On the unity of logic

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Rapports de Recherche

N° 1467

Programme 2
Calcul Symbolique, Programmation
et Génie logiciel

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Juin 1991

RR-1467

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We present a single sequent calculus common to classical, intuitionistic and linear logics. The main novelty is that classical, intuitionistic and linear logics appear as fragments, i.e. as particular classes of formulas and sequents. For instance a proof of an intuitionistic formula A may use classical or linear lemmas without any restriction : but after cut-elimination the proof of A is wholly intuitionistic, what is superficially achieved by the subformula property (only intuitionistic formulas are used) and more deeply by a very careful treatment of structural rules. This approach is radically different from the one that consists in "changing the rule of the game" when we want to change logic, e.g. pass from one style of sequent to another : here there is only one logic, which -depending on its use- may appear classical, intuitionistic or linear.

DE L'UNITE DE LA LOGIQUE

Nous présentons un calcul des séquents unifié, commun aux logiques classique, intuitionniste et linéaire. La principale nouveauté est que les logiques classique, intuitionniste et linéaire apparaissent comme des fragments, c'est à dire comme des classes particulières de formules et de séquents. Par exemple la démonstration d'un énoncé intutionniste pourra utiliser des lemmes classiques ou intuitionnistes sans limitation : simplement après élimination des coupures, la démonstration se fera entièrement dans le fragment intuitionniste, ce qui est superficiellement assuré par la propriété de la sous-formule (seulement des formules intuitionnistes sont utilisées) et plus profondément par un traitement très rigoureux des règles structurelles. Cette approche est radicalement différente de l'approche habituelle qui consiste tout bonnement à changer la règle du jeu quand on veut changer de logique, c'est à dire de style de séquent : ici il n'y a plus qu'une seule logique, qui au gré des utilisations peut apparaître classique, intuitionniste ou linéaire.

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By the turn of the century the situation concerning logic was quite simple : there was basically one logic (classical logic) which could be used (by changing the set of proper axioms) in various situations. Logic was about *pure* reasoning. Brouwer's criticism destroyed this dream of unity : classical logic was not adapted to constructive features and therefore lost its universality. By the end of the century we are now faced with an incredible number of logics –some of them only named "logics" by antiphrasis, some of them introduced on serious grounds–. Is still logic about pure reasoning ? In other terms, could there be a way to reunify logical systems –let us say those systems with a good sequent calculus– into a single sequent calculus ? Could we handle the (legitimate) distinction classical/intuitionistic not through a change of system, but through a change of formulas ? Is it possible to obtain classical effects by restricting one to classical formulas ? etc.

Of course there are surely ways to achieve this by cheating, typically by considering a disjoint union of systems... all these jokes will be made impossible if we insist on the fact that that the various systems represented should freely communicate (and for instance a classical theorem could have an intuitionistic corollary and *vice versa*).

In the unified calculus LU that we present below, classical, linear and intuitionistic logics appear as *fragments*. This means that one can define notions of *classical*, *intuitionistic* or *linear* sequents and prove that a cut–free proof of a sequent in one of these fragments is wholly inside the fragment ; of course a proof with cuts has the right to use arbitrary sequents, i.e. the fragments can freely communicate.

1. unified sequents

Standard sequent calculi essentially differ by their different maintenances of sequents :
i) classical logic accepts weakening and contraction on both sides
ii) intuitionistic (minimal) logic restricts the succedent to one formula –which has the effect of forbidding weakening and contraction to the right.
iii) linear logic refuses both, but has special connectives ! and ? which –when they prefix a formula–, allow structural rules on the left (!) and on the right (?).

Our basic unifying idea will be to define two zones in a sequent : a zone with a "classical" maintenance, and a zone with a linear maintenance ; there will be no zone with an intuitionisitic maintenance : intuitionisitic maintenance, i.e. "one formula on the right", will result from a careful linear maintenance. Typically we could use a notation $\Gamma ; \Gamma' \vdash \Delta' ; \Delta$ to indicate that Γ' and Δ' behave classically whereas Γ and Δ behave linearily. We could try to identify classical sequents with those where Γ and Δ are empty, and intuitionistic ones as those in which Γ and Δ' are empty, Δ consisting of one formula. This is roughly what will happen, with some difficulties and some surprises :

i) it must be possible to pass on both sides of the semi-column : surely one should be able to enter the central zone (we lose information), and also -with some constraint otherwise the semi-column would lose its interest- to move to the extremes. One of these constraints could be the addition of a symbol, e.g. move A from Γ' to Γ, but write it now as !A.

ii) this is not quite satisfactory, typically a formula already starting with "!" should be able to do it freely... it immediately turns out that those guys that can cross the left semi-column in both ways are closed under the linear connectives $\otimes$ and $\oplus$ and under the quantifier $\vee x$. The sensible thing to do is therefore to distinguish among formulas *positive* ones, including positive atomic formulas for problems of substitution. Symmetrically one distinguishes *negative* formulas... the remaining ones are called *neutral*: those ones must pay at both borders.

iii) the restatement of the rules of linear logic in this wider context is unproblematic and rather satisfactory, especially the treatment of "!" and "?" becomes slightly smoother.

iv) we have now define three *polarities* (classes of formulas) and we can toy with the connectives of linear logic to define synthetic connectives, built like *chimeras*, with a head of $\otimes$, a tail of &, etc. ; only good taste limits the possibilities. Typically if we want to define a conjunction we would like it to be associative (at the level of provability, but moreover at the level of denotational semantics) hence this imposes some coordination between the various parts of our chimera. In fact the connectives built have been chosen on two constraints :

- limitation of the number of connectives : for instance only one conjunction, only one disjunction, for classical and intuitionistic logics, but unfortunately two distinct implications for these logics

- maximisation of the number of remarkable isomorphisms

v) as far as classical logic is concerned, the results presented here are consistent with the previous work of the author [G2] ; in fact classical logic is obtained by limitation to formulas which are (hereditarily) non-neutral. What plays the role of classical sequents are the sequents of the form $\Gamma ; \Gamma' \vdash \Delta' ; \Delta$ when the non-permeable part of Γ, Δ consists of at most one formula (the *stoup* of [G2]). The reader is referred to this paper to check the extreme number of isomorphisms satisifed by the classical fragment (some of them, typically the De Morgan duality between $\wedge$ and $\vee$ do not extend to neutral polarities). Only one small defect : a single formula A is interpreted by $; \vdash A ;$ whereas for the other logics, it is interpreted by $; \vdash ; A$ ... however, if A is negative (right permeable) we can replace $; \vdash A ;$ with $; \vdash ; A$, and if A is positive we can replace $; \vdash A ;$ with $; \vdash ; \forall x A$ (x dummy) or $; \vdash ; A \vee (\neg V)$...

vi) as far as disjunction, existence and negation are ignored, intuitionistic logic is a quite even system in proof-theoretic terms, as shown by various relations to λ-calculus. The *neutral intuitionistic* fragment is made of (hereditarily) neutral formulas, and basically accepts intuitionistic $\supset$, $\wedge$ and $\wedge x$; besides sequents ; $\Gamma \vdash$; B which were expected, arise sequents A ; $\Gamma \vdash$; B corresponding to the notion of *headvariable*. Not only usual intuitionistic sequent calculus is recovered, but it is improved !
vii) surely less perfect is the full intuitionistic system with $\vee$, $\exists x$ and **F** (i.e. negation) ; the translation of this system into linear logic (the starting point of linear logic, see [G1]) made use of the combination !A $\oplus$!B, awfully non-associative (denotationally speaking) : compare !(!A $\oplus$!B) $\oplus$!C with !A $\oplus$!(!B $\oplus$!C) . However one could use A instead of !A if A were known to be positive... therefore there is a room for an associative disjunction provided we consider not only neutral formulas, but also positive ones. The resulting disjunction is a very complex chimera which manages to be associative and commutative, and also works in the classical case. We surely do not get as many denotational isomorphisms as we would like (typically there is no unit for the disjunction, or $A \supset B \wedge C \approx (A \supset B) \wedge (A \supset C)$ only when B and C are neutral), but the situation is incredibly better than expected. In terms of sequents, we lose the phenomenon of "headvariable", since a term may be linear in several of its variables if we perform iterated pattern-matchings.

The system presented here is rather big, for the reason that we used a two-sided version to accomodate intuitionistic features more directly, and that there are classical, intuitonistic and linear connectives ; last, but not least, rules can split into several cases depending on polarities, and for instance the rules for disjunction fill a whole page ! But this complication is rather superficial : it is more convenient to use the same symbol for nine "micro-connectives" corresponding to all possible polarities of the disjuncts. Given A and B, then we get at most two possible right rules and only one left rule, as usual. So **LU** has a very big number of connectives but apart from this it is a quite even sequent calculus.

2. polarities

Each formula is given with a *polarity* +1 (positive), 0 (neutral), −1 (negative). We use the following notational trick to indicate polarities : P, Q, R for positive formulas, S, T, U for neutral formulas, L, M, N for negative formulas. When we want to ignore the polarity, wxe shall use the letters A, B, C.

Semantically speaking a neutral formula refers to a coherent space ; a positive formula refer to a positive correlation space and a negative formula to a negative correlation space (see [G2] for a definition). Now remember that a correlation space is a coherent space plus extra structure (in fact PCS generalise spaces of the form !X, and NCS generalise spaces ?X ; both are about structural rules : a PCS is a space with left structural rules, a NCS accepts right structural rules) ; this explains the polarity table for linear logic : we first combine the underlying coherent spaces S and T to get a coherent space U (e.g. U = S ⊗ T), and if possible we try to endow U with a canonical structure of correlation space (typically if S and T are underlying coherent spaces for PCS P and Q, we equip S ⊗ T with a structure of PCS in the obvious way).

A	B	A⊗B	A⅋B	A⊸B	A&B	A⊕B	$A^{\perp}$	!A	?A	ΛxA	VxA	1	⊥	⊤	0
+1	+1	+1	0	0	0	+1	−1	+1	−1	0	+1	+1	−1	−1	+1
0	+1	0	0	0	0	0	0	+1	−1	0	0				
−1	+1	0	0	0	0	0	+1	+1	−1	−1	0				
+1	0	0	0	0	0	0									
0	0	0	0	0	0	0									
−1	0	0	0	0	0	0									
+1	−1	0	0	−1	0	0									
0	−1	0	0	0	0	0									
−1	−1	0	−1	0	−1	0									

tableau 1 : polarities for linear connectives

Before even starting, we have to make a choice about polarity 0 : do we consider that something of polarity +1 (or −1) has also the polarity 0 ? But in that case it would be normal to indicate that we decide to forget the non-zero polarity... complications, complications. In fact if we decide to answer NO, we get a quite reasonable answer : in linear logic we can forget negative polarity by forming A ⊗ 1 and a positive one by forming A ⅋ ⊥, hence if I replace A by (A ⊗ 1) ⅋ ⊥ I can change the polarity to 0. (In a similar way, V ⊃ A neutralises any intuitionistic formula.)

3. sequent calculus : identity and structure

The sequent calculus **LU** is defined as follows ;
sequents are of the form $\Gamma ; \Gamma' \vdash \Delta' ; \Delta$ where Γ, Γ', Δ and Δ' are sequences of formulas of the language. The space between the two semi-columns is a space in which usual structural rules are available ; the intended meaning of such a sequent is that of a proof which is linear in Γ and Δ, i.e. in terms of linear logic of $\Gamma, !\Gamma' \vdash ?\Delta', \Delta$.

IDENTITY

$$\frac{}{A; \vdash ;A}$$

$$\frac{\Gamma;\Gamma' \vdash \Delta';\Delta,A \qquad A,\Lambda;\Gamma' \vdash \Delta';\Pi}{\Gamma,\Lambda;\Gamma' \vdash \Delta';\Delta,\Pi}$$

$$\frac{\Gamma;\Gamma' \vdash \Delta',A;\Delta \qquad A;\Gamma' \vdash \Delta';}{\Gamma;\Gamma' \vdash \Delta';\Delta} \qquad \frac{;\Gamma' \vdash \Delta';A \qquad \Lambda;A,\Gamma' \vdash \Delta';\Pi}{\Lambda;\Gamma' \vdash \Delta';\Pi}$$

STRUCTURE

$$\frac{\Gamma;\Gamma' \vdash \Delta';\Delta}{\sigma(\Gamma);\sigma'(\Gamma') \vdash \tau'(\Delta');\tau(\Delta)}$$

$$\frac{\Gamma;\Gamma' \vdash \Delta';\Delta}{\Gamma;\Gamma' \vdash A,\Delta';\Delta} \qquad \frac{\Gamma;\Gamma' \vdash \Delta';\Delta}{\Gamma;\Gamma',A \vdash \Delta';\Delta}$$

$$\frac{\Gamma;\Gamma' \vdash A,A,\Delta';\Delta}{\Gamma;\Gamma' \vdash A,\Delta';\Delta} \qquad \frac{\Gamma;\Gamma',A,A \vdash \Delta';\Delta}{\Gamma;\Gamma',A \vdash \Delta';\Delta}$$

$$\frac{\Gamma;\Gamma' \vdash \Delta';A,\Delta}{\Gamma;\Gamma' \vdash \Delta',A;\Delta} \qquad \frac{\Gamma,A;\Gamma' \vdash \Delta';\Delta}{\Gamma;A,\Gamma' \vdash \Delta';\Delta}$$

$$\frac{\Gamma;\Gamma' \vdash \Delta',N;\Delta}{\Gamma;\Gamma' \vdash \Delta';N,\Delta} \qquad \frac{\Gamma;P,\Gamma' \vdash \Delta';\Delta}{\Gamma,P;\Gamma' \vdash \Delta';\Delta}$$

What has been presented is independent of any commitment : these rules are for all formulas, and do not refer to any distinction of the form *classical/intuitionistic/linear*. We have adopted a two-sided version, which has the effect of doubling the number of rules ; a one sided version would have been more economical, but we would have payed for this facility when considering the intuitionistic fragment that would look slightly artificial written on the right. To compensate this complication, we have decided to use an *additive* maintenance for the central part of sequents (the same Γ' and Δ' in binary rules) which is possible since structural rules are permitted in this area. Another notational trick would be (instead of the semi-column) to underline those formulas with a classical maintenance, which would simplify the schematic writing of our rules, but would not change anything deep... this is really a matter of taste.

As expected, weakening and contraction are freely performed in the central part of the sequent. Besides the exchange rules which basically allow permutation of formulas separated by a comma, we get additional *permeability* rules, which allow formulas to enter the central zone, and to exit from this zone under some restriction on polarities. The last group of rules is only one depending on polarities.

The identity axiom is written in a pure linear maintenance. The case of cut is more complex, and in fact falls into two cases, depending on the style of maintenance for the two occurences of A :

i) if they are both linear (i.e. outside the central area), then we obtain a rather expected rule
ii) if one of them is linear, the other "classical" then we obtain two symmetric form of cut ; observe that the premise containing the linear occurence of A is of the form $A \,;\, \Gamma' \vdash \Delta' \,;$ or $;\, \Gamma' \vdash \Delta' \,;\, A$, i.e. the context of A is handled classically.

There is no possibility of defining a cut between two occurences of A with a classical maintenance. By the way there is no need for that : typically in classical logic, is we get a cut on A, then A has a polarity +1 or −1 and one of the two occurences of A can be handled linearily.

4. logical rules : case of linear connectives

The calculus presented below seems rather heavy compared with usual formulation of linear logic ; but this is rather an unpleasant illusion due to the fact that we have chosen a two-sided version which is more than twice the size of the one-sided version.

$$\frac{}{;\vdash ;1} \qquad\qquad \frac{}{\bot; \vdash ;}$$

$$\frac{\Gamma;\Gamma' \vdash \Delta';\Delta,A \qquad \Lambda;\Gamma' \vdash \Delta';\Pi,B}{\Gamma,\Lambda;\Gamma' \vdash \Delta';\Delta,\Pi,A\otimes B} \qquad\qquad \frac{A,B,\Gamma;\Gamma' \vdash \Delta';\Delta}{A\otimes B,\Gamma;\Gamma' \vdash \Delta';\Delta}$$

$$\frac{\Gamma;\Gamma' \vdash \Delta';\Delta,A,B}{\Gamma;\Gamma' \vdash \Delta';\Delta,A \mathbin{⅋} B} \qquad\qquad \frac{A,\Gamma;\Gamma' \vdash \Delta';\Delta \qquad B,\Lambda;\Gamma' \vdash \Delta';\Pi}{A \mathbin{⅋} B,\Gamma,\Lambda;\Gamma' \vdash \Delta';\Delta,\Pi}$$

$$\frac{A,\Gamma;\Gamma' \vdash \Delta';\Delta,B}{\Gamma;\Gamma' \vdash \Delta';\Delta,A\multimap B} \qquad\qquad \frac{\Gamma;\Gamma' \vdash \Delta';\Delta,A \qquad B,\Lambda;\Gamma' \vdash \Delta';\Pi}{A\multimap B,\Gamma,\Lambda;\Gamma' \vdash \Delta';\Delta,\Pi}$$

$$\frac{}{\Gamma; \vdash ;\Delta,\top} \qquad\qquad \frac{}{0,\Gamma; \vdash ;\Delta}$$

$$\frac{\Gamma;\Gamma' \vdash \Delta';\Delta,A \qquad \Gamma;\Gamma' \vdash \Delta';\Delta,B}{\Gamma;\Gamma' \vdash \Delta';\Delta,A\& B} \qquad \frac{A,\Gamma;\Gamma' \vdash \Delta';\Delta}{A\& B,\Gamma;\Gamma' \vdash \Delta';\Delta} \qquad \frac{B,\Gamma;\Gamma' \vdash \Delta';\Delta}{A\& B,\Gamma;\Gamma' \vdash \Delta';\Delta}$$

$$\frac{\Gamma;\Gamma' \vdash \Delta';\Delta,A}{\Gamma;\Gamma' \vdash \Delta';\Delta,A\oplus B} \qquad \frac{\Gamma;\Gamma' \vdash \Delta';\Delta,B}{\Gamma;\Gamma' \vdash \Delta';\Delta,A\oplus B} \qquad \frac{A,\Gamma;\Gamma' \vdash \Delta';\Delta \qquad B,\Gamma;\Gamma' \vdash \Delta';\Delta}{A\oplus B,\Gamma;\Gamma' \vdash \Delta';\Delta}$$

$$\frac{\Gamma;\Gamma' \vdash \Delta';\Delta,A}{A^{\bot},\Gamma;\Gamma' \vdash \Delta';\Delta} \qquad\qquad \frac{A,\Gamma;\Gamma' \vdash \Delta';\Delta}{\Gamma;\Gamma' \vdash \Delta';\Delta,A^{\bot}}$$

$$\frac{;\Gamma' \vdash \Delta';A}{;\Gamma' \vdash \Delta';!A} \qquad\qquad \frac{\Gamma;A,\Gamma' \vdash \Delta';\Delta}{!A,\Gamma;\Gamma' \vdash \Delta';\Delta}$$

$$\frac{\Gamma;\Gamma' \vdash \Delta',A;\Delta}{\Gamma;\Gamma' \vdash \Delta';\Delta,?A} \qquad\qquad \frac{A;\Gamma' \vdash \Delta';}{?A;\Gamma' \vdash \Delta';}$$

$$\frac{\Gamma;\Gamma' \vdash \Delta';\Delta,A}{\Gamma;\Gamma' \vdash \Delta';\Delta,\bigwedge xA} \qquad\qquad \frac{A[t/x],\Gamma;\Gamma' \vdash \Delta';\Delta}{\bigwedge xA,\Gamma;\Gamma' \vdash \Delta';\Delta}$$

$$\frac{\Gamma;\Gamma' \vdash \Delta';\Delta,A[t/x]}{\Gamma;\Gamma' \vdash \Delta';\Delta,\bigvee xA} \qquad\qquad \frac{A,\Gamma;\Gamma' \vdash \Delta';\Delta}{\bigvee xA,\Gamma;\Gamma' \vdash \Delta';\Delta}$$

As expected the rules for quantifiers (right $\bigwedge x$ and left $\bigvee x$) are subject to the restriction on variables : x not free in Γ ; $\Gamma' \vdash \Delta$; Δ'.

This calculus is equivalent to usual linear logic ; more precisely we can translate usual linear logic into this new system by declaring all atomic propositions to be *neutral.* Then a sequent $\Gamma \vdash \Delta$ in usual (two-sided) linear logic becomes Γ ; $\vdash$; Δ. It is easy to translate proof to proof... the rules for the exponentials ! and ? are translated by a heavy use of structural manipulations. For instance to pass from $!\Gamma$; $\vdash$; $?\Delta$, A to $!\Gamma$; $\vdash$; $?\Delta$, !A we transit through ; $!\Gamma \vdash ?\Delta$; A, then ; $!\Gamma \vdash ?\Delta$; !A and the ultimate moves to $!\Gamma$; $\vdash$; $?\Delta$, !A use the polarities of $?\Delta$ and $!\Gamma$.

Conversely this new calculus (as long as we restrict to neutral atomic propositions) can be translated into usual linear logic as follows : a sequent Γ ; Γ'_+, $\Gamma'' \vdash \Delta''$, Δ'_- ; Δ (Γ'_+ positive, Δ'_- negative) translates as Γ , Γ'_+, $!\Gamma'' \vdash ?\Delta''$, Δ'_- , Δ in the old syntax for linear logic. Then we have to mimick all rules of the new calculus in the old one, which offers no difficulty. We have of course to prove in the old calculus a stronger form of the rule for "!", namely that one can pass from $\Gamma \vdash \Delta$, A to $\Gamma \vdash \Delta$, !A as soon as Γ is positive and Δ negative... but since our atoms are neutral, positive formulas are built from 0, 1 and formulas !A by means of $\oplus$, $\otimes$, and $\exists x$ (and symmetrically for negative formulas) and we can make an easy inductive argument.

Of all the logical rules of linear logic, only the rules for exponentials do something to the central part : the right rule for "!" assumes that the context lies wholly in the central part, whereas the left rule moves a formula from the central area to the extreme left, at the price of a symbol "!" ; the new formula !A can now pass the semi-column in both ways.

5. some chimeric connectives

It is possible to define new connectives by pattern matching, i.e. by considering polarities. We shall below only consider those connectives and quantifiers which are of interest to classical and intuitionistic logics : these connectives are $\wedge$, $\vee$, $\Rightarrow$, $\neg$, **V**, **F**, $\forall x$, $\exists x$ (classical), $\cap$, $\cup$, $\supset$, $\sim$, **V**, **F**, (x) and (Ex) (intuitionistic) . However it turns out that $\cap$, $\cup$, **V**, **F**, (x) and (Ex) can be chosen to coincide with $\bigwedge$, $\bigvee$, 1, 0, $\bigwedge x$, $\exists x$; moreover intuitionistic negation is better handled as $\sim A := A \supset 0$.

Our tables have been chosen so as to minimise the total number of connectives, and to get as many denotational isomorphisms as possible. It has not been possible to keep the same connective for implication, (conflicts of polarities). Our classical conjunction has been made up from $\neg A \vee B$ and is quite complicated ; another one built on $\neg(A \wedge \neg B)$ would be simpler, but the discussion is rather sterile since the difference cannot be noticed on classical formulas...

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A	B	A∧B	A∨B	A⇒B	A⊃B	∀xA	∃xA
+1	+1	+1	+1	−1	0	−1	+1
0	+1	+1	+1	+1	0	−1	+1
−1	+1	+1	−1	+1	0	−1	+1
+1	0	+1	+1	−1	0		
0	0	0	+1	+1	0		
−1	0	0	−1	+1	0		
+1	−1	+1	−1	−1	−1		
0	−1	0	−1	−1	−1		
−1	−1	−1	−1	−1	−1		

tableau 2 : polarities for classical and intuitionistic connectives

A	B	A∧B	A∨B	A⇒B	A⊃B	∀xA	∃xA
+1	+1	A⊗B	A⊕B	A⊸?B	A⊸B	⋀x?A	⋁xA
0	+1	!A⊗B	!A⊕B	!A$^{\perp}$⊕B	!A⊸B	⋀x?A	⋁x!A
−1	+1	!A⊗B	A⅋?B	A$^{\perp}$⊕B	!A⊸B	⋀xA	⋁x!A
+1	0	A⊗!B	A⊕!B	A⊸?!B	A⊸B		
0	0	A&B	!A⊕!B	!A$^{\perp}$⊕!B	!A⊸B		
−1	0	A&B	A⅋?!B	A$^{\perp}$⊕!B	!A⊸B		
+1	−1	A⊗!B	?A⅋B	A⊸B	A⊸B		
0	−1	A&B	?!A⅋B	?!A$^{\perp}$⅋B	!A⊸B		
−1	−1	A&B	A⅋B	!A⊸B	!A⊸B		

tableau 3 : classical and intuitionitic connectives
definition in terms of linear logic

RULES FOR CONJUNCTION

$$\frac{\Gamma;\Delta \vdash \Delta';\Delta,P \qquad \Lambda;\Gamma' \vdash \Delta';\Pi,Q}{\Gamma,\Lambda;\Gamma' \vdash \Delta';\Delta,\Pi,P\wedge Q} \qquad \frac{P,Q,\Gamma;\Gamma' \vdash \Delta';\Delta}{P\wedge Q,\Gamma;\Gamma' \vdash \Delta';\Delta}$$

$$\frac{;\Gamma' \vdash \Delta';A \qquad \Lambda;\Gamma' \vdash \Delta';\Pi,Q}{\Lambda;\Gamma' \vdash \Delta';\Pi,A\wedge Q} \qquad \frac{Q,\Gamma;\Gamma',A \vdash \Delta';\Delta}{A\wedge Q,\Gamma;\Gamma' \vdash \Delta';\Delta}$$

$$\frac{\Gamma;\Gamma' \vdash \Delta';\Delta,P \qquad ;\Gamma' \vdash \Delta';B}{\Gamma;\Gamma' \vdash \Delta';\Delta,P\wedge B} \qquad \frac{P,\Gamma;\Gamma',B \vdash \Delta';\Delta}{P\wedge B,\Gamma;\Gamma' \vdash \Delta';\Delta}$$

$$\frac{\Gamma;\Gamma' \vdash \Delta';\Delta,A \qquad \Gamma;\Gamma' \vdash \Delta';\Delta,B}{\Gamma;\Gamma' \vdash \Delta';\Delta,A\wedge B} \qquad \frac{A,\Gamma;\Gamma' \vdash \Delta';\Delta}{A\wedge B,\Gamma;\Gamma' \vdash \Delta';\Delta} \qquad \frac{B,\Gamma;\Gamma' \vdash \Delta';\Delta}{A\wedge B,\Gamma;\Gamma' \vdash \Delta';\Delta}$$

comments : P, Q positive ; A, B not positive

RULES FOR INTUITIONISTIC IMPLICATION

$$\frac{P,\Gamma;\Gamma' \vdash \Delta';\Delta,B}{\Gamma;\Gamma' \vdash \Delta';\Delta,P\supset B} \qquad \frac{\Gamma;\Gamma' \vdash \Delta';\Delta,P \qquad B,\Lambda;\Gamma' \vdash \Delta';\Pi}{P\supset B,\Gamma,\Lambda;\Gamma' \vdash \Delta';\Delta,\Pi}$$

$$\frac{\Gamma;\Gamma',A \vdash \Delta';\Delta,B}{\Gamma;\Gamma' \vdash \Delta';\Delta,A\supset B} \qquad \frac{;\Gamma' \vdash \Delta';A \qquad B,\Lambda;\Gamma' \vdash \Delta';\Pi}{A\supset B,\Lambda;\Gamma' \vdash \Delta';\Pi}$$

comments : P positive ; A not positive ; B arbitrary

RULES FOR "∀"

$$\frac{\Gamma;\Gamma' \vdash \Delta',A;\Delta}{\Gamma;\Gamma' \vdash \Delta';\Delta,\forall xA} \qquad \frac{A[t/x];\Lambda' \vdash \Pi';}{\forall xA;\Lambda' \vdash \Pi';}$$

$$\frac{\Gamma;\Gamma' \vdash \Delta';\Delta,N}{\Gamma;\Gamma' \vdash \Delta';\Delta,\forall xN} \qquad \frac{N[t/x],\Lambda;\Lambda' \vdash \Pi';\Pi}{\forall xN,\Lambda;\Lambda' \vdash \Pi';\Pi}$$

comments : A not negative ; N negative ; x not free in $\Gamma ; \Gamma' \vdash \Delta' ; \Delta$.

RULES FOR DISJUNCTION

$$\frac{\Gamma;\Gamma' \vdash \Delta';\Delta,P}{\Gamma;\Gamma' \vdash \Delta';\Delta,P\vee Q} \qquad \frac{\Gamma;\Gamma' \vdash \Delta';\Delta,Q}{\Gamma;\Gamma' \vdash \Delta';\Delta,P\vee Q} \qquad \frac{P,\Gamma;\Gamma' \vdash \Delta';\Delta \qquad Q,\Gamma;\Gamma' \vdash \Delta';\Delta}{P\vee Q,\Gamma;\Gamma' \vdash \Delta';\Delta}$$

$$\frac{;\Gamma' \vdash \Delta';S}{;\Gamma' \vdash \Delta';S\vee Q} \qquad \frac{\Gamma;\Gamma' \vdash \Delta';\Delta,Q}{\Gamma;\Gamma' \vdash \Delta';\Delta,S\vee Q} \qquad \frac{\Gamma;S,\Gamma' \vdash \Delta';\Delta \qquad Q,\Gamma;\Gamma' \vdash \Delta';\Delta}{S\vee Q,\Gamma;\Gamma' \vdash \Delta';\Delta}$$

$$\frac{\Gamma;\Gamma' \vdash \Delta',Q;\Delta,M}{\Gamma;\Gamma' \vdash \Delta';\Delta,M\vee Q} \qquad \frac{M,\Gamma;\Gamma' \vdash \Delta';\Delta \qquad ;\Gamma',Q \vdash \Delta';}{M\vee Q,\Gamma;\Gamma' \vdash \Delta';\Delta}$$

$$\frac{\Gamma;\Gamma' \vdash \Delta';\Delta,P}{\Gamma;\Gamma' \vdash \Delta';\Delta,P\vee T} \qquad \frac{;\Gamma' \vdash \Delta';T}{;\Gamma' \vdash \Delta';P\vee T} \qquad \frac{P,\Gamma;\Gamma' \vdash \Delta';\Delta \qquad \Gamma;\Gamma',T \vdash \Delta';\Delta}{P\vee T,\Gamma;\Gamma' \vdash \Delta';\Delta}$$

$$\frac{;\Gamma' \vdash \Delta';S}{;\Gamma' \vdash \Delta';S\vee T} \qquad \frac{;\Gamma' \vdash \Delta';T}{;\Gamma' \vdash \Delta';S\vee T} \qquad \frac{\Gamma;\Gamma',S \vdash \Delta';\Delta \qquad \Gamma;\Gamma',T \vdash \Delta';\Delta}{S\vee T,\Gamma;\Gamma' \vdash \Delta';\Delta}$$

$$\frac{\Gamma;\Gamma' \vdash \Delta';\Delta,M}{\Gamma;\Gamma' \vdash \Delta';\Delta,M\vee T} \qquad \frac{;\Gamma' \vdash \Delta';M,T}{;\Gamma' \vdash \Delta';M\vee T} \qquad \frac{M,\Gamma;\Gamma' \vdash \Delta';\Delta \qquad ;\Gamma',T \vdash \Delta';}{M\vee T,\Gamma,\Lambda;\Gamma' \vdash \Delta';\Delta,\Pi}$$

$$\frac{\Gamma;\Gamma' \vdash P,\Delta';\Delta,N}{\Gamma;\Gamma' \vdash \Delta';\Delta,P\vee N} \qquad \frac{P;\Gamma' \vdash \Delta'; \qquad N,\Lambda;\Gamma' \vdash \Delta';\Pi}{P\vee N,\Lambda;\Gamma' \vdash \Delta';\Pi}$$

$$\frac{;\Gamma' \vdash \Delta';S,N}{;\Gamma' \vdash \Delta';S\vee N} \qquad \frac{\Gamma;\Gamma' \vdash \Delta';\Delta,N}{\Gamma;\Gamma' \vdash \Delta';\Delta,S\vee N} \qquad \frac{;\Gamma',S \vdash \Delta'; \qquad N,\Lambda;\Gamma' \vdash \Delta';\Pi}{S\vee N,\Gamma,\Lambda;\Gamma' \vdash \Delta';\Delta,\Pi}$$

$$\frac{\Gamma;\Gamma' \vdash \Delta';\Delta,M,N}{\Gamma;\Gamma' \vdash \Delta';\Delta,M\vee N} \qquad \frac{M,\Gamma;\Gamma' \vdash \Delta';\Delta \qquad N,\Lambda;\Gamma' \vdash \Delta';\Pi}{M\vee N,\Gamma,\Lambda;\Gamma' \vdash \Delta';\Delta,\Pi}$$

comments : P, Q positive ; M, N negative ; S, T neutral.

The author will be accused of bureaucracy : even if one regroups rules, their number remains ... frightening. Surely the fact that disjunction is defined by nine independant cases is for something in this inflation. However, observe that these rules are always variations on the familiar rules for disjunction, and each line differs from the other by a slightly different structural maintenance. Given a concrete disjunction A V B, only one of these lines can work, i.e. at most three rules at usual. Moreover usual fragments use at most four lines out of nine...

Also these rules manage to unify classical and intuitionistic disjunction in the same *associative* connective, which is a non-trivial achievement.

RULES FOR "∃"

$$\frac{\Gamma;\Gamma' \vdash \Delta';\Delta,P[t/x]}{\Gamma;\Gamma' \vdash \Delta';\Delta,\exists xP} \qquad \frac{P,\Lambda;\Lambda' \vdash \Pi';\Pi}{\exists xP,\Lambda;\Lambda' \vdash \Pi';\Pi}$$

$$\frac{;\Gamma' \vdash \Delta';A[t/x]}{;\Gamma' \vdash \Delta';\exists xA} \qquad \frac{\Lambda;\Lambda',A \vdash \Pi';\Pi}{\exists xA,\Lambda;\Lambda' \vdash \Pi';\Pi}$$

comments : P positive, A not positive, x not free in Λ ; $\Lambda' \vdash \Pi'$; Π.

RULES FOR CLASSICAL IMPLICATION

$$\frac{\Gamma;\Gamma' \vdash \Delta';\Delta,P}{\Gamma;\Gamma' \vdash \Delta';\Delta,N\Rightarrow P} \quad \frac{N,\Gamma;\Gamma' \vdash \Delta';\Delta}{\Gamma;\Gamma' \vdash \Delta';\Delta,N\Rightarrow P} \quad \frac{\Gamma;\Gamma' \vdash \Delta';\Delta,N \qquad Q,\Lambda;\Gamma' \vdash \Delta';\Pi}{N\Rightarrow P,\Gamma,\Lambda;\Gamma' \vdash \Delta';\Delta;\Pi}$$

$$\frac{P,\Gamma;\Gamma' \vdash Q,\Delta';\Delta}{\Gamma;\Gamma' \vdash \Delta';\Delta,P\Rightarrow Q} \qquad \frac{\Gamma;\Gamma' \vdash \Delta';\Delta,P \qquad Q;\Gamma' \vdash \Delta';}{P\Rightarrow Q,\Gamma;\Gamma' \vdash \Delta';\Delta}$$

$$\frac{\Gamma;\Gamma',M \vdash \Delta';\Delta,N}{\Gamma;\Gamma' \vdash \Delta';\Delta,M\Rightarrow N} \qquad \frac{;\Gamma' \vdash \Delta';M \qquad N,\Lambda;\Gamma' \vdash \Delta';\Pi}{M\Rightarrow N,\Lambda;\Gamma' \vdash \Delta';\Pi}$$

$$\frac{P,\Gamma;\Gamma' \vdash \Delta';\Delta,N}{\Gamma;\Gamma' \vdash \Delta';\Delta,P\Rightarrow N} \qquad \frac{\Gamma;\Gamma' \vdash \Delta';\Delta,P \qquad N,\Lambda;\Gamma' \vdash \Delta';\Pi}{P\Rightarrow N,\Gamma,\Lambda;\Gamma' \vdash \Delta';\Delta,\Pi}$$

comments : P, Q positive, M, N negative ; this set of rules is incomplete (we have omitted the rules involving neutral formulas, for which there is no use at present ; the reader may reconstitute them from the rules of disjunction). Observe that the four rules written would have been the same if implication had been defined from conjunction.

All other usual connectives coincide with one of those already introduced, with the exception of intuitionistic negation : it is impossible to write its rules without using the constant **F** (or 0) : this minor defect comes from our very cautious treatment of structural rules ; it is therefore better to consider ~ as defined by ~A := A ⊃ **F**.

6. some properties of the calculus

First let us fix once for all a reasonable language :
- atomic predicates are given with their polarity (+1, 0, −1)
- two constants 0 and 1, both positive (also noted **F** and **V**)
- unary connectives : !, ?, $(.)^\perp$ (also noted ¬)
- binary connectives : ∧, ∨, ⇒, ⊃, ⊗, ⅋, ⊸, ⊕, &
- quantifiers : ∀x, ∃x, ⋀x, ⋁x

We now define remarkable fragments ; they are all defined by a restriction of the possible atomic formulas and of the possible connectives and quantifiers :

1° the classical fragment :
- positive and negative atoms (including **V** and **F**) ; closed under ¬, ∧, ∨, ⇒, ∀x and ∃x

2° the intuitionistic fragment :
- positive and neutral atoms (including **V** and **F**) ; closed under ∧, ∨, ⊃, ⋀x, ∃x

3° the neutral intuitionistic fragment :
- neutral atoms ; closed under ∧, ⊃, ⋀x

4° the linear fragment :
- all atoms ; closed under $(.)^\perp$, ⊗, ⅋, ⊸, ⊕, &, !, ?, ⋀x, ⋁x

The interest of these various fragments is to enable us to formalise arguments belonging to various logical systems inside **LU**, with the advantage of a unique proof-maintenance. Each fragment uses a very small part of our *kolossal* sequent calculus. But **LU** is not the union of its fragments : there must be interesting formulas outside of these fragments (and also other interesting fragments ; for instance a positive intutionistic fragment based on the implication !(A ⊸ B) should be investigated).

The classical fragment is based on the idea of staying within positive or negative formulas ; the intuitionistic fragment stays within positive and neutral formulas ; the neutral intuitionistic fragment is wholly neutral ; the linear fragment admits all three polarities.

An important property of these fragments is the *substitution property* : let a be a proper predicate symbol of arity n, and let A be a formula of the same polarity as a, in which distinct free variables $x_1,...,x_n$ have been distinguished. Then one can define for any formula B the substitution $B[\lambda x_1...x_n.A/a]$ as the result of replacing any atom $at_1...t_n$ of B by $A[t_1,...,t_n]$ (with usual precautions concerning free and bound variables, comrade Tchernienko). All the fragments considered are closed under mutual substitution.

Each fragment gets its own notion of sequent : first all formulas must belong to the fragment ; but some additional properties may be required :

i) a *classical sequent* $\Gamma ; \Gamma' \vdash \Pi' ; \Pi$ is such that if we make the sum of the number of negative

formulas in Γ and of positive formulas in Π, we get the total number 0 or 1.
ii) an *intuitionistic sequent* is of the form Γ ; Γ' $\vdash$; A
iii) a *neutral intuitionistic sequent* is a sequent Γ ; Γ' $\vdash$ A, with at most one formula in Γ.

THEOREM

If a sequent of one the fragments considered is provable, it is provable within the fragment.

proof: we limit our search to cut-free proofs ; by the subformula property, all the *formulas* occuring in the proofs belong to the fragment ; in particular this is enough for linear logic, since no additional restriction has been imposed on linear sequents. Let us consider the remaining cases : in all cases we have to check that the restriction on the shape of the sequent can be forwarded from the conclusion to the premise(s).

NEUTRAL INTUITIONISTIC FRAGMENT : first observe that the restriction "Δ' empty" will be easily forwarded (this holds for both intuitionistic fragments). Then observe that for any cut-free rule of **LU** ending with a neutral intuitionistic sequent Γ ; Γ' $\vdash$; M, then :

- all premises are of the form Λ ; Λ' $\vdash$; N
- one of these premises, say Λ ; Λ' $\vdash$; N is such that the number of formulas in Λ is greater or equal to the number of formulas in Γ, with only one exception, namely the identity axiom. In particular there is no way to prove a sequent Γ ; Γ' $\vdash$; M of formulas in this fragment when Γ has two formulas or more ; the formula of Γ (if there is one) is the analogue of the familiar *headvariable* fo typed λ-calculi, which are based on neutral intuitionisitic fragments.

This proves that all premises of the rule must be also neutral intuitionistic sequents.

CLASSICAL FRAGMENT : if S is the sequent Γ ; Γ' $\vdash$ Δ' ; Δ let us define $\mu(S)$ to be the sum of the number of negative formulas in Γ and of positive formulas in Δ. Now for any rule with a conclusion S made of classical formulas, there is a premise S' such that $\mu(S') \geq \mu(S)$, with only two exceptions : the identity axiom and the axiom **F**, Γ ; $\vdash$; Δ. Furthermore, there are only two rules with a premise S' and a conclusion S such that $\mu(S') > \mu(S)$: the two permeability rules enabling a formula to enter the central zone. Now it is an easy exercise, given any cut-free proof of a sequent S made of classical formulas with $\mu(S) > 1$, to produce another proof of any sequent S' obtained by removing as many formulas among those which contribute to $\mu(S)$. In particular, a "bad" permeability rule can be replaced with a weakening and so we stay among classical sequents.

INTUITIONISTIC FRAGMENT : if $\nu(S)$ counts the number of formulas in the part Δ of a sequent S = Γ ; Γ' $\vdash$; Δ , then the restriction $\nu(S) \leq 1$ is forwarded from the conclusion to the premises of all rules involving intuitionistic formulas but for the case of a rule

$$\frac{\Gamma;\Gamma' \vdash ;C,P \qquad B,\Lambda;\Gamma' \vdash ;}{P \supset B,\Gamma,\Lambda;\Gamma' \vdash ;C}$$

Easy commutation arguments reduce the use of this rule to the case where C is positive or atomic. From this it can be ensured that all sequents with $\nu(S) > 1$ occuring in the proof of an intuitionistic sequent have a succedent made of positive or atomic formulas. Now one can easily produce given a proof of a sequent $\Gamma ; \Gamma' \vdash ; \Delta$ with $\nu(\Delta) \neq 1$ (this includes $\nu(\Delta) = 0$) and all formulas intuitionistic, another proof of $\Gamma ; \Gamma' \vdash ; \Pi$ where Π has been obtained from Δ by adding formulas, or removing atomic or positive ones. In particular we can replace the "bad" rule above by the "good" one :

$$\frac{\Gamma;\Gamma' \vdash ;P \qquad B,\Lambda;\Gamma' \vdash ;C}{P \supset B,\Gamma,\Lambda;\Gamma' \vdash ;C}$$

and this shows that we can stay among intuitionistic sequents. QED

remarks :

i) we implicitely used a cut-elimination theorem for **LU** that is more or less obvious (but maybe a bit too long to write down explicitely

ii) the results of the theorem concern not only provability, but also proofs, in the sense of denotational semantics ; there would be nothing to prove in the classical and the intuitionistic case if the constant 0 were not allowed (only the axioms involving 0 and its negation prevent us to conclude like in the neutral fragment). Now the proofs we look at with the wrong $\mu(S)$ or $\nu(S)$ are in fact interpreted in a coherent space with an empty web : all proofs of such sequents are denotationally equal, and we therefore replace a proof by another one with the same semantics !

iii) the paper of Schellinx [S] investigates the faithfulness of the translation intuitionistic $\mapsto$ linear and our proof is roughly inspired from this paper.

Now, it remains to compare the systems **LU** restricted to various fragments with the sequent calculi for the corresponding logics :

i) the two intuitionistic fragments are OK : just translate $\Gamma ; \Gamma' \vdash ; A$ as $\Gamma, \Gamma' \vdash A$ and observe that all rules are correct. The other way around might be slightly more delicate at least if we investigate cut-free provability.

ii) the classical fragment translates not to **LK**, but to **LC** (see [G2]) ; more precisely besides the superficial difference one-sided/two sided, **LC** uses the semi-column in a different way : one tries to put as many formulas as possible in the central zone : in particular starting with $\Gamma ; \Gamma' \vdash \Delta' ; \Delta$ the idea is to move all positive formulas from Γ to the right, and all negative formulas of Δ to the left... with the result that Γ, Δ can consist of at most one formula.

conclusion

As a matter of conclusion let us observe that this attempt at unification is orthogonal to synchretic attempts of the style "logical framewotk" : too often unification is at the price of a loss of structure (we lose properties : cut-elimination, nay consistency). Here it goes the other way around. All fragments considered are better as subsystems of **LU** than they were as isolated systems :

i) classical logic is handled by **LC** which is much better than **LK**

ii) the neutral intuitionistic fragment gets a legalisation of the notion of *headvariable* and its normalisation procedure should be of the style "linear head-reduction"

iii) the intuitionistic fragment gets a subtler approach to pattern-matching, typically a denotationally associative disjunction

iv) linear logic gets a smoother sequent calculus, in particular for exponential connectives ; this formulation has some similarities with the linear sequent calculi proposed by Andréoli and Pareschi [AP].

.... not to speak of the fact that all these systems are part of the same calculus, i.e. are free to interact....

There is of course the obvious question : is this *LOGIC*, i.e. did we catch here all possible logical systems ? Surely not, and there are additional parameters on which one can play to broaden the scope of a unified appraoch to logic :

i) the consideration of additional polarities : a polarity can be abstractly seen as the permission to perform certain structural rules on the left or the right of a sequent. Many other cocktails (from the absolute non-commutative polarity to classical polarities) are possible, and all the combinations between weakening, exchange and contraction yield up to 15 possible polarities. Most of these combinations presumably make no sense, and one should not hurry to invent polarities with no concrete application. However if one absolutely wants to experiment in that way, it seems that a good criterion for the consideration of additional polarities could be the possibility of extending the definition of disjunction so as to preserve its denotational associativity.

ii) the extension of these results to systems which have always been on the border line of logic : systems of arithmetic, and more generally inductive definitions

iii) the extension to second-order ; in particular it will be possible to cope with the loss of subformula property by considering quantifications ranging on various fragments (exemple : for all positive and classical α).

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ISSN 0249 - 6399