Control of an overhead crane : stabilization of flexibilities

Brigitte d'Andréa-Novel, Fadi Boustany, Francis Conrad

UNITÉ DE RECHERCHE
INRIA-LORRAINE

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P. 105
78153 Le Chesnay Cedex
France
Tél.:(1) 39 63 55 11

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Brigitte D'ANDREA-NOVEL
Fadi BOUSTANY
Francis CONRAD

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CONTROL OF AN OVERHEAD CRANE : STABILIZATION OF FLEXIBILITIES [1]

Brigitte d'Andréa-Novel [2], Fadi Boustany [2], Francis Conrad [3]

Abstract

This paper deals with the feedback stabilization of the cable of an overhead crane, by the means of the position of the platform. The wellposedness of the closed-loop PDE system with boundary control and homogeneous Neumann condition on part of the boundary is established, the asymptotic stabilization is proved by Lasalle's Invariance Principle for a class of simple feedbacks and decay estimates are given. Illustrative simulations are displayed.

CONTROLE D'UN PONT ROULANT : STABILISATION DES FLEXIBILITES

Résumé

Nous étudions la stabilisation par bouclage du câble d'un pont roulant, au moyen de la position du chariot. Nous montrons que le problème (EDP avec contrôle frontière et condition de Neumann homogène sur une partie de la frontière) est bien posé et qu'il y a stabilisation asymptotique pour une classe de bouclages simples. Nous donnons des estimations de la décroissance de l'énergie. Les résultats sont illustrés par des simulations.

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[1] Supported by a grant from EDF-DER, Chatou

[2] Centre d'Automatique, Ecole des Mines, 35, rue St Honoré 77305 Fontainebleau, Fax : 64 69 47 01

[3] Université de Nancy I, U.R.A. CNRS 0750 and Projet NUMATH,INRIA Lorraine, Département de Mathématiques, B.P. 239 54506 Vandoeuvre les Nancy

1 Statement of the problem.

We consider an overhead crane, consisting of a motorized platform moving along an horizontal beam [1] and equipped with a winch, around which a cable is enrolled, holding a load. (see Figure 1, and [3])
Several studies ([3, 5, 7, 8, 20]) dealt with the "Rigid Case", that is the case where the cable is supposed to be rigid (and generally with negligible mass). The system can then be seen as a pendulum with variable length and mobile basis. The aim is to stabilize the load as quickly as possible, or to make it follow a given trajectory as precisely as possible. The actuators are the force u_1 applied by the motor to the platform, and the torque u_2 applied on the winch. When using the non-linear approach, one must make the assumption that the whole state is observed, i.e. : the position of the platform, the length of the cable and the angle of the cable with the vertical axis, as well as their derivatives.

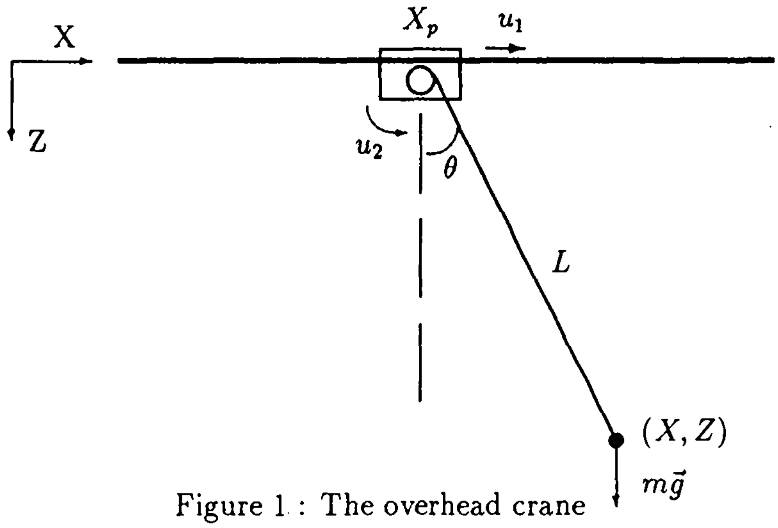

Figure 1 : The overhead crane

In [3, 5] it has been proved that this mechanical system, with less actuators than degrees of freedom, can be completely linearized by using **dynamic** non linear state feedback.

[1]This beam is assumed here to be fixed, so we are concerned with a two-dimensional problem

In this work, we focus on the cable. We take into account its mass and its flexibilities. We make the following assumptions :

Assumptions H1 :

- The cable is completely flexible and non-stretching.
- The length of the cable is constant.
- Displacements are small.
- The angle of the cable with the vertical axis is small everywhere along the cable.
- Dynamics of the platform is ignored, that is, the control is supposed to be the position or equivalently, the velocity of the platform, and not the force u_1.

Remark. For large displacements and length variation we can make use of the dynamic feedback linearization result for the rigid case [3, 5]. Then, close to the final desired position, and for the adequate length of the cable, assumptions H1 make sense and the objective is to stabilize the flexible modes of the cable.

The equation of heavy cables. We denote by s the curvilinear abscissa, and by $x(s,t)$ the position at time t of the point which has curvilinear abscissa s. (see Figure 2)

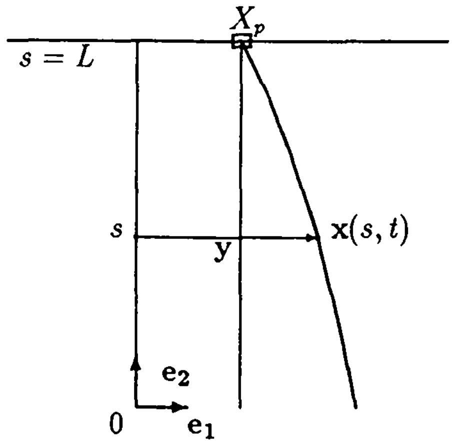

Figure 2 : Motion of the cable

Let $\tau(s,t)$ be the tension along the cable and ρ the lineic mass of the cable; the equation of heavy cables is the following ([17]) :

$$\rho x_{tt}(s,t) = \rho g + (\tau(s,t)x_s)_s \tag{1}$$

where x_s (resp. x_t) denotes the derivative of x with respect to s (resp. to t).

The non-stretching condition gives :

$$x_s(s,t).x_s(s,t) = 1 \tag{2}$$

Assumptions H1 enable us to study the tangent linearization of the system. To do this, we take the following equilibrium as reference :

$$\begin{cases} x_{ref} & = & se_2 \\ \tau_{ref}(s) & = & (m + s\rho)g \end{cases} \tag{3}$$

and we set :

$$\begin{cases} x(s,t) = x_{ref} + y \\ \tau(s,t) = \tau_{ref} + \delta \end{cases} \tag{4}$$

Replacing x and τ in (1) and (2), keeping only first-order terms and projecting upon the horizontal (e_1) axis (longitudinal vibrations are second-order quantities), we obtain the following system :

$$\begin{cases} y_{tt} - (ay_s)_s = 0 \\ y_s(0,t) = 0 \\ y(L,t) = X_p(t) \end{cases} \tag{5}$$

where

$$a(s) = g(\frac{m}{\rho} + s) \tag{6}$$

The first boundary condition expresses the verticality of the cable at its lowest end, due to the load; the second one means that the cable is, at its highest end, fixed to the platform.

Remark. Given X_p, the solution of system (5) can be explicitely written, using the eigenfunctions of the associated homogeneous problem, which are Bessel functions.

The equation describing the movement of the platform completes the dynamical model:

$$M\ddot{X}_p = (m + \rho L)g\theta + u_1 \tag{7}$$

where $(m + \rho L)g$ is an approximation of the tension, and θ is an approximation of $sin\theta$. System (5) with (7) leads to a hybrid PDE-ODE system (see [2]). Here (see Assumption H1), we study the simpler problem of stabilizing the cable, ignoring the dynamics of the platform.

Consider the following energy function :

$$E(t) = \frac{1}{2}\int_0^L (y_t^2 + ay_s^2)ds \quad + \frac{k}{2}y(L,t)^2 \ , \quad k > 0 \tag{8}$$

The integral is the internal energy of the cable, and the last factor could be replaced by $(y(L,t) - x_c)^2$, so that the platform could be steered to any equilibrium position x_c.

A formal computation of the time derivative of E gives :

$$\frac{dE}{dt} = y_t(L,t)\ [ay_s(L,t) + ky(L,t)] \tag{9}$$

which leads to the following dissipative boundary feedback laws, expressed in terms of boundary condition at $s = L$:

$$ay_s(L,t) + ky(L,t) = -g(y_t(L,t)) \tag{10}$$

where g is monotone (continuous), with $g(0) = 0$.

Consequently, the closed-loop system will be :

$$\begin{cases} y_{tt} - (ay_s)_s = 0 \\ y_s(0,t) = 0 \\ (ay_s)(L,t) + ky(L,t) = -g(y_t(L,t)) \end{cases} \tag{11}$$

Remark.

Problem (11) is similar to a model studied by Zuazua ([22, sect. 3.1]) concerning the wave equation in space dimension $n \leq 3$, in the case of active control on the whole boundary. The author also introduced the term $\frac{1}{2}y^2(L,t)$ in the energy, and obtained decay estimates for small k in the nonlinear case. In [23], it has been proved that this restriction on k is unnecessary when g is linear, and exponential uniform stability holds in that case. In our specific monodimensionnal problem, we improve the results of [22] in the sense that the coefficient of the PDE is not constant and we have an homogeneous Neumann condition on part of the boundary. These modifications are quite minor with respect to the work of [22]. More interesting is the fact that we obtain the estimates **for any** $k > 0$ even if g is nonlinear. Also, we handle both the superlinear case (as in [22]) and the sublinear case.

This paper is organized as follows :

- **Section 2:** we show that problem (11) is well-posed. This will give in particular a sense to the time-derivation (9).
- **Section 3 :** we prove that E decays to zero (strong asymptotic stabilization).
- **Section 4 :** we give estimates for this decay.
- **Section 5 :** we present some numerical results and we conclude.

2 Wellposedness

Though the results of this section hold for any $a \in L^\infty(0, L)$ satisfying :

$$a \geq \alpha > 0 \tag{12}$$

and for any maximal monotone graph g of $\mathbf{R}^2$ such that :

$$0 \in g(0) \tag{13}$$

we consider only the physical framework where a is affine and g is a continuous function.

Let H and V be the following Hilbert spaces :

$$\begin{array}{l} H = L^2(0, L) \\ V = H^1(0, L) \end{array} \tag{14}$$

Following the method of [9] it is possible to show that (11) defines a contraction semi-group on $V \times H$, associated with a nonlinear, maximal, monotone operator on $V \times H$, with dense domain. Instead, we apply an abstract result of [15].

Let $A \in \mathcal{L}(V, V')$ be defined by the following bilinear form :

$$< Au, v >_{V'V} = \int_0^L a u_s v_s ds + k u(L) v(L), \ \forall u, v \in V \tag{15}$$

Considering A as an unbounded operator on H, we have :

$$dom\ A = \{u \in V; (au_s)_s \in H; au_s(0) = 0; au_s(L) + ku(L) = 0\} \tag{16}$$

and, if u is in $dom\ A$:

$$Au = -(au_s)_s \tag{17}$$

A is a positive, self-adjoint, operator on H, and $V = dom\ A^{\frac{1}{2}}$. Moreover, $dom\ A$ is dense in V and in H. In the sequel, V will be equipped with the following scalar product :

$$(u, v)_V \doteq \int_0^L a u_s v_s ds + k u(L) v(L) \tag{18}$$

defining on V a norm, equivalent to the usual norm. $V \times H$ is equipped with the scalar product :

$$((u,w),(v,z))_{V\times H} = \int_0^L au_{,s}v_{,s}ds + ku(L)v(L) + \int_0^L wzds \tag{19}$$

Next, we introduce :

$$Cv = v(L), \quad \forall v \in V \tag{20}$$

C belongs to $\mathcal{L}(V, \mathbf{R})$, and is surjective.

Multiplying the PDE in (11) by an element φ of V, and integrating by parts, we obtain the following abstract formulation :

$$y_{tt} + Ay + C^*g(Cy_t) = 0 \quad \text{in } V'; \; y \in V \tag{21}$$

where g, which is continuous and monotone, is the sub-differential of a convex, l.s.c (continuous) function on $\mathbf{R}$.

Hence, it is possible to use Theorem 2.2 from [15] : (21) written as a first-order system :

$$\begin{pmatrix} y \\ y_t \end{pmatrix}_t + \begin{pmatrix} 0 & -I \\ A & C^*g(C.) \end{pmatrix} \begin{pmatrix} y \\ y_t \end{pmatrix} = 0 \tag{22}$$

defines a nonlinear semigroup of contractions $S(t)$ on $V \times H$, associated with the following maximal monotone operator :

$$\begin{aligned} dom\ \mathcal{A} = \{ \; & (u,v) \in V \times V, (au_{,s})_{,s} \in H, \\ & au_{,s}(0) = 0, au_{,s}(L) + ku(L) = -g(v(L))\} \end{aligned} \tag{23}$$

and, if $(u,v) \in dom\ \mathcal{A}$:

$$\mathcal{A}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -v \\ -(au_{,s})_{,s} \end{pmatrix} \tag{24}$$

For initial data $(y(0), y_t(0))$ in $dom\ \mathcal{A}$,

$$(y(t), y_t(t)) = S(t)(y(0), y_t(0)) \tag{25}$$

is a strong solution of (22) i.e. $(y(t), y_t(t))$ is in *dom* $\mathcal{A}$ and $t \rightarrow (y(t), y_t(t))$ is Lipschitz continuous, almost everywhere differentiable, and (11) is satisfied almost everywhere for t positive (see [6]).

It is possible to extend the semi-group property to $V \times H$ by density of *dom* $\mathcal{A}$ in $V \times H$. Indeed, let :

$$E = dom\ A \times \{v \in V; v(L) = 0\} \tag{26}$$

On the one hand :

$$dom\ A = \{u \in V; (au_s)_s \in H; au_s(0) = 0; au_s(L) + ku(L) = 0\} \tag{27}$$

is dense in $V = dom\ A^{\frac{1}{2}}$. On the other hand, $\{v \in V; v(L) = 0\}$ is dense in $H = L^2$. Then E is dense in $V \times H$, and since E is a subset of *dom* $\mathcal{A}$ (because $g(0) = 0$), the result follows.

3 Strong asymptotic stabilization

We prove the strong asymptotic stability of system (11) in $V \times H$ by using *Lasalle's Invariance Principle* ([12]).

The results in this section can be established in the case when a is any function in $L^\infty(0, L)$ satisfying (12), and g is a maximal monotone graph, satisfying :

$$g(0) = \{0\}, \quad 0 \in g(\xi).\xi \Rightarrow \xi = 0 \tag{28}$$

Once again, we keep the physical framework.

Let (y, y_t) be a solution of (22). The energy associated with this solution is given by :

$$E(t) = \frac{1}{2}\int_0^L (y_t^2 + ay_s^2)ds \quad + \frac{k}{2}y(L,t)^2 \tag{29}$$

Lemma 1
(i) E is nonincreasing w.r.t. t
(ii) if the initial condition is in dom $\mathcal{A}$ then :

$$\frac{dE}{dt} = -g(y_t(L,t))y_t(L,t) \le 0 \tag{30}$$

Proof
Direct computation gives (ii). Then (i) is proved by density of *dom* $\mathcal{A}$ and contraction property of $S(t)$. □

E is then a Lyapunov Function. In order to use the Invariance Principle in infinite dimension, we need the following result :

Lemma 2 *The resolvent* $(I+\lambda\mathcal{A})^{-1}$ *is compact from* $V \times H$ *to* $V \times H$*, and* $0 \in R(\mathcal{A})$

Proof
We have immediatly :

$$\mathcal{A}(0) = 0 \tag{31}$$

The resolvent $(I + A)^{-1}$ is compact on H. Indeed, consider $f \in H$, $u = (I + A)^{-1}(f)$ satisfies the following system :

$$\begin{cases} u - (au_s)_s = f \\ au_s(0) = 0 \\ au_s(L) + ku(L) = 0 \end{cases} \tag{32}$$

Multiplying by u and integrating from 0 to L leads to the following estimate :

$$|u|_V^2 \leq Const.|f|_H^2 \tag{33}$$

ensuring the compactness. Since g is obviously compact from $\mathbf{R}$ to $\mathbf{R}$, using [15, Th. 2.3] the resolvent $(I+\lambda\mathcal{A})^{-1}$ is then compact from $V \times H$ to $V \times H$. □

From Lemma 2 and classical results ([6, 12]) we have :

Proposition 1
For any $(y(0), y_t(0)) \in V \times H$*, the trajectories* $\{S(t)(y(0), y_t(0)), t \geq 0\}$ *are precompact in* $V \times H$*, and the* ω*-limit set* $\omega(y(0), y_t(0))$ *exists.*

Asymptotic stabilization can now be obtained, using another abstract result from [15], which is in fact the application of Lasalle's Invariance Principle [12].

Theorem 1 *Assume that $g(\xi) \neq 0$ for $\xi \neq 0$. Then for any $(y(0), y_t(0)) \in V \times H$, we have :*

$$(y(t), y_t(t)) \to 0, \quad in \ V \times H, \quad when \ t \to \infty \tag{34}$$

Proof

We show that $\omega(y(0), y_t(0)) = \{0\}$. In fact, it is enough to prove this for $(y(0), y_t(0)) \in Dom\ \mathcal{A}$. Let $\bar{w}_0$ be an element of $\omega(y(0), y_t(0))$, and let

$$\bar{w}(t) = (w, w_t)(t) = S(t)\bar{w}_0 \in \omega(y(0), y_t(0)) \tag{35}$$

First note that, by the Invariance Principle applied to $\bar{w}$, we have :

$$\frac{dE}{dt} = 0 \tag{36}$$

thus

$$g(w_t(L,t))w_t(L,t) = 0 \ a.e. \tag{37}$$

We use the assumption on g to deduce

$$w_t(L,t) = (aw_s)(L,t) + kw(L,t) = 0 \ a.e. \tag{38}$$

Proceeding as in [9] (see also [15, Thm 2.4 (ii)]), in order to show that $\omega(y(0), y_t(0)) = \{0\}$, we are led to prove the following uniqueness property :

Lemma 3 *Let $\bar{w} = (w, w_t)$ be a solution of the following overdetermined system :*

$$\begin{cases} w_{tt} - (aw_s)_s = 0 \\ (aw_s)(0,t) = 0 \\ (aw_s)(L,t) + kw(L,t) = 0 \\ w_t(L,t) = 0 \\ \bar{w}_0 \in \omega(y(0), y_t(0)) \subset Dom\ \mathcal{A} \end{cases} \tag{39}$$

Then

$$\bar{w} = 0 \tag{40}$$

Proof

We prove this result using Fourier series. We consider the eigenvalue problem :

$$\begin{cases} -(a\varphi_s)_s = \lambda\varphi \\ a\varphi_s(0) = 0 \\ a\varphi_s(L) + k\varphi(L) = 0 \end{cases} \tag{41}$$

Let (λ_n, φ_n) be the eigenvalues and eigenfunctions, with φ_n normalized in H. The solution of (39) for initial condition $\bar{w}_0 = (w(s,0), w_t(s,0))$ in *dom* $\mathcal{A}$ can be written as :

$$\begin{cases} w(t) = \sum_{i=1}^{\infty}(a_n cos\sqrt{\lambda_n}t + b_n sin\sqrt{\lambda_n}t)\frac{\varphi_n}{\sqrt{\lambda_n}} \\ \\ w_t(t) = \sum_{i=1}^{\infty}(b_n cos\sqrt{\lambda_n}t - a_n sin\sqrt{\lambda_n}t)\varphi_n \end{cases} \tag{42}$$

The first series converges in V and the second converges in H, with

$$\begin{cases} a_n = \int_0^L aw(s,0)_s \frac{\varphi_{n_s}}{\sqrt{\lambda_n}}ds + kw(L,0)\frac{\varphi_n(L)}{\sqrt{\lambda_n}} \\ \\ b_n = \int_0^L w_t(s,0)\varphi_n ds \end{cases} \tag{43}$$

As in [9], one can prove by a classical argument that $w(s,0)$ and $w_t(s,0)$ satisfy in fact the homogeneous boundary conditions, and consequently, integrating by parts (43):

$$\begin{cases} \sqrt{\lambda_n}a_n = \int_0^L (aw(s,0)_s)_s\varphi_n(s)ds \in l^2(\mathbf{N}) \\ \\ \sqrt{\lambda_n}b_n = \int_0^L aw_{ts}(s,0)\frac{\varphi_{n_s}(s)}{\sqrt{\lambda_n}}ds + kw_t(L,0)\frac{\varphi_n(L)}{\sqrt{\lambda_n}} \in l^2(\mathbf{N}) \end{cases} \tag{44}$$

since $w(s,0)$ and $w_t(s,0)$ lie in a suitable domain. Then, the second series in (42) converges in V and taking the value at L we have for almost every positive t :

$$0 = \sum_{i=1}^{\infty}(b_n cos\sqrt{\lambda_n}t - a_n sin\sqrt{\lambda_n}t)\varphi_n(L) \tag{45}$$

with an uniformly convergent series. As in [9], it is easy to see that (45) implies :

$$a_n \varphi_n(L) = b_n \varphi_n(L) = 0 \tag{46}$$

But we have $\varphi_n(L) \neq 0, \forall n \in \mathbb{N}$. Indeed, otherwise $\varphi = \varphi_n$ is solution of the following Cauchy problem :

$$\begin{cases} -(a\varphi_s)_s = \lambda\varphi \\ a\varphi_s(0) = 0; \;\; a\varphi_s(L) = 0 \\ \varphi(L) = 0 \end{cases} \qquad \Rightarrow \varphi = 0 \; ! \tag{47}$$

Thus (46) implies :

$$a_n = b_n = 0, \; \forall n \tag{48}$$

and consequently :

$$(w, w_t) = 0 \tag{49}$$

which completes the proofs of both Lemma 3 and Theorem 1. □

4 Decay estimates

Here, we assume a satisfies (6), in order to obtain the observability lemma.

Let $y(t)$ be a solution of (11), corresponding to initial conditions (y_0, y_1) in $V \times H$, and consider the associated energy :

$$E(t) = \frac{1}{2}\int_0^L (y_t^2 + a y_s^2) ds \;\; + \frac{k}{2} y(L,t)^2 \tag{50}$$

Theorem 2

(i) If

$$C_2|\xi| \leq |g(\xi)| \leq C_1|\xi|, \quad C_1, C_2 > 0 \tag{51}$$

then

$$E(t) \leq M e^{-\mu t} E(0) \tag{52}$$

where M, μ are positive constants.

(ii) Superlinear case : if

$$C_2 \, min\{|\xi|, |\xi|^p\} \leq |g(\xi)| \leq C_1|\xi|, \quad C_1, C_2 > 0 \tag{53}$$

with $p > 1$, then

$$E(t) \leq M(\frac{1}{1+\mu t})^{\frac{2}{p-1}} E(0) \tag{54}$$

where M is a positive constant, and $\mu > 0$ depends on $E(0)$.
(iii) Sublinear case : if

$$C_2|\xi| \leq |g(\xi)| \leq C_1 \, max\{|\xi|, |\xi|^p\}, \quad C_1, C_2 > 0 \tag{55}$$

with $p < 1$, then

$$E(t) \leq M(\frac{1}{1+\mu t})^{\frac{2p}{p-1}} E(0) \tag{56}$$

where M is a positive constant, and $\mu > 0$ depends on $E(0)$.

Remarks

- Case (i) (in particular if g is linear) corresponds to uniform exponential stabilization.
- It would be of interest to study the decay rate with respect to k (at least numerically, and for linear g) .
- The proof of Theorem 2 is different from the one used in [22] or [23]. It is based upon an abstract result from [10], which adapts, to the nonlinear case and for boundary control, an idea of [13].

Proof.

First, we need the wellposedness of system (11), which has already been proved in Section 2.

Then, we need the following lemmas (the proofs will be given in the sequel) :

Lemma 4 *The following open-loop system :*

$$\begin{cases} z_{tt} - (az_s)_s = 0 \\ z_s(0,t) = 0 \\ (az_s)(L,t) + kz(L,t) = u(t) \end{cases} \tag{57}$$

is well posed, i.e. the mapping :

$$(z_0, z_1, u) \in V{\times}H \times L^2(0,T) \to (z, z_t, z_t(L,.)) \in L^2(0,T; V{\times}H){\times}L^2(0,T) \tag{58}$$

is well defined and is continuous, $\forall u \in L^2(0,T)$, $\forall T > 0$, $\forall (z_0, z_1) \in V \times H$ $(H2)$.

Lemma 5 *The uncontrolled problem :*

$$\begin{cases} \varphi_{tt} - (a\varphi_s)_s = 0 \\ \varphi_s(0,t) = 0 \\ (a\varphi_s)(L,t) + k\varphi(L,t) = 0 \end{cases} \tag{59}$$

satisfies the **strong observability condition** *:*

$$\exists M_0 > 0, \exists T_0 > 0 \; s.t. \; |\varphi_0|_V^2 + |\varphi_1|_H^2 \leq M_0 \int_0^{T_0} \varphi_t^2(L,t)dt \;\; (H3) \tag{60}$$

We use then Theorem 1 from [10] : the wellposedness of system (11), (H2), (H3) and the strucural assumptions on g made in Theorem 2 ensure the estimates of this Theorem. □

We give now the proofs of Lemmas 4 and 5.

Proof of Lemma 4.
It all amounts to get the a priori estimate :
$\forall T > 0$, $\exists C_1(T), C_2(T)$ such that :

$$\int_0^T E(t)dt + \int_0^T z_t^2(L,t)dt \leq C_1(T)E(0) + C_2(T)\int_0^T u^2(t)dt \tag{61}$$

where E is given by (8).
To obtain (61) we first establish that :

$$E(t) \leq E(0) + \frac{\lambda}{2}\int_0^t z_t^2(L,\tau)d\tau + \frac{1}{2\lambda}\int_0^t u^2(\tau)d\tau, \; \forall \lambda > 0 \tag{62}$$

integrating the identity :

$$\frac{dE}{dt} = u(t)z_t(L,t) \tag{63}$$

and using Young's inequality.

Then we establish the following inequality :

$$\frac{1}{2}\int_0^t z_t^2(L,\tau)d\tau \leq C_1\int_0^t E(\tau)d\tau + C_2E(0) + C_2E(t) \qquad (64)$$

by multiplication of the equation in (57) by saz_s and integration over space and time.

Plugging (64) into (62), applying Gronwall's lemma, and integrating from 0 to T leads to the following :

$$\int_0^T E(t)dt \leq C_1(T)E(0) + C_2(T)\int_0^T u^2(t)dt \qquad (65)$$

From (64) and (65) it is easy to get (61).□

Proof of Lemma 5.

Condition (H3) can be proved by Fourier analysis, using an inequality established by Ingham and improved by Ball and Slemrod. The result, based on such an approach, can be stated as follows :

Consider again the following eigenvalue problem ($\omega > 0$) :

$$\begin{cases} -(a\varphi_s)_s = \omega^2\varphi \\ \varphi_s(0) = 0 \\ (a\varphi_s)(L) + k\varphi(L) = 0 \end{cases} \qquad (66)$$

Let (ω_n^2) be the sequence of (simple) eigenvalues of (66), φ_n a sequence of associated eigenfunctions, normalized in $H = L^2(0,L)$. If :

$$\begin{array}{l} (i)\ \ |\varphi_n(L)| \geq \beta > 0, \quad \forall n \in \mathbb{N} \\ (ii)\ \ \underline{lim}(\omega_{n+1} - \omega_n) = \alpha > 0, \ \ n \to \infty \end{array} \qquad (67)$$

then condition (H3) is true (for any $T_0 > \frac{2\pi}{\alpha}$)

We show now that (i) and (ii) are satisfied.

Proof of (i).

We multiply the equation in (66) by φ and integrate :

$$\int_0^L a\varphi_s^2 ds + k\varphi^2(L) = \omega^2 \qquad (68)$$

Then, using a multiplier $\psi\varphi_s$, one gets

$$\int_0^L a(\frac{\psi}{a})_s a\varphi_s^2 ds + \omega^2 \int_0^L \psi_s \varphi^2 ds \leq [\omega^2 + \frac{k^2}{a}]\psi(L)\dot{\varphi}^2(L) \tag{69}$$

Combining (69) with (68), and choosing $\psi \geq 0$ such that

$$a(\frac{\psi}{a})_s \geq 1 \text{ and } \psi_s \geq 1 \tag{70}$$

leads to :

$$\varphi^2(L) \geq \frac{2\omega^2}{(w^2 + \frac{k^2}{a})\psi(L) + k} > 0 \ \ \forall\omega = \omega_n. \tag{71}$$

When n tends to infinity, the fraction tends to 2. So (i) is true.

Proof of (ii).

We first define the spectral elements of problem (66) in a more precise way.

Let σ be defined by :

$$\sigma^2 = \frac{4\omega^2}{g}(s + \frac{m}{\rho}) \tag{72}$$

and let

$$\psi(\sigma) = \varphi(s) \tag{73}$$

Then, ψ is given by the following equation :

$$\psi''(\sigma) + \frac{\psi'(\sigma)}{\sigma} + \psi(\sigma) = 0 \tag{74}$$

The solutions of (74) can be written in the following form :

$$\psi(\sigma) = AJ_0(\sigma) + BY_0(\sigma) \tag{75}$$

where J_0 is the Bessel function of first kind, and Y_0 is the Neumann Bessel function of second kind (see [1]).

The boundary conditions give :

$$\begin{aligned} &AJ_0'(\sigma(0)) + BY_0'(\sigma(0)) = 0 \\ &a(L)\frac{d\sigma}{ds}(L)[AJ_0'(\sigma(L)) + BY_0'(\sigma(L))] + k[AJ_0(\sigma(L)) + BY_0(\sigma(L))] = 0 \end{aligned} \tag{76}$$

This system must have solutions other than $A = B = 0$. Thus, the ω_i are the zeros of the determinant :

$$\begin{aligned} F(\omega) &= a(L)\tfrac{d\sigma}{ds}[Y_0'(\sigma(L))J_0'(\sigma(0)) - Y_0'(\sigma(0))J_0'(\sigma(L))] \\ &+ k[J_0'(\sigma(0))Y_0(\sigma(L)) - J_0(\sigma(L))Y_0'(\sigma(0))] \end{aligned} \tag{77}$$

We use now the well-known asymptotic expansions of Bessel functions (see [19]) :

$$\begin{aligned} J_0(z) &= \sqrt{\frac{2}{\pi z}}cos(z - \frac{\pi}{4}) + O(\frac{1}{z^{\frac{3}{2}}}) \\ Y_0(z) &= \sqrt{\frac{2}{\pi z}}sin(z - \frac{\pi}{4}) + O(\frac{1}{z^{\frac{3}{2}}}) \\ J_0'(z) &= -\sqrt{\frac{2}{\pi z}}sin(z - \frac{\pi}{4}) + O(\frac{1}{z^{\frac{3}{2}}}) \\ Y_0'(z) &= \sqrt{\frac{2}{\pi z}}cos(z - \frac{\pi}{4}) + O(\frac{1}{z^{\frac{3}{2}}}) \end{aligned} \tag{78}$$

and we obtain :

$$\begin{aligned} F(\omega) &= F_0 sin(\sigma(L) - \sigma(0)) + O(\tfrac{1}{\omega}) \\ &= F_0 sin(\omega\tau) + O(\tfrac{1}{\omega}) \end{aligned} \tag{79}$$

where F_0 is a constant, and :

$$\tau = 2\sqrt{\frac{\frac{m}{\rho} + L}{g}} - 2\sqrt{\frac{m}{\rho g}} \tag{80}$$

In order to claim that the zeros of $F(\omega)$ are the zeros of its principal term, asymptotically, it is enough to prove that they are simple. We compute $F'(\omega)$ using (77), and eliminate the second order derivatives using (74). Using the asymptotic expansions (78), we finally obtain :

$$F'(\omega) = F_1.cos(\omega\tau) + O(\frac{1}{\omega}) \tag{81}$$

We denote by ω_i the strictly increasing sequence defined by (66) (i.e. of positive zeros of F), and by ω_i^0 the increasing sequence of positive zeros of $sin(\sigma(L) - \sigma(0))$:

$$\omega_i^0 = \frac{i\pi}{\tau} \tag{82}$$

Let ω_i and ω_{i+1} be two consecutive zeros of F. We have :

$$\begin{array}{l} \omega_i = \omega_j^0 + O(\frac{1}{\omega_i}) \\ \omega_{i+1} = \omega_k^0 + O(\frac{1}{\omega_i}) \end{array} \tag{83}$$

We use (81) to show that necessarily :

$$j \neq k \tag{84}$$

then :

$$\omega_{i+1} - \omega_i \geq |\omega_k^0 - \omega_j^0| - |O(\frac{1}{\omega_i})| = \frac{\pi}{\tau} - |O(\frac{1}{\omega_i})| \tag{85}$$

implying (ii), with :

$$\alpha = \frac{\pi}{\tau} \tag{86}$$

Finally, we have the observability condition (H3) for any T_0 such that :

$$T_0 > 2\tau \tag{87}$$

□

5 Numerical results

We have simulated the closed-loop system (11) in the case of a linear function $g(\xi) = \gamma\xi$:

$$\begin{cases} y_{tt} - (ay_s)_s = 0 \\ y_s(0,t) = 0 \\ (ay_s)(L,t) + ky(L,t) = -\gamma y_t(L,t) \end{cases} \tag{88}$$

and we used a modal decomposition.

Modal study. Our goal here is to expand the solution of (88) in terms of the eigenfunctions of the open-loop system (see [21]). To do this, we have to consider homogeneous boundary conditions. Therefore we introduce a new function $z(s,t)$:

$$z(s,t) = y(s,t) - y(L,t) \tag{89}$$

which leads to the following PDE system :

$$\begin{cases} z_{tt} - (az_s)_s = -y_{tt}(L,t) = \frac{1}{\gamma}(az_{st}(L,t) + ky_t(L,t)) \\ z(L,t) = 0 \\ z_s(0,t) = 0 \\ I.C. \end{cases} \quad (90)$$

The eigenvalue problem associated with the open-loop system is the following :

$$\begin{cases} -(a\varphi_s)_s = \omega^2\varphi \\ a\varphi_s(0) = 0 \\ \varphi(L) = 0 \end{cases} \quad (91)$$

Let σ be defined by (72) and let ψ be defined from φ as in (73). Then, ψ satisfies (74) and can be written in the following form :

$$\psi(\sigma) = AJ_0(\sigma) + BY_0(\sigma) \quad (92)$$

where J_0 is the Bessel function of first kind, and Y_0 is the Neumann Bessel function of second kind (see [1]). To find the eigenfrequencies, as well as A and B, we write the boundary conditions :

$$\begin{cases} z(L,t) = 0 \quad \Rightarrow \quad AJ_0(\sigma(L)) + BY_0(\sigma(L)) = 0 \\ z_s(0,t) = 0 \quad \Rightarrow \quad AJ_0'(\sigma(0)) + BY_0'(\sigma(0)) = 0 \end{cases} \quad (93)$$

Thus, in order to obtain nonzero solutions we must have :

$$J_0(\sigma(L))Y_0'(\sigma(0)) - Y_0(\sigma(L))J_0'(\sigma(0)) = 0 \quad (94)$$

Using (72) and [1, §9.5.32], we find the solutions ω_i of this equation. The ω_i's are real, simple and increase indefinitely with i.
With every ω_i, we associate (see (72)) :

$$\sigma_i^2 = \frac{4\omega_i^2}{g}(s + \frac{m}{\rho}) \quad (95)$$

and the eigenfunction (see (73)) :

$$\varphi_i(s) = A_iJ_0(\sigma_i) + B_iY_0(\sigma_i) \quad (96)$$

where A_i and B_i are chosen to satisfy one of the boundary conditions and the normalization :

$$\int_0^L \varphi_i^2(s)ds = 1 \tag{97}$$

Then, we can write formally the solution of (90) in the following way :

$$z(s,t) = \sum_{i=1}^{\infty} \alpha_i(t)\varphi_i(s) \tag{98}$$

where the α_i satisfy :

$$\ddot{\alpha}_i + \omega_i^2\alpha_i = \frac{K_i}{\gamma}(az_{st}(L,t) + ky_t(L,t)) \tag{99}$$

with

$$K_i = \int_0^L \varphi_i(s)ds \tag{100}$$

Noticing that the evolution of $y(L,t)$ is given by :

$$y_t(L,t) = -\frac{1}{\gamma}(az_s(L,t) + ky(L,t)) \tag{101}$$

(99) gives a linear infinite dimensional state formulation of the system. In order to perform numerical computations, we kept the first five modes only, obtaining the following system :

$$\begin{cases} \ddot{\alpha}_i + \omega_i^2\alpha_i = \dfrac{K_i}{\gamma}(az_{st}(L,t) + ky_t(L,t)) \; i = 1,\ldots,5 \\ y_t(L,t) = -\dfrac{1}{\gamma}(az_s(L,t) + ky(L,t)) \end{cases} \tag{102}$$

or, equivalently :

$$\begin{cases} \ddot{\alpha}_i + \omega_i^2\alpha_i = \dfrac{K_i}{\gamma}(a\sum_{i=1}^{5} \dot{\alpha}_i(t)\dfrac{d\varphi_i}{ds}(L) + ky_t(L,t)), \quad i = 1,\ldots,5 \\ y_t(L,t) = -\dfrac{1}{\gamma}(a\sum_{i=1}^{5} \alpha_i(t)\dfrac{d\varphi_i}{ds}(L) + ky(L,t)) \end{cases} \tag{103}$$

Remark : The angular variable θ (in the rigid case) is nothing but :

$$\theta = -y_s(L,t) = -z_s(L,t) = -\sum_{i=1}^{\infty} \frac{d\varphi_i}{ds}(L)\alpha_i(t) \tag{104}$$

We used Basile to simulate system (103), results are displayed in Figure 3.

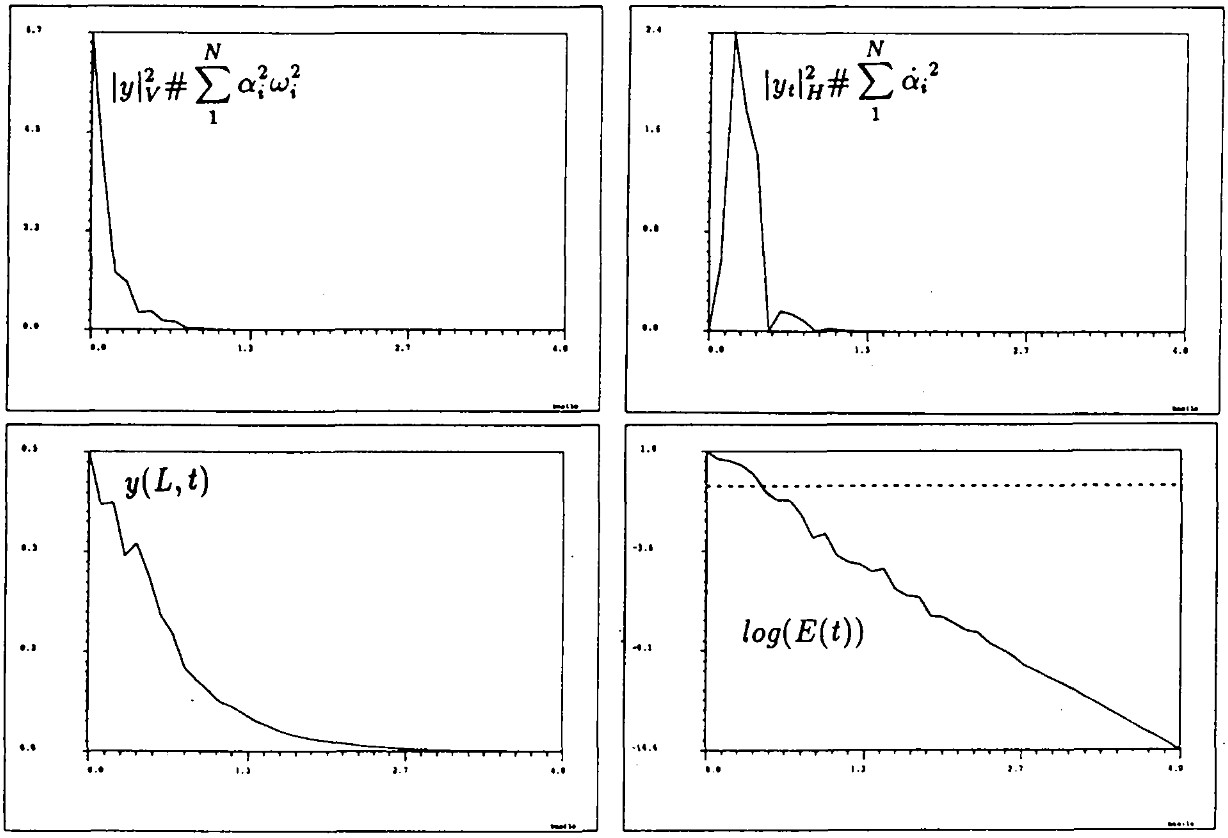

Figure 3 : Time evolution for a modal approximation with linear feedback.

As a conclusion we can stress the simplicity of the feedback. It involves only quantities usually observed in the rigid case. Practically, sensors are not able to take into account all the modes, very high frequencies are neglected. As a model case, simulations with a feedback not including all the modes (e.g. considering only the first three α_i) were performed. The residual modes (here fourth and fifth) were not stabilized, but they remained bounded. This fact is not surprising, and was already mentionned in several papers (see [4] for instance).

The complete "hybrid" PDE-ODE system, taking into account the dynamical equation (7) governing the platform, as well as the effect of observation delays on the performances of the feedback, is the subject of present investigations.

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