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Abstract

In this paper, we consider a family of irreducible, ergodic and aperiodic Markov chains $X^{(\varepsilon)} = \{X_n^{(\varepsilon)}, n \geq 0\}$, depending on a parameter $\varepsilon > 0$, so that the local drifts have a critical behaviour (in terms of Pakes lemma). The purpose is to analyze the steady state distributions of these chains (in the sense of weak convergence), when $\varepsilon \downarrow 0$. Under assumptions involving at most the existence of moments of order $2+\gamma$ for the jumps, we show that, whenever $X^{(0)}$ is not ergodic, it is possible to caracterize accurately these limit distributions. Connections with the gamma and uniform distributions are revealed.

ERGODICITY; MARKOV CHAIN; RANDOM ACCESS; STABILITY; TRANSIENT; WEAK CONVERGENCE

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PHENOMENES TRANSITOIRES POUR LES CHAINES DE MARKOV ET APPLICATIONS

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Résumé

Dans cet article, nous considérons une famille de chaines de Markov $X^{(\varepsilon)} = \{X_n^{(\varepsilon)}, n \geq 0\}$, irréductibles, ergodiques et apériodiques, dépendant d'un paramètre $\varepsilon > 0$, de façon que les dérives locales aient un comportement critique (vis à vis du lemme de Pakes, par exemple). On veut analyser les distributions limites stationnaires de ces chaines (au sens de la convergence faible), lorsque $\varepsilon \downarrow 0$. Sous des hypothèses d'existence de moments d'ordre $2+\gamma$ (au plus) pour les sauts, nous montrons que, lorsque $X^{(0)}$ n'est pas ergodique, il est possible de caractériser très précisément ces distributions limites. Des relations avec les distributions gamma et uniforme sont ainsi mises en évidence.

ERGODICITÉ; CHAINES DE MARKOV; ACCES ALÉATOIRE; STABILITÉ; TRANSITOIRE; CONVERGENCE FAIBLE.

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I. INTRODUCTION

Let $X^{(\varepsilon)} = \{X_n^{(\varepsilon)}, n \geq 0\}$ be a sequence (with respect to ε) of irreducible aperiodic and ergodic Markov chain (M.C.), for $\varepsilon > 0$, with state space {0,1,2...}. We shall suppose that the transition probabilities of these chains have a property of convergence $p_{ij}^{(\varepsilon)} \to p_{ij}$ as $\varepsilon \downarrow 0$, where $\| p_{ij} \|$ is the matrix of transition probabilities of an irreducible aperiodic M.C. $X^{(0)}$.

Let $\pi_j^{(\varepsilon)}$ be stationary probabilities for the matrix $\| p_{ij}^{(\varepsilon)} \|$.

$$\pi_i^{(\varepsilon)} = \sum_{j \geq 0} \pi_j^{(\varepsilon)} p_{ji}^{(\varepsilon)} , \quad i \geq 0 \tag{1}$$

and $\zeta^{(\varepsilon)}$ a random variable with the distribution $\mathbb{P}[\zeta^{(\varepsilon)} = j] = \pi_j^{(\varepsilon)}$.
In sections II and III , we consider the asymptotic behaviour of the distribution of $\zeta^{(\varepsilon)}$ as $\varepsilon \downarrow 0$ under some assumptions for $p_{ij}^{(\varepsilon)}$, which shall be explained later. If $X^{(0)}$ is ergodic, then the problem mentioned above concerns the stability of the ergodic distributions $X^{(\varepsilon)}$ with respect to the parameter $\varepsilon \downarrow 0$. However, the conditions that will be exposed below are related, generally speaking, to another situation, the "critical" one, when $X^{(0)}$ can be non ergodic. Section III proposes an application of these results to a M.C. arising from the analysis of the well known basic *Aloha* algorithm, which is employed to solve the contention in some distributed random access communication systems.

II. TRANSIENT PHENOMENA THEOREMS FOR MARKOV CHAINS

We shall use the following notation

$$m_i^{(\varepsilon)} = \mathbb{E}[X_1^{(\varepsilon)} - i \,/\, X_0^{(\varepsilon)} = i]$$

$$b_i^{(\varepsilon)} = \mathbb{E}[(X_1^{(\varepsilon)} - i)^2 \,/\, X_0^{(\varepsilon)} = i]$$

To introduce a reasonable classification of the transient phenomena for the M.C. $X^{(\varepsilon)}$, we will suppose that the asymptotic behaviour of $m_i^{(\varepsilon)}$ and $b_i^{(\varepsilon)}$, as $i \to \infty$, $\varepsilon \downarrow 0$ is quite regular.
In particular, we suppose the existence of the following limits :

$$\lim_{i\to\infty} m_i^{(0)} = 0 ,$$

$$\lim_{i\to\infty} i\, m_i^{(0)} = -\mu \quad , \quad -\infty \le \mu \le \infty ,$$

$$\lim_{i\to\infty} b_i^{(0)} = b \quad , \quad 0 < b < \infty .$$

It turns out that the parameters μ , b are very essential for the ergodicity of $X^{(0)}$ and later on for the asymptotic behaviour of $\zeta^{(\varepsilon)}$ also. In particular it was proved in [12], [16] that $X^{(0)}$ is ergodic if $2\mu > b$, and one can see in this paper that the M.C. $X^{(0)}$ under consideration will be nonergodic if $2\mu \le b$. We will consider the convergence of $\| p_{ij}^{(\varepsilon)} \|$ to $\| p_{ij} \|$ ensuring that the drifts $m_i^{(\varepsilon)}$ have a negative bias $-\varepsilon$ with respect to $m_i^{(0)}$, so that $\lim_{i\to\infty} m_i^{(\varepsilon)} = -\varepsilon$. Thus $X^{(\varepsilon)}$ is ergodic for $\varepsilon > 0$, (see [14], [16]). To be more exact, we make the following assumptions :

$$\lim_{i\to\infty,\, \varepsilon\downarrow 0} m_i^{(\varepsilon)} = 0 ,$$

$$\lim_{i\to\infty,\, \varepsilon\downarrow 0} i\,(m_i^{(\varepsilon)} + \varepsilon) = -\mu \ , \ -\infty \le \mu \le \infty , \tag{2}$$

$$\lim_{i\to\infty,\, \varepsilon\downarrow 0} b_i^{(\varepsilon)} = b \ , \ 0 < b < \infty \ , \ \text{and} \quad \sup_{i\ge 0,\, \varepsilon\ge 0} b_i^{(\varepsilon)} < \infty \tag{3}$$

A very particular example of such M.C. is the sequence X_n, defined by the equalities $X_{n+1} = (X_n + \xi_n)^+$, where the ξ_n are i.i.d. random variables with $\mathbb{E}[\xi_n] = -\varepsilon$. Transient phenomena for this type of random walks (in this case $\mu = 0$) and for some more general ones were considered in [2], [3], [10], [15].

THEOREM 1 *(Stability).*

If (2), (3) hold , $2\mu > b$ *and*

$$\sup_{i\ge 0,\, \varepsilon\ge 0} \mathbb{E}[\,|X_1^{(\varepsilon)} - i|^{2+\gamma} / X_0^{(\varepsilon)} = i] < C < \infty \quad , \tag{4}$$

where γ *is an arbitrary but strictly positive number , then* $\zeta^{(\varepsilon)} \to \zeta^{(0)}$ *as* $\varepsilon \downarrow 0$ *(the chain* $X^{(0)}$ *is ergodic and the stability takes place as* $\varepsilon \downarrow 0$ *).*

THEOREM 2 *(Convergence to a Γ – distribution).*

If (2) and (3) hold and $-\infty < 2\mu < b$, then $X^{(0)}$ is transient.
If, moreover, the series representing $b_i^{(\varepsilon)} = \sum_{u \geq 0} p_{iu}^{(\varepsilon)} (u-i)^2$ (5)

converges uniformly with respect to i and ε, then

$$2\varepsilon \zeta^{(\varepsilon)} \to \Gamma_{\frac{1}{b}, 1-\frac{2\mu}{b}} \quad \text{as} \quad \varepsilon \downarrow 0.$$

It is worth mentioning that condition (5) will be satisfied, if we merely assume the more stringent condition (4).

Remark.: In fact, theorems 1 and 2 will be proved under slightly more general assumptions. In theorem 1, (2) can be replaced by the inequalities $m_i^{(\varepsilon)} \leq m_i^{(0)}$, for $i \geq i_0$, where i_0 is some positive integer. In Theorem 2, (2) can be replaced by the representation

$$m_i^{(\varepsilon)} = -\varepsilon - \frac{\mu}{i} + o(\varepsilon + \frac{1}{i}) \quad \text{as} \quad i \to \infty \;,\; \varepsilon \downarrow 0 .$$

It remains to consider the critical case $2\mu = b$.

THEOREM 3

If (2), (3), (4) hold, $2\mu = b$ *and*

$$2i\,(m_i^{(\varepsilon)} + \varepsilon) + b_i^{(\varepsilon)} = o(\varepsilon + \frac{1}{i}) \qquad (6)$$

then

$$\log(\zeta^{(\varepsilon)}) / \log(1/\varepsilon) \to U[0,1] \;,\; \text{as} \quad \varepsilon \downarrow 0,$$

where $U[0,1]$ *denotes the uniform distribution on* $[0,1]$.

Remark.: If the rate of convergence to 0 in (6) is taken different, then the asymptotic behaviour of the distribution of $\log(\zeta^{(\varepsilon)})$ can be different too.

The following auxiliary lemmas will be useful in the sequel.

LEMMA 1 If the irreducible aperiodic transition matrices $\| p_{ij} \|$ and $\| p^*_{ij} \|$ differ only by a finite number of transition probabilities, then the existence of an invariant probabilistic measure for $\| p^*_{ij} \|$ follows from the existence of an invariant measure for $\| p_{ij} \|$.

Proof. We consider the following situation :

$p_{ij} = p^*_{ij}$, for $i \neq i_0$ and $j > J$, where i_0 or J are fixed.

It is known, see[9], that an irreducible aperiodic M.C. X is ergodic iff there exists an invariant probabilistic measure for the matrix $\| p_{ij} \|$. On the other hand, X is ergodic iff it is positive recurrent .

Let $r_{ij} = \min \{n > 0 : X_n = j / X_0 = i\}$ be the length of the path from the state i to the state j . Positive recurrence means that $\mathbb{E}[r_{ii}] < \infty$. Besides, see [9],

$$\pi_i = \lim \mathbb{P}\{X_n = i\} = 1/\mathbb{E}[r_{ii}] .$$

We consider the chains $\{X_n\}$,with $X_0 = i_0$ and $\{X_n^*\}$, with $X_0^* = i_0$. One can easily see that the following equality holds for $\mathbb{E}[r_{i_0 i_0}]$:

$$\mathbb{E}[r_{i_0 i_0}] = \mathbb{P}[r_{i_0 i_0} = 1] + 2\mathbb{P}[r_{i_0 i_0} = 2] + \ldots$$

$$+ N\,\mathbb{P}[r_{i_0 i_0} = N] + \sum_{j \neq i_0} \mathbb{P}[r_{i_0 i_0} > N, X_N = j]\,(N + \mathbb{E}[r_{j i_0}]) < \infty .$$

It is easy to see that the value of $\mathbb{E}[r_{ji_0}]$, for $j \neq i_0$, does not depend on the values of $p_{i_0 j}$.

It follows from the irreducibility and aperiodicity of X that, for any J, there exists a number N such that

$$\mathbb{P}[X_N = j, r_{i_0 i_0} > N] > 0 \quad , \quad \text{for } j \neq i_0, j > J .$$

Therefore,

$$\mathbb{E}[r_{j i_0}] < \infty \text{ , for } j \neq i_0 \text{ , } j > J .$$

Hence,

$$\mathbb{E}[r^*_{i_0 i_0}] = p^*_{i_0 i_0} + \sum_{j \leq J,\, i \neq i_0} p^*_{i_0 j}(1 + \mathbb{E}[r_{ji_0}]) + \sum_{j > J,\, i \neq i_0} p_{i_0 j}(1 + \mathbb{E}[r_{ji_0}]) < \infty .$$

Lemma 1 is proved . ●

LEMMA 2 If $\pi_i^{(\varepsilon)}$ does not tend to 0 as $\varepsilon \downarrow 0$, then the system (1) has a probabilistic solution.

Proof. Under the assumption of the lemma, there exists a subsequence $\varepsilon_{k'} \downarrow 0$ such that $\pi_0^{(\varepsilon_{k'})} \to \pi_0 > 0$. It follows from Helly's theorem, see [13], that there also exists a subsequence $\varepsilon_k \subseteq \varepsilon_{k'}$ such that, for any $i \geq 0$,

$$\lim_{\varepsilon_k \downarrow 0} \pi_i^{(\varepsilon_k)} = \pi_i > 0 , \quad \sum_{i \geq 0} \pi_i \leq 1 .$$

We have now to show that the sequence $\{\pi_i , i \geq 0\}$ satisfies system (1), corresponding to $\varepsilon = 0$.

Instantiating $\varepsilon = \varepsilon_k$ in system (1), we obtain

$$|\pi_i - \sum_{j\geq 0} \pi_j\, p_{ji}| = |\pi_i - \pi_i^{(\varepsilon_k)} - \sum_{j\geq 0}(\pi_j p_{ji} - \pi_j^{(\varepsilon_k)} p_{ji}^{(\varepsilon_k)})| \leq$$

$$\leq |\pi_i - \pi_i^{(\varepsilon_k)}| + \sum_{j\leq N} |\pi_j p_{ji} - \pi_j^{(\varepsilon_k)} p_{ji}^{(\varepsilon_k)}| + \sum_{j>N}(\pi_j p_{ji} + \pi_j^{(\varepsilon_k)} p_{ji}^{(\varepsilon_k)}) .$$

The first and second terms in the right member tend to zero because $\pi_i^{(\varepsilon_k)} \to \pi_i$ an. $p_{ji}^{(\varepsilon_k)} \to p_{ji}$, as $\varepsilon_k \downarrow 0$.

From the inequality $\pi_i < 1$ and from the Chebyshev's inequality , it follows that

$$\sum_{j>N}(\pi_j\, p_{ji} + \pi_j^{(\varepsilon_k)} p_{ji}^{(\varepsilon_k)}) \leq \sum_{j>N}(p_{ji} + p_{ji}^{(\varepsilon_k)}) \leq 2\sum_{j>N} \frac{\sup b_i^{(\varepsilon)}}{(j-i)^2} \to 0 ,$$

as $N \to \infty$. Therefore $\pi_i = \sum_{j\geq 0} \pi_j p_{ji}$ and the proof of lemma 2 is concluded •

The main point of the proof of the stability in theorem 1 is the weak compactness for $\varepsilon \geq 0$ of the family $\{\pi_i^{(\varepsilon)}, i \geq 0\}$. In this connection, we prove the following

LEMMA 3 Suppose that $2\mu > b$. Choose γ (defined in the statement of theorem 1) arbitrarily small and positive, satisfying

$$0 < \gamma < \frac{2\mu}{b} - 1 . \tag{7}$$

Then, under the assumptions of theorem 1, the moment of order γ of the stationary distributions $\{\pi_i^{(\varepsilon)}, i \geq 0\}$ is uniformly bounded with respect to $\varepsilon \geq 0$:

$$\sup_{\varepsilon\geq 0} \mathbb{E}[\zeta^{(\varepsilon)\gamma}] = \sup_{\varepsilon\geq 0} \sum_{i\geq 0} \pi_i^{(\varepsilon)} i^{\gamma} < \infty .$$

Proof. Consider the following series

$$\sum_{i\geq 0} \pi_i^{(\varepsilon)}\, \mathbb{E}[X_{n+1}^{(\varepsilon)\gamma+2} - i^{\gamma+2} / X_n^{(\varepsilon)} = i] . \tag{8}$$

We will prove the series (8) converges and its values is nonnegative.

Define

$$W_n^{(\varepsilon)} = X_{n+1}^{(\varepsilon)} - X_n^{(\varepsilon)} .$$

We apply the well known Taylor's formula its the function $(1+y)^{\gamma+2}$. Then

$$(1+y)^{\gamma+2} = 1 + y(\gamma+2) + \frac{y^2}{2}(\gamma+1)(1+y\theta(y))^{\gamma} ,$$

where $\theta(y)$ is a continuous functions of y satisfying

$$0 < \theta(y) < 1 \quad , \quad \theta(-1) = 1 - \left(\frac{2}{\gamma+2}\right)^{\frac{1}{\gamma}} < 1 \ . \tag{9}$$

Hence

$$\mathbb{E}[X_{n+1}^{(\varepsilon)\,\gamma+2} - i^{\gamma+2} / X_n^{(\varepsilon)} = i] = i^{\gamma+2}\mathbb{E}[\left(1 + \frac{W_n^{(\varepsilon)}}{i}\right)^{\gamma+2} - 1 / X_n^{(\varepsilon)} = i]$$

$$= (\gamma+2)\, i^{\gamma}\, \mathbb{E}[i\, W_n^{(\varepsilon)} + \frac{(\gamma+1)}{2}\, W_n^{(\varepsilon)2}\left(1+\frac{\theta_{ni} W_n^{(\varepsilon)}}{i}\right)^{\gamma} / X_n^{(\varepsilon)} = i] , \tag{10}$$

where θ_{ni} is a function of $W_n^{(\varepsilon)}$ satisfying $0 < \theta_{ni} < 1$.

Upon applying the elementary inequality

$$|1 + a|^{\gamma} \le 1 + |a|^{\gamma}, \quad 0 \le \gamma \le 1 ,$$

we obtain, since the random variable

$$1 + \frac{\theta_{ni} W_n^{(\varepsilon)}}{i} \text{ is always positive,}$$

$$\mathbb{E}[W_n^{(\varepsilon)\,2}\left(1+\frac{\theta_{ni} W_n^{(\varepsilon)}}{i}\right)^{\gamma} / X_n^{(\varepsilon)} = i] \le b_i^{(\varepsilon)} + i^{-\gamma}\, \mathbb{E}[\,|W_n^{(\varepsilon)}|^{\gamma+2} / X_n^{(\varepsilon)} = i] . \tag{11}$$

According to the assumption (4) in theorem 1, the expectation in the right side member of (11) is uniformly bounded with respect to i and ε. Thus, combining (10) and (11), we obtain

$$\mathbb{E}[\, X_{n+1}^{(\varepsilon)\,\gamma+2} - i^{\gamma+2} / X_n^{(\varepsilon)} = i] \le (\gamma+2)\, i^{\gamma}[\, i\, m_i^{(\varepsilon)} + \frac{(\gamma+1)}{2} b_i^{(\varepsilon)} + C\, i^{-\gamma}] , \tag{12}$$

where C is the constant introduced in (4) .

It follows from (7) and condition

$$m_i^{(\varepsilon)} \le m_i^{(0)} \approx -\frac{\mu}{i}\ , \ \text{for}\ i \ge i_0,$$

that the following inequality takes place, for sufficiently large i and small ε,

$$\mathbb{E}\,[X_{n+1}^{(\varepsilon)\,\gamma+2} - i^{\gamma+2} / X_n^{(\varepsilon)} = i] \le -\mathcal{K}\, i^{\gamma} < 0\ , \quad \text{for}\ i \ge i_*,\ \varepsilon \le \varepsilon_*\ . \tag{13}$$

This inequality plays a basic role in our proof. It follows from (13) that the partial sums S_{i_1} of series (8) are decreasing for $i \ge i_*$, $\varepsilon \le \varepsilon_*$. Thus, the limit of these partial sums does exist and is either finite or equal to $-\infty$.

In fact, we shall prove the following stronger statement : the sum of the series (8) is non negative. Indeed , suppose that this sum is negative. Then , there exists a number $i_1 \ge i_*$, such that $S_{i_1} < 0$. Consider the chain $X_n^{(\varepsilon)}$, with $X_0^{(\varepsilon)} = 0$. Under the assumptions of theorem 1, we have

$$\mathbb{E}\,[X_n^{(\varepsilon)\gamma+2}] < \infty\ , \ \text{for any}\ \ n \ge 0\ .$$

Denote $\pi_{n\,i}^{(\varepsilon)} = \mathbb{P}[X_n^{(\varepsilon)} = i]$.

It follows from the ergodicity that

$$\lim_{n\to\infty} \pi_{ni}^{(\varepsilon)} = \pi_i^{(\varepsilon)}\ , \quad \text{for any}\ \ i \ge 0\ .$$

This, in turn, implies

$$\sum_{i \le i_1} \pi_{ni}^{(\varepsilon)}\ \mathbb{E}\,[X_n^{(\varepsilon)\,\gamma+2} - i^{\gamma+2} / X_n^{(\varepsilon)} = i] \ \to S_{i_1} < 0,\ \text{as}\ n \to \infty\ .$$

Therefore we have, for n sufficiently large,

$$\mathbb{E}\,[X_{n+1}^{(\varepsilon)\,\gamma+2} - X_n^{(\varepsilon)\,\gamma+2}\ ;\ X_n^{(\varepsilon)} \le i_1] = -\,\delta < 0\ .$$

It follows directly from (13) that

$$\sum_{i > i_1} \pi_{ni}^{(\varepsilon)}\, \mathbb{E}[X_{n+1}^{(\varepsilon)\,\gamma+2} - i^{\gamma+2} / X_n^{(\varepsilon)} = i] =$$

$$= \mathbb{E}\,[X_{n+1}^{(\varepsilon)\,\gamma+2} - X_n^{(\varepsilon)\,\gamma+2}\ ;\ X_n^{(\varepsilon)} > i_1] < 0\ , \quad \text{for any}\ n \ge 0\ .$$

Hence, we obtain

$$E\,[X_{n+1}^{(\varepsilon)\,\gamma+2}] \le E\,[X_n^{(\varepsilon)\,\gamma+2}] - \delta\ , \ \text{for}\ n \ge n_0\ \text{large enough}\ .$$

Since $a = \mathbb{E}\,[X_{n_0}^{(\varepsilon)\,\gamma+2}] < \infty$, we get finally

$$0 < \mathbb{E}[X^{(\varepsilon)\,\gamma+2}_{n_0+\frac{a}{\delta}+1}] \le \mathbb{E}[X^{(\varepsilon)\,\gamma+2}_{n_0}] - \delta(\frac{a}{\delta}+1) = -\delta < 0 ,$$

which is a contradiction. Therefore the sum of series (8) is non-negative. Hence,

$$\sum_{i\ge i_*} \pi_i^{(\varepsilon)} \mathbb{E}[X^{(\varepsilon)\,\gamma+2}_{n+1} - i^{\gamma+2} / X^{(\varepsilon)}_n = i]$$

$$\ge - \sum_{i< i_*} \pi_i^{(\varepsilon)} \mathbb{E}[X^{(\varepsilon)\,\gamma+2}_{n+1} - i^{\gamma+2} / X^{(\varepsilon)}_n = i] .$$

Using (13), it follows that, for $\varepsilon \le \varepsilon_*$,

$$\sum_{i\ge i_*} \pi_i^{(\varepsilon)} i^{\gamma} \le \frac{-A(i_*)}{\mathcal{H}} < \infty$$

The proof of lemma 3 is concluded. ●

Remark: It is indeed easy to verify that, if $2\mu > b$, the Markov chain $X^{(0)}$ is ergodic.

Indeed, upon introducing the process

$$Y_n = X_n^{(0)\,2} ,$$

which is again an irreducible Markov chain on the positive integers, we obtain

$$\mathbb{E}[Y_{n+1} - Y_n / Y_n = i^2] = 2\,i\,m_i^{(0)} + b_i^{(0)} < -\varepsilon ,$$

for $i \ge I_0$ sufficiently large.

The conclusion follows from Pakes lemma [14].

Proof of Theorem 1

According to lemma 3, $\{ \pi_i^{(\varepsilon)} , i \ge 0 \}$ is a weakly compact family of distributions and the weak convergence becomes now straightforward . Let $\{\pi_i, i \ge 0\}$ be a limit point such that (as $\varepsilon \downarrow 0$)

$$\pi_i = \lim \pi_i^{(\varepsilon_k)} , \quad \varepsilon_{k'} \to 0 .$$

The proof of Lemma 2 entails that $\{\pi_i, i \ge 0\}$ satisfies the system (1) corresponding to ε=0 and, from the weak compactness, that $\sum_{i\ge 0} \pi_i = 1$. Moreover, it is known that system (1) has a unique solution (see [9]) with the property $\sum_{i\ge 0} \pi_i = 1$. Thus $\pi_i = \pi_i^{(0)}$.

Theorem 1 is proved. •

Proof of theorem 2

First of all, let us give a brief description of the main ideas used in the sequel.
In Lemma 4, it will be proved that $2\varepsilon\zeta^{(\varepsilon)}$ is a weakly compact family of random variables. Next, from system (1), we will obtain an equation for the generating function of the stationary distribution of $2\varepsilon\,\zeta^{(\varepsilon)}$. Then we shall overcome the main difficulty : the convergence of $\pi_0^{(\varepsilon)}$ to zero as $\varepsilon \downarrow 0$ will be shown, allowing to neglect the remainding terms in the equation for the generating function. Finally a change of variables yields the convergence of the Laplace transform of r.v.'s $2\varepsilon\zeta^{(\varepsilon)}$ to the Laplace transform of a Γ– distribution.

Define

$$m_i^{(\varepsilon)} = -\varepsilon - \frac{\mu}{i} + \Delta(i,\varepsilon)\,(\varepsilon + \frac{1}{i}) \,, \tag{14}$$

where $\Delta(i, \varepsilon) \to 0$ as $i \to \infty$, $\varepsilon \downarrow 0$. The ergodicity of $X_n^{(\varepsilon)}$ follows at once from the inequality $m_i^{(\varepsilon)} \leq -\varepsilon/2$, for i large enough (see [14].

LEMMA 4 The family of r.v.'s $2\,\varepsilon\,\zeta^{(\varepsilon)}$ is a weakly compact family and

$$\sup_{0<\varepsilon\leq\varepsilon_0} 2\,\varepsilon\,\mathbb{E}\,[\zeta^{(\varepsilon)}] < \infty ,$$

for sufficiently small ε_0 .

Proof. Let us consider the series

$$\sum_{i\geq 0} \pi_i^{(\varepsilon)}\,\mathbb{E}[X_{n+1}^{(\varepsilon)\,2} - i^2 / X_n^{(\varepsilon)} = i] \,. \tag{15}$$

Now we proceed to compute the mathematical expectation in (15) :

$$\mathbb{E}[X_{n+1}^{(\varepsilon)\,2} - i^2 / X_n^{(\varepsilon)} = i] = \mathbb{E}[\,2\,i\,W_n^{(\varepsilon)} + W_n^{(\varepsilon)\,2} / X_n^{(\varepsilon)} = i]$$

$$= 2i\,m_i^{(\varepsilon)} + b_i^{(\varepsilon)} \leq -\frac{\varepsilon}{2}\,i < 0 \,,$$

for i sufficiently large. Following the argument of lemma 3, we can assert that series (15) is summable and its sum is non negative. Hence,

$$\sum_{i\geq 0}\pi_i^{(\varepsilon)}[-2\,i\,\varepsilon\,(1-\Delta(i,\varepsilon))-2\,\mu+2\,\Delta(i,\varepsilon)+b_i^{(\varepsilon)}]\;\geq\;0\,,$$

or, equivalently,

$$2\,\varepsilon\sum_{i\geq 0}\pi_i^{(\varepsilon)}\,i\,(1-\Delta(i,\varepsilon))\;\leq\;\sum_{i\geq 0}\pi_i^{(\varepsilon)}\,(-2\,\mu+2\,\Delta(i,\varepsilon)+b_i^{(\varepsilon)})$$

$$\leq \sup_{i\,,\,\varepsilon}|-2\,\mu+2\,\Delta(i,\varepsilon)+b_i^{(\varepsilon)}|=A_*\;<\infty\,.$$

There exist i_0 and ε_0 such that

$$1-\Delta(i,\;\varepsilon)\;>1/2\;\;,\quad\text{for } i\;\geqslant\;i_0\;\;,\;\;\varepsilon\;\leqslant\;\varepsilon_0\,.$$

Therefore

$$\varepsilon\sum_{i\geq i_0}i\,\pi_i^{(\varepsilon)}\;\leq\;A_*+2\,\varepsilon\sum_{i<i_0}\pi_i^{(\varepsilon)}\,i\,|1-\Delta(i,\varepsilon)|\;\leq\;A_{**}<\infty\;,\quad\text{for }\varepsilon\leq\varepsilon_0\,.$$

The proof of lemma 4 is concluded. •

Denoting by $u_i^{(\varepsilon)}(z)$ the generating function of the distribution $\{p_{i,i+k}^{(\varepsilon)}\}_{k\geq -i}$, we have

$$u_i^{(\varepsilon)}(z)=\sum_{k\geq -i}p_{i,i+k}^{(\varepsilon)}z^k\;,\;|z|\leq 1\,.$$

Turning back to the proof of Theorem 2 ,we shall derive from system (1) an equation for the generating function of the stationary distribution associated to $\zeta^{(\varepsilon)}$.

$$\pi^{(\varepsilon)}(z)=\sum_{i\geq 0}\pi_i^{(\varepsilon)}\,z^i\,u_i^{(\varepsilon)}(z)\,. \qquad (16)$$

Unless otherwise stated , z will always denote, from now on, a real number with $0\leq z\leq 1$.

LEMMA 5 If (2), (3) hold together with $2\mu<b$, we have

$$\pi_0^{(\varepsilon)} = \mathbb{P}[\zeta^{(\varepsilon)} = 0] \to 0 \ , \quad \text{as } \varepsilon \downarrow 0 .$$

Proof. Lemma 2 implies that it is sufficient to check non existence of the probabilitic solution of system (1) corresponding to $\varepsilon = 0$. Suppose that system (1) corresponding to $\varepsilon = 0$, has a probabilistic solution.

For the sake of brevity, we shall omit the supercript 0 (whenever it should occur). Substituting $\varepsilon = 0$ in (16) and differenciating with respect to z, we obtain

$$\sum_{i \geq 0} \pi_i z^{i-1} f_i(z) = 0 \tag{17}$$

where

$$f_i(z) = i\,(1 - u_i(z)) - z\, u_i'(z) .$$

By Taylor's formula, we can write

$$f_i(z) = - u_i'(1) + (1-z)\,[(1+i)\, u_i'(\theta_i) + \theta_i u_i''(\theta_i)] , \tag{18}$$

with

$$z < \theta_i < 1 , \quad \forall\, i \geq 0.$$

Instantiating (18) in (17) yields

$$\sum_{i \geq 0} \pi_i z^{i-1} u_i'(1) = (1-z) \sum_{i \geq 0} \pi_i z^{i-1} [(i+1)\, u_i'(\theta_i) + \theta_i u_i''(\theta_i)] , \tag{19}$$

On the other hand, using again Taylor's formula in (16), we get

$$\pi(z) = \sum_{i \geq 0} \pi_i z^i [1 + (z-1)\, u_i'(\beta_i)] ,$$

which in turn implies

$$\sum_{i \geq 0} \pi_i z^i u_i'(\beta_i) = 0 \quad , \quad 0 < z < 1 , \tag{20}$$

where

$$z < \beta_i < 1 , \quad \forall\, i \geq 0.$$

Since

$$u_i'(\beta_i) = u_i'(1) + (\beta_i - 1)\, u_i''(\gamma_i) \ , \quad \beta_i < \gamma_i < 1 ,$$

we have

$$0 = \sum_{i\geq 0} \pi_i z^i u_i'(1) + \sum_{i\geq 0} \pi_i \left(\frac{z}{\gamma_i}\right)^i (\beta_i - 1)\, \gamma_i^i\, u_i''(\gamma_i) \ . \qquad (21)$$

The function $u_i''(z)$ has a power series expansion with positive coefficients and, from the basic assumptions made in theorem 2, $z^{i+2} u_i''(z) < u_i''(1) < \infty$, $\forall\, i \geq 0$.

Hence, the second term in the right member of (21) is a uniformly convergent series of continuous functions for $0 \leq z \leq 1$, which tend to 0 as $z \to 1$. Consequently

$$\sum_{i\geq 0} \pi_i u_i'(1) = 0 \ . \qquad (22)$$

Note that (22) is in general not valid for Markov chains without further assumptions.

Now, using (22), we rewrite (19) in the form

$$\sum_{i\geq 0} \pi_i \left(\frac{z^{i-1}-1}{1-z}\right) u_i'(1) = \sum_{i\geq 0} \left[(i+1)\, u_i'(\theta_i) + \theta_i u_i''(\theta_i)\right] \pi_i z^{i-1} \ . \qquad (23)$$

Observe first that

$$\frac{(1-z^i)\,(u_i'(1))}{1-z} \leq i\,|\,u_i'(1)\,| \ \approx\ |\mu| \quad \text{as} \quad i \to \infty \ , \qquad 0 \leq z \leq 1 \ .$$

Moreover, all the terms of the second series in the right member of (23) are positive continuous functions of z. Thus, by the well known Dini 's theorem, this series, which is convergent, is also uniformly convergent. Consequently, the first series in the right member of (23) is, by difference, also uniformly convergent. Hence, using (22) and letting $z \to 1$, we obtain the following basic equality

$$\sum_{i \geq 0} a_i \pi_i = 0 \ , \qquad (24)$$

where

$$a_i = b_i + 2\, i\, m_i \ .$$

Hence, $a_i \approx b - 2\mu > 0$, for $i \geq i_0$, i_0 being chosen large enough.

The positivity of the numbers $\{a_i,\ i < i_0\}$ depends only on a finite number of transition probabilities $\{p_{ij},\ 0 \leq i \leq i_0\}$. It then follows by lemma 1, that we always can assume $a_i > 0$ for every $i \geq 0$. But this contradicts (24).

The proof of lemma 5 (and at the same time of the first part of theorem 2) is concluded. •

We proceed now to the derivation of a functional (in fact differential) equation, for a suitable Laplace transform . To that end, substitute

$$z = e^{-2\varepsilon t} \quad , \quad t \in \mathbb{R}^+ .$$

Introduce the notation

$$\pi^{(\varepsilon)}(z) = \mathbb{E}[\, e^{-2\varepsilon t \zeta^{(\varepsilon)}} \,] \equiv \beta_{2\varepsilon \zeta^{(\varepsilon)}}(t) \equiv \beta^{(\varepsilon)}(t) .$$

The function $\beta^{(\varepsilon)}$ is the Laplace transform of the random variable $2\,\varepsilon\zeta^{(\varepsilon)}$. Moreover

$$-2\,\varepsilon\, z \frac{d}{dz} \pi^{(\varepsilon)}(z) = \frac{d\,\beta^{(\varepsilon)}(t)}{dt} .$$

Differenciating (16) with respect to z, just like in (17) leads to

$$\sum_{i \geq 0} \pi_i^{(\varepsilon)} z^{i-1} i\,(1 - u_i^{(\varepsilon)}(z)) = \sum_{i \geq 0} \pi_i^{(\varepsilon)} u_i^{(\varepsilon)'}(z)\, z^i \quad . \tag{25}$$

Upon repeatedly using Taylor's formula, we obtain

$$1 - u_i^{(\varepsilon)}(z) = (1-z)\,[\, m_i^{(\varepsilon)} + \frac{(z-1)}{2} u_i^{(\varepsilon)''}(\theta_i)\,] \,, \tag{26}$$

$$u_i^{(\varepsilon)'}(z) = m_i^{(\varepsilon)} + (z-1)\, u_i^{(\varepsilon)''}(\gamma_i) \quad ,$$

where $z < \gamma_i ,\ \theta_i < 1$.

Hence, from (16), we get

$$\pi^{(\varepsilon)}(z) = \sum_{i \geq 0} \pi_i^{(\varepsilon)} z^i\,[1 + (z-1)\, m_i^{(\varepsilon)} + \frac{(z-1)^2}{2} u_i^{(\varepsilon)''}(\theta_i)],$$

or

$$\sum_{i \geq 0} \pi_i^{(\varepsilon)} z^i m_i^{(\varepsilon)} = \frac{1-z}{2} \sum_{i \geq 0} \pi_i^{(\varepsilon)} z^i u_i^{(\varepsilon)''}(\theta_i) \ . \tag{27}$$

Replacing (26) and (27) in (25), and dividing by (1-z), we obtain easily the main equality

$$\sum_{i \geq 0} \pi_i^{(\varepsilon)} z^{i-1} i\,[m_i^{(\varepsilon)} + \frac{(z-1)}{2} u_i^{(\varepsilon)''}(\theta_i)]$$

$$= \sum_{i \geq 0} \pi_i^{(\varepsilon)} z^i \,[\frac{1}{2} u_i^{(\varepsilon)''}(\theta_i) - u_i^{(\varepsilon)''}(\gamma_i)] \quad . \tag{28}$$

In the above equation (28), θ_i and γ_i are in fact (continuous) functions of z.

By using (14) and $z = 1 - 2\,\varepsilon\, t + o\,(\varepsilon\, t)$, we rewrite (28) in a form which explicitly reveals the functions $\beta^{(\varepsilon)}(t)$ and $\beta^{(\varepsilon)'}(t)$ as follows

$$\beta^{(\varepsilon)'}(t)\,[1+\ bt+o\,(1)] + \sum_{i\geq 0} i\,\pi_i^{(\varepsilon)} z^{i-1}\,[\Delta(i,\varepsilon)\,(\varepsilon+\frac{1}{i}) + \frac{(z-1)}{2}\,(u_i^{(\varepsilon)''}(\theta_i) - b)]$$

$$= \beta^{(\varepsilon)}(t)\,[2\mu - b + o\,(1)] + \sum_{i\geq 0} \pi_i^{(\varepsilon)} z^i\,[\frac{1}{2}u_i^{(\varepsilon)''}(\theta_i) - u_i^{(\varepsilon)''}(\gamma_i) + \frac{b}{2}]\,. \qquad (29)$$

The next step consists in showing that, in (29), the two sums tend to zero as $\varepsilon \to 0$.

We first prove that, for any bounded function $\varphi(i, \varepsilon)$ satisfying the conditions

$$|\,\varphi(i,\varepsilon)| < K\,, \quad \forall\, i, \varepsilon \quad ,$$

$$\sup_{i\geq i_0,\ \varepsilon\leq\varepsilon_0} |\,\varphi(i,\varepsilon)| \to 0\,, \quad \text{as} \quad i_0 \to \infty \quad , \quad \varepsilon_0 \to 0 \quad ,$$

the following equality holds

$$\lim_{\varepsilon\to 0} \sum_{i\geq 0} \pi_i^{(\varepsilon)} z^i\,\varphi(i,\varepsilon) = 0 \quad . \qquad (30)$$

This emerges directly from the decomposition

$$\sup_{\varepsilon\leq\varepsilon_0} \sum_{i\geq 0} \pi_i^{(\varepsilon)} z^i\ \varphi(i,\varepsilon) \ \leq\ K \sum_{i\leq i_0} \pi_i^{(\varepsilon)} + \sup_{i>i_0,\ \varepsilon\leq\varepsilon_0} \varphi(i,\varepsilon)\ ,$$

and from lemma 5, by letting $i_0 \to \infty$, $\varepsilon_0 \to 0$.

In a similar way, upon writing $1-z = O\,(\varepsilon t)$, and the decomposition which lead to (30), we get, from lemmas 4 and 5,

$$\lim_{\varepsilon\to 0} \sum_{i\geq 0} (1-z)\, i\, \pi_i^{(\varepsilon)}\ \varphi(i,\varepsilon)\ = 0 \quad . \qquad (31)$$

Hence, in order to prove that the sums in (29) are o (1), it suffices, according to (30) and (31), to show that

$$z^i\,[u_i^{(\varepsilon)''}(\theta_i) - b\,] \to 0 \ \ , \ \ \text{as} \ \ i \to \infty, \ \ \varepsilon \to 0\ , \qquad (32)$$

uniformly in ε (or equivalently in z), for all $z \le \theta_i \le 1$.

Since $u_i^{(\varepsilon)}(1) = b_i^{(\varepsilon)} - m_i^{(\varepsilon)}$, it follows from condition (3) that

$$\sup_{i,\varepsilon,z\le y\le 1} z^i u_i^{(\varepsilon)''}(y) \le \sup_{i,\varepsilon,y} y^i u_i^{(\varepsilon)''}(y) < \infty ,$$

which in turn implies that (32) will hold if we can check

$$\Big| \sum_{u\ge 0} p_{iu}^{(\varepsilon)} (u-i)^2 (z^i - z^u) \Big| \to 0, \quad \text{as } i \to \infty,\ \varepsilon \to 0 \quad . \tag{33}$$

Write, for any fixed i,

$$\Big| \sum_{u\ge 0} p_{iu}^{(\varepsilon)} (u-i)^2 (z^i - z^u) \Big| \le \sum_{|u-i|>J} p_{iu}^{(\varepsilon)} (u-i)^2 + \sum_{|u-i|<J} p_{iu}^{(\varepsilon)} (u-i)^2 |z^i - z^u| , \tag{34}$$

where J is an arbitrary positive number.

Let T denote the first term in the right side of (34).

Then under condition (5), given $\delta > 0$, it is possible to choose J, depending on δ but neither on ε nor on i, such that

$$T < \delta . \tag{35}$$

On the other hand, by

$$z^i - z^u = (i-u)\ z^{i+\theta(u-i)} \operatorname{Log} z , \quad 0 < \theta < 1 ,$$

we get the following rough bound for the second term in the right member of (34)

$$\sum_{|u-i|<J} p_{iu}^{(\varepsilon)} (u-i)^2 |z^i - z^u| < J^3 z^{i-J} |\operatorname{Log} z| . \tag{36}$$

Now, J (function of δ) being fixed to satisfy (34) and (35), we choose $z = 1 - o(J^{-3})$ to render the right side of (36) smaller than any given positive δ_1.
Consequently, we have proved that, uniformly with respect to ε and i, the left member of (33) becomes smaller than $\delta + \delta_1$, for $z = 1 - o(J^{-3})$, which was the assertion, since δ and δ_1 are arbitrary.

Hence (32) holds and the sum in the right side of (29) is $o(1)$, when $\varepsilon \to 0$.
Quite similar arguments show that the sum in the left side of (29) is also $o(1)$.

Finally, from the preceding estimates, the sought equation for the Laplace transform becomes

$$\beta^{(\varepsilon)'}(t)\,(1+bt+o(1)) = \beta^{(\varepsilon)}(t)\,(2\mu - b + o(1)) + o(1) \quad , \quad \text{as } \varepsilon \downarrow 0 \ .$$

It emerges from lemma 4 that for any $t > 0$, we have

$$\inf_{\varepsilon \geq 0} \beta^{(\varepsilon)}(t) > 0 \ .$$

Therefore the following equation takes place, uniformly with respect to t taken from an arbitrary compact set on the positive real line,

$$\frac{\beta^{(\varepsilon)'}(t)}{\beta^{(\varepsilon)}(t)} = \frac{2\mu - b}{1+bt} + o(1) \ ,$$

$$\frac{d}{dt} \ln \beta^{(\varepsilon)}(t) = \frac{2\mu - b}{1+bt} + o(1) \ ,$$

whence

$$\lim_{\varepsilon \downarrow 0} \beta^{(\varepsilon)}(t) = (1+bt)^{2\mu/b-1} \ .$$

Thus, we get the announced convergence

$$\beta_{2\varepsilon}\, \zeta^{(\varepsilon)}(t) \rightarrow \beta_{\Gamma_{\frac{1}{\beta},\, 1-\frac{2\mu}{b}}}(t) \quad \text{as } \varepsilon \downarrow 0 .$$

The proof of theorem 2 is terminated. ●

Proof of theorem 3

First of all, let us give a brief description of the main ideas used in the sequel. It will be proved in lemma 8 that $\pi_0^{(\varepsilon)} \rightarrow 0$. Then, with the help of the dirichlet series $\mathbb{E}[(1+\zeta^{(\varepsilon)})^{\sigma}]$ and of their derivatives with respect to σ, we obtain funtional differential equations, which now cannot be solved in explicit form, but allow to derive relations for the moments of the random variable

$$\frac{\log(1+\zeta^{(\varepsilon)})}{\log(1/\varepsilon)} \ .$$

More exactly, the following convergence will be proved

$$\mathbb{E}\Big[\Big(\frac{\log(1+\zeta^{(\varepsilon)})}{\log(1/\varepsilon)}\Big)^m\Big] \to \frac{1}{m+1} = \mathbb{E}[\xi^m], \quad \varepsilon \downarrow 0, \quad m = 0, 1, 2, \ldots,$$

where ξ denotes a random variable having a uniform distribution on [0,1].

LEMMA 6

For $\varepsilon > 0$ and any positive measurable function f on the positive integers $\{0,1,\ldots\}$, if $\mathbb{E}[f(\zeta^{(\varepsilon)})]$ exits, then the following relation holds:

$$O = \sum_{i\geq 0} \pi_i^{(\varepsilon)} \mathbb{E}[f(X_1^{(\varepsilon)}) - f(i) / X_0^{(\varepsilon)} = i] \quad . \tag{37}$$

Proof. Immediate by considering the ergodic Markov chain

$$X_n^{(\varepsilon)}, X_0^{(\varepsilon)} \overset{d}{=} \zeta^{(\varepsilon)} \quad . \text{ Then } X_1^{(\varepsilon)} \overset{d}{=} X_0^{(\varepsilon)} \; .$$

Lemma 6 is proved. ●

LEMMA 7

1) For $\varepsilon > 0$ arbitrarily small,

$$\mathbb{E}[\zeta^{(\varepsilon)\,1+\gamma}] < \infty \; .$$

2)

$$\sup_{\varepsilon\geq 0} \mathbb{E}[(\varepsilon\,\zeta^{(\varepsilon)})^{1+\gamma}] \leq M < \infty \; .$$

Proof.

It follows directly from the arguments presented in lemmas 3 and 4. Let α be a real number with $0 \leq \alpha \leq \gamma$. Using (4) and (6), the inequation (12) now takes the form

$$\mathbb{E}[X_{n+1}^{(\varepsilon)\,\alpha+2} - i^{\alpha+2} / X_n^{(\varepsilon)} = i] \leq -(\alpha+2)\Big[\varepsilon i^{\alpha+1} - i^{\alpha}\Big(\frac{\alpha b_i^{(\varepsilon)}}{2} + o\,(\varepsilon + \frac{1}{i})\Big) + C\Big] \; .$$

Thus

$$\sum_{i\geq 0} \pi_i^{(\varepsilon)} \mathbb{E}[X_{n+1}^{(\varepsilon)\,\alpha+2} - i^{\alpha+2} / X_n^{(\varepsilon)} = i] \geq 0 \; ,$$

which yields

$$\varepsilon\,\mathbb{E}[\zeta^{(\varepsilon)\,1+\alpha}] \leq \frac{\alpha b}{2} \mathbb{E}[\zeta^{(\varepsilon)\,\alpha}(1 + o\,(1))] + C \quad . \tag{38}$$

On the other hand, by Hölder's inequality, we have

$$\mathbb{E}[(\varepsilon\,\zeta^{(\varepsilon)})^{1+\alpha}] \geq \mathbb{E}^{1+\frac{1}{\alpha}}[(\varepsilon\,\zeta^{(\varepsilon)})^{\alpha}] \; .$$

Hence, setting $G = \mathbb{E}[\zeta^{(\varepsilon)\alpha}]$ and combining the last two inequalities, we get

$$G^{1+\frac{1}{\alpha}} \leq DG + C ,$$

where C is the constant introduced in (4) and D is a constant independent of ε. Thus $G < \infty$ and the two assertions of the lemma are obtained by multiplying (38) by ε^{α}, $0 \leq \alpha \leq \gamma$.

Lemma 7 is proved. ●

LEMMA 8 If (2), (3), (4) hold, then

$$\pi_0^{(\varepsilon)} = \mathbb{P}\{\zeta^{(\varepsilon)} = 0\} \to 0 \ , \text{as } \varepsilon \downarrow 0 .$$

Proof. Lemma 2 implies that it suffices again to check the non existence of a probabilistic solution of system (1) corresponding to $\varepsilon = 0$. Suppose this is not the case. Then, it has been shown in lemma 5, after using (22) and (23), that (with the notation of lemma 5)

$$\sum_{i\geq 0} \pi_i [\, 2\, i\, m_i + u_i^{''}(0)\,] \ .$$

But , since

$$2\, i\, m_i + u_i^{''}(0) = \frac{b}{i} + o\left(\frac{1}{i}\right), \ i \geq i_0 ,$$

where i_0 is taken large enough, the arguments of lemma 5 do apply.

The proof is concluded . ●

LEMMA 9 For any k = 1, 2,...,

$$2\,\varepsilon\,\mathbb{E}\,[(1+\zeta^{(\varepsilon)})\log^k(1+\zeta^{(\varepsilon)})] = \sum_{i\geq 0} \pi_i^{(\varepsilon)}\, k\,[\,b + o\,(1)]\,\log^{k-1}(1+i) \ , \qquad (39)$$

where, as usual, o (1) denotes a quantity tending to 0 when ε and i tend to 0.

In particular, instantiating k =1 in (39) yields

$$\lim_{\varepsilon\to 0} 2\,\varepsilon\,\mathbb{E}\,[(1+\zeta^{(\varepsilon)})\log(1+\zeta^{(\varepsilon)})] = b . \qquad (40)$$

Proof. The assumptions of lemma 6 are satisfied by choosing

$$f(x) = (1+x)^{\sigma} \log^k (1+x) \ , \ \mathbb{R}_e(\sigma) < 1+\gamma, \ k \geq 0 .$$

Using Taylor's expansion and lemma 7, we get

$$0 = \sum_{i\geq 0} \pi_i^{(\varepsilon)} \Big[m_i^{(\varepsilon)} f'(i) + \frac{b_i^{(\varepsilon)}}{2} f''(i) + \mathbb{E}\,\Big[\frac{W_0^{(\varepsilon)\,2}}{2} (f''(i+\theta_i W_0^{(\varepsilon)}) - f''(i)) / X_0^{(\varepsilon)} = i\Big]\Big] , \quad (41)$$

where

$$W_0^{(\varepsilon)} = X_1^{(\varepsilon)} - X_0^{(\varepsilon)} ,$$

$$0 < \theta_i < 1 ,$$

$$f'(i) = (1+i)^{\sigma-1} \log^{k-1}(1+i)\, [\sigma \log(1+i) + k] \ ,$$

$$f''(i) = (1+i)^{\sigma-2} [\sigma(\sigma-1) \log^k(1+i) + k(2\sigma-1) \log^{k-1}(1+i) + k(k-1) \log^{k-2}(1+i)\,] .$$

But all the functions coming in the right member of (41) are analytic with respect to σ. Thus, by using lemma 7 and the principle of analytic continuation, (41) is indeed valid for $\mathbb{R}e(\sigma) < 2+\gamma$. For our purpose, it suffices to take $\sigma = 2$. With this choice of σ, we have now

$$f'(i) = (1+i)\log^{k-1}(1+i)\,[2\log(1+i)+k] \;,$$
$$f''(i) = 2\log^{k}(1+i) + 3k\log^{k-1}(1+i) + k(k-1)\log^{k-2}(1+i) \;.$$

It is easy to check that the first two terms in the right member of (41) produce (39). It remains to estimate the last term in (41). To this end, we note first that, for any real $y \neq 0$,

$$y^2\,\mathbb{P}(|W_0^{(\varepsilon)}| > |y|) < C|y|^{-\alpha}, \quad 0 \le \alpha \le \gamma .$$

Then, taking $|y| = i^{\frac{1-\beta}{3}}$, $\beta \ge 0$ arbitrarily small, we can write

$$|y^2[\log^k(1+i+\theta_i y) - \log^k(1+i)]| \le \frac{ky^3\log^{k-1}(1+i)}{1+i}\Big[1+o\big(i^{\frac{-(2+\beta)}{3}}\big)\Big]$$
$$\le k\,i^{-\beta}\log^{k-1}(1+i)\,[1+o\big(i^{\frac{-(2+\beta)}{3}}\big)]$$

The last two inequalities yield at once

$$\mathbb{E}\Big[\frac{W_0^{(\varepsilon)\,2}}{2}\big(f''(i+\theta_i W_0^{(\varepsilon)}) - f''(i)\big) \,/\, X_0^{(\varepsilon)} = i\Big] = o(i^{-\delta}) \;, \text{ for some } \delta > 0 \;.$$

The proof of lemma 9 is concluded. ●

LEMMA 10 For any $k = 1, 2, \ldots,$

$$\lim_{\varepsilon\to 0} 2\,\varepsilon\,\mathbb{E}\Big[\frac{(1+\zeta^{(\varepsilon)})\log^{k+1}(1+\zeta^{(\varepsilon)})}{\log^k(1/\varepsilon)}\Big] = b \;.$$

Proof. We proceed in two steps

i) The following estimate takes place

$$\varepsilon\,\mathbb{E}\Big[(1+\zeta^{(\varepsilon)})\log^{k+1}(1+\zeta^{(\varepsilon)})\,;\,1+\zeta^{(\varepsilon)} \ge \varepsilon^{-(x+1)}\Big] \le \varepsilon\,\mathbb{E}\Big[\frac{(1+\zeta^{(\varepsilon)})^{1+\alpha}}{(1+\zeta^{(\varepsilon)})^{\alpha-\beta}}\,;\,1+\zeta^{(\varepsilon)} \ge \varepsilon^{-(x+1)}\Big],$$

for some β, where $0 < \beta < \alpha \le \gamma$ and x is a fixed strictly positive real number. Hence, it follows from lemma 7 that

$$\varepsilon\,\mathbb{E}\Big[(1+\zeta^{(\varepsilon)})\log^{k+1}(1+\zeta^{(\varepsilon)})\,;\,1+\zeta^{(\varepsilon)} \ge \varepsilon^{-(x+1)}\Big] \le M\varepsilon^{\delta} \;,$$

where we can choose β such that

$$\delta = \alpha x - \beta(1+x) > 0\,, \quad \text{i.e } \beta < \frac{\alpha x}{1+x} \;.$$

On the other hand, we have

$$2\,\varepsilon\,\mathbb{E}\Big[\frac{(1+\zeta^{(\varepsilon)})\log^{k+1}(1+\zeta^{(\varepsilon)})}{\log^k(1/\varepsilon)}\Big] = 2\,\varepsilon\,\mathbb{E}\Big[\frac{(1+\zeta^{(\varepsilon)})\log^{k+1}(1+\zeta^{(\varepsilon)})}{\log^k(1/\varepsilon)}\,;\,1+\zeta^{(\varepsilon)} \le \varepsilon^{-(x+1)}\Big]$$

$$+\,2\,\varepsilon\,\mathbb{E}\Big[\frac{(1+\zeta^{(\varepsilon)})\log^{k+1}(1+\zeta^{(\varepsilon)})}{\log^k(1/\varepsilon)}\,;\,1+\zeta^{(\varepsilon)} > \varepsilon^{-(x+1)}\Big]$$

$$\le 2\varepsilon\,\mathbb{E}\,[(1+\zeta^{(\varepsilon)})\log(1+\zeta^{(\varepsilon)})]\,(1+x)^k + 2\,M\,\varepsilon^{\delta}\log^{-k}(1/\varepsilon)\ .$$

Thus, by (40) in lemma 9 and since x is arbitrary, we infer that

$$\limsup_{\varepsilon\to 0} 2\,\varepsilon\,\mathbb{E}\Big[\frac{(1+\zeta^{(\varepsilon)})\log^{k+1}(1+\zeta^{(\varepsilon)})}{\log^k(1/\varepsilon)}\Big] \le b\ . \qquad (42)$$

ii) We have

$$\varepsilon\,\mathbb{E}\Big[\frac{(1+\zeta^{(\varepsilon)})\log^{k+1}(1+\zeta^{(\varepsilon)})}{\log^k(1/\varepsilon)}\Big] \ge \varepsilon\,\mathbb{E}\Big[(1+\zeta^{(\varepsilon)})\log(1+\zeta^{(\varepsilon)})\Big] +$$

$$+\,\varepsilon\,\mathbb{E}\Big[(1+\zeta^{(\varepsilon)})\log(1+\zeta^{(\varepsilon)})\Big(\frac{\log^k(1+\zeta^{(\varepsilon)})}{\log^k(1/\varepsilon)} - 1\Big)\,;\,1+\zeta^{(\varepsilon)} \le \frac{1}{\varepsilon}\Big].$$

For any positive number a, the function $y\,[(y/a)^k - 1]$, $y \le a$, reaches its minimum at the point y_0, where

$$\Big(\frac{y_0}{a}\Big)^k = \frac{1+y_0}{1+y_0+k}\ .$$

Hence, one easily sees that the second term, in the right member of the above inequality, has its modulus bounded by

$$\varepsilon\,\mathbb{E}\Big[(1+\zeta^{(\varepsilon)})\frac{k\,e^{y_0}}{1+e^{y_0+k}}\Big] \le k\,\varepsilon\,\mathbb{E}[\,1+\zeta^{(\varepsilon)}\,]\ ,$$

which, from the arguments used in lemma 9, tends to zero when $\varepsilon \to 0$. Thus

$$\liminf_{\varepsilon\to 0} 2\,\varepsilon\,\mathbb{E}\Big[\frac{(1+\zeta^{(\varepsilon)})\log^{k+1}(1+\zeta^{(\varepsilon)})}{\log^k(1/\varepsilon)}\Big] \ge b\ . \qquad (43)$$

The proof of (40) is completed by combining (42) and (43). •

Dividing (39) by k in lemma 9 and using lemma 10 yield

$$\mathbb{E}\Big[\Big(\frac{\log(1+\zeta^{(\varepsilon)})}{\log(1/\varepsilon)}\Big)^k\Big] \to \frac{1}{k+1} = \mathbb{E}[\xi^k],\quad \varepsilon \downarrow 0\,,\quad k = 0, 1, 2, \ldots$$

The proof of theorem 3 is terminated. •

III. APPLICATION TO THE ALOHA NETWORK

We illustrate the preceding sections with the Markov chain arising from the model of the original ALOHA packet switching network, originally proposed by Abramson [1], and which was indeed the motivation of our study. Let us first briefly recall the salient features of the system.

a) A single error-free channel is shared among an infinite population of users (or stations), which retransmit messages of constant length (packets). Time is slotted and may be considered discrete. Users are synchronized with respect to the slots, so that packets are transmitted at the beginning of slots only. Each slot is equal to the time required to transmit a packet.

b) Each transmission is within reception range of every user. When more than one user transmit simultaneously, packets collide (interfere) and none is received correctly. These collisions are treated as transmission errors and each user must strive to retransmit its colliding packet until it is correctly received. The users all employ the same algorithm for this purpose and have to resolve the contention without the benefit of any other source of information on other user's activity save the common channel .

c) Each user with a colliding packet will repeatedly transmit each time with a certain probability, until it hits a free slot and thus succeeds.

The main drawback of the above described Aloha protocol is that, left to their own device, the nodes congest the channel which, in absence of additional control, is non ergodic.The approach suggested in [11] was to let retransmission probabilities be a function of the number of blocked stations at time t. Such a retransmission control policy (RCP) can stabilize the channel. This facts have been proved in [5], [6], under "Markov" assumptions, but remain true when the external input process is only stationary [4], [7]. We shall here deal with the Markov situation.

Let A_k be the number of new packets generated by the stations during the k-th slot.We shall assume the $\{A_k , k \geq 0\}$ form a sequence of i.i.d random variables, with

$$\mathbb{P}(A_k = i) = c_i , i \geq 0, \quad \text{and} \quad \mathbb{E}(A_k) = \lambda , k \geq 0.$$

Let $X_k , k \geq 0$, be the number of blocked stations at time k (i.e observed at the beginning of the k-th slot) and $f(X_k)$ the probability that a blocked station retransmits during this k-th slot . Given $\{X_k = n\}$, the random number of messages in the k-th slot has a binomial distribution. Hence, $X = \{X_k , k \geq 0\}$ form a Markov chain .

Define

$$d_n = c_0\, n\, f(n)\, (1- f(n))^{n-1} + c_1 (1- f(n))^n . \qquad (44)$$

Thus d_n represents, in the wide sense, the mean downward drift of X in the k-th slot, given the event $\{X_k = n\}$. We recall the main result (see [4],[5],[6],[7]) :

THEOREM 4

i) If $\lambda < \liminf_{i\to\infty} d_i$, X is ergodic;

ii) If $\lambda > \limsup_{i\to\infty} d_i$, X is transient.

Our goal is to analyze the stability and ergodicity of a class of RCP's, in the limit "zero drift" case. These RCP's are chosen (see [5],[6]) such that there exists

$$r = \lim_{i\to\infty} i\, f(i) \,. \tag{45}$$

Then, (44) gives

$$d = \lim_{i\to\infty} d_i = e^{-r}(r\, c_0 + c_1) \,. \tag{46}$$

For the problem to be meaningful, we have to choose $c_0 > c_1$. Otherwise d, given by (46), would be a decreasing function of r ; then, $\lambda > c_1 > d$ and the system could never be ergodic. In fact, we do not restrict the generality by taking

$$f(i) = \frac{r}{i} \;,\; i > 0 \,.$$

Now,with the notation of section I, combining (44), (45), (46), we get by a direct computation

$$m_i = \lambda - d - \frac{\mu}{i} + O\left(\frac{1}{i^2}\right), \tag{47}$$

$$b_i = \mathbb{E}[(X_{k+1} - X_k)^2 / X_k = i] = \varphi + c_0\, i\, f(i)\,(1 - f(i))^{i-1} - c_1\,(1 - f(i))^i \,,$$

where

$$\mu = \frac{r^2 e^{-r}}{2}[c_0\,(2 - r) - c_1] \,,$$

$$\varphi = \mathbb{E}[A_k^2] \,.$$

Thus

$$b = \lim_{i\to\infty} b_i = \varphi + c_0\, r\, e^{-r} - c_1\, e^{-r} .$$

From now on, we assume $\lambda = d$.

THEOREM 5

If $\lambda = d$, the Markov chain X of the Aloha protocol is transient.

Proof.

Putting $s = b - 2\mu$, it simply suffices to show that s is stictly positive. We have

$$s = e^{-r}\varphi - (1 - r)[\, c_1\,(1 + r) + c_0\, r\,(r - 1)] \,.$$

For $r \geq 1$, the result is obvious. When $0 \leq r \leq 1$, it follows, since

$$\varphi > \lambda = d = e^{-r}(r\, c_0 + c_1) \,,$$

that

$$\frac{e^r\varphi}{1 - r} - [\, c_1\,(1 + r) + c_0\, r\,(r - 1)] > \frac{r\, c_0 + c_1}{1 - r} - [\, c_1\,(1 + r) + c_0\, r\,(r - 1)] \,,$$

which yields

$$\frac{r\, c_0[1 + (1 - r)^2] + c_1 r^2}{1 - r} > 0 .$$

Hence, $s > 0$ andthe proof of the theorem is concluded. •

Remark:

i) The transience of X could be directly obtained by introducing the M.C $\frac{1}{X_n^2}$, which outside some compact set including the origin can be shown to be a positive supermartingale, since the downward jumps are bounded see[8]...

ii) An interesting situation is the *"optimal policy "* [5],[6], which aims at maximizing d_i (the throughput of the system), with respect to the retransmission probability f, for any $0 < i \leq \infty$. In this case,

$$f(i) = \frac{c_0 - c_1}{ic_0 - c_1}$$

and, from (45), (46),

$$r = \frac{c_0 - c_1}{c_0} , \quad d = c_0 e^{-r} . \tag{48}$$

Moreover, one can check that the expansion in (47) is still valid, provided that r is replaced by its value specified in (48). Then

$$\mu = \frac{c_0 r^2 e^{-r}}{2} .$$

Coming back to the general RCP's introduced in this section, let us suppose the input sequence $\{A_k^{(\varepsilon)}, k \geq 0\}$ is perturbated and depends on some parameter ε, so that

$$\lambda^{(\varepsilon)} = \mathbb{E}[A_k^{(\varepsilon)}] = d - \varepsilon , \quad \varepsilon \geq 0 .$$

Assume also conditions (2) and (3) hold and that

$$m_i^{(\varepsilon)} = -\varepsilon - \frac{\mu}{i} + O\left(\varepsilon + \frac{1}{i^2}\right) ,$$

which, from a practical point of view, is not a drastic restriction. By theorem 5, the M.C. $X^{(0)}$ is transient, whence it follows that the sequence $X^{(\varepsilon)}$ is justiciable of theorem 2 •

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