 = 4$; $n_2 = 5$ $M_2 = 6$; $N_2 = 7$ 
Curved C^1-compatible element with ARGYRIS triangle $n = 3$ $m_1 = n_1 = 5$ $M_1 = N_1 = 7$	$m = 3$; $0(h^4)$, $\forall \hat{\phi} \in P_{12}$, $\hat{E}(\hat{\phi}) = 0$, Scheme with 33 nodes [18] $A_{IJ} \in W^{4,\infty}(\tilde{\Omega})$, $\tilde{A}\tilde{\vec{u}} \in (W^{4,q}(\tilde{\Omega}))^3$, $q \geq 2$, $\tilde{\vec{u}} \in (H^5(\tilde{\Omega}))^2 \times H^6(\tilde{\Omega})$.	
Curved C^1-compatible element with BELL triangle $n = 3$ $m_1 = 4$; $n_1 = 5$ $M_1 = 6$; $N_1 = 7$		$m = 2$; $0(h^3)$, $\forall \hat{\phi} \in P_{12}$, $\hat{E}(\hat{\phi}) = 0$, Scheme with 33 nodes [18] $A_{IJ} \in W^{3,\infty}(\tilde{\Omega})$, $\tilde{A}\tilde{\vec{u}} \in (W^{3,q}(\tilde{\Omega}))^3$, $q \geq 2$, $\tilde{\vec{u}} \in (H^4(\tilde{\Omega}))^2 \times H^5(\tilde{\Omega})$.
Curved C^0-compatible element with cubic HERMITE element $n = m_1 = M_1 = n_1 = N_1 = 3$ 	$m = 2$; $0(h^3)$, $\forall \hat{\phi} \in P_{10}$, $\hat{E}(\hat{\phi}) = 0$, Scheme with 25 nodes [18,19] $A_{IJ} \in W^{3,\infty}(\tilde{\Omega})$, $\tilde{A}\tilde{\vec{u}} \in (W^{3,q}(\tilde{\Omega}))^3$, $q \geq 2$, $\tilde{\vec{u}} \in (H^4(\tilde{\Omega}))^2 \times H^5(\tilde{\Omega})$.	$m = 2$; $0(h^3)$, $\forall \hat{\phi} \in P_{10}$, $\hat{E}(\hat{\phi}) = 0$, Scheme with 25 nodes [18,19] $A_{IJ} \in W^{3,\infty}(\tilde{\Omega})$, $\tilde{A}\tilde{\vec{u}} \in (W^{3,q}(\tilde{\Omega}))^3$, $q \geq 2$, $\tilde{\vec{u}} \in (H^4(\tilde{\Omega}))^2 \times H^5(\tilde{\Omega})$.

Figure 9.2 : Case of curved finite elements, i.e., $K \in \mathcal{T}_h^2$, $n = 3$.
These elements can be used in case of homogeneous boundary conditions of Dirichlet type.
(we indicate successively : (i) $m = -1 + \min(m_1, m_2 - 1)$; (ii) error estimate $0(h^{m+1})$;
(iii) criterion on the numerical integration scheme ;
(iv) references of appropriate numerical integration schemes ;
(v) regularity hypotheses upon $A_{IJ}, \tilde{A}\tilde{\vec{u}}$, $\tilde{\vec{u}}$).

Finite elements used to construct V_{h2} / Finite elements used to construct V_{h1}	Curved C^1-compatible element with ARGYRIS triangle $n = 5$; $m_2 = n_2 = 5$ $M_2 = N_2 = 9$ 	Curved C^1-compatible element with BELL triangle $n = 5$; $m_2 = 4$; $n_2 = 5$ $M_2 = 8$; $N_2 = 9$ 
Curved C^1-compatible element with ARGYRIS triangle $n = 5$ $m_1 = n_1 = 5$ $M_1 = N_1 = 9$	$m = 3$; $0(h^4)$, $\forall \hat{\phi} \in P_{16}$, $\hat{E}(\hat{\phi}) = 0$, Scheme with 52 nodes [18] (*) $A_{IJ} \in W^{4,\infty}(\tilde{\Omega})$, $\tilde{A}\tilde{\vec{u}} \in (W^{4,q}(\tilde{\Omega}))^3$, $q \geq 2$, $\tilde{\vec{u}} \in (H^5(\tilde{\Omega}))^2 \times H^6(\tilde{\Omega})$.	
Curved C^1-compatible element with BELL triangle $n = 5$ $m_1 = 4$; $n_1 = 5$ $M_1 = 8$; $N_1 = 9$		$m = 2$; $0(h^3)$, $\forall \hat{\phi} \in P_{16}$, $\hat{E}(\hat{\phi}) = 0$, Scheme with 52 nodes [18] (*) $A_{IJ} \in W^{3,\infty}(\tilde{\Omega})$, $\tilde{A}\tilde{\vec{u}} \in (W^{3,q}(\tilde{\Omega}))^3$, $q \geq 2$, $\tilde{\vec{u}} \in (H^4(\tilde{\Omega}))^2 \times H^5(\tilde{\Omega})$.
Curved C^0-compatible element with cubic HERMITE element $n = 5$ $m_1 = M_1 = n_1 = N_1 = 3$ 	$m = 2$; $0(h^3)$, $\forall \hat{\phi} \in P_{14}$, $\hat{E}(\hat{\phi}) = 0$, Scheme with 42 nodes [18] $A_{IJ} \in W^{3,\infty}(\tilde{\Omega})$, $\tilde{A}\tilde{\vec{u}} \in (W^{3,q}(\tilde{\Omega}))^3$, $q \geq 2$, $\tilde{\vec{u}} \in (H^4(\tilde{\Omega}))^2 \times H^5(\tilde{\Omega})$.	$m = 2$; $0(h^3)$, $\forall \hat{\phi} \in P_{14}$, $\hat{E}(\hat{\phi}) = 0$, Scheme with 42 nodes [18] $A_{IJ} \in W^{3,\infty}(\tilde{\Omega})$, $\tilde{A}\tilde{\vec{u}} \in (W^{3,q}(\tilde{\Omega}))^3$, $q \geq 2$, $\tilde{\vec{u}} \in (H^4(\tilde{\Omega}))^2 \times H^5(\tilde{\Omega})$.

(*) This scheme has 3 nodes outside of the triangles. In order to respect hypothesis 3.1, we can use the scheme exact for polynomials of degree 17 involving 61 nodes [18].

Figure 9.3 : Case of curved finite elements, i.e., $K \in \mathcal{T}_h^2$, $n = 5$. (we indicate successively : (i) $m = -1 + \min(m_1, m_2 - 1)$; (ii) error estimate $0(h^{m+1})$; (iii) criterion on the numerical integration scheme ; (iv) references of appropriate numerical integration schemes ; (v) regularity hypotheses upon $A_{IJ}, \tilde{A}\tilde{\vec{u}}, \tilde{\vec{u}}$).

References

[1] M. BERNADOU, Convergence of conforming finite element methods for general shell problems, Internat. J. Engng. Sci., 18 (1980), pp. 249-276.

[2] W.T. KOITER, On the foundations of the linear theory of thin elastic shells. Proc. Kon. Ned. Akad. Wetensch. B73 (1970), pp. 169-195.

[3] M. BERNADOU, $\mathcal{C}^1$-curved finite elements with numerical integration for thin plate and thin shell problems. Part 1 : construction and interpolation properties of curved $\mathcal{C}^1$ finite elements (to appear).

[4] J.H. ARGYRIS, I. FRIED, D.W. SCHARPF, The TUBA family of plate elements for the matrix displacement method, The Aeronautical Journal of the Royal Aeronautical Society, 72 (1968), pp.701-709.

[5] K. BELL, A refined triangular plate bending element, Internat. J. Numer. Methods. Engrg., 1 (1969), pp. 101-122.

[6] M. BERNADOU, J.M. BOISSERIE, Curved finite elements of class $\mathcal{C}^1$: Implementation and numerical experiments (to appear).

[7] M. BERNADOU, P.G. CIARLET, Sur l'ellipticité du modèle linéaire de coques de W.T. Koiter. In "Computing Methods in Applied Sciences and Engineering" (R. GLOWINSKI and J.L. LIONS, Editors), Lectures notes in Economics and Mathematical Systems, Vol. 134, pp. 89-136, Springer Verlag, Berlin, 1976.

[8] M. BERNADOU, J.M. BOISSERIE, The Finite Element Method in Thin Shell Theory : Application to Arch Dam Simulations, Birkhäuser, Boston, 1982.

[9] P.G. CIARLET, Conforming finite element methods for thin shell problems. In : Mathematics of Finite Elements and Applications II (J.R. Whiteman, Editor), Academic Press, London, pp. 105-123, 1976.

[10] A. ZENISEK, Nonhomogeneous boundary conditions and curved triangular finite elements. Apl. Mat. 26 (1981), pp. 121-141.

[11] Discrete forms of Friedrich's inequalities in the finite element method. RAIRO Analyse Numérique, Vol. 15, n^o 3 (1981), pp. 265-286.

[12] K. YOSIDA, Functional Analysis. Grundleheren B.123, 2nd Edition, Springer, Berlin, 1968.

[13] J.H. BRAMBLE, S.R. HILBERT, Estimation of linear functionals on Sobolev spaces with application to Fourier transforms and spline interpolation, SIAM J. Numer. Anal. 7, n^0 1 (1970), pp. 112-124.

[14] P.G. CIARLET, The Finite Element Method for Elliptic Problems. North-Holland, Amsterdam, 1978.

[15] P.G. CIARLET, Basic Error Estimates for Elliptic Problems. In : Handbook of Numerical Analysis (P.G. CIARLET and J.L. LIONS editors). Vol. II : Finite Element Methods, Part 1, pp. 17-351, North-Holland, Amsterdam, 1991.

[16] G.R. COWPER, Gaussian quadrature formulas for triangles . Internat. J. Numer. Methods Engrg., 7, n^0 3, (1973), pp. 405-408.

[17] J.N. LYNESS, D. JESPERSEN, Moderate degree symmetric quadrature rules for the triangle. J. Inst. Maths. Applics., 15 (1975), pp. 19-32.

[18] D.A. DUNAVANT, High degree efficient symmetrical gaussian quadrature rules for the triangle, Internat. J. Numer. Methods Engrg., 21 (1985), pp. 1129-1148.

[19] M.E. LAURSEN, M. GELLERT, Some criteria for numerically integrated matrices and quadrature formulas for triangles. Internat. J. Numer. Methods Engrg., 12 (1978), pp. 67-76.

[20] M. BERNADOU, A. HASSIM, F. PALMA, Approximation of general thin shell problems in an Argyris-Ganev finite element space (to appear).

[21] H.G. GANEV, Tch. T. DIMITROV, Calculation of arch dams as a shell using IBM-370 Computer and curved finite elements. In "Theory of Shells", W.T. Koiter and G.K. Mikhailov Eds., North-Holland Pub. Comp., 1980, pp. 691-696.

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