Stabilized finite element methods : I. application to the advective-diffusive model

L.P. Franca, S.L. Frey, T.J.R. Hughes

UNITÉ DE RECHERCHE
INRIA-ROCQUENCOURT

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P. 105
78153 Le Chesnay Cedex
France

Tél.:(1) 39 63 55 11

Rapports de Recherche

N° 1300

Programme 7
Calcul Scientifique,
Logiciels Numériques et Ingénierie Assistée
par Ordinateur

STABILIZED FINITE ELEMENT METHODS : I. APPLICATION TO THE ADVECTIVE-DIFFUSIVE MODEL

Leopoldo P. FRANCA
Sérgio L. FREY
Thomas J.R. HUGHES

Octobre 1990

STABILIZED FINITE ELEMENT METHODS : I. APPLICATION TO THE ADVECTIVE-DIFFUSIVE MODEL

Leopoldo P. FRANCA[1]
Sérgio L. FREY[2]
Thomas J.R. HUGHES[3]

Abstract

Some stabilized finite element methods for the Stokes problem are reviewed. The Douglas-Wang approach confirms better stability features for high order interpolations. Next, the advective-diffusive model is approximated in the light of various stabilized methods, a global convergence analysis is presented and numerical experiments are performed. Biquadratic elements produce better numerical results under all stabilized methods examined. The design of the stability parameter is confirmed to be a crucial ingredient for simulating the advective-diffusive model, and some improved possibilities are suggested. Combinations of these methodologies are given in the conclusions and will be examined in detail in the sequel to this paper applied to the incompressible Navier-Stokes equations.

METHODES DES ELEMENTS FINIS STABILISEES : I. APPLICATION AU MODELE ADVECTION-DIFFUSION.

Résumé

Quelques méthodes d'éléments finis stabilisées sont présentées. La méthode de Douglas-Wang a les meilleures caractéristiques de stabilité pour des interpolations d'ordre élevé. Ensuite, le modèle advection-diffusion est approximé par plusieurs méthodes stabilisées, une analyse de convergence globale est présentée et des essais numériques sont faits. Les éléments biquadratiques donnent de meilleurs résultats numériques pour toutes les méthodes stabilisées considérées. Le design d'un paramètre de stabilité est crucial pour la simulation du modèle d'advection-diffusion et quelques possibilités sont suggérées pour l'améliorer. Les combinaisons de ces méthodologies sont données en conclusion. Elles seront examinées et appliquées aux équations incompressibles de Navier-Stokes en détail dans la suite de cet article.

1. Projet Modulef, INRIA B.P. 105, 78153 LE CHESNAY Cedex France et, LNCC/CNPq, Rua Lauro Müller 455, 22290 Rio de Janeiro, RJ-Brazil
2. Instituto de Matematica Universidade Federal 32 (UFF), Pr. Valonguindo s\n° 24210 Niteroi, RJ-BRASIL
3. Division of Applied Mechanics, Durand Building, Stanford University, Stanford, California 94305 USA.

Contents

1. Introduction

Simple strategies have been proposed that may overcome most of the limitations found in the Galerkin method when applied to fluid flows, deformations in incompressible media, etc. The methodology we are concerned with, consists of adding mesh-dependent terms to the usual Galerkin method, which are functions of the residuals of the Euler-Lagrange equations evaluated elementwise. Since the residuals of the Euler-Lagrange equations are satisfied by the exact solutions, consistency is preserved in these methods. The perturbation terms are designed to enhance stability of the original Galerkin formulation without upsetting consistency. Convergence results may be derived for a wide family of simple finite element interpolations.

Stabilized methods for the advective model were introduced by Hughes and Brooks in [2,3,13,14] and referred to as SUPG, that stands for Streamline-Upwind-Petrov-Galerkin methods. Johnson and Nävert [23,28] made a convergence analysis of this method, a very important first step to establish various extensions to other fluid applications, (cf. [18-26,29] and references therein). These methods are referred by Johnson and co-workers as Streamline Diffusion Methods and currently by Hughes et al. [17] as Galerkin-least-squares methods. There is already a substantial literature on these methods and we refer to [7,8,20,22] and references therein for background.

In this paper we extend to the advective-diffusive model context, the idea contained in the method proposed by Douglas-Wang [5] for Stokes flow. In Section 2, the Douglas-Wang method for Stokes problem is reviewed and compared to the Galerkin-least-squares method [15] with respect to their different stability features [5,9] and with respect to numerical performance. By the virtue that the Douglas-Wang method has improved stability characteristics employing high order interpolations, it seems natural to examine the consequences of its application to the incompressible Navier-Stokes equations. As a first step, we analyze herein the effect of this approach on the simple advective-diffusive model. In Section 3 the method is presented and compared to other stabilized methods, a global error estimate is included and numerical tests are performed. Herein we do not deal with shock capturing or nonlinear models for the advective-diffusive problem (cf. [6,10,19,26] and references therein).

In addition, we readdress the quest for a careful design of the stability parameter, crucial for the good performance of these methods in the entire spectrum varying from advective to diffusive dominated flows (cf. [13,17,19]). The design of the stability parameter is slightly improved upon by taking into account recent computations of inverse estimate constants [12]. We modify the usual definition of the *element Peclet number* to include the effect of the specific finite element polynomial employed.

The problems considered herein are defined on a bounded domain $\Omega \subset \mathbb{R}^N, N = 2,3$, with a polygonal or polyhedral boundary Γ. A partition $\mathcal{C}_h$ of $\overline{\Omega}$ into elements consisting of triangles (tetrahedrons in $\mathbb{R}^3$) or convex quadrilaterals (hexahedrons) is performed in the usual way (i.e., no overlapping is allowed between any two elements of the partition; the union of all element domains K reproduces $\overline{\Omega}$) and a combination of triangles and quadrilaterals for the two-dimensional case can be accommodated. Quasiuniformity is *not* assumed.

In what follows $C, C_k, \widetilde{C}_k, C_1, C_2, \ldots$ denote various positive constants independent of physical properties (diffusivity, viscosity, etc ...) and independent of any element diameter h_K, $K \in \mathcal{C}_h$. As usual, $L^2(\Omega)$ is the space of square-integrable functions in Ω; $L_0^2(\Omega)$, the space of L^2-functions with zero mean value in Ω; $(\cdot,\cdot)$ denotes the L^2-inner product in Ω and $\|\cdot\|_0$, the $L^2(\Omega)$-norm. We also employ $(\cdot,\cdot)_K$ and $\|\cdot\|_{0,K}$ to denote the L^2-inner product and norm in the element domain K, respectively. Also, $C^0(\Omega)$ is the space of continuous functions in Ω, $H_0^1(\Omega)$ is the Sobolev space of functions with square-integrable value and derivatives in Ω with zero value on the boundary Γ, and, the H^1-norm is denoted by $\|\cdot\|_1$.

For convenience we adopt the following notation

$$R_m(K) = \begin{cases} P_m(K) & \text{if } K \text{ is a triangle or tetrahedron,} \\ Q_m(K) & \text{if } K \text{ is a quadrilateral or hexahedron,} \end{cases}$$

where for each integer $m \geq 0$, P_m and Q_m have the usual meaning.

2. A motivation: Stokes problem studies

2.1 The Douglas-Wang method for Stokes

Let us without loss of generality consider the viscosity equal to 1/2, such that the equations of Stokes flow are given by:

$$-\nabla \cdot \boldsymbol{\varepsilon}(\mathbf{u}) + \nabla p = \mathbf{f} \qquad \text{in } \Omega, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Omega, \tag{2}$$

$$\mathbf{u} = \mathbf{0} \qquad \text{on } \Gamma \tag{3}$$

where $\mathbf{u}$ is the velocity, p is the pressure, $\boldsymbol{\varepsilon}(\mathbf{u})$ is the symmetric part of the velocity gradient and $\mathbf{f}$ is the body force. For the discussion below, the homogeneous

Dirichlet boundary condition (3) sufffices (see [9] for more general boundary conditions).

In principle, the finite element spaces we wish to work with are the usual conforming spaces, given by:

$$\mathbf{V}_h = \{\mathbf{v} \in H_0^1(\Omega)^N \,|\, \mathbf{v}_{|K} \in R_k(K)^N,\ K \in \mathcal{C}_h\}, \tag{4}$$

$$P_h = \{p \in C^0(\Omega) \cap L_0^2(\Omega) \,|\, p_{|K} \in R_l(K),\ K \in \mathcal{C}_h\}, \tag{5a}$$

or

$$P_h = \{p \in L_0^2(\Omega) \,|\, p_{|K} \in R_l(K),\ K \in \mathcal{C}_h\}, \tag{5b}$$

where the integers $k \geq 1$ in (4), $l \geq 1$ in (5a) and $l \geq 0$ in (5b).

The (simplified) Douglas-Wang method presented in [5] can be written as: Find $\mathbf{u}_h \in \mathbf{V}_h$ and $p_h \in P_h$ such that

$$B_{DW}(\mathbf{u}_h, p_h; \mathbf{v}, q) = F_{DW}(\mathbf{v}, q) \qquad (\mathbf{v}, q) \in \mathbf{V}_h \times P_h \tag{6}$$

with

$$\begin{aligned} B_{DW}(\mathbf{u}, p; \mathbf{v}, q) &= (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v})) - (\nabla \cdot \mathbf{v}, p) - (\nabla \cdot \mathbf{u}, q) \\ &\quad - \alpha \sum_{K \in \mathcal{C}_h} h_K^2 (-\nabla \cdot \boldsymbol{\varepsilon}(\mathbf{u}) + \nabla p, \nabla \cdot \boldsymbol{\varepsilon}(\mathbf{v}) + \nabla q)_K \end{aligned} \tag{7}$$

and

$$F_{DW}(\mathbf{v}, q) = (\mathbf{f}, \mathbf{v}) - \alpha \sum_{K \in \mathcal{C}_h} h_K^2 (\mathbf{f}, \nabla \cdot \boldsymbol{\varepsilon}(\mathbf{v}) + \nabla q)_K, \tag{8}$$

where $\alpha > 0$ is a stability parameter, discussed below.

This method can be considered as an alternative to the method introduced in [15], referred to as GLS here, and written as: Find $\mathbf{u}_h \in \mathbf{V}_h$ and $p_h \in P_h$ such that

$$B_{GLS}(\mathbf{u}_h, p_h; \mathbf{v}, q) = F_{GLS}(\mathbf{v}, q) \qquad (\mathbf{v}, q) \in \mathbf{V}_h \times P_h \tag{9}$$

with

$$B_{GLS}(\mathbf{u},p;\mathbf{v},q) = (\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}(\mathbf{v})) - (\nabla\cdot\mathbf{v},p) - (\nabla\cdot\mathbf{u},q)$$
$$- \alpha \sum_{K\in\mathcal{C}_h} h_K^2(-\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{u}) + \nabla p, -\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{v}) + \nabla q)_K \tag{10}$$

and

$$F_{GLS}(\mathbf{v},q) = (\mathbf{f},\mathbf{v}) - \alpha \sum_{K\in\mathcal{C}_h} h_K^2(\mathbf{f}, -\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{v}) + \nabla q)_K. \tag{11}$$

Remarks

1. By comparing the equations (7)-(8) with (10)-(11), we note that their only difference is in the sign in front of $\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{v})$. This apparent slight change produces nicer stability characteristics for the Douglas-Wang alternative when we employ high order interpolations (see discussion below). For linear velocities, $\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{v}) = 0$, and the methods coincide with each other, reducing to the method considered in [16] for this case. If, in addition, $\mathbf{f} = \mathbf{0}$, then they are all equal to the method proposed by Brezzi and Pitkäranta [1].
2. The original methods, as proposed in [5,15], have an additional "jump" term on pressures, which is not important to our discussion (See Kechkar and Silvester [27] for further developments using "jump" terms).

2.2 Discussion of stability characteristics

In [9] improved error analysis are obtained for the methods given by equations (6) and (9). Let us recall in this section the main results derived therein and focus our attention to how differently stability is achieved in each method.

Let us first recall an inverse estimate [4], given by

$$C_k \sum_{K\in\mathcal{C}_h} h_K^2\|\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{v})\|_{0,K}^2 \leq \|\boldsymbol{\varepsilon}(\mathbf{v})\|_0^2 \qquad \mathbf{v}\in\mathbf{V}_h. \tag{12}$$

where C_k is a positive constant independent of $\mathcal{C}_h$.

Consider the following notation (we recall that N is the space dimension)

$$S(K) = \begin{cases} P_N(K) & \text{if } K \text{ is a triangle or tetrahedron,} \\ Q_2(K) & \text{if } K \text{ is a quadrilateral or hexahedron.} \end{cases} \tag{13}$$

and

$$\mathbf{S}_h = \{\mathbf{v} \in H_0^1(\Omega)^N \,|\, \mathbf{v}_{|K} \in S(K)^N, \; K \in \mathcal{C}_h\} \tag{14}$$

We may rewrite the Theorem 3.2 of [9] for the Douglas-Wang method as follows:

THEOREM 2.1. *Suppose that the solution to (1)-(3) satisfies* $\mathbf{u} \in H^{k+1}(\Omega)^N$ *and* $p \in H^{l+1}(\Omega) \cap L_0^2(\Omega)$, *and that one of the following conditions is satisfied*

I. $\mathbf{S}_h \subset \mathbf{V}_h$,

II. $P_h \subset C^0(\Omega)$,

Then for $\alpha > 0$, *(6)-(8) has a unique solution satisfying*

$$\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0 \leq C(h^k|\mathbf{u}|_{k+1} + h^{l+1}|p|_{l+1}). \tag{15}$$

In addition, by the regularity of the Stokes problem on a convex polygon Ω

$$\|\mathbf{u}\|_2 + \|p\|_1 \leq C\|\mathbf{f}\|_0, \tag{16}$$

the following estimate is also established

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \leq C(h^{k+1}|\mathbf{u}|_{k+1} + h^{l+2}|p|_{l+1}). \quad \blacksquare \tag{17}$$

The proof of Theorem 2.1 for the original method (including the jump terms on pressure) can be found in [9].

In other words, Theorem 2.1 states that the method given by equations (6)-(8) converges for velocity and pressure at the rates given by (15) and (17) if *either*

i) the velocity interpolation is sufficiently high order (*regardless of the degree of the pressure interpolation* $l \geq 0$ *and whether the pressure is continuous or not*),

ii) the pressure is interpolated continuously (*regardless of the degree of the velocity interpolation* $k \geq 1$).

Thus, almost *any* combination of velocity and pressure interpolations is convergent for this method, which is a result that does *not* hold for the Galerkin method ($\alpha = 0$) for such a wide choice of combinations.

For GLS (equations (9)-(11)), an identical convergence result holds (see [9]), provided the assumption of $\alpha > 0$ is restricted to $0 < \alpha < C_k$, where C_k is the constant in equation (12). This restriction in the GLS method compared to the Douglas-Wang alternative can be understood, if we examine the stability characteristics of each method as follows (see [9] for further details):

For GLS:

$$\begin{aligned} B_{GLS}(\mathbf{u}, p; \mathbf{u}, -p) &= \|\boldsymbol{\varepsilon}(\mathbf{u})\|_0^2 + \alpha \sum_{K \in \mathcal{C}_h} h_K^2 \|\nabla p\|_{0,K}^2 - \alpha \sum_{K \in \mathcal{C}_h} h_K^2 \|\nabla \cdot \boldsymbol{\varepsilon}(\mathbf{u})\|_{0,K}^2 \\ &\geq (1 - \alpha C_k^{-1}) \|\boldsymbol{\varepsilon}(\mathbf{u})\|_0^2 + \alpha \sum_{K \in \mathcal{C}_h} h_K^2 \|\nabla p\|_{0,K}^2 \\ &\geq C_1 \|\boldsymbol{\varepsilon}(\mathbf{u})\|_0^2 + \alpha \sum_{K \in \mathcal{C}_h} h_K^2 \|\nabla p\|_{0,K}^2 \end{aligned} \tag{18}$$

since we select $0 < \alpha < C_k$.

For Douglas-Wang: Let $\gamma > 1$ be a parameter to be set below:

$$\begin{aligned}
B_{DW}(\mathbf{u},p;\mathbf{u},-p) &= \|\boldsymbol{\varepsilon}(\mathbf{u})\|_0^2 + \alpha \sum_{K\in\mathcal{C}_h} h_K^2 \|-\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{u}) + \nabla p\|_{0,K}^2 \\
&= \|\boldsymbol{\varepsilon}(\mathbf{u})\|_0^2 + \alpha \sum_{K\in\mathcal{C}_h} h_K^2 \|\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{u})\|_{0,K}^2 \\
&\quad + 2\alpha \sum_{K\in\mathcal{C}_h} h_K^2 (-\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{u}), \nabla p)_K + \alpha \sum_{K\in\mathcal{C}_h} h_K^2 \|\nabla p\|_{0,K}^2 \\
&\geq \|\boldsymbol{\varepsilon}(\mathbf{u})\|_0^2 + \alpha(1-\gamma) \sum_{K\in\mathcal{C}_h} h_K^2 \|\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{u})\|_{0,K}^2 \\
&\quad + \left(1-\frac{1}{\gamma}\right)\alpha \sum_{K\in\mathcal{C}_h} h_K^2 \|\nabla p\|_{0,K}^2 \\
&\geq \left(1+\alpha(1-\gamma)C_k^{-1}\right)\|\boldsymbol{\varepsilon}(\mathbf{u})\|_0^2 + \left(1-\frac{1}{\gamma}\right)\alpha \sum_{K\in\mathcal{C}_h} h_K^2 \|\nabla p\|_{0,K}^2 \\
&\geq C_2 \|\boldsymbol{\varepsilon}(\mathbf{u})\|_0^2 + C_3\alpha \sum_{K\in\mathcal{C}_h} h_K^2 \|\nabla p\|_{0,K}^2
\end{aligned} \tag{19}$$

if we set $1 < \gamma < (1 + C_k\alpha^{-1})$, for every $\alpha > 0$.

Therefore, the Douglas-Wang alternative is stable for *any* positive value of the parameter α, whereas this parameter in GLS is bounded by C_k.

This brings us to the practical question of what the value of α should be. From a strict theoretical point-of-view, this is an irrelevant question, since once α is fixed according to the rules (namely, any positive value for Douglas-Wang and $0 < \alpha < C_k$ for GLS), then by Theorem 2.1, when the mesh is refined, we will have to observe convergence, regardless of our guesswork. Clearly, this is not a satisfactory answer, given that certain choices of α look unreasonable from start: for example, take $\alpha << 1$, then both methods will behave, in practice, as if they were the Galerkin method, and, for certain combinations of velocity and pressure interpolations, the pathological behavior of the Galerkin method will contaminate and render useless most of the computations.

To partially address these remarks, let us test various values of α for a fixed mesh and observe the different numerical solutions obtained. The numerical experiments are performed for the "leaky" cavity problem: a unit square with velocity boundary condition $u_1 = 1, u_2 = 0$ at $y = 1$ $(0 \leq x \leq 1)$ and $\mathbf{u} = \mathbf{0}$ on the other boundaries. Biquadratic velocity combined with continuous biquadratic pressure (Q2/Q2C) is used on a uniform mesh with 17x17 nodes (For the detail on how we treat the upper corner boundary condition to benefit from the Q2 approximation see, e.g., Figure 3 of [16]). In Figure 1 we show the pressure results for GLS with a fixed ("nice") value of $\alpha = 0.01$. In Figure 2 we compare the pressure elevation for two values of α using GLS and Douglas-Wang. Note that for $\alpha = 0.0001$, our concerns are numerically confirmed, and both methods perform poorly in this "Galerkin limit". The surprise is left for a rather "reasonable" value of α, $\alpha = 1$, when GLS fails and Douglas-Wang still produces a nice solution. This can be explained by examining the value of C_k in (12). According to recent results derived by Harari [12], for Q2, $C_{k=2} = 11/270$ for square elements. Therefore in our experiments we took GLS beyond its limit of applicability, namely, $\alpha = 1 > \alpha_{Lim} = C_{k=2} = 0.0407$.

Summing up, the Douglas-Wang alternative seems more attractive for high order methods, because it produces reasonable numerical solutions for a wide range of the α-parameter. This prompted us to examine extensions of this idea to the incompressible Navier-Stokes equations. To do this, in the next section, we examine the application of this idea on a simpler model: the advection-diffusion equation. Navier-Stokes will be treated in the sequel to this paper.

3. Stabilized methods for the advection-diffusion equation

3.1 A stabilized method

Consider the (homogeneous-Dirichlet) advective-diffusive problem of finding a scalar field $u = u(\mathbf{x})$, such that

$$\mathbf{a} \cdot \nabla u - \kappa \Delta u = f \qquad \text{in } \Omega, \tag{20}$$

$$u = 0 \qquad \text{on } \Gamma \tag{21}$$

where $\mathbf{a}(\mathbf{x})$ is the given flow velocity with $\nabla \cdot \mathbf{a} = 0$ in Ω, $\kappa = \kappa(\mathbf{x}) > 0$ is the diffusivity and $f(\mathbf{x})$ is a prescribed source function. The results presented herein can be extended without any conceptual difficulty to accommodate more general boundary conditions, following e.g. [17].

The scalar field u is approximated with the following standard conforming approximation,

$$W_h = \{ v \in H_0^1(\Omega) \,|\, v_{|K} \in R_k(K), \ K \in \mathcal{C}_h \} \tag{22}$$

The finite element we wish to consider is: Find $u_h \in W_h$ such that

$$B(u_h, v) = F(v) \qquad v \in W_h \tag{23}$$

with

$$B(u, v) = (\mathbf{a} \cdot \nabla u, v) + (\kappa \nabla u, \nabla v) + \sum_{K \in \mathcal{C}_h} (\mathbf{a} \cdot \nabla u - \kappa \Delta u, \tau(\mathbf{a} \cdot \nabla v + \kappa \Delta v))_K \tag{24}$$

and

$$F(v) = (f, v) + \sum_{K \in \mathcal{C}_h} (f, \tau(\mathbf{a} \cdot \nabla v + \kappa \Delta v))_K \tag{25}$$

where the stability parameter τ is defined from error analysis considerations as follows:

$$\tau(\mathbf{x}, \mathrm{Pe}_K(\mathbf{x})) = \frac{h_K}{2|\mathbf{a}(\mathbf{x})|_p}\xi(\mathrm{Pe}_K(\mathbf{x})) \tag{26}$$

$$\mathrm{Pe}_K(\mathbf{x}) = \frac{m_k|\mathbf{a}(\mathbf{x})|_p h_K}{2\kappa(\mathbf{x})} \tag{27}$$

$$\xi(\mathrm{Pe}_K(\mathbf{x})) = \begin{cases} \mathrm{Pe}_K(\mathbf{x}) & , 0 \leq \mathrm{Pe}_K(\mathbf{x}) < 1 \\ 1 & , \mathrm{Pe}_K(\mathbf{x}) \geq 1 \end{cases} \tag{28}$$

$$|\mathbf{a}(\mathbf{x})|_p = \begin{cases} \left(\sum_{i=1}^{N}|a_i(\mathbf{x})|^p\right)^{1/p} & , \quad 1 \leq p < \infty \\ \max\limits_{i=1,N}|a_i(\mathbf{x})| & , \quad p = \infty \end{cases} \tag{29}$$

$$m_k = \min\left\{\frac{1}{3}, \ 2\widetilde{C}_k\right\} \tag{30}$$

$$\widetilde{C}_k \sum_{K\in\mathcal{C}_h} h_K^2\|\Delta v\|_{0,K}^2 \leq \|\nabla v\|_0^2 \qquad v \in W_h \tag{31}$$

For comparison purposes, let us also review the SUPG and GLS methods for this model [3,17]:

SUPG:

$$B_{SUPG}(u_h, v) = F_{SUPG}(v) \qquad v \in W_h \tag{32}$$

with

$$B_{SUPG}(u,v) = (\mathbf{a}\cdot\nabla u, v) + (\kappa\nabla u, \nabla v) + \sum_{K\in\mathcal{C}_h}(\mathbf{a}\cdot\nabla u - \kappa\Delta u, \tau\,\mathbf{a}\cdot\nabla v)_K \tag{33}$$

and

$$F_{SUPG}(v) = (f, v) + \sum_{K\in\mathcal{C}_h}(f, \tau\mathbf{a}\cdot\nabla v)_K, \tag{34}$$

GLS:

$$B_{GLS}(u_h, v) = F_{GLS}(v) \qquad v \in W_h \tag{35}$$

with

$$B_{GLS}(u,v) = (\mathbf{a}\cdot\nabla u, v) + (\kappa\nabla u, \nabla v) + \sum_{K\in\mathcal{C}_h} (\mathbf{a}\cdot\nabla u - \kappa\Delta u, \tau\,(\mathbf{a}\cdot\nabla v - \kappa\Delta v))_K \quad (36)$$

and

$$F_{GLS}(v) = (f,v) + \sum_{K\in\mathcal{C}_h} (f, \tau\,(\mathbf{a}\cdot\nabla v - \kappa\Delta v))_K, \quad (37)$$

Remarks

1. Glancing through the methods, we note that their difference appears in the τ-terms through $\kappa\Delta v$ which is: added in (24)-(25); absent in SUPG; and with a minus sign for GLS. Obviously, when we employ linear elements $\Delta v = 0$ on element interiors, and therefore, the three methods coincide. By introducing the method (23)-(25) we wish to evaluate the consequences of the sign change of $\kappa\Delta v$, which plays an important role in improving the stability characteristics of high order elements in the Stokes problem (cf. Section 2).

2. The formulae defining τ, (26)-(31), are different than the versions of SUPG proposed in [17,19]. We wish to combine the advantages of both approaches. First the p-norm of $\mathbf{a}$ is included in the τ-definition without changing the standard definition of h_K. Herein we consider the constant $\widetilde{C}_k$ to be the largest possible value satisfying equation (31). Existence of such constants follows from standard inverse estimates (cf. [4]) with h_K being the usual element diameter. As defined the method *needs* the specific value of such constants. Recently, Harari [12] suggested to modify the usual definition of h_K to $h_K = \sqrt{2}\,\text{Area}(K)/\text{Diagonal}(K)$, for rectangular elements, so that for biquadratic elements, $\widetilde{C}_{k=2} = 1/24$ is obtained independent of the relative size and shape of all rectangles in the partition (within the scope of the usual

regularity assumption, [4]). In the numerical computations presented herein we adopt Harari's suggestions for h_K and the computed value $\widetilde{C}_{k=2} = 1/24$. Further improved explicit computations of $\widetilde{C}_k$ are desirable, since this is an ingredient built-in in these finite element formulations.

3. We are modifying the usual definition of the element Peclet number Pe_K, by including m_k, which takes into account the effect of the degree of the interpolation used in each method (For example, in 2-D, for bilinear elements $\widetilde{C}_{k=1} = \infty$ and for biquadratic elements $\widetilde{C}_{k=2} = 1/24$ (see [12]) and, therefore, by eq. (30), $m_{k=1} = 1/3$ and $m_{k=2} = 1/12$). This definition may be viewed as a rescaling of the usual element Peclet number, such that, for all interpolations employed, the locally advection dominated flows are given by $\mathrm{Pe}_K(\mathbf{x}) > 1$ and the locally diffusion dominated flows are given by $\mathrm{Pe}_K(\mathbf{x}) < 1$. (See also discussion in Remark 4 in Section 3.2). This behavior follows immediately from (26)-(28) since $\tau = \mathcal{O}(h_K)$ for $\mathrm{Pe}_K(\mathbf{x}) > 1$ and $\tau = \mathcal{O}(h_K^2)$ for $\mathrm{Pe}_K(\mathbf{x}) < 1$ simulates the correct order of τ in those limits (see [19]). Furthermore, the formulae (26)-(30) preclude the need of further requirements on the τ parameter for deriving the error estimates presented in the next section.

3.2 Error analysis

In this section we study the convergence features of the method given by equations (23)-(25) with stability parameter defined by (26)-(31).

The definition of τ was originally designed inspecting a one-dimensional problem, with constant flow and diffusivity, absence of source, uniform mesh, linear elements, so that SUPG produce nodal exact solutions ([13]). It has been verified that an asymptotic behavior simulating advection dominated flows and diffusion dominated flows (herein given by eq. (28)) produces a stable convergent class of methods for the advection-diffusion model (see [3,25,28] for SUPG, [17] for GLS, and references therein). We show in this section that similar convergence results hold for the method proposed in this work (cf. (23)-(25)) and readdress the stability analysis for SUPG and GLS, under the new definition of τ given by (26)-(31).

Let us initially collect immediate consequences of (26) to (30). By (30) either

$$m_k = \frac{1}{3} \quad \Rightarrow \quad m_k \leq 2\widetilde{C}_k$$

or

$$m_k = 2\widetilde{C}_k$$

Thus, in both cases,

$$\frac{m_k}{4\widetilde{C}_k} \leq \frac{1}{2} \tag{38}$$

Next, by (28) either

$$\frac{\xi(\mathrm{Pe}_K(\mathbf{x}))}{\mathrm{Pe}_K(\mathbf{x})} = 1$$

or

$$\frac{\xi(\mathrm{Pe}_K(\mathbf{x}))}{\mathrm{Pe}_K(\mathbf{x})} = \frac{1}{\mathrm{Pe}_K(\mathbf{x})} \leq 1$$

Thus

$$\frac{\xi(\mathrm{Pe}_K(\mathbf{x}))}{\mathrm{Pe}_K(\mathbf{x})} \leq 1 \tag{39}$$

Let us now present some preliminary results:

LEMMA 3.1 (Stability) : *Assuming the given data* $\mathbf{a}(\mathbf{x})$ *and* $\kappa(\mathbf{x})$ *to satisfy*

I. $\nabla \cdot \mathbf{a}(\mathbf{x}) = 0$

II. $\kappa(\mathbf{x}) = \kappa = constant > 0$,

$$B(v,v) \geq \frac{1}{2}(\kappa\|\nabla v\|_0^2 + \|\tau^{1/2}\,\mathbf{a}\cdot\nabla v\|_0^2) \quad \blacksquare$$

Proof: First note that by integration by parts, by assumption I and by (22)

$$(\mathbf{a}\cdot\nabla v, v) = 0 \qquad v \in W_h$$

since $v = 0$ on Γ for $v \in W_h$.

Next, by (24)

$$B(v,v) = 0 + \kappa\|\nabla v\|_0^2 + \|\tau^{1/2}\,\mathbf{a}\cdot\nabla v\|_0^2 - \sum_{K\in\mathcal{C}_h}\|\tau^{1/2}\,\kappa\Delta v\|_{0,K}^2 \tag{40}$$

Note that by (26)-(27) and (39)

$$\begin{aligned}\tau(\mathbf{x},\mathrm{Pe}_K(\mathbf{x})) &= \frac{h_K}{2|\mathbf{a}(\mathbf{x})|_p}\xi(\mathrm{Pe}_K(\mathbf{x}))\\ &= \frac{m_k h_K^2}{4\kappa}\frac{\xi(\mathrm{Pe}_K(\mathbf{x}))}{\mathrm{Pe}_K(\mathbf{x})}\\ &\leq \frac{m_k h_K^2}{4\kappa}\end{aligned} \tag{41}$$

Thus, for each element domain K, τ is bounded by a constant. Substituting in the last term of (40) gives

$$\begin{aligned}\sum_{K\in\mathcal{C}_h}\|\tau^{1/2}\,\kappa\Delta v\|_{0,K}^2 &\leq \frac{m_k\kappa}{4}\sum_{K\in\mathcal{C}_h} h_K^2\|\Delta v\|_{0,K}^2 && \text{(by assumption II and (41))}\\ &\leq \frac{m_k}{4\tilde{C}_k}\kappa\|\nabla v\|_0^2 && \text{(by (31))}\\ &\leq \frac{\kappa}{2}\|\nabla v\|_0^2 && \text{(by (38))}\end{aligned} \tag{42}$$

Combining this estimate with (40) yields the desired result. $\blacksquare$

Before starting the next Lemma let us recall that by standard approximation theory [4], for a regular family of elements, there exists an interpolant $\widetilde{u}_{h|K} \in R_k(K)$ such that

$$\|u - \widetilde{u}_{h|K}\|_{m,K} \leq C h_K^{k+1-m} |u|_{k+1,K} \qquad \forall u \in H^{k+1}(K)\,,\ 0 \leq m \leq k+1 \quad (43)$$

(The expression *regular family of elements* has the usual meaning [4]).

The following *interpolation estimate* can now be estabilished:

LEMMA 3.2: *Assume that the solution to (20)-(21) satisfies* $u \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$. *Denoting by* $\eta = \widetilde{u}_h - u$ *the interpolation error, for each* $K \in \mathcal{C}_h$

a) *if* $\mathrm{Pe}_K(\mathbf{x}) \geq 1\,, \quad \forall \mathbf{x} \in K$ *then*

$$\begin{aligned} \|\tau^{-1/2}\,\eta\|_{0,K}^2 + \kappa\|\nabla\eta\|_{0,K}^2 + \|\tau^{1/2}\,\mathbf{a}\cdot\nabla\eta\|_{0,K}^2 + \|\tau^{1/2}\,\kappa\Delta\eta\|_{0,K}^2 \\ \leq C \sup_{\mathbf{x}\in K} |\mathbf{a}|_p \, h_K^{2k+1} |u|_{k+1,K}^2 \end{aligned} \quad (44)$$

b) *if* $0 \leq \mathrm{Pe}_K(\mathbf{x}) < 1\,, \quad \forall \mathbf{x} \in K$ *then*

$$\|\tau^{-1/2}\,\eta\|_{0,K}^2 + \kappa\|\nabla\eta\|_{0,K}^2 + \|\tau^{1/2}\,\mathbf{a}\cdot\nabla\eta\|_{0,K}^2 + \|\tau^{1/2}\,\kappa\Delta\eta\|_{0,K}^2 \leq C\,\kappa\,h_K^{2k} |u|_{k+1,K}^2 \quad (45)$$

Therefore,

$$\begin{aligned} \|\tau^{-1/2}\,\eta\|_0^2 + \kappa\|\nabla\eta\|_0^2 + \|\tau^{1/2}\,\mathbf{a}\cdot\nabla\eta\|_0^2 + \sum_{K\in\mathcal{C}_h} \|\tau^{1/2}\,\kappa\Delta\eta\|_0^2 \\ \leq C \sum_{K\in\mathcal{C}_h} h_K^{2k} |u|_{k+1,K}^2 (\mathrm{H}(\mathrm{Pe}_K - 1) h_K \sup_{\mathbf{x}\in K} |\mathbf{a}|_p + \mathrm{H}(1 - \mathrm{Pe}_K)\kappa) \end{aligned} \quad (46)$$

where $\mathrm{H}(\cdot)$ *is the Heaviside function given by*

$$\mathrm{H}(x-y) = \begin{cases} 0, & x < y; \\ 1, & x > y. \end{cases} \qquad \blacksquare \quad (47)$$

Proof: Let us first note that

$$\begin{aligned}
\|\tau^{1/2}\,\mathbf{a}\cdot\nabla\eta\|_{0,K}^2 &= \|\left(\frac{h_K}{2|\mathbf{a}|_p}\xi\right)^{1/2}\mathbf{a}\cdot\nabla\eta\|_{0,K}^2 \\
&\le \frac{h_K}{2}\|\left(\frac{\xi}{|\mathbf{a}|_p}\right)^{1/2}|\mathbf{a}|_2\,|\nabla\eta|_2\|_{0,K}^2 \qquad \text{(by Cauchy-Schwarz)} \\
&\le \widehat{C}\,\frac{h_K}{2}\|(\xi|\mathbf{a}|_p)^{1/2}\,|\nabla\eta|_2\|_{0,K}^2 \quad \text{(by equivalence of norms on } \mathrm{R}^N)
\end{aligned} \tag{48}$$

Now, we divide the proof in two parts:

(a) Let $\mathrm{Pe}_K(\mathbf{x}) \ge 1, \forall \mathbf{x} \in K$. Then

$$\begin{aligned}
&\|\tau^{-1/2}\,\eta\|_{0,K}^2 + \kappa\|\nabla\eta\|_{0,K}^2 + \|\tau^{1/2}\,\mathbf{a}\cdot\nabla\eta\|_{0,K}^2 + \|\tau^{1/2}\,\kappa\Delta\eta\|_{0,K}^2 \\
&\le \frac{2}{h_K}\||\mathbf{a}|_p^{1/2}\,\eta\|_{0,K}^2 + \|\left(\frac{m_k|\mathbf{a}(\mathbf{x})|_p h_K}{2\mathrm{Pe}_K}\right)^{1/2}\nabla\eta\|_{0,K}^2 + \widehat{C}\,\frac{h_K}{2}\sup_{\mathbf{x}\in K}|\mathbf{a}|_p\|\nabla\eta\|_{0,K}^2 \\
&\quad + \frac{m_k^2 h_K^3}{8}\|\left(\frac{|\mathbf{a}|_p}{\mathrm{Pe}_K}\right)^{1/2}\Delta\eta\|_{0,K}^2 \\
&\le \frac{2}{h_K}\sup_{\mathbf{x}\in K}|\mathbf{a}|_p\,\|\eta\|_{0,K}^2 + h_K\sup_{\mathbf{x}\in K}|\mathbf{a}|_p\|\nabla\eta\|_{0,K}^2 + \widehat{C}\,\frac{h_K}{2}\sup_{\mathbf{x}\in K}|\mathbf{a}|_p\|\nabla\eta\|_{0,K}^2 \\
&\quad + h_K^3\sup_{\mathbf{x}\in K}|\mathbf{a}|_p\|\Delta\eta\|_{0,K}^2 \\
&\le C\sup_{\mathbf{x}\in K}|\mathbf{a}|_p h_K^{2k+1}|u|_{k+1,K}^2
\end{aligned} \tag{49}$$

(b) Let $0 < \mathrm{Pe}_K(\mathbf{x}) \le 1, \forall \mathbf{x} \in K$. Then

$$\begin{aligned}
&\|\tau^{-1/2}\,\eta\|_{0,K}^2 + \kappa\|\nabla\eta\|_{0,K}^2 + \|\tau^{1/2}\,\mathbf{a}\cdot\nabla\eta\|_{0,K}^2 + \|\tau^{1/2}\,\kappa\Delta\eta\|_{0,K}^2 \\
&\le \frac{4\kappa}{m_k h_K^2}\|\eta\|_{0,K}^2 + \kappa\|\nabla\eta\|_{0,K}^2 + \widehat{C}\,\|\left(\frac{\mathrm{Pe}_K^2\kappa}{m_k}\right)^{1/2}|\nabla\eta|_2\|_{0,K}^2 + \frac{m_k h_K^2\kappa}{4}\|\Delta\eta\|_{0,K}^2 \\
&\le \frac{4\kappa}{m_k h_K^2}\|\eta\|_{0,K}^2 + \kappa\|\nabla\eta\|_{0,K}^2 + \widehat{C}\,\frac{\kappa}{m_k}\|\nabla\eta\|_{0,K}^2 + h_K^2\kappa\|\Delta\eta\|_{0,K}^2 \\
&\le C\,\kappa\,h_K^{2k}\,|u|_{k+1,K}^2
\end{aligned} \tag{50}$$

Therefore, (44)-(46) follows from (49)-(50). ■

We may now establish the following convergence estimate:

THEOREM 3.1. *Under the same hypotheses as in Lemma 3.1 and Lemma 3.2, the solution* u_h *of the method given by eqs. (23)-(30) converges to* u*, solution of eqs. (20)-(21) as follows:*

$$\kappa\|\nabla(u_h-u)\|_0^2+\|\tau^{1/2}\,\mathbf{a}\cdot\nabla(u_h-u)\|_{0,K}^2$$
$$\leq C\sum_{K\in\mathcal{C}_h} h_K^{2k}|u|_{k+1,K}^2\Big(\mathrm{H}(\mathrm{Pe}_K-1)h_K\sup_{\mathbf{x}\in K}|\mathbf{a}|_p+\mathrm{H}(1-\mathrm{Pe}_K)\kappa\Big)$$

where $\mathrm{H}(\,\cdot\,)$ *is defined in eq. (47).* ■

Proof: Let $e_h = u_h - \widetilde{u}_h$ and $e = e_h + \eta$. The convergence proof goes as follows:

$$\frac{1}{2}\Big(\kappa\|\nabla e_h\|_0^2+\|\tau^{1/2}\,\mathbf{a}\cdot\nabla e_h\|_0^2\Big)$$
$$\leq B(e_h,e_h) \qquad \text{(by Lemma 3.1)}$$
$$= B(e-\eta,e_h)$$
$$= -B(\eta,e_h) \qquad \text{(consistency)}$$
$$\leq |B(\eta,e_h)|$$
$$= |(\mathbf{a}\cdot\nabla\eta,e_h)+\kappa(\nabla\eta,\nabla e_h)+\sum_{K\in\mathcal{C}_h}(\mathbf{a}\cdot\nabla\eta-\kappa\Delta\eta,\tau(\mathbf{a}\cdot\nabla e_h+\kappa\Delta e_h))_K|$$
$$\leq |(\eta,\mathbf{a}\cdot\nabla e_h)|+\frac{3\kappa}{2}\|\nabla\eta\|_0^2+\frac{\kappa}{6}\|\nabla e_h\|_0^2+\sum_{K\in\mathcal{C}_h}(\|\tau^{1/2}\,\mathbf{a}\cdot\nabla\eta\|_{0,K}$$
$$+\|\tau^{1/2}\,\kappa\Delta\eta\|_{0,K})(\|\tau^{1/2}\,\mathbf{a}\cdot\nabla e_h\|_{0,K}+\|\tau^{1/2}\,\kappa\Delta e_h\|_{0,K})$$
$$\leq \frac{1}{8}\|\tau^{1/2}\,\mathbf{a}\cdot\nabla e_h\|_0^2+2\|\tau^{-1/2}\,\eta\|_0^2+\frac{\kappa}{6}\|\nabla e_h\|_0^2+\frac{3\kappa}{2}\|\nabla\eta\|_0^2$$
$$+\frac{1}{8}\|\tau^{1/2}\,\mathbf{a}\cdot\nabla e_h\|_0^2+4\|\tau^{1/2}\,\mathbf{a}\cdot\nabla\eta\|_0^2+4\sum_{K\in\mathcal{C}_h}\|\tau^{1/2}\,\kappa\Delta\eta\|_{0,K}^2$$
$$+\frac{1}{6}\sum_{K\in\mathcal{C}_h}\|\tau^{1/2}\,\kappa\Delta e_h\|_{0,K}^2+3\|\tau^{1/2}\,\mathbf{a}\cdot\nabla\eta\|_0^2+3\sum_{K\in\mathcal{C}_h}\|\tau^{1/2}\,\kappa\Delta\eta\|_{0,K}^2$$

$$\leq \frac{1}{4}\|\tau^{1/2}\,\mathbf{a}\cdot\nabla e_h\|_0^2 + \frac{\kappa}{6}\|\nabla e_h\|_0^2 + \frac{\kappa}{12}\sum_{K\in\mathcal{C}_h} h_K^2 \widetilde{C}_k \|\Delta e_h\|_{0,K}^2$$

$$+ 2\|\tau^{-1/2}\,\eta\|_0^2 + \frac{3\kappa}{2}\|\nabla\eta\|_0^2 + 7\|\tau^{1/2}\,\cdot\nabla\eta\|_0^2$$

$$+ 7\sum_{K\in\mathcal{C}_h}\|\tau^{1/2}\,\kappa\Delta\eta\|_{0,K}^2 \qquad \text{(by (38) and (41))}$$

Therefore, by the inverse estimate (31) and by combining with Lemma (3.2) yields

$$\frac{1}{4}\left(\kappa\|\nabla e_h\|_0^2 + \|\tau^{1/2}\,\mathbf{a}\cdot\nabla e_h\|_0^2\right)$$
$$\leq C\sum_{K\in\mathcal{C}_h} h_K^{2k}|u|_{k+1,K}^2\Big(\mathrm{H}(\mathrm{Pe}_K-1)h_K\sup_{\mathbf{x}\in K}|\mathbf{a}|_p + \mathrm{H}(1-\mathrm{Pe}_K)\kappa\Big) \tag{51}$$

Since by Lemma 3.2

$$\kappa\|\nabla\eta\|_0^2 + \|\tau^{1/2}\,\mathbf{a}\cdot\nabla\eta\|_0^2$$
$$\leq C\sum_{K\in\mathcal{C}_h} h_K^{2k}\,|u|_{k+1,K}^2\Big(\mathrm{H}(\mathrm{Pe}_K-1)h_K\sup_{\mathbf{x}\in K}|\mathbf{a}|_p + \mathrm{H}(1-\mathrm{Pe}_K)\kappa\Big) \tag{52}$$

the result follows by redefining constants in (51) and (52) and using the triangle inequality. ∎

The convergence of SUPG and GLS with τ defined by the formulae (26)-(31) can be established similarly to the analysis above. Under the same hypotheses as in the Lemma 3.1, let us sketch the stability arguments (error estimates can be done as above):

SUPG

$$B_{SUPG}(v,v) = 0 + \kappa\|\nabla v\|_0^2 + \|\tau^{1/2}\,\mathbf{a}\cdot\nabla v\|_0^2 - \sum_{K\in\mathcal{C}_h}(\kappa\Delta v, \tau\,\mathbf{a}\cdot\nabla v)_K$$
$$\geq \kappa\|\nabla v\|_0^2 + \frac{1}{2}\|\tau^{1/2}\,\mathbf{a}\cdot\nabla v\|_0^2 - \frac{1}{2}\sum_{K\in\mathcal{C}_h}\|\tau^{1/2}\,\kappa\Delta v\|_{0,K}^2$$
$$\geq \frac{3\kappa}{4}\|\nabla v\|_0^2 + \frac{1}{2}\|\tau^{1/2}\,\mathbf{a}\cdot\nabla v\|_0^2 \tag{53}$$
$$\geq \frac{1}{2}(\kappa\|\nabla v\|_0^2 + \|\tau^{1/2}\,\mathbf{a}\cdot\nabla v\|_0^2)$$

The second to the third line follows from eq. (42), which yields stability in the norm used in Theorem 3.1. ∎

GLS: Let $\gamma > 1$ to be set below:

$$\begin{aligned} B_{GLS}(v,v) &= 0 + \kappa\|\nabla v\|_0^2 + \|\tau^{1/2}\,\mathbf{a}\cdot\nabla v\|_0^2 - 2\sum_{K\in\mathcal{C}_h}(\kappa\Delta v, \tau\,\mathbf{a}\cdot\nabla v)_K \\ &\quad + \sum_{K\in\mathcal{C}_h}\|\tau^{1/2}\,\kappa\Delta v\|_{0,K}^2 \\ &\geq \kappa\|\nabla v\|_0^2 + \left(1-\frac{1}{\gamma}\right)\|\tau^{1/2}\,\mathbf{a}\cdot\nabla v\|_0^2 + (1-\gamma)\sum_{K\in\mathcal{C}_h}\|\tau^{1/2}\,\kappa\Delta v\|_{0,K}^2 \\ &\geq \left(\frac{3}{2}-\frac{\gamma}{2}\right)\kappa\|\nabla v\|_0^2 + \left(1-\frac{1}{\gamma}\right)\|\tau^{1/2}\,\mathbf{a}\cdot\nabla v\|_0^2 \\ &\geq \frac{1}{2}(\kappa\|\nabla v\|_0^2 + \|\tau^{1/2}\,\mathbf{a}\cdot\nabla v\|_0^2) \end{aligned} \tag{54}$$

and a result similar to the other methods follows by choosing $\gamma = 2$. ∎

Remarks

1. Using the techniques presented in [17,22,25], it is not difficult to extend the present method and convergence results to the time-dependent advective-diffusive model and to advective-diffusive systems.

2. Note that according to the definitions (26)-(27), when $|\mathbf{a}|_p \to 0 \Rightarrow \mathrm{Pe}_K \to 0$, even though τ remains positive. Indeed by (26)-(28) for $0 \leq \mathrm{Pe}_K < 1$ we have

$$\tau(\mathbf{x}, \mathrm{Pe}_K(\mathbf{x})) = \frac{m_k h_K^2}{4\kappa(\mathbf{x})} > 0$$

For the degenerate situation when $\mathbf{a} = \mathbf{0}$, the SUPG method reduces to the Galerkin method for the Poisson equation, and GLS and the method (23)-(25)

will be given by, respectively,

$$(\kappa\nabla u_h, \nabla v) + \sum_{K\in\mathcal{C}_h} \frac{m_k h_K^2}{4}(\kappa\Delta u_h, \Delta v)_K = (f, v) - \sum_{K\in\mathcal{C}_h} \frac{m_k h_K^2}{4}(f, \Delta v)_K \qquad v \in W_h \tag{55}$$

$$(\kappa\nabla u_h, \nabla v) - \sum_{K\in\mathcal{C}_h} \frac{m_k h_K^2}{4}(\kappa\Delta u_h, \Delta v)_K = (f, v) + \sum_{K\in\mathcal{C}_h} \frac{m_k h_K^2}{4}(f, \Delta v)_K \qquad v \in W_h \tag{56}$$

Replacing v by u_h in (55)-(56), note that the second term is less or equal to half of the first term (cf. eq. (42)), and therefore stability follows for both methods, as before. The error analysis is considerably simplified and both methods will converge with $\mathcal{O}(h^k)$ in the H^1-norm, as the standard Galerkin method. Whether these methods will produce improved results, or not, only a L^∞-estimate and numerical results could tell, which are questions left for future works.

Nevertheless, such degenerate case becomes a very important issue when we extend this stability parameter approach to the linearized Navier-Stokes equations. Therein when the convective term goes to zero, the methods will automatically reduce to GLS and Douglas-Wang for Stokes (cf. Section 4).

3. Johnson has proposed the following design for the stability parameter τ (cf. Remark 9.5, pp. 186, [22]):

$$\begin{cases} \tau_J = \overline{C}\dfrac{h_K}{|\mathbf{a}(\mathbf{x})|} & , \dfrac{h_K|\mathbf{a}(\mathbf{x})|}{\kappa} > 1; \\ \tau_J = 0 & , \dfrac{h_K|\mathbf{a}(\mathbf{x})|}{\kappa} \leq 1. \end{cases} \tag{57}$$

where $\overline{C}$ is a constant that should be small enough to attain stability, following the analysis in Remark 9.4, pp. 186, [22] (see also [25]). With these definitions the stabilized methods discussed reduce to the Galerkin method for locally diffusive dominated flows.

Let us slightly change Johnson's design of τ as follows:

$$\begin{cases} \overline{\tau} = \dfrac{h_K}{2|\mathbf{a}(\mathbf{x})|_p} & , \mathrm{Pe}_K(\mathbf{x}) > 1; \\ \overline{\tau} = 0 & , \mathrm{Pe}_K(\mathbf{x}) \le 1. \end{cases} \tag{58}$$

which can be written in the form of (26)-(31) by replacing equation (28) with $\xi(\mathrm{Pe}_K(\mathbf{x})) = \mathrm{H}(\mathrm{Pe}_K(\mathbf{x}) - 1)$, where $\mathrm{H}(\cdot)$ is given in (47).

Let us examine the consequences of this choice. First note that the estimate (38) is still valid. Next, for $\mathrm{Pe}_K(\mathbf{x}) > 1$

$$\overline{\tau} = \frac{h_K}{2|\mathbf{a}(\mathbf{x})|_p} \frac{1}{\mathrm{Pe}_K(\mathbf{x})} \frac{m_k |\mathbf{a}(\mathbf{x})|_p h_K}{2\kappa(\mathbf{x})} \le \frac{m_k h_K^2}{4\kappa(\mathbf{x})}$$

which reproduces the estimate (41). Therefore, for $\mathrm{Pe}_K(\mathbf{x}) > 1$ and under the assumptions of Lemma 3.1, eq. (42) follows immediately, which implies the stability estimates (Lemma 3.1, eqs. (53) and (54)) for all stabilized methods examined. Furthermore, the error analysis considered in Theorem 3.1 is still valid for $\mathrm{Pe}_K(\mathbf{x}) > 1$.

Note that the difference of alternative (57) with respect to (58) is that when the element Peclet number is large, $\overline{\tau} > \tau_J$ if $\overline{C} < 0.5$ (From Johnson's analysis (cf. Remark 9.4, pp. 186, [22]) $\overline{C} < \widetilde{C}_k$, where $\widetilde{C}_k$ is the inverse estimate constant in (31). Employing biquadratic elements, $\widetilde{C}_{k=2} = 1/24$ according to [12], and therefore, $\overline{C} < 0.05$).

4. The expressions for τ may be considered too restrictive. It is also possible to define (or "tune") τ replacing expressions (26) and (27) by

$$\tau_\alpha(\mathbf{x}, \mathrm{Pe}_K^\alpha(\mathbf{x})) = \frac{\alpha h_K}{2|\mathbf{a}(\mathbf{x})|_p} \xi(\mathrm{Pe}_K^\alpha(\mathbf{x}))$$

$$\mathrm{Pe}_K^\alpha(\mathbf{x}) = \frac{m_k|\mathbf{a}(\mathbf{x})|_p h_K}{2\alpha\kappa(\mathbf{x})}$$

where α is a constant to be chosen, in principle, $\alpha \geq 1$. In other words, this alternative gives the possibility of increasing τ_α for locally advective dominated flows ($\mathrm{Pe}_K^\alpha \geq 1$), without compromising the delicate balance of stability for locally diffusive dominated flows ($0 \leq \mathrm{Pe}_K^\alpha \leq 1$) due to the additional perturbation term. This can be immediately seen by combining these formulas with the remaining definition in (28)-(31), which yields:

$$\tau_\alpha(\mathbf{x}, \mathrm{Pe}_K^\alpha(\mathbf{x})) = \frac{\alpha h_K}{2|\mathbf{a}(\mathbf{x})|_p} \qquad \mathrm{Pe}_K^\alpha(\mathbf{x}) \geq 1$$

$$\tau_\alpha(\mathbf{x}, \mathrm{Pe}_K^\alpha(\mathbf{x})) = \frac{m_k h_K^2}{4\kappa(\mathbf{x})} \qquad 0 \leq \mathrm{Pe}_K^\alpha(\mathbf{x}) \leq 1$$

Note that estimate (41) holds for this choice as well, since for $\mathrm{Pe}_K^\alpha(\mathbf{x}) \geq 1$

$$\tau_\alpha(\mathbf{x}, \mathrm{Pe}_K^\alpha(\mathbf{x})) = \frac{\alpha h_K}{2|\mathbf{a}(\mathbf{x})|_p}\frac{1}{\mathrm{Pe}_K^\alpha(\mathbf{x})}\frac{m_k|\mathbf{a}(\mathbf{x})|_p h_K}{2\alpha\kappa(\mathbf{x})} \leq \frac{m_k h_K^2}{4\kappa(\mathbf{x})}$$

Thus the bound (42) holds for all values of $\mathrm{Pe}_K^\alpha(\mathbf{x}) \in [0, \infty)$, and stability (in the sense of Lemma 3.1) follows for all stabilized methods examined. In checking Lemma 3.2 for this alternative, it follows that the bound on the interpolation error is premultiplied by α. Therefore if α is not arbitrarily large (bounded by a "reasonable" constant), Theorem 3.1 may be established as before for this alternative.

However, also note that selecting large values of α apparently produces a (purely) least-squares method, which has the well-known disatvantages of giving too dissipative numerical results for this advective-diffusive model. For very large α's the τ-formula switches earlier on from the expression for locally advective flow ($\mathrm{Pe}_K^\alpha(\mathbf{x}) \geq 1$) to the expression for locally diffusive flow.

The latter tends to zero as $\mathrm{Pe}_K^\alpha(\mathbf{x}) \to 0$. Thus, very large values of α should be avoided in this situation (otherwise we are back to the unstable Galerkin method). Summing up, it seems that α should be understood as a "tuning" parameter, not too different than one, and numerical experiments should determine its range or a "best" value according to some criterion yet to be determined. In the next section, we present numerical experiments for formulae (26)-(31), i.e., for $\alpha = 1$. Further experiments with $\alpha > 1$ shall be reported in the future.

3.3 Numerical Results

In this section we examine the various finite element methods discussed herein employing bilinear (Q1) and biquadratic (Q2) interpolations. First considered in [13], the first two numerical tests are subjected to very high Peclet numbers and are used to assess solutions which are essentially of pure advection flows. The last numerical test is designed to assess the performance of these methods when simultaneously subjected to an advection-dominated subdomain and to a diffusion-dominated subdomain, which is also a situation of practical interest that occurs, e.g., in resolving thermal boundary layers.

3.3.1 Advection in a rotating flow field

This problem is defined on a unit square of coordinates $-0.5 \leq x, y \leq +0.5$, where the flow velocity components are given by

$$a_1 = -y \quad , \quad a_2 = x \tag{59}$$

Along the external boundary $u = 0$ and along the internal boundary (OA),

$$u = \frac{1}{2}(\cos(4\pi y + \pi) + 1) \quad , \quad -0.5 \leq y \leq 0 \tag{60}$$

(See Figure 3 for the problem statement). The diffusivity is $\kappa = 10^{-6}$. A uniform mesh with 31x31 nodes is employed for bilinear interpolation Q1 (30x30 elements) and for biquadratic interpolations Q2 (15 x 15 elements). For this problem results are shown using the p-norm with $p = 2$ (Euclidean norm). The results are practically identical to the results with $p = \infty$ (max norm), and, consequently, comparisons between them are not shown. Elevation plots of u are shown in Figure 4 for the Galerkin and the SUPG methods. In this problem, which has a *smooth* exact solution, both methods perform well, despite small amplitude oscillations in the Galerkin method. There is a slight improvement in passing from Q1 to Q2, *keeping the same number of unknowns.* Since the element Peclet number is very high in this problem, the other methods (GLS and the present method (23)-(25)) give virtually indistinguishable solutions compared to the SUPG method.

3.3.2 Advection skew to the mesh

Let us consider $\kappa = 10^{-6}$ and $|\mathbf{a}|_2 = 1$ on a unit square, so that once more we have a problem with a very large Peclet number. In these problems a discontinuity in the data at the inflow boundary is propagated into the domain, which creates an internal layer. In addition, the problem is subjected to homogeneous essential boundary conditions at the outflow boundary, which gives rise to outflow boundary layers (see Figure 5 for the problem statement). These are rough numerical tests in that various layers are present in the exact solution. In these problems, results for the Galerkin method are highly oscillatory and, consequently, not shown here.

As in the previous numerical test, by the virtue of the high element Peclet number employed, the results are practically identical among all stabilized methods examined (SUPG, GLS and the present method given by eqs. (23)-(25)). A uniform mesh with 21x21 nodes is employed for Q1 (20x20 elements) and for Q2 (10x10 elements). Results for $\theta = \arctan 0.5, \arctan 1.0, \arctan 2.0$ are shown in Figures 6, 7 and 8, respectively. In each Figure results using the 2-norm and max norm for Q1 and Q2 interpolations are compared. Notice that all solutions present oscillations on thin layer regions, as expected from local error analysis and numerical results introduced earlier [13,19,25]. We should point out the considerable improvement by selecting a Q2 interpolation, for a mesh *with the same number of unknowns* as the mesh tested for Q1. Thus, the results of Q2, with a comparable computational effort as the results of Q1, produce a more desirable set of numerical solutions for these test cases, employing the stabilized methods discussed, with $p = \infty$.

3.3.3 Thermal boundary layer problem

Let us consider a rectangular domain of sides $L_x = 1.0$ and $L_y = 0.5$, subjected to the following boundary conditions (see Figure 9 for problem statement):

$$u = 1 \begin{cases} x = 0 & , 0 \leq y \leq 0.5; \\ y = 1 & , 0 \leq x \leq 1. \end{cases} \tag{61}$$

$$u = 0 \quad , \quad y = 0 \, , \, 0 < x \leq 1.0 \tag{62}$$

$$u = 2y \quad , \quad x = 1 \, , \, 0 < y < 0.5 \tag{63}$$

The flow components are

$$a_1 = 2y \quad , \quad a_2 = 0 \qquad \text{in} \quad \Omega \tag{64}$$

and the diffusivity is $\kappa = 7 \times 10^{-4}$. This problem may be viewed as the simulation at the outset of a thermal boundary layer on a fully developed flow between two parallel plates, where the top plate is moving with velocity equals to one and the bottom plate is fixed. Taking the top plate velocity as the characteristic flow velocity, we have for this flow a Peclet number $\text{Pe} = VL_y/\kappa = 714$.

We employ a mesh consisting of 21 equally-spaced nodes in the x-direction, 11 nodes uniformly distributed in the interval $0 \leq y \leq 0.1$ and the same number of nodes equally-spaced on $0.1 \leq y \leq 0.5$. This amounts to subdivide the domain into two different regions. Employing Q2 interpolations, the lower region, referred to by the superscript (I), is constituted by rectangles with sides $h_x^{(\text{I})} = 0.1$ and $h_y^{(\text{I})} = 0.02$ and in the upper region, (II), $h_x^{(\text{II})} = h_x^{(\text{I})} = 0.1$ and $h_y^{(\text{II})} = 0.08$. Computing the element parameters (cf. Remark 2, Section 3.1)

$$h_K^{(\text{I})} = \frac{\sqrt{2}\,\text{Area}(K)}{\text{Diagonal}(K)} = 0.027735 \quad ; \quad h_K^{(\text{II})} = 0.088345$$

and since $m_{k=2} = 1/12$ (cf. Remark 3, Section 3.1) and $|\mathbf{a}|_p = a_1 = 2y$ we have the following element Peclet numbers (cf. eq. (27)):

$$\text{Pe}_K^{(\text{I})} = \frac{1}{12}\frac{2yh_K^{(\text{I})}}{2\kappa} = 3.302y \quad , \quad 0 \leq y \leq 0.1$$

$$\text{Pe}_K^{(\text{II})} = 10.52y \quad , \quad 0.1 \leq y \leq 0.5$$

Therefore, from the numerical standpoint the domain is advectively-dominated in (II) and diffusively-dominated in (I) since $\text{Pe}_K^{(\text{I})} < 1$ and $\text{Pe}_K^{(\text{II})} > 1$.

In Figure 10, u-profiles are shown at $x = 0.2, 0.5$ and 0.8 employing SUPG, GLS and the present method. The performances of all methods are comparable. An attentive reader may quickly protest that this is a very simple flow, even from the numerical standpoint, since the flow is parallel to the mesh, and even the poorly stable Galerkin method (or central difference scheme in the finite difference context) might work. This conjecture is not confirmed, as seen in Figure 11, where the Galerkin method oscillates from one node to the next in the x-direction (profiles are shown at consecutive nodes at $x = 0.50$ and $x = 0.55$). The poor performance of the Galerkin method is due to the essential outflow boundary condition employed that creates an outflow boundary layer which contaminates the Galerkin solution.

Finally, in Figure 12 the present method employing Q2 with $m_{k=2} = 1/12$ is compared to a selection of $m_{k=2} = 1/3$, which is a value beyond the scope of our analysis (cf. eq. (30) and Remark 3 of Section 3.1). Oscillations pollute the numerical solution for $m_{k=2} = 1/3$, confirming the importance of appropriately designing τ.

4. Conclusions

Various stabilized methods are discussed and a new one is proposed for the advective-diffusive model. Abiding by the design of the stability parameter, similar convergence properties and comparable numerical performance are obtained.

The importance of the τ-design is readdressed and slightly improved upon by considering a different notion of the *element Peclet number.* In rescaling the element Peclet number, a natural and precise division between locally advective dominated flows and locally diffusive dominated flows are achieved. In particular, it is now possible to vary the definition of τ for the advective dominated part of the flow, without upsetting the definition on the diffusive part (cf. Remark 4, Section 3.2).

These findings are important for careful generalizations in other contexts. In particular, we are interested in combining these methodologies for the incompressible Navier-Stokes equations. Hansbo and Szepessy [11] have recently considered a velocity-pressure formulation for Navier-Stokes which combines linear velocity with continuous linear pressure. (A method for these equations in streamfunction-vorticity is proposed and analyzed by Johnson and Saranen in [24]). In the sequel to this paper an analysis and numerical results of the linearized Navier-Stokes equations will be presented in detail that accommodates Hansbo-Szepessy method and various high order methods. To antecipate some of the contents, let us consider the following formulations: Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times P_h$ such that

$$B_{\pm}(\mathbf{u}_h, p_h; \mathbf{v}, q) = F_{\pm}(\mathbf{v}, q) \qquad (\mathbf{v}, q) \in \mathbf{V}_h \times P_h$$

with

$$\begin{aligned} B_{\pm}(\mathbf{u}, p; \mathbf{v}, q) &= ((\nabla \mathbf{u})\mathbf{a}, \mathbf{v}) + (\nu \boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v})) - (\nabla \cdot \mathbf{v}, p) - (\nabla \cdot \mathbf{u}, q) \\ &+ \sum_{K \in \mathcal{C}_h} \Big((\nabla \mathbf{u})\mathbf{a} + \nabla p - \nu \nabla \cdot \boldsymbol{\varepsilon}(\mathbf{u}), \tau(\mathbf{x}, \mathrm{Re}_K)((\nabla \mathbf{v})\mathbf{a} - \nabla q \pm \nu \nabla \cdot \boldsymbol{\varepsilon}(\mathbf{v})) \Big)_K \\ &+ \delta(\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}) \end{aligned}$$

and

$$F_{\pm}(\mathbf{v}, q) = (\mathbf{f}, \mathbf{v}) + \sum_{K \in \mathcal{C}_h} \Big(\mathbf{f}, \tau(\mathbf{x}, \mathrm{Re}_K)((\nabla \mathbf{v})\mathbf{a} - \nabla q \pm \nu \nabla \cdot \boldsymbol{\varepsilon}(\mathbf{v})) \Big)_K$$

and

$$\tau(\mathbf{x}, \mathrm{Re}_K(\mathbf{x})) = \frac{h_K}{2|\mathbf{a}(\mathbf{x})|_p} \xi(\mathrm{Re}_K(\mathbf{x}))$$

$$\mathrm{Re}_K(\mathbf{x}) = \frac{m_k |\mathbf{a}(\mathbf{x})|_p h_K}{2\nu(\mathbf{x})}$$

$$\xi(\mathrm{Re}_K(\mathbf{x})) = \begin{cases} \mathrm{Re}_K(\mathbf{x}) & , 0 \le \mathrm{Re}_K(\mathbf{x}) < 1; \\ 1 & , \mathrm{Re}_K(\mathbf{x}) \ge 1. \end{cases}$$

$$|\mathbf{a}(\mathbf{x})|_p = \left(\sum_{i=1}^{N} |a_i(\mathbf{x})|^p \right)^{1/p}, \quad 1 \leq p < \infty$$

$$m_k = \min\left\{ \frac{1}{3}, \; 2C_k \right\}$$

$$C_k \sum_{K \in \mathcal{C}_h} h_K^2 \|\nabla \cdot \boldsymbol{\varepsilon}(\mathbf{v})\|_{0,K}^2 \leq \|\boldsymbol{\varepsilon}(\mathbf{v})\|_0^2 \qquad \mathbf{v} \in \mathbf{V}_h$$

The notion of element Peclet number is replaced by *element Reynolds number* and the formulae defining τ is similar to (26)-(31). In passing, this element Reynolds number definition also includes the information on the finite element polynomial employed through m_k. Recently, Tezduyar-Ganjoo-Shih [30] have proposed to combine the SUPG strategy with the stabilized methodology for Stokes to produce a method for Navier-Stokes, using a related design of τ. The differences between the present approach and their formulation are also discussed in detail in the sequel to the present work.

Acknowledgements

The authors wish to thank Jim Douglas, Jr., Isaac Harari, Claes Johnson and Rolf Stenberg for useful comments and discussions.

References

[1] F. Brezzi and J. Pitkäranta, On the stabilization of finite element approximations of the Stokes problem, in: W. Hackbusch, ed., Efficient Solutions of Elliptic Systems, Notes on Numerical Fluid Mechanics, Vol. 10, pp. 11-19 (Viewig, Wiesbaden, 1984).

[2] A.N. Brooks and T.J.R. Hughes, Streamline upwind/Petrov-Galerkin methods for advection dominated flows, in: Third International Conference on Finite Element Methods in Fluid Flows, Banff, Canada, (1980).

[3] A.N. Brooks and T.J.R. Hughes, Streamline upwind/Petrov-Galerkin formulations for convective dominated flows with particular emphasis on the incompressible Navier-Stokes equations, Comput. Methods Appl. Mech. Engrg. 32 (1982) 199-259.

[4] P.G. Ciarlet, The finite element method for elliptic problems (North-Holland, Amsterdam, 1978).

[5] J. Douglas and J. Wang, An absolutely stabilized finite element method for the Stokes problem, Math. Comp. (1989) 495-508.

[6] E.G. Dutra do Carmo and A.C. Galeão, High order P-G finite elements for convection-dominated problems, in: R. Gruber, J. Periaux and R.P. Shaw, eds., Proc. Fifth International Symposium on Numerical Methods in Engineering, Vol. 2, pp 151-156 (Computational Mechanics Publications and Springer-Verlag, Lausanne, 1989).

[7] L.P. Franca and E.G. Dutra do Carmo, The Galerkin-gradient-least-squares method, Comput. Methods Appl. Mech. Engrg. 74 (1989) 41-54.

[8] L.P. Franca and T.J.R. Hughes, Two classes of mixed finite element methods, Comput. Methods Appl. Mech. Engrg. 69 (1988) 89-129.

[9] L.P. Franca and R. Stenberg, Error analysis of some Galerkin-least-squares methods for the elasticity equations, Rapports de Recherche-INRIA 1054, Rocquencourt, France, 1989.

[10] A.C. Galeão and E.G. Dutra do Carmo, A consistent approximate upwind Petrov-Galerkin method for convection-dominated problems, Comput. Methods Appl. Mech. Engrg. 68 (1988) 83-95.

[11] P.Hansbo and A. Szepessy, A velocity-pressure streamline diffusion finite element method for the incompressible Navier-Stokes equations, Preprint.

[12] I. Harari, Ph.D. Thesis (forthcoming), Division of Applied Mechanics, Stanford University, Stanford, California.

[13] T.J.R. Hughes and A.N. Brooks, A multidimensional upwind scheme with no crosswind diffusion, in: T.J.R.Hughes, ed., Finite Element Methods for

Convection Dominated Flows (ASME, New York, 1979) 19-35.

[14] T.J.R. Hughes and A.N. Brooks, A theoretical framework for Petrov-Galerkin methods with discontinuous weighting functions: application to the streamline upwind procedure, in: R.H. Gallagher, G.F. Carey, J.T. Oden and O.C. Zienkiewicz, eds., Finite Element in Fluids IV (Wiley, Chichester, 1982) 46-65.

[15] T.J.R. Hughes and L.P. Franca, A new finite element formulation for computational fluid dynamics: VII. The Stokes problem with various well-posed boundary conditions: symmetric formulations that converge for all velocity/pressure spaces, Comput. Methods Appl. Mech. Engrg. 65 (1987) 85-96.

[16] T.J.R. Hughes, L.P. Franca and M. Balestra, A new finite element formulation for computational fluid dynamics: V. Circumventing the Babuška-Brezzi condition: A stable Petrov-Galerkin formulation of the Stokes problem accomodating equal-order interpolations, Comput. Methods Appl. Mech. Engrg. 59 (1986) 85-99.

[17] T.J.R. Hughes, L.P. Franca and G.M. Hulbert, A new finite element formulation for computational fluid dynamics: VIII. The Galerkin-least-squares method for advective-diffusive equations, Comput. Methods Appl. Mech. Engrg., 73 (1989) 173-189.

[18] T.J.R. Hughes, M. Mallet and L.P. Franca, Entropy-stable finite element methods for compressible fluids: application to high order Mach number flows with shocks, in: P.G. Bergan, K.J. Bathe and W. Wunderlich, eds., Finite Element Methods for Nonlinear Problems (Springer, Berlin-Heidelberg, 1986) 761-773.

[19] T.J.R. Hughes, M. Mallet and A. Mizukami, A new finite element method for computational fluid dynamics: II. Beyond SUPG, Comput. Methods Appl. Mech. Engrg. 54 (1986) 341-355.

[20] T.J.R. Hughes and F. Shakib, Computational aerodynamics and the finite element method, in: Proc. AIAA 26th Aerospace Sciences Meeting (Reno, Nevada, 1988).

[21] C. Johnson, Finite element methods for convection-diffusion problems, in: R. Glowinski and J.L. Lions, eds., Computing Methods in Engineering and Applied Sciences V (North-Holland, Amsterdam, 1982) 311-323.

[22] C. Johnson, Numerical solution of partial differential equations by the finite element method (Studentlitteratur, Sweden, 1987).

[23] C. Johnson and U. Nävert, An analysis of some finite element methods for advection-diffusion problems, in: O. Axelsson, L. S. Frank and A. Van Der Sluis, eds., Analytical and Numerical Approaches to Asymptotic Problems in Analysis, (North-Holland, Amsterdan, 1981) 99-116.

[24] C. Johnson and J. Saranen, Streamline diffusion methods for the incompressible Euler and Navier-Stokes equations, Math. Comp. 47 (1986) 1-18.

[25] C. Johnson, U. Nävert and J. Pitkäranta, Finite element methods for linear hyperbolic problem, Comput. Methods Appl. Mech. Engrg. 45 (1984) 285-312.

[26] C. Johnson, A. Szepessy and P. Hansbo, On the convergence of shock-capturing streamline diffusion finite element methods for hyperbolic conservation laws, Math. Comp. 54 (1990) 107-129.

[27] N. Kechkar and D.J. Silvester, The stabilization of low order mixed finite element methods for incompressible flows, in: R. Gruber, J. Periaux and R.P. Shaw, eds., Proc. Fifth International Symposium on Numerical Methods in Engineering, Vol. 2, pp 111-116 (Computational Mechanics Publications and Springer-Verlag, Lausanne, 1989).

[28] U. Nävert, A finite element method for convection-diffusion problem, Ph.D. Thesis, Department of Computer Science, Chalmers University of Technology, Göteborg, Sweden (1982).

[29] A. Szepessy, Convergence of the streamline diffusion finite element method for conservation laws, Ph.D. Thesis, Department of Mathematics, Chalmers University of Technology, Göteborg, Sweden.(1989).

[30] T.E. Tezduyar, D.K. Ganjoo and R.Shih, Incompressible flow computations using various velocity-pressure elements, Preprint UMSI 89/207 (University of Minnesota/Supercomputer Institute, Minneapolis, 1989).

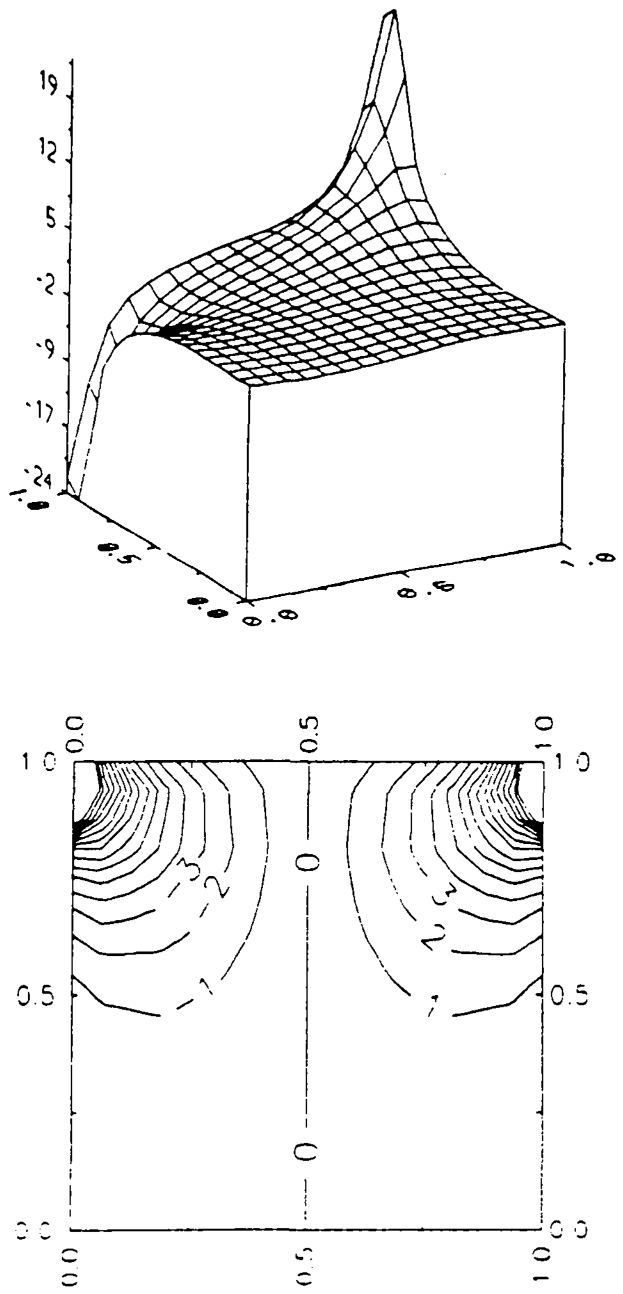


Figure 1. Pressure elevation and contours for the "leaky" cavity flow: 8x8 Q2/Q2C elements employing GLS with $\alpha = 0.01$.

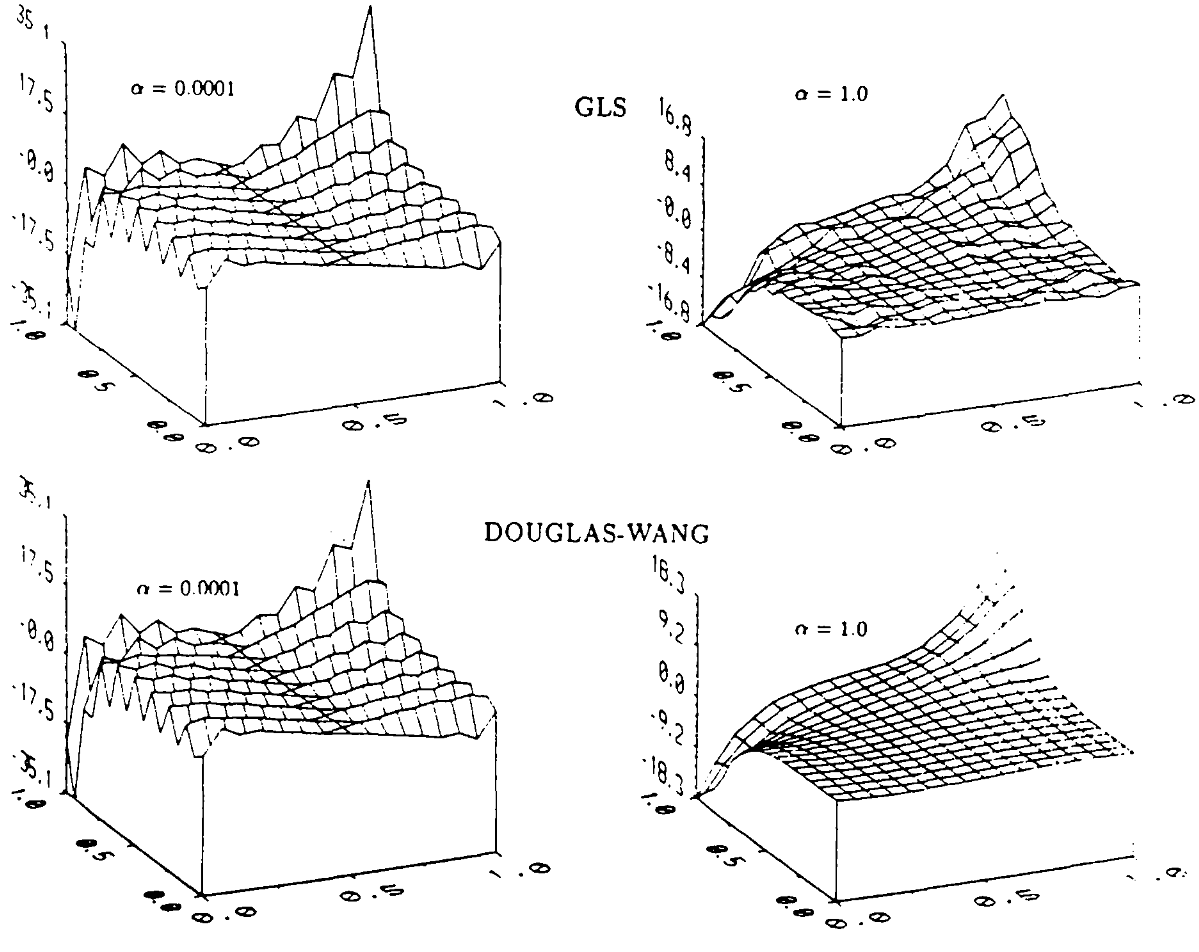


Figure 2. Pressure elevations for the "leaky" cavity flow: 8x8 Q2/Q2C elements.

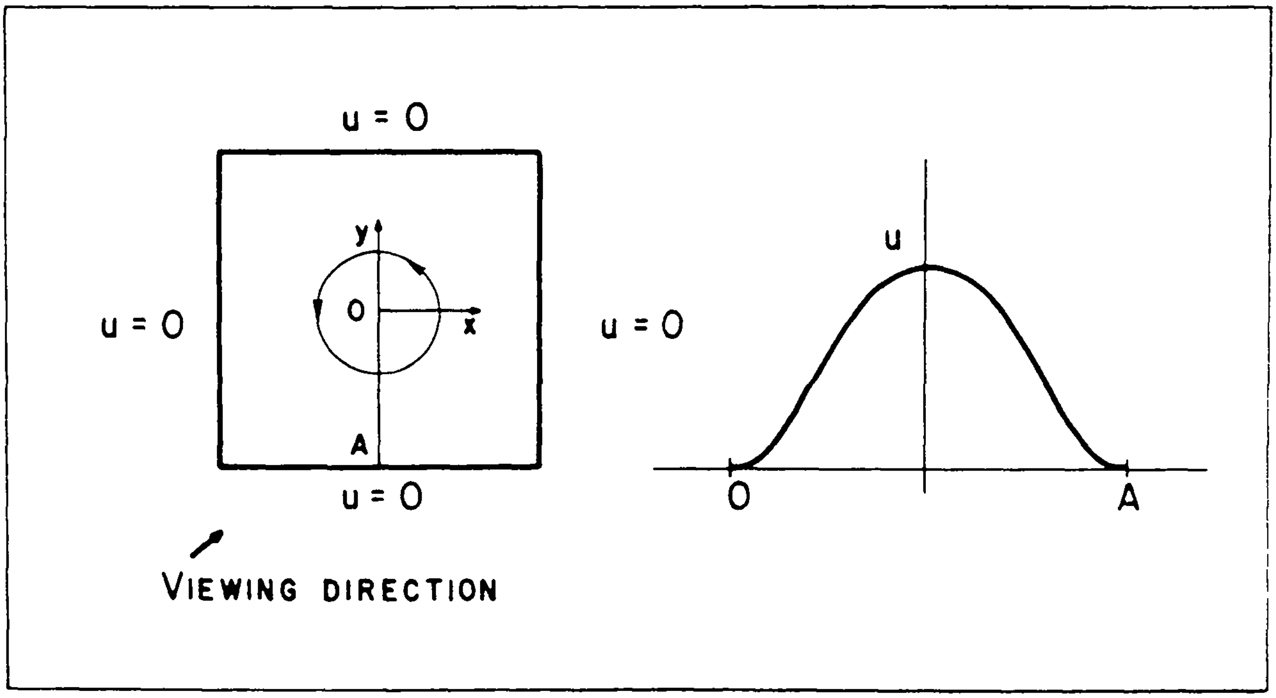


Figure 3. Advection in a rotating flow field: Problem statement.

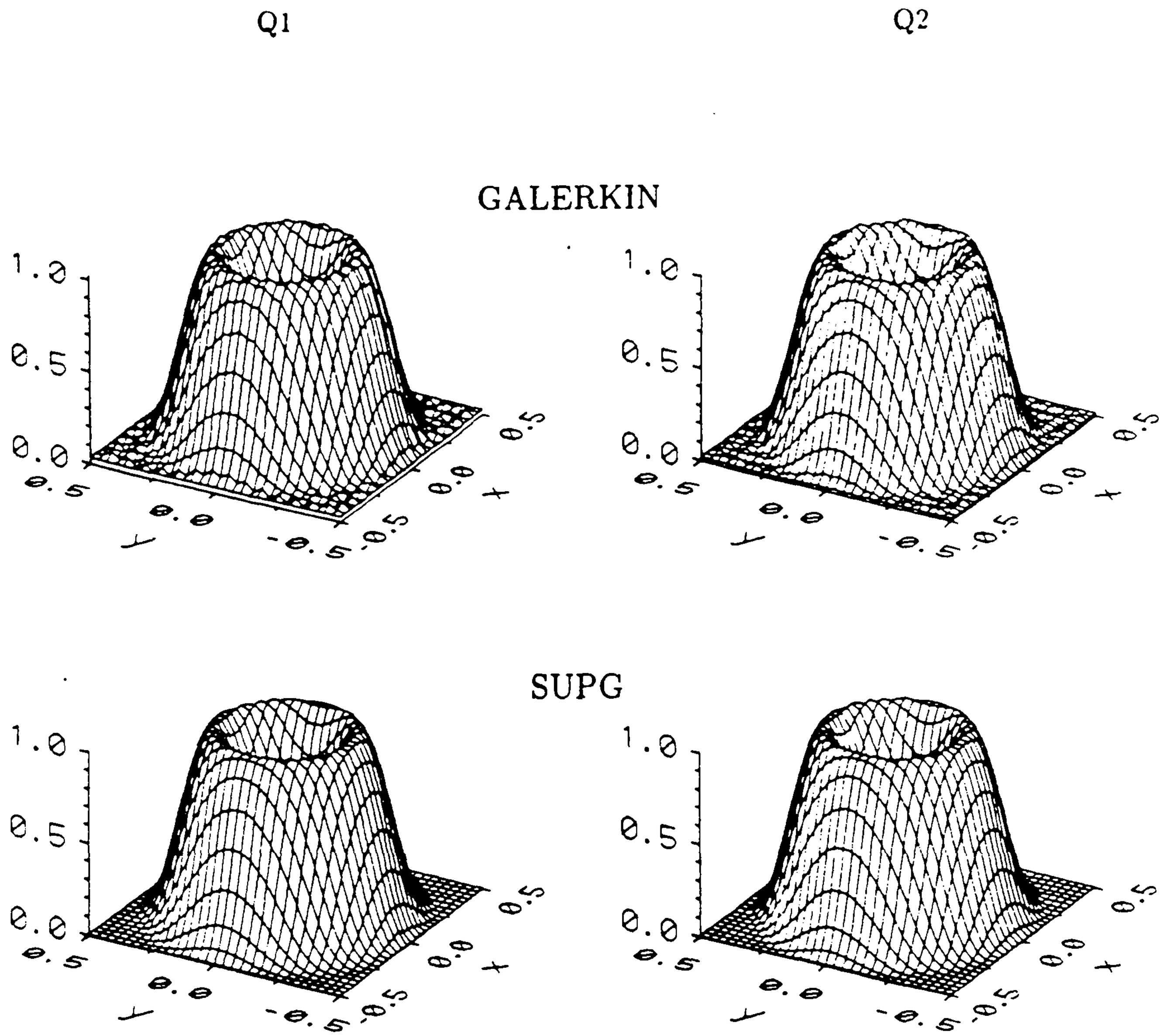


Figure 4. Advection in a rotating flow field: Results employing $p = 2$ in the definition of τ.

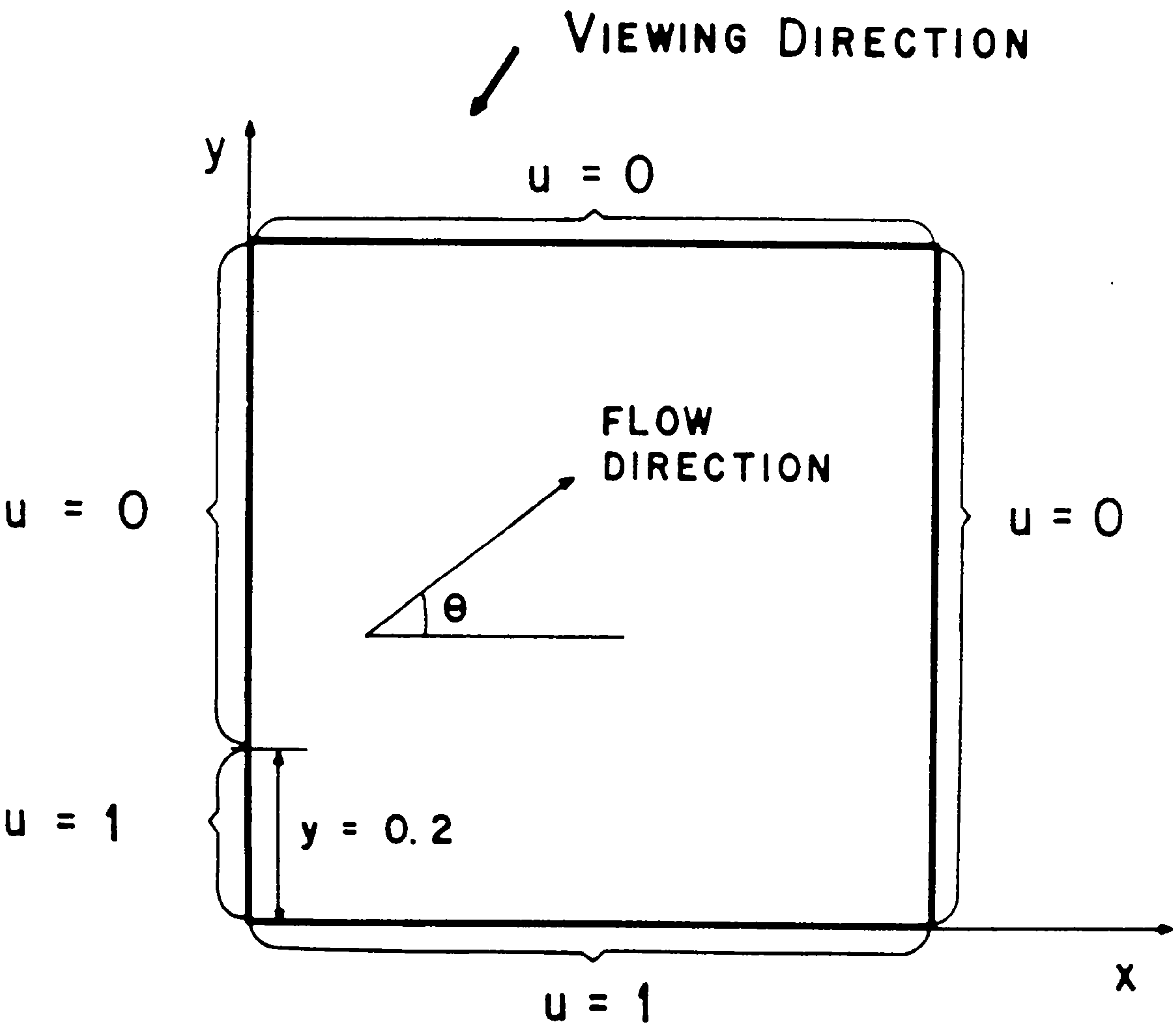


Figure 5. Advection skew to the mesh: Problem statement.

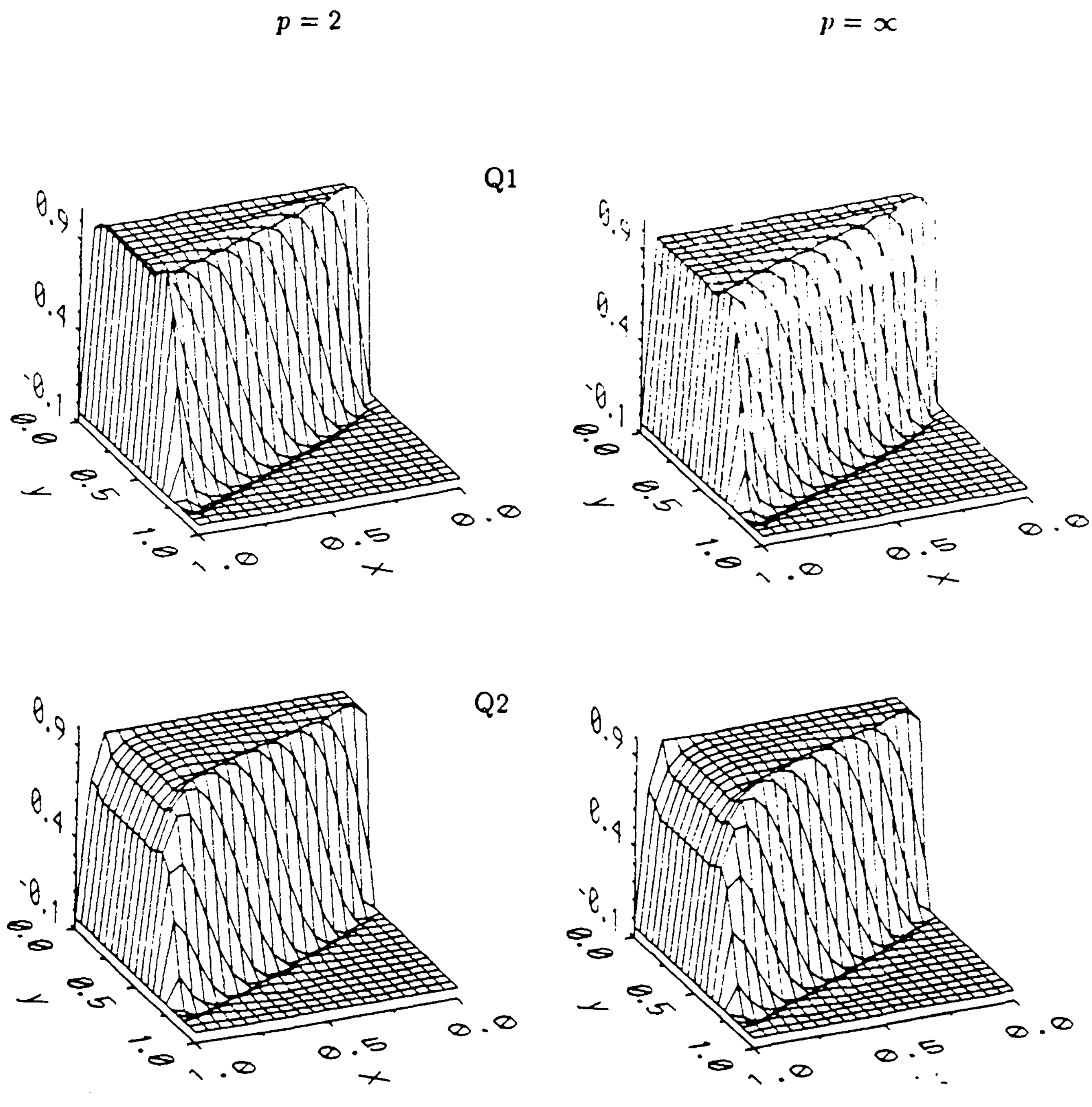


Figure 6. Advection skew to the mesh: Results for any stabilized method with $\theta = \arctan 0.5$.

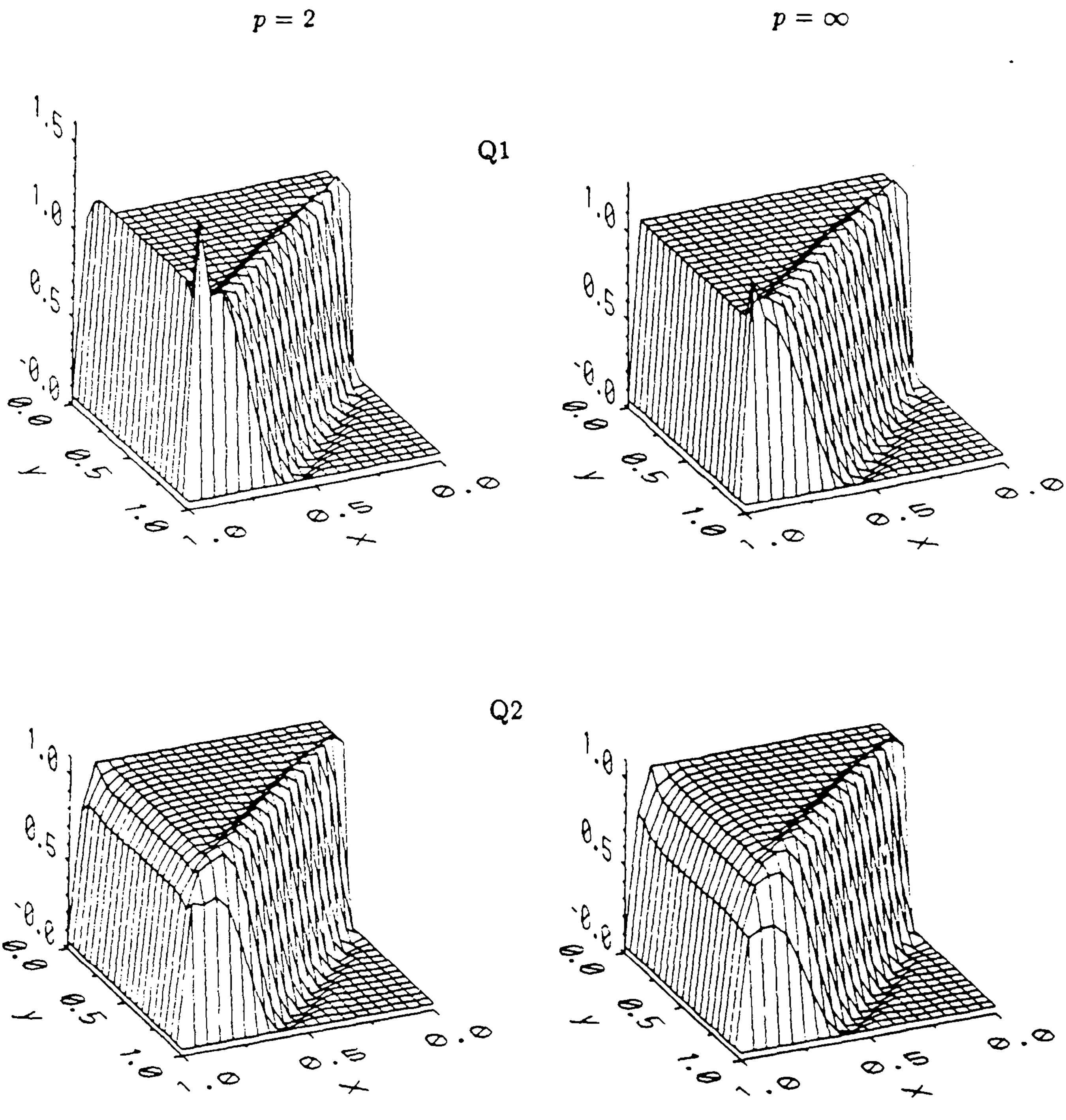


Figure 7. Advection skew to the mesh: Results for any stabilized method with $\theta = \arctan 1.0$.

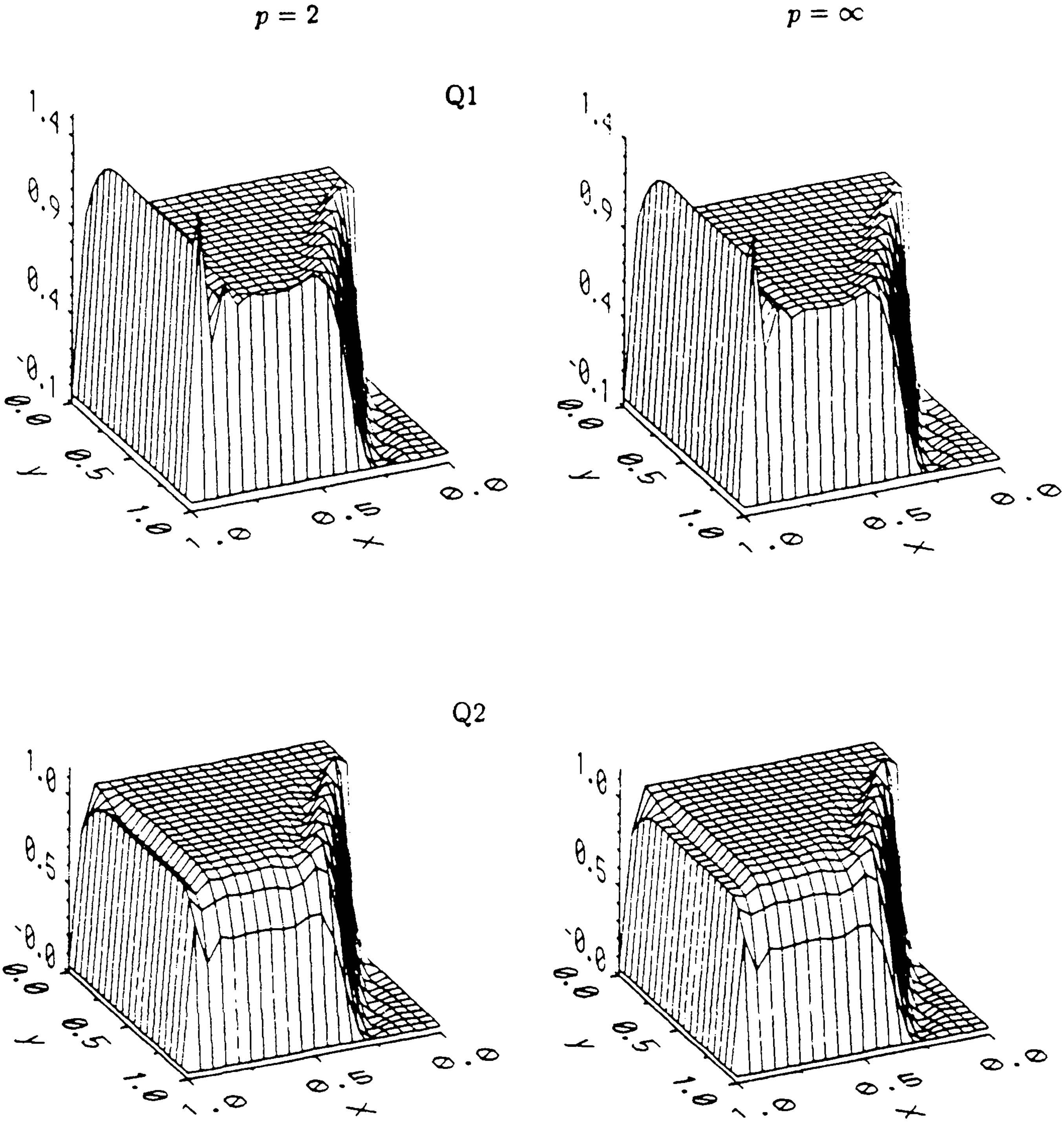


Figure 8. Advection skew to the mesh: Results for any stabilized method with $\theta = \arctan 2.0$.

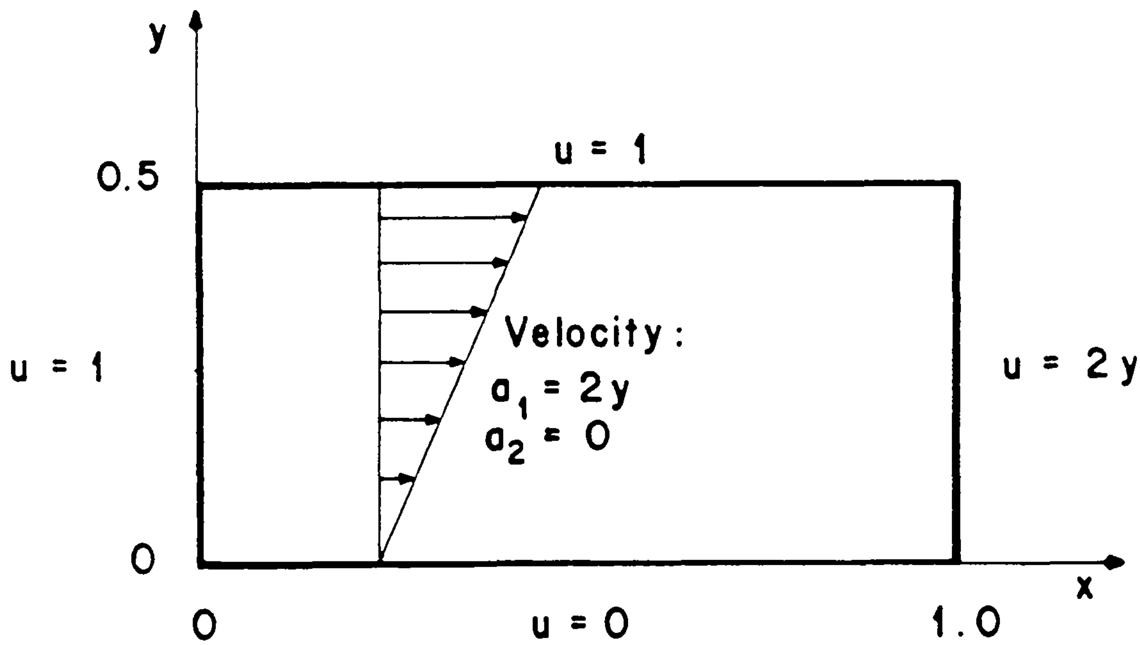


Figure 9. Thermal boundary layer problem: Problem statement.

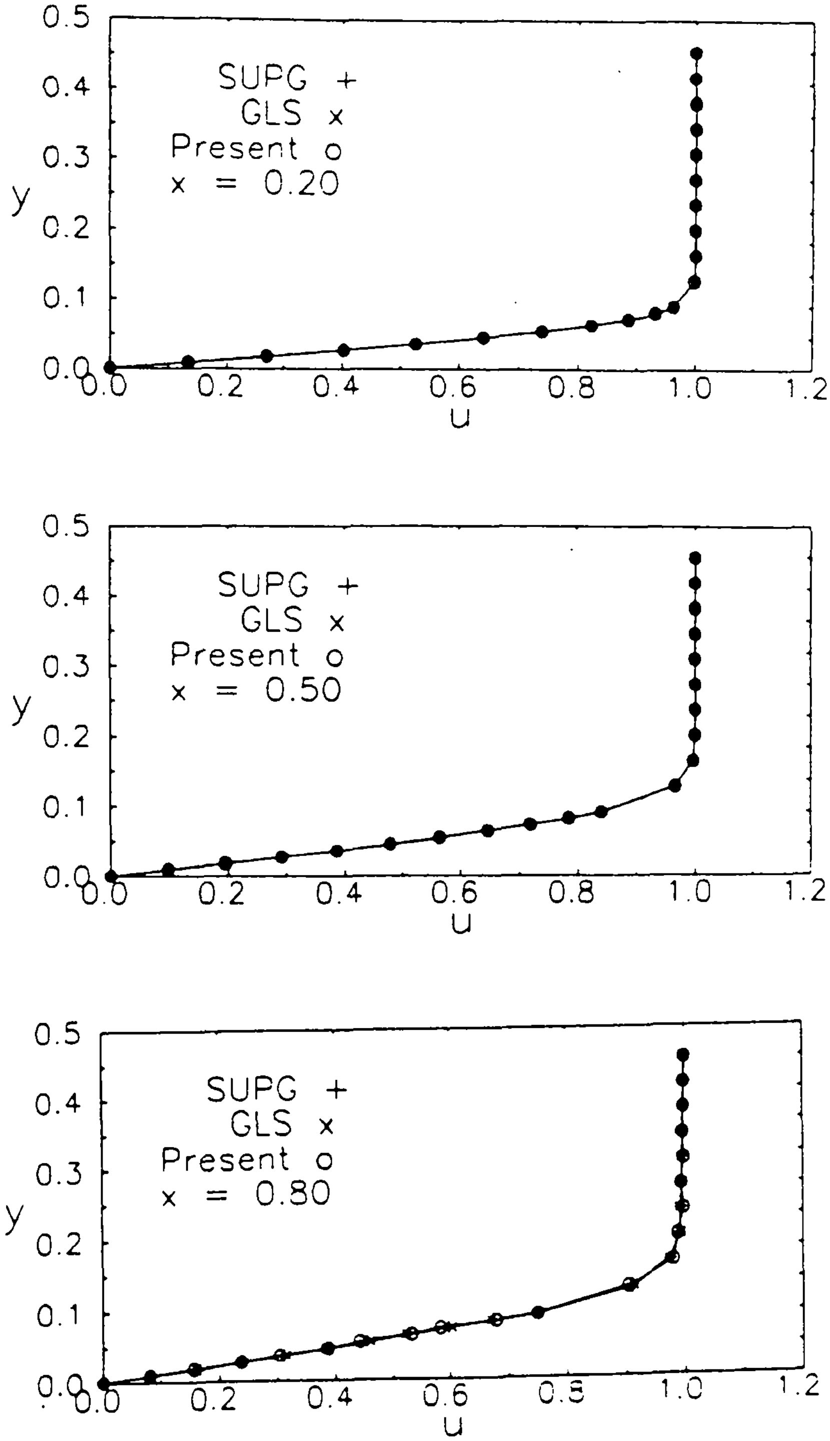
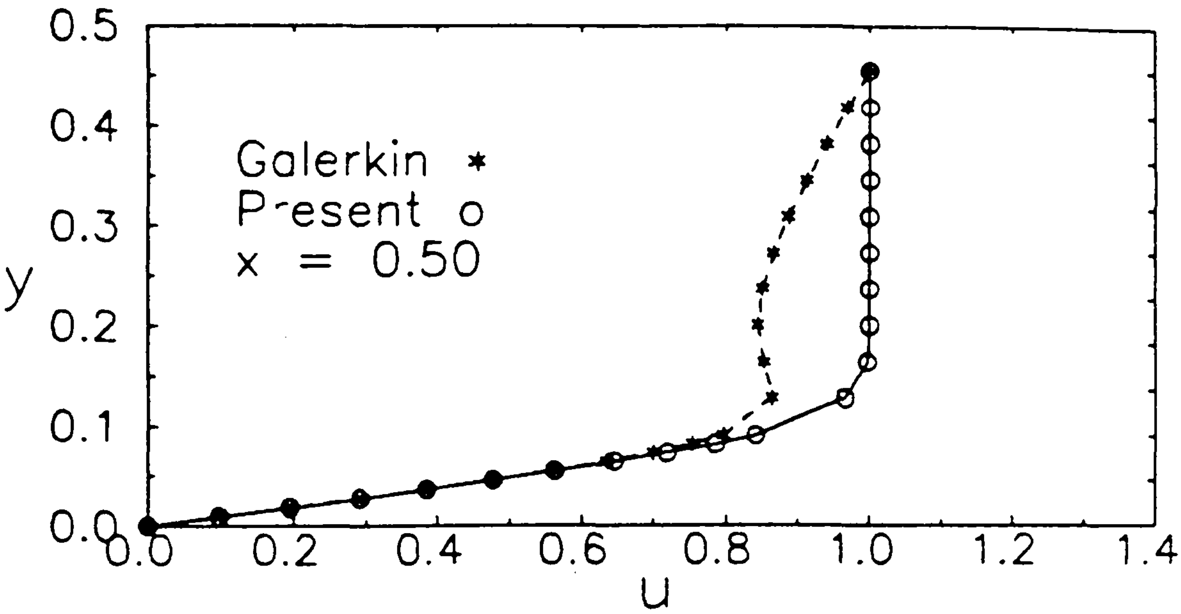


Figure 10. Thermal boundary layer problem: Comparisons between all stabilized methods employing Q2 elements.



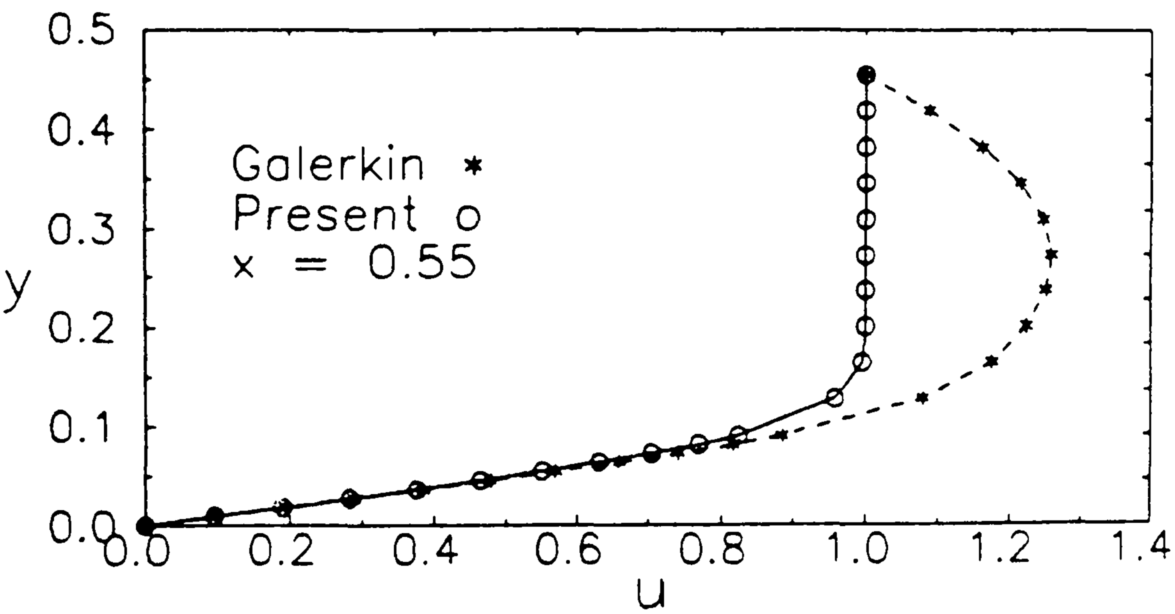


Figure 11. Thermal boundary layer problem: Comparison between the present stabilized method and the Galerkin method.

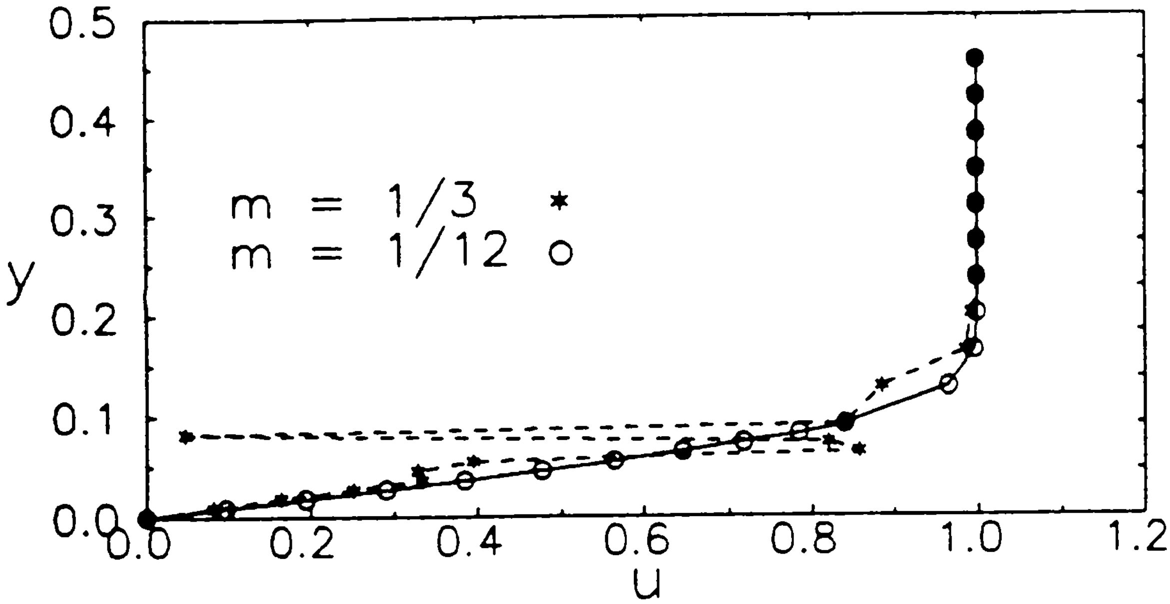


Figure 12. Thermal boundary layer problem: Comparison between "wrong" and "correct" choices of parameters in the τ-design.

Imprimé en France
par
l'Institut National de Recherche en Informatique et en Automatique

ISSN 0249-6399