Probabilistic analysis of some distributed algorithms

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Abstract
In this paper, we analyze :
i) a storage allocation algorithm which permits to maintain two stacks inside a shared (contiguous) memory area of a fixed size,
ii) the well-known banker algorithm which plays a fundamental role in parallel processing.
The natural formulation of these problems is in terms of constrained random walks. Our results rely on diffusion techniques.

ANALYSE PROBABILISTE DE QUELQUES ALGORITHMES DISTRIBUES

Résumé
Dans cet article, nous analysons :
i) un algorithme de gestion dynamique de deux piles avec mémoire partagée,
ii) l'algorithme du banquier qui joue un rôle important en parallélisme.
Ces problèmes se modélisent en termes de marches aléatoires contraintes. Les résultats sont obtenus à l'aide de techniques probabilistes liées aux diffusions.

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ABSTRACT

In this paper, we analyse :

i) a storage allocation algorithm (Knuth [15] Ex.2.2.2.13) which permits to maintain two stacks inside a shared (contiguous) memory area of a fixed size,

ii) the well-known banker algorithm which plays a fundamental role in parallel processing (Françon [10], Habermann [12], Peterson, Silberschatz [18]).

The natural formulation of the problems to be solved here is in terms of random walks. For (i) the random walk $Y_m(.)$ takes place in a triangle in a 2-dimensional lattice space with two reflecting barriers along the axes (a deletion takes no effect on an empty stack) and one absorbing barrier parallel to the second diagonal (the algorithm stops when the combined sizes of the stacks exhaust the available storage).

For (ii) the random walk takes place in a rectangle with breaked corner and has four reflecting barriers and one absorbing barrier (see Figure 8).

With the help of diffusions techniques, we obtain, asymptotically:

- the hitting place (Z_m) and time (T_m) distributions on the absorbing boundary
- the joined distribution of Z_m and T_m
- the distribution $P[Y_m(n) \leq y_m, n < T_m)$

The two stacks problem has been partially investigated by Yao [19] and more recently by Flajolet [9] with different tools. We provide here an analysis of the general case with new limiting distributions. At our knowledge such kind of analysis has never been done before for the banker algorithm.

1. INTRODUCTION

The analysis of distributed algorithms is often the source of difficult mathematical questions. The two problems to be investigated here confirm this assertion.
First we consider the evolution of two stacks inside a shared, contiguous memory area of a fixed size m. The shared storage allocation algorithm studied in this paper lets them grow from both ends of the memory until the cumulated sizes of the stacks exhaust the available storage. That algorithm is to be compared to another strategy, namely allocating separate zones of size m/2 to each stack. This problem of Knuth [15] has been investigated by Yao [19] and more recently by Flajolet [9] to obtain some probability distributions and expectations of the sizes of the stacks and the time until the system runs out of memory. Flajolet's approach is based on generating functions and continued fractions techniques. As noticed, the natural formulation of this problem is in terms of random walks inside a triangle with two reflecting barriers along the axes (a deletion takes no effect on an empty stack) and one absorbing barrier parallel to the second diagonal (the algorithm stops when the combined sizes of the stacks exhaust the available storage).
See Figure 2 and 8 for a more detailed presentation of this problem. Diffusions appear to be a powerful tool for the analysis of this algorithm and also for the banker algorithm, another simple distributed algorithm whose complete description can be found in (Françon [10], Habermann [12], Peterson and Silberschatz [18]).
Let us just mention that this algorithm permits to avoid deadlock in parallel processing and works as follows consider p processes $P_1, P_2, \ldots, P_p$ who have access to r entities $R_1, R_2, \ldots, R_r$ (card (Ri)≥ 1 for i $\in \{1\ldots,r\}$).
Initially, each process Pi gives to the banker (i.e. the controller) an upper bound for the maximum number of examplars of each entity needed for the execution of a given transaction.
Knowing these numbers, the banker decides to affect to the process Pi the required entities only if the remaining entities are sufficient in order to fulfill the requirements of the working processes.
The paper is organized as follows : the analysis of the two stacks storage allocation algorithm is treated in Section II, basic notations and definitions are contained in Subsection II.1, the trend-free case is examined in II.2, the general case is investigated in the part II.3. Section III concerns the banker algorithm: do to the difficulty of this problem, we consider here two processes P_1, P_2 sharing

one entity R (card (R) = $m \geqslant 1$). As explained in Section III.1. the formulation of this problem is in terms of random walks inside a rectangle with breaked corner (see Figure 8).
Results concerning the hitting place and time distributions on the absorbing boundary are obtained with the same tools as previously. Detailed limiting distributions shall be included in the full version of this paper.

II. THE EVOLUTION OF TWO STACKS IN BOUNDED SPACE

II.1. Basic notations and definitions

This problem was initially posed by Knuth [15] and investigated successively by Yao [19] and Flajolet [9]. It's easy to see that the natural formulation of the shared storage allocation algorithm is in terms of random walks in a triangle in a 2-dimensional lattice space: a state is the couple formed with the size of both stacks. The random walk $Y_m(.)$ has two reflecting barriers along the axes and one absorbing barrier parallel to the second diagonal (see Figure 2).
The steps distribution (steps of unit length): ΔY is given by Figure 1

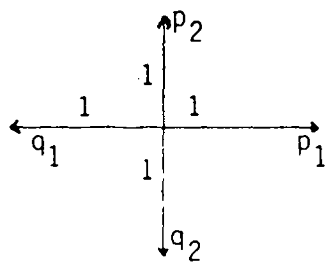


Figure 1

with the boundary conditions :

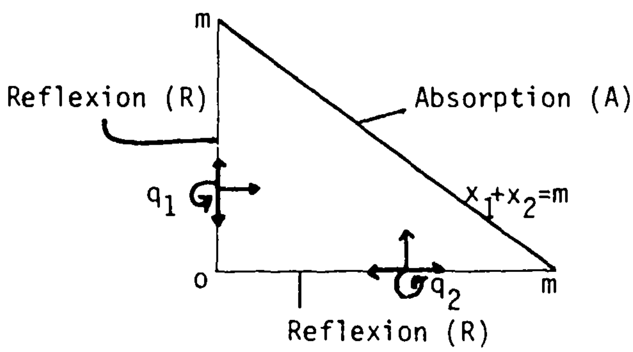


Figure 2

The following questions will be treated in this paper.

With initial condition $Y_m(0)=x_m$, what are asymptotically $(m\to\infty)$

- the hitting place (Z_m) distribution on the (A) boundary
- the hitting time (T_m) distribution on the (A) boundary
- the joined distribution of Z_m and T_m
- the distribution $P\{Y_m(n)\leq y_m,\ n<T_m\}$

If we let the drift (or trend) μ be $(\mu_1,\mu_2)=(p_1-q_1,p_2-q_2)$, we have two fundamental different limit behaviours, according to $\mu=0$ or $\mu\neq 0$.

We will need the covariance matrix of one step :

$$CY:=E(\Delta Y.\Delta Y^T)=\begin{pmatrix}\sigma y_1^2 & Cy_{12}\\ Cy_{12} & \sigma y_2^2\end{pmatrix}$$. This is easily computed as

$$CY=\begin{pmatrix}p_1+q_1-(p_1-q_1)^2 & -(p_1-q_1)(p_2-q_2)\\ -(p_1-q_1)(p_2-q_2) & p_2+q_2-(p_2-q_2)^2\end{pmatrix} \tag{1}$$

The following notations will be used in the sequel

- $\sim$: asymptotic to, for $m\to\infty$
- $\underset{m\to\infty}{\Rightarrow}$: weak convergence of random functions in the space of all right continuous functions having left limits and endowed with the Skorohod metric (see Billingsley [3] Ch.III.)
- $\mathcal{N}(M,V)$:= the Normal (or Gaussian) random variable with mean M and variance V

II.2. The trend-free case : $\mu=0$

To simplify the analysis, assume that $p_1=q_1=p_2=q_2=\frac{1}{4}$.

(The case $p_1=q_1$, $p_2=q_2$ can be treated along the same lines). The asymptotic distribution of $Y_m(.)$ is given by the following Theorem

Theorem 1

$$\frac{\sqrt{2}\, Y_m([m^2t])}{m} \Rightarrow W(t) \qquad (2)$$

where W(t) is a two-dimensional Brownian Motion (B.M.) with the following boundary conditions

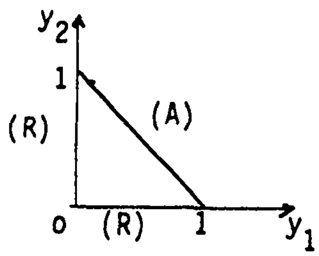


Figure 3

Let the hitting time for W(.) be T. The density of W(.) is given by

$$P_x[W(t) \in dy,\ t < T) =$$

$$\{\ 2\sum_{k=1}^{\infty} \exp(-k^2\pi^2t/2)\cos(k\pi x_1)[\cos(k\pi y_1)-\cos[k\pi(1-y_2)]]$$

$$+\ 2\sum_{\ell=1}^{\infty} \exp(-\ell^2\pi^2t/2)\cos(\ell\pi x_2)[\cos(\ell\pi y_2)-\cos[\ell\pi(1-y_1)]] \qquad (3)$$

$$+\ 4\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} \exp[-(k^2+\ell^2)\pi^2t/2]\cos(k\pi x_1)\cos(\ell\pi x_2)$$

$$.[\cos(k\pi y_1)\cos(\ell\pi y_2)-\cos[k\pi(1-y_2)]\cos[\ell\pi(1-y_1)]]\}\ dy_1dy_2$$

Proof

The weak convergence of (2) is easily deduced from Chung-Williams [4] Th.8.4. (The $\sqrt{2}$ factor is derived from σy^2 as given by (1)). The limiting process is not sticking at the R boundary (RB) as the probability of the random walk staying at RB during one step is Θ (probability of leaving RB). The classical one-dimensional B.M. with reflecting boundaries at 0 and 1 is well known : see Feller [8] Ex.X.5.e and prob.XIX.9.11. Its density is given by (call this process W_1):

$P_{x_1}[W_1(t) \in dy_1] =$

$\{1+2\sum_{k=1}^{\infty} \exp[-k^2\pi^2 t/2]\cos(k\pi x_1)\cos(k\pi y_1)\}dy_1 = [1+\Sigma_1(x_1,y_1)]dy_1$, say, and similarly for W_2.

By a classical reflection principle across the absorbing boundary, we see that

$$P_x[W(t) \in dy,\ t<T] = \{[1+\Sigma_1(x_1,y_1)][1+\Sigma_2(x_2,y_2)] - [1+\Sigma_1(x_1,1-y_2)][1+\Sigma_2(x_2,1-y_1)]\}dy_1dy_2$$

hence (3). ■

We now have all necessary ingredients to derive the limiting hitting distributions. Let a new coordonate system z be given by the following translation and rotation

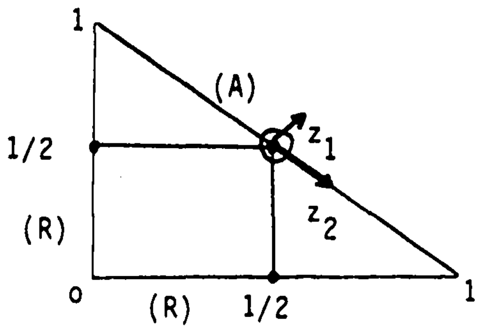


Figure 4

We have the following theorem for the hitting place (Z) and hitting time (T) distributions.

Theorem 2

$P_x[T \in dt,\ Z(T) \in dz_2] =$

$$\{\sum_{k=1}^{\infty} \exp(-k^2\pi^2 t/2)\cos(k\pi x_1)\,\frac{2k\pi}{\sqrt{2}}\,\sin[k\pi(\tfrac{1}{2}+\tfrac{z_2}{\sqrt{2}})]$$

$$+\sum_{\ell=1}^{\infty} \exp(-\ell^2\pi^2 t/2)\cos(\ell\pi x_2)\,\frac{2\ell\pi}{\sqrt{2}}\,\sin[\ell\pi(\tfrac{1}{2}-\tfrac{z_2}{\sqrt{2}})]$$

$$+\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} \exp[-(k^2+\ell^2)\pi^2 t/2]\cos(k\pi x_1)\cos(\ell\pi x_2)\,\frac{4\pi}{\sqrt{2}}\,[k\sin[k\pi(\tfrac{1}{2}+\tfrac{z_2}{\sqrt{2}})]\cos[\ell\pi(\tfrac{1}{2}-\tfrac{z_2}{\sqrt{2}})]$$

$$+\ \ell\cos[k\pi(\tfrac{1}{2}+\tfrac{z_2}{\sqrt{2}})]\sin[\ell\pi(\tfrac{1}{2}-\tfrac{z_2}{\sqrt{2}})]]\}\ dtdz_2 \qquad (4)$$

The marginal densities are given by

$$P_x[Z(t) \in dz_2] =$$

$$\{\frac{4}{\pi\sqrt{2}} [\sum_{k=1}^{\infty} \frac{\cos(k\pi x_1)}{k} \sin[k\pi(\frac{1}{2} + \frac{z_2}{\sqrt{2}})]$$

$$+ \sum_{\ell=1}^{\infty} \frac{\cos(\ell\pi x_2)}{\ell} \sin[\ell\pi(\frac{1}{2} - \frac{z_2}{\sqrt{2}})]$$

$$+ \frac{8}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{\cos(k\pi x_1)\cos(\ell\pi x_2)}{(k^2+\ell^2)} [k\sin[k\pi(\frac{1}{2} + \frac{z_2}{\sqrt{2}})]\cos[\ell\pi(\frac{1}{2} - \frac{z_2}{\sqrt{2}})]$$

$$+ \ell\cos[k\pi(\frac{1}{2} + \frac{z_2}{\sqrt{2}})]\sin[\ell\pi(\frac{1}{2} - \frac{z_2}{\sqrt{2}})]]]\} \, dz_2 \tag{5}$$

$$P_x[T \in dt] =$$

$$\{4 \sum_{k \text{ odd}>0} \exp(-k^2\pi^2 t/2)\cos(k\pi x_1)$$

$$+4 \sum_{\ell \text{ odd}>0} \exp(-\ell^2\pi^2 t/2)\cos(\ell\pi x_2) \tag{6}$$

$$+8 \sum_{k \text{ odd}>0} \sum_{\ell \text{ even}>0} \exp[-(k^2+\ell^2)\pi^2 t/2](k^2+\ell^2)/(k^2-\ell^2)\cos(k\pi x_1)\cos(\ell\pi x_2)$$

$$-8 \sum_{k \text{ even}>0} \sum_{\ell \text{ odd}>0} \exp[-(k^2+\ell^2)\pi^2 t/2] (k^2+\ell^2)/(k^2-\ell^2)\cos(k\pi x_1)\cos(\ell\pi x_2)\} \, dt$$

Proof

The new coordonate system $(z_1 z_2)$ is given by

$$\begin{cases} y_1 = \frac{1}{2} + (z_1+z_2)/\sqrt{2} \\ y_2 = \frac{1}{2} + (z_1-z_2)/\sqrt{2} \end{cases} \quad \text{or} \quad \begin{cases} z_1 = \frac{-\sqrt{2}}{2}(1-y_1-y_2) \\ z_2 = \frac{\sqrt{2}}{2}(y_1-y_2) \end{cases} \tag{7}$$

In this new system, let

$$\varphi_x(t,z_1,z_2) := P_x[W(t) \in dz, \; t < T]$$

It is well known that the hitting density is given by

$$P_x[T \in dt, \; Z(t) \in dz_2] = -\frac{1}{2} [\partial_{z_1}\varphi_x(t,z_1,z_2)]_{z_1=0}$$

(this is a classical outward heat flow). This gives (4) after standard manipulations. Note that permuting x_1 with x_2 in (4) changes z_2 into $-z_2$ (as it should be). Integrating on t gives (5).

(6) is obtained from (4) after some tedious but simple computations. One could wonder if (6) is an honest density.

This can be checked as follows. Integrating (6) on t gives, for the first term

$$4 \sum_{k\ \text{odd}>0} \cos(k\pi x_1) 4 \cdot \frac{2}{\pi^2 k^2} = -2x_1 + 1 \quad \text{by (A.3)}$$

The second term gives $-2x_2 + 1$,

the fourth term leads to

$$(-8)\ \frac{2}{\pi^2} \sum_{k\ \text{even}>0} \frac{\cos(k\pi x_1)}{2k} \sum_{\ell\ \text{odd}\neq 0} \frac{1}{(k-\ell)} \cos(\ell\pi x_2) \tag{8}$$

The last Σ in (8) easily gives

$$\sin(k\pi x_2) \sum_{\ell'\text{odd}\neq 0} \frac{\sin(\ell'\pi x_2)}{\ell'} = \frac{\pi}{2} \sin(k\pi x_2) \quad \text{by (A.5)}.$$

(8) becomes

$$-\frac{4}{\pi} \sum_{k\ \text{even}>0} \frac{\cos(k\pi x_1)\sin(k\pi x_2)}{k}$$

$$= -\frac{2}{\pi} \sum_{k\ \text{even}>0} [\sin[k\pi(x_1+x_2)] + \sin[k\pi(x_2-x_1)]]/k \tag{9}$$

Assume, without loss of generality, that $x_1 > x_2$

By (A.4) (9) gives $2x_2$.

Similarly, the fourth term of (6) leads to $(-1+2x_1)$. Collecting our results, we see that (6) is indeed an honest density integrating into 1. ■

By (2) the original hitting place (Z_m) and hitting time (T_m) are given by

$$\frac{\sqrt{2}\ Z_m}{m} \Rightarrow Z\ , \quad \frac{T_m}{m^2} \Rightarrow T$$

Note also that the particular case $x_1=x_2=0$ transforms, after some simple manipulations, formula (5) into Flajolet [9] Th.2 and formula (6) into the bivariate theta distribution announced in Flajolet [9] Th.4.

II.3. The case $\mu \neq 0$

μ is defined by (μ_1,μ_2) where

$\mu_1 = p_1-q_1, \quad \mu_2 = p_2-q_2.$

In the new system of Fig.4, we have by (7)

$$\begin{cases} \mu z_1 = \frac{\sqrt{2}}{2}(\mu_1+\mu_2) = \frac{\sqrt{2}}{2}(p_1+p_2-q_1-q_2) \\ \mu z_2 = \frac{\sqrt{2}}{2}(\mu_1-\mu_2) = \frac{\sqrt{2}}{2}(p_1-q_1-p_2+q_2) \end{cases} \tag{10}$$

Let θ be given by : $tg\theta := \mu z_2/\mu z_1$ (11)

Three different subcases must be considered, depending on the relative values of x and θ. The following diagram will help to understand the different situations

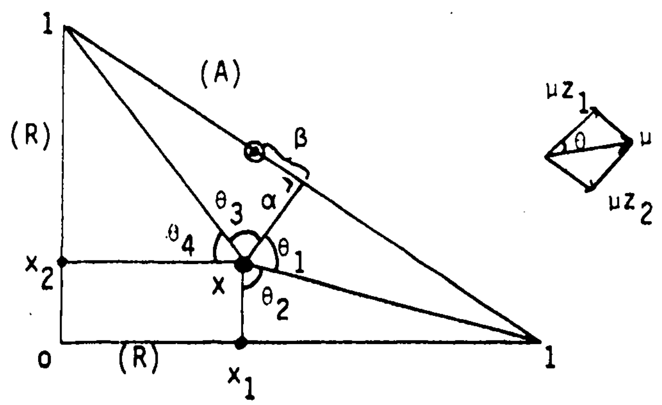


Figure 5

We immediately check by (7) that

$\alpha = \frac{\sqrt{2}}{2}(1-x_1-x_2)$

$\beta = \frac{\sqrt{2}}{2}(x_1-x_2)$

$(\frac{\sqrt{2}}{2} - \beta) = \alpha tg\theta_1$, $(\frac{\sqrt{2}}{2} + \beta) = -\alpha tg\theta_2$

$1-x_1 = x_2 tg\theta_2$ $\quad$ $1-x_2 = -x_1 tg\theta_4$

The different subcases are obviously related to the direction θ of the trend with respect to θ_i, $i = 1...4$.

Subcase 3.1: $-\theta_1 \leq \theta \leq \theta_3$

The asymptotic distribution of Y_m is now derived in the following theorem

Theorem 3

$$\sqrt{m}\left(\frac{Y([mt])}{m} - \mu t\right) \Rightarrow W(t) \quad (12)$$

where

W(t) is a B.M. with covariance matrix CY as given by (1)

Proof

Were it not for the reflection boundaries, the convergence is deduced from Billingsley [3]

But proceeding exactly as in Yao [19] Lemma 2 and 3, it is easily shown that we can indeed asymptotically ignore the (R) boundaries (we omit the details).

■

Subcase 3.1 can now be described by the following diagram

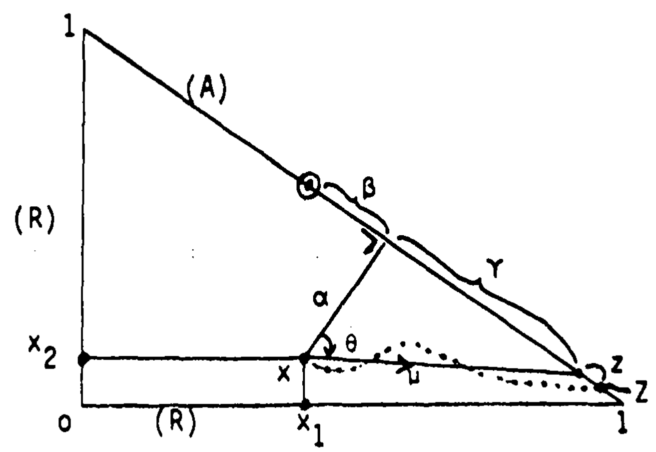


Figure 6

where, by (11), $\gamma = \alpha\mu z_2/\mu z_1$ and the process $\mu t + W(t)/\sqrt{m}$ is (grossly) pictured by the dotted line ••••

Hitting place Z and hitting time T are now characterized by the following theorem

Theorem 4

$$\tau := (T-\alpha/\mu z_1) \sim \mathcal{IP}[o,\alpha.\sigma z_1^2/(m.\mu z_1^3)]. \tag{13}$$

Conditioned on τ, we have

$$(z + \mu z_1.Cz_{12}.\tau/\sigma z_1^2) \sim \mathcal{IP}[o,\alpha(\sigma z_2^2-Cz_{12}^2/\sigma z_1^2)/(m.\mu z_1)] \tag{14}$$

where Cz_{12} is given by (15) below

Following Fig.6, we have

$Z = \beta+\gamma+z$

Proof

We have already computed $(\mu z_1,\mu z_2)$ (see (10)) in the new system of Fig.4.

The new covariance matrix $CZ := E(\Delta Z.\Delta Z^T)$ is given from (7) as

$CZ = \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} CY \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}/2.$ By (1) this gives

$$CZ = \begin{pmatrix} 1-(p_1-q_1+p_2-q_2)^2 & p_1+q_1-p_2-q_2-(p_1-q_1)^2+(p_2-q_2)^2 \\ p_1+q_1-p_2-q_2-(p_1-q_1)^2+(p_2-q_2)^2 & 1-(p_1-q_1-p_2+q_2)^2 \end{pmatrix}/2$$

$$= \begin{pmatrix} \sigma z_1^2 & Cz_{12} \\ Cz_{12} & \sigma z_2^2 \end{pmatrix}, \text{ say.} \tag{15}$$

We can now consider the asymptotic T distribution.

We have, by (12), for some classical B.M. B_1

$$P_x[\min(u' : \mu z_1 u' + B_1(u')\sigma z_1/\sqrt{m} = \alpha) \in du]$$

$$= P_x[\min(u' : B_1(u') = (\alpha - \mu z_1 u')\sqrt{m}/\sigma z_1) \in du] \tag{16}$$

$$= \frac{\alpha\sqrt{m}}{\sigma z_1\sqrt{2\pi}\, u^{3/2}} \exp\left[\frac{-m(\alpha-\mu z_1 u)^2}{2u\sigma z_1^2}\right] du$$

by a classical result on the crossing time of a B.M. and a straight line : see Cox and Miller [5] p. 221.

Let $u = \alpha/\mu z_1 + \tau$ (17)

We obtain the following asymptotic density for τ

$$\frac{\sqrt{m}\,\mu z_1^{3/2}}{\sqrt{2\pi\alpha}\, z_1} \exp[-m\tau^2\mu z_1^3/(2\alpha\sigma z_1^2)]$$

which proves (13).

Let now $\rho z_{12} := Cz_{12}/(\sigma z_1 . \sigma z_2)$ (correlation coefficient).

Conditioned on $B_1(u)$, it is well known that the projection of $\frac{W(u)}{\sqrt{m}}$ along z_2 is given by

$$z := \frac{B_2(u)\sigma z_2}{\sqrt{m}} \quad \text{with}$$

$$\frac{B_2(u)}{\sqrt{u}} = \rho z_{12}\frac{B_1(u)}{\sqrt{u}} + \sqrt{1-\rho z_{12}^2}\;\mathcal{N}(0,1)$$

But, by (16) and (17), $B_1(u) = -\mu z_1\sqrt{m}\tau/\sigma z_1$.

(14) is now easily deduced. ■

Note that, here, the original hitting place (Z_m) and hitting time (T_m) are given by

$$\frac{Z_m}{m} \to Z, \quad \frac{T_m}{m} \to T$$

Note also that the particular case $x_1=x_2=0$, $p_1=p_2$, $q_1=q_2$ leads to

$B=\gamma=0$, $Cz_{12} = 0$, $\mu z_1 = \sqrt{2}(p_1-q_1)$, $\alpha=\frac{\sqrt{2}}{2}$, $\sigma z_1^2 = \frac{1}{2} - 2(p_1-q_1)^2$, $\sigma z_2^2 = 1/2$

So that $(T - \frac{1}{2(p_1-q_1)}) \sim \mathcal{N}[0, \frac{[1-4(p_1-q_1)^2]}{8m(p_1-q_1)^3}]$

and $z \sim \mathcal{N}(0, \frac{1}{4m(p_1-q_1)})$

This gives $E(|z|) \sim 1/\sqrt{2\pi m(p_1-q_1)}$ which confirms Yao [14] Th.1.

Subcase 3.2 : $\theta \in [-\theta_1-\theta_2,-\theta_1]$, $\theta \in [\theta_3,\theta_3+\theta_4]$

Let us consider the second case (the other one is similar). The process can be described by the following diagram

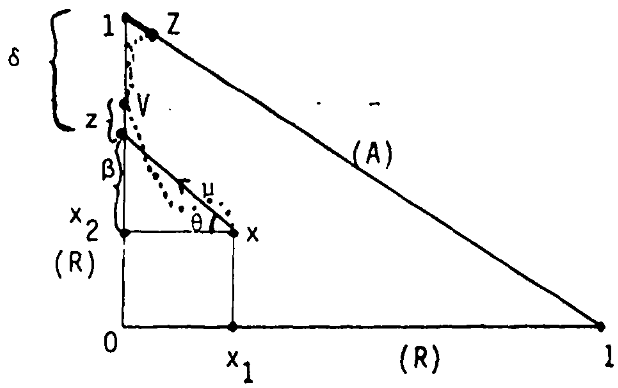


Figure 7

where $\beta = -\mu_2 x_1/\mu_1$, $\delta := 1-x_2-\beta$

Indeed, by the same argument as in Theorem 3's proof, the absorbing boundary can be asymptotically ignored in the first part of the process, leading to hitting place V (at time T, say) on the R-boundary.

The weak convergence of Theorem 3 is still valid. From V on, the y_2-component of the process obeys Theorem'3 law and the y_1-component is given by a classical reflecting random walk.

The following theorem gives the asymptotic laws for Z and T

Theorem 5

$T = T_1 + T_2$ where

$\tau_1 := (T_1 - x_1/|\mu_1|) \sim \mathcal{N}[0,\ x_1 . \sigma y_1^2/(m|\mu_1|^3)$

$\tau_2 := (T_2 - (\delta - z)/\mu_2) \sim \mathcal{N}(0,\ \delta . \sigma y_2^2/(m\mu_2^3))$ (18)

with, conditioned on τ_1,

$(z + |\mu_1|\ Cy_{12} .\ \tau_1/\sigma y_1^2) \sim \mathcal{N}[0,\ x_1(\sigma y_2^2 - Cy_{12}^2/\sigma y_1^2)/(m|\mu_1|)]$

$Z_1 \sim 1 - k/m,$ (19)

where the distribution of k is given by (20) below.

Proof

The first part of the Theorem is derived similarly to Theorem's 4 proof, we omit the details. We only have to check that the reflected y_1- random walk contribution is asymptotically negligible.

Indeed, in the original Fig.2 random walk,

$T_{2,m} \sim m\bar{T}_2 = O(m\delta/\mu_2)$.

But is well known (see Cox and Miller [5] p.45) that the reflected random walk has an asymptotic stationary distribution (even when $p_1 + q_1 < 1$)

$$P\ [y_1 = k] = [1 - \frac{p_1}{q_1}]\ \ [\frac{p_1}{q_1}]^k \quad (20)$$

This distribution is in force after $T_{2,m}$.

So, in (18), δ should be changed into $\delta - \frac{\Delta}{m}$

with $\Delta = O(1)$. But τ_2 and z are both $O(1/\sqrt{m})$. The contribution of Δ is asymptotically negligible.

(19) is now obvious from (20).

Again,

$\frac{T_m}{m} \to T, \quad \frac{Z_m}{m} \to Z$

Subcase 3.3 : $\theta_3 + \theta_4 \leqslant \theta \leqslant -\theta_1 - \theta_2$

This subcase will itself lead to different possibilities.

Subcase 3.3.1 : $p_1=p_2=p$, $q_1=q_2=q$, $p<q$

This is the simplest random walk, which can be analysed as follows.

A first theorem is obtained on the limiting hitting place Z_m distribution.

Theorem 6

The hitting place Z_m is asymptotically uniformly distributed on the absorbing boundary.

Proof

First of all, proceeding similarly to subcase 3.2, we see that, asymptotically, we reach the origin in time $T_m = O(m)$.

From there on, the reflecting random walks on y_1 and y_2 reach their stationary distribution (20) with exponential speed : from Feller [7] p.438, equ.3.16, we see that the extra terms converge to 0 as $[2\sqrt{pq}\,]^n$.

The joined stationary distribution is thus given by

$(1-\frac{p}{q})^2 \, (\frac{p}{q})^{k_1+k_2}$, constant on the line $k_1+k_2 = k$

If we can show that the absorbing time is large enough, (this is proved in the next theorem), the asymptotic hitting place Z_m is indeed uniformly distributed on the absorbing barrier. ■

Theorem 6 is identical to Flajolet [9] Th.3, but our direct probabilistic proof is much simpler.

To analyse the hitting time T_m, we firstly make the following approximation: given that the random walk is on the diagonal $y_1+y_2=j$, we assume that it is uniformly distributed on that diagonal. In the following we call this the Basic assumption.

All quantities related to this new process will be indexed by the diagonal value and affected by a star: *.

In this section, we set $p':=2p$, $q':=2q$. We firstly need the probably $\varphi^*_{j,m}$ that starting at j, we reach 0 before m (see B.7). This is given in the following theorem.

Theorem 7

$$\varphi^*_{j,m} = 1- \Psi(j-1)/\Psi(m-1) \tag{21}$$

with

$$\Psi(j-1):= \sum_1^j (\frac{q}{p})^{i-1}/i$$

$$\Psi(j-1) \underset{j\to\infty}{\sim} (\frac{q}{p})^j/[j(\frac{q}{p} - 1)][1 + 0(\frac{1}{j})] \tag{22}$$

Proof

φ^* satisfies the equations (we drop the m to ease notations)

$$\begin{cases} \varphi^*_j = p' \, \varphi^*_{j+1} + \frac{q'}{j+1} \varphi^*_j + q' \frac{j}{j+1} \varphi^*_{j-1}, \quad j\neq 0,m \\ \varphi^*_0 = 1, \; \varphi^*_m = 0 \end{cases}$$

To solve this difference equation, set

$\Delta_j := \varphi^*_j - \varphi^*_{j-1}$ and

$$\Delta_{j+1} := \frac{y_j}{j+1} (\frac{q}{p})^j$$

This readily gives

$$y_j = y_{j-1} = y_0$$

so that

$$\varphi_j^* = 1 + \sum_i^j \frac{y_o}{i} \left(\frac{q}{p}\right)^{i-1} = 1 + y_o\, \Psi(j-1), \text{ hence (21).}$$

The asymptotic form (22) is obtained as follows: let $\theta := q/p$

$$\Psi(j-1) = \theta^{j-1} \sum_1^j \frac{\theta^{i-j}}{i} = \theta^{j-1} \sum_0^{j-1} \frac{\theta^{-k}}{j-k} = \frac{\theta^{j-1}}{j} \sum_0^{j-1} \frac{\theta^{-k}}{1-\frac{k}{j}}$$

$$= \frac{\theta^{j-1}}{j} \sum_0^{j-1} \theta^{-k}[1+\frac{k}{j}+\frac{k^2}{j^2}\ldots]$$

$$\sim \frac{\theta^{j-1}}{j}[\frac{1}{1-\theta^{-1}} + \frac{1}{\theta j(1-\theta^{-1})^2} + \ldots]$$

$$\sim \frac{\theta^j}{j}[\frac{1}{\theta-1} + \frac{1}{j(\theta-1)^2} + \ldots]$$

■

Proceeding now as in Sec.B.2, we analyse the probability generating function (P.G.F.) $H_{j,m}^*(s)$ of the hitting time to 0 <u>before m</u>.

Let the double G.F.: $G^*(u,s) := \sum_0^{\infty} u^j\, H_j^*(s)$. Its general expression is given in the following theorem.

<u>Theorem 8</u>

$$G^*(u,s) = (1-u/\tilde{\lambda}_1)^A(1-u/\tilde{\lambda}_2)^B + D(u)(u-\tilde{\lambda}_1)^A(u-\tilde{\lambda}_2)^B \tag{23}$$

with

$\tilde{\lambda}_1 := 1/\lambda_1$, $\tilde{\lambda}_2 := 1/\lambda_2$ (with λ_1, λ_2 given by (B.1), with p', q')

$A := (\tilde{\lambda}_2 - 1)/(\tilde{\lambda}_1 - \tilde{\lambda}_2)$, $B := (1-\tilde{\lambda}_1)(\tilde{\lambda}_1 - \tilde{\lambda}_2)$

$$D(u) := C(s)/(-sq') \int_0^u [(v-\tilde{\lambda}_1)^{A+1}(v-\tilde{\lambda}_2)^{B+1}]^{-1}\, dv \tag{24}$$

$C(s) := [1-p's\, H_1^*(s) - sq']$

Proof

$H_j^*(s)$ satisfies the equations

$$\begin{cases} H_j^* = s\,[p'H_{j+1}^* + \frac{q'}{j+1} H_j^* + q' \frac{j}{j+1} H_{j-1}^*] \ , \ j \neq 0,m \\ H_0^* = 1, \ H_m^* = 0 \end{cases}$$

This leads to the following differential equation

$$G_u^{*'}(u-p's-q's\ u^2) = G^*(-1+q's+q'u\ s) + (1-p's\ H_1^* - sq') \qquad (25)$$

The roots of the first polynomial are obviously $\tilde{\lambda}_1$, $\tilde{\lambda}_2$.

Routine computation leads to (23). ■

We must now derive $H_1^*(s)$ from the condition $H_m^*(s) = 0$.

But according to Sec. B.2, all we need is to use this condition with $m \to \infty$ (this amounts to ignore the barrier at m) and limit ourselves to the neighbourhood of s=1 (i.e. $\eta=0$). We then obtain the following result.

Theorem 9

$$H_1^*(s) \sim 1 + \gamma\eta \qquad \eta \to 0 \qquad (26)$$

with $\gamma := (2q-p)/(q-p)$

Proof

From (B.3) we infer, with $\delta := 1-p/q$

$$\tilde{\lambda}_1 \sim \frac{p}{q}(1+\eta), \quad \tilde{\lambda}_2 \sim 1-\eta, \quad A \sim \eta/\delta, \quad B \sim -1-\eta/\delta. \qquad (27)$$

Call G_1^* the first part of (23).

By Darboux'theorem (see Greene and Knuth [11] equ.4.117), we derive ($\langle u^n\rangle F$ is the coefficient of u^n in the expansion of F)

$$\langle u^n\rangle \quad G_1^* \sim \left(1-\frac{\tilde\lambda_1}{\tilde\lambda_2}\right)^B \frac{1}{\tilde\lambda_1^n\, \Gamma(-A) n^{A+1}} \cdot \qquad (28)$$

Let G_2^* be the second part of (23). Assume firstly that $2\tilde\lambda_1 < \tilde\lambda_2$. Expanding (24) in the neighbourhood of $v=\tilde\lambda_1$ ($\tilde\lambda_1<\tilde\lambda_2$), and using again Darboux' theorem, we deduce, after some tedious manipulations that G_2^* is made of two parts:

$G_{2,1}^* + G_{2,2}^*$ where

$$\langle u^n\rangle \quad G_{2,1}^* \sim \frac{n^A[-\frac{1}{A}+\varphi_1(A)]}{\Gamma(-B) s\, q'\, \tilde\lambda_2} \cdot \frac{1}{\tilde\lambda_2^n}$$

$$G_{2,2}^* = G_1^* \frac{C(s)\, \tilde\lambda_2^B}{s\, q'(\tilde\lambda_1-\tilde\lambda_2)(\tilde\lambda_2-\tilde\lambda_1)^B} \left[\frac{-1}{A} + \varphi_2(A)\right]$$

$$\varphi_1(A) := \int_0^1 \frac{(1-v)^A-1}{v^{A+1}}\, dv$$

$$\varphi_2(A) := \int_0^{\tilde\lambda_1/(\tilde\lambda_2-\tilde\lambda_1)} \frac{(1+v)^A-1}{v^{A+1}}\, dv.$$

Clearly $\langle u^m\rangle G^* \underset{m\to\infty}{\to} 0$ iff

$$\frac{-C(s)\, \tilde\lambda_2^B}{s\, q'(\tilde\lambda_1-\tilde\lambda_2)(\tilde\lambda_2-\tilde\lambda_1)^B A} + 1 = 0$$

which easily leads to (26).

If now $2\tilde{\lambda}_1 > \tilde{\lambda}_2$, we decompose (24) into $\int_0^{2\tilde{\lambda}_1-\tilde{\lambda}_2} + \int_{2\tilde{\lambda}_1-\tilde{\lambda}_2}^{u}$ and a similar analysis again yields (26). ■

We now have all necessary ingredients to proceed as in (B.9): starting from

$$J_m^*(s) = \sum_{k=0}^{\infty} \frac{p's}{1-q's} \varphi_{1,m}^{*k} [H_1^*(s) \frac{p's}{1-q's}]^k [1-\varphi_{1,m}^*]$$

it is not too difficult to deduce, for the hitting time T_m^* P.G.R.

$$F_0^*(s) \sim \frac{\eta_2}{\eta-\eta_2} = \frac{-\varepsilon_2}{\varepsilon-\varepsilon_2} = \frac{-\varepsilon_2}{s-s_2} \qquad (29)$$

with $\eta_2 = m\,\delta^2(\frac{p}{q})^m$ and $\varepsilon_2 = m\,q'\,\delta^3(\frac{p}{q})^m$, hence the following result on T_m^*.

<u>Theorem 10</u>

$\varepsilon_2 T_m^*$ is asymptotically distributed as a negative exponential random variable (with density e^{-x}). ■

<u>Remark 1</u>

At this time of our analysis, one could wonder if it is not possible to derive $F_j^*(s)$ from a direct computation similar to our Theorem 8. Indeed F_j^* satisfies the equation

$$\begin{cases} F_0^* = s[q'F_0^* + p'F_1^*],\ F_m^* = 1 \\ F_j^* = s[p'F_{j+1}^* + \frac{q'}{j+1} F_j^* + q' \frac{j}{j+1} F_{j-1}^*],\ j \neq 0,m. \end{cases}$$

Defining again $G^*(u,s) := \sum_0^{\infty} u^j F_j^*(s)$, the associated differential equation leads now to

$$G^*(u,s) = F_0^*(s)(1-u/\tilde{\lambda}_1)^A\ (1-u/\tilde{\lambda}_2)^B$$

and $<u^n> G^*(u,s) \sim 1 + (1-\frac{\tilde{\lambda}_1}{\tilde{\lambda}_2})^B \frac{1}{\tilde{\lambda}_1^n \Gamma(-A)\ n^{A+1}}$

(we cannot neglegt here the constant term as we did in (28)).

But $<u^m> G^*(u,s)=0$, which by asymptotic analysis again leads to (29). ■

Remark 2

Theorem 7 can also be deduced from Theorem 8's techniques.

Indeed $\varphi^*_{j,m} = \lim_{s\to\infty} H^*_{j,m}(s)$.

Letting $s\to 1$ in the differential equation (25) and solving, we obtain

$$G^*(u) = \frac{1}{1-u} + \frac{1}{1-u}\frac{p'(1-\varphi_1^*)}{q'} \ln[1-\frac{q'u}{p'}]$$

which leads to (21) by standard expansion. ■

Remark 3

To analyse the effect of the reflecting boundary, consider two absorbing (diagonal) barriers: at d (d>>1) and m.

Starting from d+j, what is the P.G.F. $H^*_{j,m,d}(s)$ of the hitting time to d before m.

Proceeding as in Theorem's 8 proof, and omitting the details, we can check that we again derive an explicit solution for the double G.F. $G^*(u,s,d)$.

For large d, we can show that G^* is asymptotically equivalent to the G.F. of an ordinary one-dimensional random walk (with step probabilities p', q' and absorbing barriers at d and m). The reflecting affect is asymptotically negligible. ■

It is now time to discuss our Basic assumption of uniform distribution on the diagonal.

First of all, let us check the mean hitting time α from 1 to 0, as given by (26) and the relation $s-1=(q'-p')n$ (see Sec. B.1).

This leads to $\alpha = (2q-p)/[2(q-p)^2]$. (29)

But this can be checked by a formula dating back to Smoluchowski:

let γ be the recurrence time, for an initial state of some set Ω of a recurrent Markov chain to another state of Ω, with a stationary distribution π.

Smoluchowski's formula tells us that

$$\alpha := E(\gamma) = (1-\pi^+ 1)/(\pi^+ P^- 1) \tag{30}$$

where P is the associated transition matrix and

$(A^+B)_{ij}$ means $\sum_{i\in\Omega} A_{ir}b_{rj}$ (similarly $\sum_{i\notin\Omega}$ for A^-B). (For a simple proof: see Louchard [ ]).

In our case, let $\Omega = (0,0)$. It is easily checked that $\pi_{(j,i)} = (1-\frac{p}{q})^2(\frac{p}{q})^j$ where, for each state (x_1,x_2), j is given by the diagonal value $j:=x_1+x_2$ and i $(i=1...j+1)$ denotes the position of the state along that diagonal (starting form above). (29) is now immediate for (30).

Let us mention that $E(\gamma^2)$ can be associated to the capacity of Ω in the potential associated to P (see Louchard [ ], Sec.III). This analysis is presently under investigation.

The second aspect of our Basic assumption deals with the hitting probability $\varphi_{(j,i),m}$ approximated by $\varphi^*_{j,m}$ in Theorem 7.

Dropping m to ease the notations, we define $\xi_{(j,i)}$ by $\xi_{(j,i)} := \varphi_{(j,i)} - \varphi^*_j$.

We will derive the following result.

Theorem 11

There exists some constant K and, for any sufficiently large m, some small τ such that

$$|\xi_{(j,i)}| \leqslant \frac{\Psi(j-1)}{\Psi(m-1)} \frac{K}{m^{1-\tau}} \quad \text{for } j=1$$

$$\text{and } j=m-1,\ i \in [m^{\tau/2}, m-m^{\tau/2}]$$

Proof

The proof is rather long and will be divided into 4 steps

i) After a detailed analysis of our random walk, it appears that $\xi_{(j,i)}$ satisfies the equations

$$\left\{\begin{array}{l} \xi_{(j,i)} = q\, \xi_{(j,1)} + p[\xi_{(j+1,1)} + \xi_{(j+1,2)}] + q\, \xi_{j-1,1} - (j-1)\, \beta_j \\ \qquad\qquad\qquad\qquad\qquad\qquad j=1\ldots m-1 \qquad (32) \\ \text{(similar equation for } \xi_{(j,j+1)}) \\ \xi_{(j,i)} = q[\xi_{(j-1,i)} + \xi_{(j-1,i-1)}] + p[\xi_{(j+1,i)} + \xi_{(j+1,i+1)}] + 2\beta_j \\ \qquad\qquad\qquad\qquad\qquad\qquad j=1\ldots m-1,\ i \neq 1, j+1 \\ \xi_{(o,o)} = \xi_{(m,.)} \equiv 0 \end{array}\right.$$

where $\beta_j := [q(\frac{q}{p})^{j-1}]/[(j+1)j\ \varphi(m-1)]$.

Set $\sigma_{(j,i)} := (\frac{p}{q})^j\ \xi_{(j,i)}$.

(For the interested reader, let us mention that this amounts to use the dual Markov Process (M.P.): see Kemeny et al. [14] p.136 for more details).

(32) implies

$$\sigma = \sigma P^{-} + v \qquad (33)$$

where P^- is the transition matrix of the transient M.P. with absorbing barriers at 0 and m and

$$\begin{cases} v_{(j,1)} = v_{(j,j+1)} := -(\frac{p}{q})^j \, (j-1) \, \beta_j \\ v_{(j,i)} := 2(\frac{p}{q})^j \, \beta_j, \quad i=2..j. \end{cases} \tag{34}$$

(32) is now solved by

$$\sigma = v[1-P^-]^{-1} \tag{35}$$

Note that $\sum_i v_{(j,i)} = 0$. This allows us to use Lemma (2.3) from Haviv et al. [13] which states that, under this condition, and setting $k := (\ell, v)$.

$$\begin{aligned} |\xi_k| &= (\tfrac{q}{p})^\ell \, |\sigma_k| \\ &\leq \sum_{j=2}^{m-1} 4(\tfrac{q}{p})^{\ell-j} \, (j-1) \, \beta_j \, [\max_i \, \#_{(j,i),k} - \min_i \#_{(j,i),k}]/2 \\ &= \sum_{j=2}^{m-1} [4(\tfrac{q}{p})^\ell \, \frac{p(j-1)}{(j+1) \, j\Psi(m-1)}] \, [\max_i \rho_{(j,i),k} - \min_i \rho_{(j,i),k}] \, \frac{\#_{k,k}}{2} \end{aligned} \tag{36}$$

where

$\rho_{u,r}$ is the probability of reaching state r from state u in the M.P. P^-, before absorption at 0 or m.

and $\#_{k,r}$ is the mean number of passages from k to r under the same conditions.

ii) Let us now analyse $\#_{k,k}$. From now on, K, will denote constants independent of m.

- for $k=(1.)$, it is clear, by replacing the absorbing barrier at m by a reflecting barrier, that $\#_{k,k} \leq \sum_{0}^{\infty} \ell(1-q)\, q^{\ell} = \frac{1-q}{q}$

- for $k=(m-1,.)$, we replace the absorbing barrier at 0 by a reflecting barrier and we consider the mean number of returns to diagonal $(m-1)$. We are led to a geometric distribution with mean $\frac{1-p'}{p'}$

- for $k=(j,i)$, $j \in [2,m-2]$, we replace the absorbing barrier at m by a reflecting barrier. Then we firstly consider the mean number of returns to diagonal j, excluding any excursion to $(\ell,.)$, $\ell < j$. This is obviously bounded by $\frac{q}{p'+q}$

  We now count the mean number of returns from any state $(j-1,.)$ to diagonal j (with reflexion at j) before absorption at 0. It is clear that this quantity is bounded by the mean number of returns to j from $j-1$ in a one-dimensional random walk with absorption at 0 and probabilities $[p',q,q]$ for steps $[+1,+0,-1]$. By (B.7) we obtain the bound $(1-p'/q)/(p'/q)$. Combining our two bounds yields $q^2(1-p'/q)/[p'(p'+q)]$.

- to conclude, there exists some K_1 such that $\#_{k,k} \leq K_1$.

iii) As a first crude bound, it is not difficult to deduce from (22) and (36) that

$$|\xi_{(\ell,v)}| \leq K_2 \left(\frac{p}{q}\right)^{m-\ell} m \log m \tag{37}$$

For k = (1,.), we derive, from (35) that, by symmetry,

$$\sum_i v_{(j,i)} \; \#_{(j,i),k} = \frac{1}{2} \sum_i v_{(j,i)} [\#_{(j,i),k} + \#_{(j,j+2-i),k}]$$

$$= \frac{1}{2} [1 + \rho_{(1,1),(1,2)}] \left(\sum_i v_{(j,i)} \; \bar{\rho}_{(j,i),1}\right) \#_{k,k}$$

where $\bar{\rho}_{u,1}$ is now the probability of reaching the diagonal 1 from state u, before absorption at m.

In (36), we can thus use $\Delta_j := [\max_i \bar{\rho}_{(j,i),1} - \min_i \bar{\rho}_{(j,i),1}]$.

But repeating the reasoning which led to (37), with, this time, the M.P. with absorption at diagonal 1 and m, we can check that

$$\Delta_j \leq K_3 \left(\frac{p}{q}\right)^{m-j} m \log m \tag{38}$$

We are now ready to use (36): let the Σ be decomposed, for some small τ, into $\Sigma_1 = \sum_{j=2}^{m-m^\tau-1}$, $\Sigma_2 = \sum_{j=m-m^\tau}^{m-1}$. (36) and (38) imply, after some elementary manipulations

$$|\xi_{(1,.)}| \leq \frac{K_4}{\Psi(m-1)} \left[\frac{m \log m}{m-m^\tau} \left(\frac{p}{q}\right)^{m^\tau} + \frac{1}{m^{1-\tau}}\right]$$

which easily leads to (31).

iv) for $k=(m-1,i)$, $i \in [m^{\tau/2}, m-m^{\tau/2})$, we again decompose the Σ in (36) Σ_1 and Σ_2. Let $\tau' := \tau/2$

In Σ_1, we deduce from (22) and (37)

$\rho_{(j,u),k} \leqslant K_5 (\frac{p}{q})^{m-j} [\frac{m}{j} + m \log m]$. Hence, from (36)

$\Sigma_1 \leqslant K_6 \frac{m^2 \log m}{m-m^{\tau'}} (\frac{p}{q})^{m^{\tau'}}$.

In Σ_2, we return to the exact form (35), which yields

$$\left(\sum_{u=1}^{j+1} \nu_{(j,u)} \rho_{(j,u),k}\right)^{\#}_{k,k}.$$

From (B.7), we infer,

$$\rho_{(j,u),k} \leqslant K_7 \left(\frac{p}{q}\right)^{m^{\tau'}} \quad \text{for } u \in [1, i-1-m^{\tau'}] \cup [i+1+m^{\tau'}, j+1].$$

The corresponding summation in $|\xi_{(m-1,i)}|$ leads from (34) to a bound

$$K_8 \sum_{j=m-m^{\tau'}}^{m-1} \frac{m}{j} \left(\frac{p}{q}\right)^{m^{\tau'}} \tag{39}$$

For $u \in [i-m^{\tau'}, i+m^{\tau'}]$, we obtain in $|\xi_{(m-1,i)}|$ a bound

$$K_9 \sum_{j=m-m^{\tau'}}^{m-1} \frac{m.m^{\tau'}}{j^2} \tag{40}$$

(39) and (40) easily lead to (31). ■

Repeating now the proof of Theorem 10, with $\varphi_{(1,.)}$ as given by Theorems 7 and 11, we derive the following result.

Theorem 12

$\varepsilon_2 T_m$ is asymptotically distributed as a negative exponential random variable.

Remark 4

Theorem 6 tells us that the asymptotic hitting place is uniformly distributed on the diagonal. This allows us to use the first passage time approach (see Sec.B.3) in the following way: let the diagonal ℓ be a reflecting barrier. What is the stationary distribution π of this new recurrent M.P.?

It is easily checked that $\pi_{(j,i)} = C(\frac{p}{q})^j$.

This gives $C^{-1} = \sum_0^{\ell} (j+1)(\frac{p}{q})^j \sim \frac{1}{\delta^2}$, $\ell >> 1$. Let Ω be the diagonal ℓ.

Smoluchowski's formula (30) gives

$$E(\gamma_\ell) = [1-C(\frac{p}{q})^\ell (\ell+1)]/[\ell q' C(\frac{p}{q})^\ell] \sim \frac{(\frac{q}{p})^\ell}{\delta^2 \ell q'}$$

But γ_ℓ is nothing but the hitting time to diagonal ℓ, starting from an uniform distribution on diagonal $\ell-1$. The total mean hitting time is given by

$\sum_{\ell=1}^{m} E(\gamma_\ell) \sim \frac{(\frac{q}{p})^m}{\delta^3 m q'}$ which is exactly the mean $1/\varepsilon_2$ computed in Theorem 12. ■

Remark 5

We can also construct another recurrent M.P. from Fig.2: replace the absorbing barrier at m by letting the random walk move from diagonal m to (o,o), with probability r and returning to diagonal m with probability 1-r.

Let the corresponding stationary distribution π be given by

$$\pi_{(j,.)} := \varphi_{(j,i)} (\frac{p}{q})^j \pi_o, \quad j = 1..m-1$$

and
$$\begin{cases} \pi_o = q'\ \pi_o + q'\ \pi_{(1.)} + r\ \pi_{(m.)} & (41) \\ \pi_{(m.)} = p' \sum_{1}^{m} \pi_{(m-1,i)} + (1-r)\ \pi_{(m,.)} & (42) \end{cases}$$

It is readily checked that $\varphi_{(j,i)}$ satisfies the equations characterizing the exact probability of hitting 0 before m, starting from (j,i).
(42), theorems 7 and 11 imply

$$r\ \pi_{(m.)} \sim m\ p'\pi_o\ (\frac{p}{q})^{m-1}\ \delta. \qquad (43)$$

But we must have $\pi_o \sum_{j=o}^{m-1} \sum_{i} \varphi_{(j,i)} (\frac{p}{q})^j + \pi_{(m,.)} = 1$

with leads to $\pi_o \sim \delta^2$. Let Ω be the diagonal m.
(30) leads to $E(\gamma) \sim 1/r\pi_{(m.)}$ which, with (43) yields the mean $1/\varepsilon_2$ given by Theorem 12.
(41) is asymptotically checked by immediate computations. ■

Subcase 3.3.2. : $p_1 \neq p_2$

This more difficult situation is under investigation: the distribution is still of exponential type but the analysis of α is more intricate.

III. THE BANKER ALGORITHM

Here we restrict our attention to the following subproblem : consider two processes P_1, P_2 sharing m' examplars of an entity R. P_1 (resp. P_2) needs at most m_1 (resp. m_2) items.
An execution of the system (P_1, P_2) is a random walk inside the rectangle OA_1BA_2 below (see Figure 8) and is correct only if the random walk remains inside the rectangle with breaked corner $OA_1B_1B_2A_2$.

$[0,A_1]$, $[A_1,B_1]$, $[B_2,A_2]$, $[A_2,0]$ are reflecting barriers, $[B_1,B_2]$ is an absorbing boundary

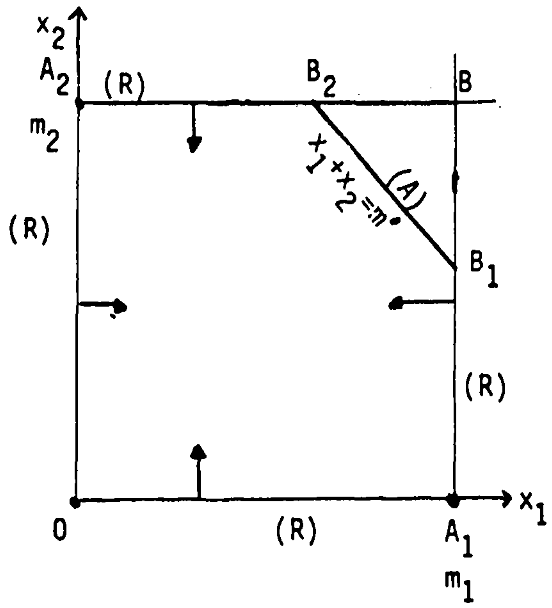


Figure 8

The absorption time T_m now corresponds to a deadlock detection.

The prevention algorithm leads to the following boundary

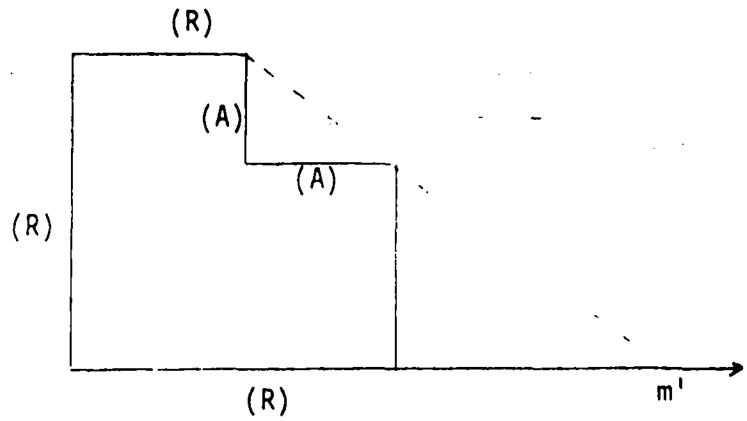
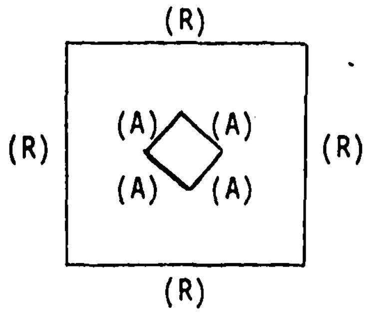


Figure 9

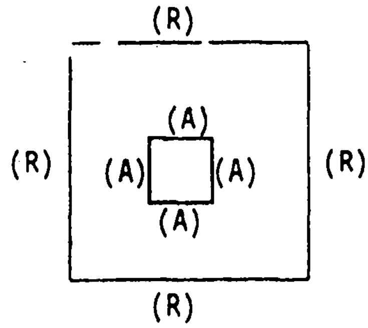
As in the storage allocation algorithm, several sub-cases must be considered. To simplify the analysis, we will set $m_1=m_2=m$.

III.1 The trend-free case : $\mu=0$

It is easily seen that our random walk is weakly convergent to a B.M. $W(t)$ with the following boundary conditions



deadlock



prevention

The density $f(x,y,t) := P_x[W(t) \in dy,\ t<T]$ satisfies the equation

$\partial_t f = \frac{1}{2} \Delta f$

$f(x,.,t) = 0$ on the absorbing boundary

$\bar{\partial}_x f = 0$ on the reflecting boundary ($\bar{\partial}_x$ is the normal derivative)

$f(x,y,0) = \delta(x-y)$ (Dirac distribution).

The analysis of f is presently under investigation in cooperation with M. Tolley (see Descloux and Tolley [6] for a typical approach.

III.2 The case $\mu\neq 0$

As far as μ is included in the following range (see Sec.II.3 for notations): $\theta \in [-\frac{\pi}{4}, \frac{\pi}{4}]$, the analysis proceeds exactly as in sub-case II.3.1 and II.3.2 (we omit the details). The other sub-cases are more difficult. We will limit ourselves to the case (similar to sub-case II.3.3.1): $p_1=p_2=p$, $q_1=q_2=q$, $p<q$. We obtain the following theorem.

Theorem 13

In the detection algorithm, the hitting place Z_m is asymptotically uniformly distributed on the absorbing boundary; $\varepsilon_3 T_m$ is asymptotical distributed as a negative exponential random variable

where $\varepsilon_3 := (2m-m')q'\delta^3 \left(\frac{p}{q}\right)^{m'}$. (44)

Proof

For a two reflecting barriers, one-dimensional random walk, the stationary distribution (20) is again reached with exponential speed (we just have to change $P(y_1=0)$ by suitable normalisation).

If we ignore the absorbing boundary, the mean return time to 0 is still given by Theorem 9.

For $j \leq m$, the hitting probability φ_j^* satisfies the same equations as in Theorems'7 proof, so that (with this proof's notation)

$$\varphi_j^* = 1 + y_0 \, \Psi(j-1)$$

and $$\varphi_m^* = \varphi_{m-1}^* + \frac{y_0}{m} \left(\frac{q}{p}\right)^{m-1}.$$

At m, we have, by our random walk reflecting properties

$$\varphi_m^* = \frac{m}{m+1} [p'\varphi_{m+1}^* + q' \varphi_{m-1}^*] + \frac{1}{m+1} \varphi_m^*$$

which yields

$$\Delta_{m+1} := \varphi_{m+1}^* - \varphi_m^* = \frac{y_0}{m} \left(\frac{q}{p}\right)^m$$

For $j > m$, the equations are

$$\varphi_j^* = q'\varphi_{j-1}^* + \frac{p'}{2m-j+1} \varphi_j^* + p' \frac{2m-j}{2m-j+1} \varphi_{j+1}^*$$

and, after some difference equation manipulation,

$$\varphi_j^* = \varphi_b^* + y_0 \, \bar{\Psi}(j-1)$$

where $\bar{\Psi}(j) = \sum_{i=m}^{j} (\frac{q}{p})^i/(2m-i)$. We have

$$\Psi(j-1) \sim (\frac{q}{p})^j / [(2m-j+1)(\frac{q}{p} -1)][1+ O(\frac{1}{2m-j})]$$

The condition $\varphi_{m'}^* = 0$ now leads to $y_0 \sim - (2m-m')(\frac{q}{p} -1)(\frac{p}{q})^m$ and we finally obtain $\varphi_1^* \sim 1 - (2m-m')(\frac{q}{p} -1)(\frac{p}{q})^m$.

We now follow Theorem 10's proof: (44) is readily derived. ■

The detection algorithm is more difficult to analyse: this problem is under investigation.

IV. CONCLUSION AND FURTHER ASPECTS

For the two stacks problem, we have recovered in a more simpler way results obtained previously by Yao and Flajolet with different techniques and we have obtained new limiting distributions.

The banker algorithm has been analysed with the same tools. The case were the number of processes is greater than two and the number of entities greater than one is the object of a work in process.

Diffusions appear to be a powerful tool for the analysis of distributed algorithms (see Louchard [12] for other typical applications of these techniques).

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APPENDIX A : some Fourier Analysis

By Abramowitz and Stegun (A.S.) [1] (23.1.18) we know that, for $0 \leqslant x \leqslant 1$:

$$\frac{1}{\pi^2} \sum_{k=1}^{\infty} \frac{\cos(2k\pi x)}{k^2} = B_2(x) = \frac{1}{6} - x + x^2 \qquad (A.1)$$

So that

$$\sum_{k \text{ even}>0} \frac{\cos(k\pi x)}{k^2} = \frac{\pi^2}{4} B_2(x) \qquad (A.2)$$

Substituting $x \to \frac{x}{2}$ into (A.1) and substracting (A.2) gives

$$\sum_{k \text{ odd}>0} \frac{\cos(k\pi x)}{k^2} = \frac{\pi^2}{4} [-x + \frac{1}{2}] \qquad (A.3)$$

By A.S [1] (23.1.17), we have, for $0 \leqslant x \leqslant 1$:

$$\sum_{k=1}^{\infty} \frac{\sin(2k\pi x)}{k} = -\pi \, B_1(x) = -\pi \, (x - \frac{1}{2})$$

Proceeding as previously, this gives

$$\sum_{k \text{ even}>0} \frac{\sin(k\pi x)}{k} = -\frac{\pi}{2} B_1(x) \qquad (A.4)$$

and

$$\sum_{k \text{ odd}>0} \frac{\sin(k\pi x)}{k} = \frac{\pi}{4} \qquad (A.5)$$

APPENDIX B : some results on one-dimensional random walk

B.1 Hitting times

Let us analyse the hitting time distribution in a one-dimensional random walk with reflecting barrier at the origin and absorbing barrier at m. Forward and backward steps probabilities are given by p and q:=1-p. The probability generating functions (P.G.F.), $F_j(s)$ of the hitting time T_m, starting at j, is characterized by the following equations

$$\begin{cases} F_m(s) = 1 \\ F_0(s) = s[q\,F_0(s) + p\,F_1(s)] \\ F_j(s) = s[p\,F_{j+1}(s) + q\,F_{j-1}(s)],\ j\neq 0,m. \end{cases}$$

The solution is easily seen to be (see Feller [7] p.496, ex.12)

$$F_j(s) = [\lambda_1^{j+1}(1-\lambda_2)-\lambda_2^{j+1}(1-\lambda_1)]/[\lambda_1^{m+1}(1-\lambda_2)-\lambda_2^{m+1}(1-\lambda_1)] \qquad (B.1)$$

with $\lambda_1,\lambda_2 := [1\pm\sqrt{1-4pqs^2}]/2ps$.

Note that $\lambda_1\lambda_2 = q/p$, $\lambda_1+\lambda_2 = 1/ps$, $\lambda_1>\lambda_2$.

Let $j:=m-1$ and $m\to\infty$.

(B.1) readily gives $F_{m-1} \to \frac{1}{\lambda_1} = \frac{p}{q}\lambda_2$.

Interchanging p and q and replacing j by m-j yields (the absorbing barrier is now at 0 and we ignore the barrier at m)

$$F_1 \to \lambda_2 \qquad (B.2)$$

From now on assume: $p<q$.

The asymptotic behaviour of F_j for $m\to\infty$ is related to the dominant singularity of (B.1). Following Feller [7] p.352, we set

$$s := 1/[2\sqrt{pq}\cos\phi].$$

This gives $\lambda_1, \lambda_2 = \sqrt{\frac{q}{p}}\, e^{\pm i\phi}$. It is easily checked that the dominant singularity of (B.1) is given by $s_1 = 1 + \varepsilon_1$, with $\varepsilon_1 > 0$, $\varepsilon_1 \ll 1$. Let ϕ_1 be the corresponding value of ϕ. Set ϕ: $-i\Psi$. We verify : $e^{\Psi} \underset{s\to 1}{\to} \frac{q}{p}$. Let $\varepsilon = s-1 = (q-p)\eta$

We can check that, for $\eta \to 0$: $\lambda_1 \sim \frac{q}{p}(1-\eta)$, $\lambda_2 \sim 1+\eta$, $e^{\Psi} \sim \sqrt{\frac{q}{p}}\,(1-\eta)$ (B.3)

An asymptotic analysis of (B.1) readily gives

$$F_o(s) \sim \frac{-\eta_1}{\eta-\eta_1} = \frac{-\varepsilon_1}{\varepsilon-\varepsilon_1} = \frac{-\varepsilon_1}{s-s_1} \qquad (B.4)$$

with $\eta_1 := \delta(\frac{p}{q})^m$ and $\delta := 1 - \frac{p}{q}$.

This leads to $\varepsilon_1 = q\,\delta^2(\frac{p}{q})^m$. (B.4) yields now

$$P_o[T_m = n] \sim \varepsilon_1\, e^{-\varepsilon_1 . n} \quad \text{(with mean } 1/\varepsilon_1) \qquad (B.5)$$

In the following, we will need another P.G.F. Let us consider a one-dimensional random walk with absorbing barriers both at 0 and m. The P.G.F. $H_{j,m}(s)$ of the hitting time to 0 before absorption at m is easily computed as

$$H_{j,m}(s) = [-\lambda_2^m \lambda_1^j + \lambda_1^m \lambda_2^j]/[\lambda_1^m - \lambda_2^m]. \qquad (B.6)$$

Remark that $H_{j,m}(s) \underset{m\to\infty}{\to} \lambda_2^j$ which is of course identical to (B.2).

As $s \to 1$, $H_{j,m}(s) \to \varphi_{j,m} = [1-(\frac{p}{q})^{m-j}]/[1-(\frac{p}{q})^m]$ (B.7)

which is nothing but the probability of hitting 0 before m.

B.2 Last leaving times

We will now analyse another way of getting (B.4). Assume we are at position $m-\ell$.., $\ell \neq 0, m$. The P.G.F. of the last leaving time from $m-\ell$ given that we never return to $m-\ell-1$ in obviously given, from (B.7), by

$$J_\ell(s) = \sum_{k=0}^{\infty} ps\ \varphi_{1,\ell}^{k}[\bar{\lambda}_2(\ell)\ ps]^k\ [1-\varphi_{1,\ell}]/[1-\varphi_{1,\ell+1}] \qquad (B.8)$$

where $\bar{\lambda}_2(\ell)$ the P.G.F. of the hitting time to $m-\ell$, starting at $m-\ell+1$, given that there is no absorption at m.

Similarly, the P.G.F. of the last leaving time from 0 is given by

$$J_m(s) = \sum_{k=0}^{\infty} \frac{ps}{1-qs}\ \varphi_{1,m}^{k}[\bar{\lambda}_2(m)\ \frac{ps}{1-qs}]^k\ [1-\varphi_{1,m}]. \qquad (B.9)$$

As a first approximation, and following (B.2), we use λ_2 instead of $\bar{\lambda}_2$ in (B.8) and (B.9).

The total P.G.F. of the hitting T_m is now given by

$$F_0(s) = J_m(s) \prod_{\ell=1}^{m-1} J_\ell(s). \qquad (B.10)$$

A detailed asymptotic analysis shows (we omit the details), that the dominant contribution of (B.10) is only due to $J_m(s)$ and, asymptotically, is identical to (B.4).

Now the exact P.G.F. $\bar{\lambda}_2(\ell)$ is given (from (B.6)) by $H_{1,\ell}(s)/\varphi_{1,\ell}$ instead of λ_2. After some tedious manipulations, it can be checked that this doesn't affect the asymptotic behaviour of (B.9).

It is also remarkable that λ_2 enter into (B.8) only though its first approximation $\lambda_2 \sim 1+\eta$.

By (B.3) this is identical to $\lambda_2 \sim 1 + \frac{1}{q-p}(s-1)$, which shows that the mean hitting time from 1 to 0 is given by $1/(q-p)$. This is well-knownand can easily be checked.

B.3 First passage times

Still another way of deriving (B.4) is to analyse the first passage time form ℓ-1 to ℓ for $\ell = 1..m$.

Letting $\bar{F}_{j,\ell}(s)$ be $F_j(s)$ from (B.1) with m replaced by ℓ, we immediately deduce (this is the classical ladder technique)

$$F_0(s) = \prod_{\ell=1}^{m} \bar{F}_{\ell-1,\ell}(s) \qquad (B.11)$$

Again, an asymptotic analysis shows that the dominant singularity is derived from $\bar{F}_{m-1,m}(s)$ but, this time, all terms of (B.11) contribute to the final asymptotic form, which is of course identical to (B.4).

We also observe that these contributions are exponentially decreasing from ℓ=m to 1.

As a final remark, the asymptotic mean of (B.5): $1/\varepsilon_1$, can also be recovered by summing all asymptotic means related to each term of (B.11)(again with exponentially decreasing contribution from ℓ = m to 1).

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