



Application of stochastic control to portfolio selection with transaction costs

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APPLICATION OF STOCHASTIC CONTROL TO PORTFOLIO SELECTION WITH TRANSACTION COSTS

Agnès SULEM

Août 1989



APPLICATION OF STOCHASTIC CONTROL TO PORTFOLIO SELECTION WITH
TRANSACTION COSTS

APPLICATION DU CONTROLE STOCHASTIQUE AU PROBLEME DE GESTION DE
PORTEFEUILLE AVEC COUTS DE TRANSACTION

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Abstract

We consider a problem of portfolio selection formulated by Davis and Norman in a recent paper [2]. We study optimal consumption and investment policy for an investor who has available a bank account paying a fixed rate of interest and a risky asset ("stock"). We suppose that each transaction between bank and stock involves a cost proportional to the amount transacted. We show that the problem can become one-dimensional, the new state variable being the fraction of wealth placed in the risky investment, and we determine the optimal policy by using an expert system for the treatment of stochastic control problems called "Pandore" that we have developed at INRIA.

Résumé

On considère le problème de gestion de portefeuille suivant : il s'agit de déterminer la politique optimale de consommation et d'investissement d'une personne ayant un compte en banque à taux d'intérêt fixe et un compte en actions modélisé par un processus aléatoire. On suppose que chaque transaction entre les comptes entraîne un coût proportionnel à la quantité d'argent transférée. On montre que l'on peut se ramener à un problème à une seule variable d'état, le nouvel état étant la proportion d'argent investie dans le compte en actions.

La politique optimale est déterminée en utilisant "Pandore" : système expert pour le contrôle stochastique développé à l'INRIA.

I - INTRODUCTION

We investigate the optimal consumption and investment policy of an individual who has available a bank account paying a fixed interest rate r , and a risky asset modelled by a geometric Brownian motion. The investor consumes at rate $c(t)$ from the bank account with the constraint that the investor has non-negative net worth, at all times.

The objective is to maximize the expected discounted utility of consumption

$$(1.1) \quad E \int_0^{\infty} e^{-\delta t} u(c(t)) dt$$

where δ is the discount factor and $u(c)$ is the utility function defined by :

$$(1.2) \quad u(c) = \frac{c^\gamma}{\gamma} \text{ where } \gamma \in \Gamma := \{\gamma \in \mathbb{R}, \gamma < 1, \gamma \neq 0\}$$

In the absence of any costs of transaction between the two investments, the problem can be solved explicitly [2,3]. When there are charges on all transactions equal to a fixed percentage of the amount transacted, Davis and Norman [2] have shown that the optimal buying and selling policies are the local times of the two-dimensional process of bank and stock holdings at the boundaries of a wedge-shaped region.

The contribution of our paper is to show that a homogeneous model for the transaction costs case leads to a one-dimensional problem, (the new state variable being the fraction of total wealth held in stock) and to determine the optimal policy by using "Pandore" : an expert system designed to treat stochastic control problems [1].

The paper is organized as follows : the system Pandore is described in §2. In §3 we briefly recall Merton results for the no transaction costs problem. The transaction costs case is formulated in §4 . In §5, we use Pandore to solve the transaction cost problem. In appendix A and B we join two reports which are automatically generated by Pandore.

In the first report, the transaction cost problem formulated as a 2-dimensional problem is solved. We determine numerically the optimal policy : we find, like Davis in [2], that the no-transaction region is a wedge, the regions above and below it being the sell and buy regions (see figures 7 and 8 of the first report).

In the second report, the one-dimensional reduced problem is solved. The optimal policy is of the following type : let X denote the fraction of total wealth held in stock, if $X \leq X_0$, it is optimal to

buy, if $X \geq X_T$, it is optimal to sell, if $X_0 < X < X_T$ no transaction is optimal. The critical levels X_0 and X_T are determined numerically (see figures 3 and 4 of the second report).

2 - AN EXPERT SYSTEM FOR STOCHASTIC CONTROL : PANDORE

Pandore is a system designed to treat stochastic and deterministic control problems [1]. It is able to carry out basic analysis of problems of the type :

optimize criteria of the following form with respect to control variables U_t

$$E \int_0^T c(X_t, U_t) dt + f(X_T)$$

or

$$E \int_0^{\infty} e^{-\lambda t} c(X_t, U_t) dt$$

where the state X_t of the system evolves according to the Ito stochastic differential equation :

$$dX_t = b(X_t, U_t)dt + \sigma dZ_t$$

Z_t is a Wiener process.

Different methods are implemented in the system to treat these problems :

- Dynamic Programming method which leads to numerical integration of Bellman equations
- Stochastic gradient techniques
- Regular Perturbation methods when the noise intensity is small
- Optimization in the class of local feedbacks for systems which have uncoupled dynamics
- Pontryagin method in the deterministic case.

The system has also a module for identification of diffusion processes with jumps.

The user has only to specify his problem in terms of stochastic control problem through an interactive interface. The system checks well-posedness of the problem, chooses, if possible, a method among those listed above, and generates Fortran code for the numerical solution. Finally a complete report, written in TEX, including the complete analysis of the method and graphs of the solution is automatically generated. This system is written in Lisp, Macsyma (formal calculus system) and Prolog, and runs on a Symbolics Lisp Machine.

3 - NO TRANSACTION COSTS : THE MERTON PROBLEM

In the absence of any transaction costs between stock and bank, the investor's holdings $s_0(t)$ and $s_1(t)$ in the bank and in the risky asset respectively, evolve according to the following equations :

$$(3.1) \quad ds_0(t) = (r s_0(t) - c(t))dt$$

$$(3.2) \quad ds_1(t) = \alpha s_1(t) + \sigma s_1(t) dz(t)$$

with the initial endowments

$$s_0(0) = x, s_1(0) = y$$

$z(t)$ is a Wiener process.

We assume that transfers between the two investments can be made instantly and are costless. In this case, we can reparametrize the problem by introducing the new state variable $w(t) = s_0(t) + s_1(t)$ (the total wealth), the decision variables being $\pi(t) = s_1(t) / w(t)$ (the fraction of total wealth held in stock) and $c(t)$ (the rate of consumption). We obtain the basic wealth equation :

$$(3.3) \quad \begin{aligned} dw(t) &= (rw(t) + (\alpha - r) \pi(t) w(t) - c(t))dt + \sigma \pi(t) w(t) dz(t), \\ w(0) &= x + y \end{aligned}$$

The problem is to choose $\pi(t)$ and $c(t)$ so as to maximize (1.1) subject to (3.3) and $w(t) \geq 0$ for all t .

This stochastic control problem can be solved explicitly . The optimal policy is :

$$\pi(t) = \pi^* \quad \text{and} \quad c(t) = Cw(t)$$

for some constant π^* and C defined by

$$\pi^* = \frac{\beta}{(1-\gamma)\sigma}, \quad C = \frac{1}{1-\gamma} \left[\delta - r - \frac{\gamma}{2\beta^2} \right]$$

with

$$\beta = \frac{\alpha - r}{\sigma}$$

The optimal policy is thus to keep the portfolio holdings on the "Merton line" of equation $s_1 = [\pi^* / (1 - \pi^*)] w$.

These results hold under a basic well-posedness condition [2]

$$(3.4) \quad \delta > \gamma \left[r + \frac{(\alpha-r)^2}{\sigma^2(1-\gamma)} \right]$$

The optimal utility is then :

$$V(w) = \sup_{c, \pi} J_w(c, \pi) \equiv \frac{1}{\gamma} C^{\gamma-1} w^\gamma$$

4 - THE TRANSACTION COSTS PROBLEM

4.1. The 2-dimensional formulation

We now consider the case of proportional transaction costs : the investor pays fractions λ and μ of the amount transacted on purchase and sale of stock respectively. All such charges are paid from the bank account.

The evolution equations are :

$$(4.1) \quad ds_0(t) = (r s_0(t) - c(t))dt - (1 + \lambda) dL_t + (1 - \mu) dU_t, \quad s_0(0) = x$$

$$(4.2) \quad ds_1(t) = \alpha s_1(t)dt + \sigma s_1(t) dz(t) + dL_t - dU_t, \quad s_1(0) = y$$

where L_t and U_t represent cumulative purchase and sale of stock on $[0, t]$ respectively.

We consider a restricted class of policies in which L and U are constrained to be absolutely continuous with bounded derivatives, i.e.

$$L_t = \int_0^t l_s ds$$

$$U_t = \int_0^t u_s ds \quad \text{with} \quad 0 \leq l_s, u_s \leq 1.$$

The Bellman equation to be solved for the value function is :

$$(4.3) \quad \max_{c, l, u} \left\{ \frac{1}{2} \sigma^2 y^2 V_{yy} + r x V_x + \alpha y V_y + \frac{1}{\gamma} c^\gamma - c V_x \right. \\ \left. + [-(1+\lambda)V_x + V_y]l + [(1-\mu)V_x - V_y]u - \delta v \right\} = 0$$

4.2. Reduction to a one-dimensional problem

We would like to have the homothetic property for the solution of the Bellman equation (4.3) :

$$(4.4) \quad V(\rho x, \rho y) = \rho^\gamma V(x, y).$$

This property does not hold with the previous formulation of the problem, when the control variables are not bounded.

To get the homothetic property we have to consider as control variables the fraction of wealth dedicated to consumption , purchase of stock and sale of stock. Namely, consider the new control variables :

$$(4.5) \quad \begin{aligned} U_1 &= \frac{c}{s_0 + s_1} \\ U_2 &= \frac{l}{s_0 + s_1} \\ U_3 &= \frac{u}{s_0 + s_1} \end{aligned}$$

In this case, the problem becomes one - dimensional, the single state variable being the fraction of wealth invested in stock.

Indeed, let us do the change of variables :

$$(4.6) \quad \rho = s_0 + s_1 \quad (\text{total wealth})$$

$$(4.7) \quad X = \frac{s_1}{s_0 + s_1} \quad (\text{fraction of wealth invested in stock})$$

We can rewrite V as a function \tilde{V} of (ρ, X)

$$\begin{aligned} V(s_0, s_1) &= V(\rho(1-X), \rho X) \\ &\equiv \tilde{V}(\rho, X). \end{aligned}$$

The function \tilde{V} satisfies the equation :

$$(4.8) \quad \begin{aligned} -\delta \tilde{V} + \max_{U_1, U_2, U_3} \left\{ \frac{\sigma^2}{2} (1-X)^2 X^2 \frac{\partial^2 \tilde{V}}{\partial X^2} + \frac{\sigma^2}{2} \rho^2 X^2 \frac{\partial^2 \tilde{V}}{\partial \rho^2} + \rho \sigma^2 (1-X) X^2 \frac{\partial^2 \tilde{V}}{\partial \rho \partial X} \right. \\ + (\sigma^2(X-1)X^2 + U_3(\mu X - 1) + U_2(\lambda X + 1) + (\alpha - r)(1-X)X + U_1 X) \frac{\partial \tilde{V}}{\partial X} \\ \left. + \rho(-\mu U_3 - \lambda U_2 + \alpha X + r(1-X) - U_1) \frac{\partial \tilde{V}}{\partial \rho} + \rho^\gamma \frac{U_1^\gamma}{\gamma} \right\} = 0 \end{aligned}$$

We easily check that the function \tilde{V} has the property

$$(4.9) \quad \tilde{V}(\rho, X) = \rho^\gamma \tilde{V}(1, X)$$

(Indeed, $\tilde{V}(\rho, X) / \rho^\gamma$ satisfies the same equation as $\tilde{V}(1, X)$, and by unicity of the solution of (4.8), we deduce (4.9)).

Let us denote

$$(4.10) \quad W(X) = \tilde{V}(1, X)$$

W is the solution of :

$$(4.11) \quad (\gamma r - \delta)W + \max_{U1, U2, U3} H(U1, U2, U3) = 0$$

with

$$(4.12) \quad \begin{aligned} H(U1, U2, U3) = & \frac{\sigma^2}{2} (1-X)^2 X^2 W''(X) \\ & + ((\gamma-1)\sigma^2(1-X) X^2 + (\mu U3 + \lambda U2 + U1) X \\ & + (\alpha-r)(1-X) X - U3 + U2) W'(X) \\ & + \left(\frac{\sigma^2}{2}(\gamma-1)\gamma X^2 - \gamma((\mu U3 + \lambda U2 + U1) + (\alpha-r)\gamma X)W \right. \\ & \left. + \frac{U1\gamma}{\gamma} \right) \end{aligned}$$

We must have the following well-posedness condition which implies that the coefficient of W is negative

$$(4.13) \quad \delta > \gamma \left(\frac{(\alpha-r)^2}{2\sigma^2(1-\gamma)} + r \right)$$

Note that this condition is implied by (3.4) which is also the one formulated by Davis for a well-posed problem [2].

We thus have to solve the equation in one variable (4.11).

The maxima are achieved as follows :

$$\begin{aligned} U1 &= (\gamma W - X W')^{1/\gamma-1} \\ U2 &= \begin{cases} 1 & \text{if } (\lambda X + 1)W' - \lambda\gamma W \geq 0 \\ 0 & \text{if } (\lambda X + 1)W' - \lambda\gamma W < 0 \end{cases} \\ U3 &= \begin{cases} 1 & \text{if } (\mu X - 1)W' - \mu\gamma W \geq 0 \\ 0 & \text{if } (\mu X - 1)W' - \mu\gamma W < 0 \end{cases} \end{aligned}$$

This indicates that the optimal transaction policies are bang - bang :

- if $X \leq X_0$, buy (B) ($U_2 = 1$ and $U_3 = 0$)
- if $X_0 < X < X_T$, no transaction (NT) ($U_2 = U_3 = 0$)
- if $X \geq X_T$, sell ($U_3 = 1$ and $U_2 = 0$)

(see figure 1 below)

X_0 is defined by

$$(\lambda X_0 + 1)W'(X_0) - \lambda\gamma W(X_0) = 0$$

and X_T by :

$$(\mu X_T - 1)W'(X_T) - \gamma\mu W(X_T) = 0$$

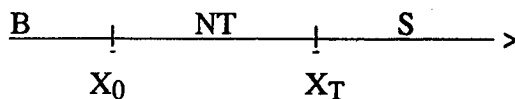


Figure 1

5 - PANDORE REPORTS

The two joined reports have been automatically generated by Pandore (except §3 relative to change of variables).

- The first report deals with the transaction costs problems, formulated as a 2-dimensional problem. The Bellman eq. (4.3) is solved numerically after changes of variables. The equation is discretized using finite differences. (Multigrid techniques will soon be integrated into the system.) The complete analysis is described in the report.

- The second report deals with the transaction costs problem reduced to a one-dimensional problem. Eq. (4.11) is solved numerically.

The optimal utility and optimal controls are displayed on graphs at the end of the reports. (Note that we have minimized the negative of the utility instead of maximizing the utility).

Let us just briefly interpret the shape of the consumption rate U_1 in the second report. When X is small, most of the money is in the bank and the consumption is high. As X increases, consumption decreases just to the point where it is necessary to sell stock to have money available in the bank.

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APPENDIX A

FIRST REPORT AUTOMATICALLY GENERATED BY PANDORE

Portfolio selection with transaction costs

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July 4, 1989

Abstract

This paper deals with portfolio selection with transaction costs. The purpose is to optimize a utility function. The optimal cost satisfies a two-dimensional Bellman equation derived from the optimal principle of Dynamic Programming which is solved numerically.

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1 Notation

- State variables : s_0, s_1
 s_0 : bank account
 s_1 : risky asset
- Control variables : U_1, U_2, U_3
 U_1 : rate of consumption
 U_2 : purchase of stock

$U3$: sale of stock

- V : the utility function
- Time variable : t
- State dimension : n
- i -th state variable : x_i
- Derivative Operator with respect to x_i : ∂_i

2 Formulation of the problem

2.1 Evolution equation of the system

- Evolution of the bank account

$$(ds_0)_t = (rs_0(t) + U3(1 - \mu) - U2(\lambda + 1) - U1)dt$$

- Evolution of the risky asset

$$(ds_1)_t = (\alpha s_1(t) - U3 + U2)dt + \sigma s_1(t)dW_t$$

where

- $U1 \in [0.10, 10]$
- $U2 \in [0, 1]$
- $U3 \in [0, 1]$

2.2 Value function

The value function is defined as the negative of the maximum of the utility function . The problem is to minimise the expectation of the function:

$$J(s) = \int_0^{+\infty} e^{-\delta t} \left(\frac{-U1^\gamma}{\gamma} \right)_t dt$$

in the feedback class , i.e. the applications $S: [s_0, s_1] \mapsto [U1, U2, U3]$

2.3 Optimality conditions

The Bellman function V is defined by:

$$V(x, y) = \min_S (E[J(s) | s_0(0) = x, s_1(0) = y])$$

V satisfies the Dynamic Programming equation [5,3]

$$-\delta V + \min_{U1, U2, U3} (H(U1, U2, U3)) = 0 \quad (1)$$

with

$$H(U1, U2, U3) = \frac{\sigma^2}{2} y^2 \partial_2^2 V + \alpha y \partial_2 V + U3((1 - \mu) \partial_1 V - \partial_2 V) + (rx - U1) \partial_1 V - U2((\lambda + 1) \partial_1 V - \partial_2 V) - \frac{U1^\gamma}{\gamma}$$

3 Change of variables

We wish to have the state variables in $[0, 1]$ Let us first suppose that s_0 and s_1 are positive. (This can always be done by setting: $\tilde{s}_0 = s_0 + (1 + \lambda)s_1$ et $\tilde{s}_1 = s_0 + (1 - \mu)s_1$).

We then state:

$$s_0 = \frac{X1}{(1 - X1)}$$

$$s_1 = \frac{X2}{(1 - X2)}$$

$$X1 \in [0, 1]$$

$$X2 \in [0, 1]$$

We have the homothetic property of V at infinity

$$V(\rho x, \rho y) \sim \rho^\gamma V(x, y)$$

The function V is not bounded, so we make the change of function

$$W(X1, X2) = (1 - X1)(1 - X2)V(X1, X2)$$

4 Evolution equation of the new state variables

For the numerical study of the Bellman equation , a small noise ϵ will be added to the evolution equations $X1$ and $X2$. This will not distort the solution W but will slightly translate it by ϵ .

We takes the following numerical values:

$$\alpha = 0.5, r = 0.2, \lambda = \mu = 0.45, \gamma = 0.3, \delta = 0.9, \sigma = 1, \epsilon = 0.4$$

We thus consider the control diffusion process defined by the dynamic differential equation:

$$\begin{aligned} dX1_t &= e_4 + e_3 + e_2 + e_1 \\ e_1 &= -dZ1_t^1 \\ e_2 &= dZ1_t^0 \end{aligned} \tag{2}$$

$$e_3 = (0.20(1 - X1)X1 + (-1.5U3 + 0.55U2 - U1)(1 - X1)^2)dt$$

$$e_4 = 0.45dW1_t$$

$$dX2_t = e_8 + e_7 + e_6 + e_5$$

$$e_5 = -dZ2_t^1$$

$$e_6 = dZ2_t^0 \tag{3}$$

$$e_7 = ((0.50 - X2)(1 - X2)X2 + (U2 - U3)(1 - X2)^2)dt$$

$$e_8 = (1.4 - X2)(X2 + 0.40)dW2_t$$

$Z_{i,t}^j$ is an increasing process, strictly increasing when x_i is on the boundary $x_i = j$
 $W_{i,t}$ is a Wiener process, i.e. a continuous gaussian process with independant increments.

This process is well defined [16] . It is the limit when the time step h goes to 0 of a Markovian discrete process $X1_n^h$ which satisfies:

$$E(X1_{n+1}^h - X1_n^h | \mathcal{F}_n) = h \begin{pmatrix} 0.20(1 - X1)X1 + (-1.5U3 + 0.55U2 - U1)(1 - X1)^2 \\ 0.50(1 - X2)X2 + (U2 - U3)(1 - X2)^2 \end{pmatrix} + o(h)$$

$$E((X1_{n+1}^h - X1_n^h)^{\otimes 2} | \mathcal{F}_n) = 2h \begin{pmatrix} 0.10 & 0 \\ 0 & 0.50(1.4 - X2)^2(X2 + 0.40)^2 \end{pmatrix} + o(h)$$

- a uniform integrability condition of the increment $X1_{n+1}^h - X1_n^h$.

\mathcal{F}_n denotes the σ -algebra generated by $X1_0, X1_1, \dots, X1_n$.

Limit conditions :

If the little noise ϵ had not been added, the limit conditions would have been natural, i.e. the process would have been singular on the boundary. As we are very closed to this situation, we put reflections on the boundary.

5 Bellman equation

The function W satisfies the equation

$$\min_{U1, U2, U3} (A(U1, U2, U3)W + c(U1, U2, U3)) - 0.90W = 0 \quad (4)$$

$$\partial_1 W(0, X2) = 0$$

$$\partial_2 W(X1, 0) = 0$$

$$\partial_1 W(1, X2) = 0$$

$$\partial_2 W(X1, 1) = 0$$

with

$$\begin{aligned} A(U1, U2, U3)W &= e_{16} + e_{15} + e_{14} + e_{13} + e_{12} \\ e_{12} &= ((U3 - U2 + 0.5)X2 - (0.55U3 - 1.5U2 - U1 - 0.2)X1 - 0.45(U3 - U2) - U1)W \\ e_{13} &= 0.10\partial_1^2 W \\ e_{14} &= 0.50(1.4 - X2)^2(X2 + 0.40)^2\partial_2^2 W \\ e_{15} &= (0.50(1 - X2)X2 + (U2 - U3)(1 - X2)^2)\partial_2 W \\ e_{16} &= (0.20(1 - X1)X1 + (-1.5U3 + 0.55U2 - U1)(1 - X1)^2)\partial_1 W \end{aligned}$$

et

$$c(U1, U2, U3) = -3.3U1^{0.30}(1 - X1)(1 - X2) \quad (5)$$

Using the Maximum principle, P.L.Lions [11] has proved the existence of a solution of Bellman equation. The discretization methods that we use further are also based on the Maximum principle and its probabilistic interpretation.

6 Dynamic Programming method

Our purpose is to solve equation (4) after discretization. This is possible because the state dimension is small.[13,14,6,8]

6.1 Discretization

We denote:

h_i : i-th space variable discretization step

We define the following operators:

$s_i : W(x_1, \dots, x_i, \dots, x_n) \mapsto W(x_1, \dots, x_i + h_i, \dots, x_n) \quad i = 1, \dots, n$

$$\partial_i^h = \frac{(s_i - s_i^{-1})}{2h_i}$$

$$\Delta_i^h = \frac{(s_i^{-1} + s_i - 2)}{h_i^2}$$

We then approximate

$\partial_i^2 W$ by $\Delta_i^h W$

$\partial_1 W$ by $\partial_1^h W$

$\partial_2 W$ by $\partial_2^h W$

The discretized Bellman equation is

$$\min_{U1, U2, U3} (c(U1, U2, U3) + A^h(U1, U2, U3)W^h) - 0.90W^h = 0$$

with

$$\begin{aligned} A^h(U1, U2, U3)W &= e_{20} + e_{19} + e_{18} + e_{17} + e_{12} \\ e_{12} &= ((U3 - U2 + 0.5)X2 - (0.55U3 - 1.5U2 - U1 - 0.2)X1 - 0.45(U3 - U2) + U1)W \\ e_{17} &= 0.50(1.4 - X2)^2(X2 + 0.40)^2\Delta_2^h W \\ e_{18} &= (0.50(1 - X2)X2 + (U2 - U3)(1 - X2)^2)\partial_2^h W \\ e_{19} &= (0.20(1 - X1)X1 + (-1.5U3 + 0.55U2 - U1)(1 - X1)^2)\partial_1^h W \\ e_{20} &= 0.10\Delta_1^h W \end{aligned}$$

6.2 Probabilistic interpretation of the discretized equation

The discretization of the Bellman equation :

$$\min_{U_1, U_2, U_3} (A(U_1, U_2, U_3)W + c(U_1, U_2, U_3)) - \lambda W = 0$$

can be interpreted as a control problem of Markov chain with discount factor λk and instantaneous cost kC . The associated cost function is

$$\sum_{n=0}^{\infty} k(\lambda k + 1)^{-1-n} C(X_n, U_n)$$

and the Markov matrix M can be written:

$$M = kA + I$$

where I is the Identity matrix, λ the discount factor and k the inverse of the diagonal of A .
 M is defined by:

| <i>Init - pt</i> | <i>Final - pt</i> | <i>Transition - probability</i> |
|------------------|----------------------------|---------------------------------|
| [X1, X2] | [X1, X2] | 0 |
| [X1, X2] | [X1 + h ₁ , X2] | $\frac{0.10h_1^{-2}}{R}$ |
| [X1, X2] | [X1, X2 + h ₂] | $\frac{0.16h_2^{-2}}{R}$ |
| [X1, X2] | [X1 - h ₁ , X2] | $\frac{0.10h_1^{-2}}{R}$ |
| [X1, X2] | [X1, X2 - h ₂] | $\frac{0.16h_2^{-2}}{R}$ |
| [0, X2] | [0, X2] | 0 |
| [0, X2] | [h ₁ , X2] | $\frac{h_1^{-2}}{5R_{10}}$ |
| [1, X2] | [1, X2] | 0 |
| [1, X2] | [1 - h ₁ , X2] | $\frac{h_1^{-2}}{5R_{11}}$ |
| [X1, 0] | [X1, 0] | 0 |
| [X1, 0] | [X1, h ₂] | $\frac{196h_2^{-2}}{625R_{20}}$ |
| [X1, 1] | [X1, 1] | 0 |
| [X1, 1] | [X1, 1 - h ₂] | $\frac{196h_2^{-2}}{625R_{21}}$ |

with

$$R = (-1.0U3 + U2 - 0.50)X2 + (0.55U3 - 1.5U2 - 1.0U1 - 0.20)X1 + 0.45U3 + 0.45U2 + U1$$

$$R_{10} = -U3X2 + U2X2 - 0.50X2 + 0.45U3 + 0.45U2 + U1 + 0.31h_2^{-2} + 0.20h_1^{-2}$$

$$R_{11} = -U3X2 + U2X2 - 0.50X2 + U3 - U2 + 0.31h_2^{-2} + 0.20h_1^{-2} - 0.20$$

$$R_{20} = 0.55U3X1 - 1.5U2X1 - U1X1 - 0.20X1 + 0.45U3 + 0.45U2 + U1 + 0.31h_2^{-2} + 0.20h_1^{-2}$$

$$R_{21} = 0.55U3X1 - 1.5U2X1 - U1X1 - 0.20X1 - 0.55U3 + 1.5U2 + U1 + 0.31h_2^{-2} + 0.20h_1^{-2} - 0.50$$

if the following stability conditions are satisfied

$$h_1 \leq \min_{U1, U2, U3, X1} \left(\frac{0.20}{|0.20(1 - X1)X1 + (-1.5U3 + 0.55U2 - U1)(1 - X1)^2|} \right)$$

$$h_2 \leq \min_{U1, U2, U3, X2} \left(\frac{(1.4 - X2)^2(X2 + 0.40)^2}{|0.50(1 - X2)X2 + (U2 - U3)(1 - X2)^2|} \right)$$

the matrix coefficients are positive and the sum of the coefficients on a same line is equal to 1. The matrix M is thus a transition matrix of a Markov chain. Moreover the optimal cost obeys

$$(\lambda k + 1)W^h = \min_{U1, U2, U3} \left(M(U1, U2, U3)W^h + kC(U1, U2, U3) \right)$$

We use the contraction iteration

$$W_{n+1}^h = \frac{\min_{U1, U2, U3} \left(M(U1, U2, U3)W_n^h + kC(U1, U2, U3) \right)}{(\lambda k + 1)}$$

6.3 Optimization of the Hamiltonian

We minimize the control-dependant part \mathcal{H}^h of the discretized hamiltonian with respect to $U = [U1, U2, U3]$ by a projected gradient method.

$$U_{n+1} = \mathcal{P}_{[0,1,10] \otimes [0,1] \otimes [0,1]} \left(U_n - \rho \frac{d\mathcal{H}^h(U_n)}{dU_n} \right)$$

that is

$$\left(\begin{array}{l} U1_{n+1} = \mathcal{P}_{[0,1,10]} \left(U1_n - \rho_1 \left((X1 - 1)W + U1_n^{-7/10} (X2 + X1 - 1 - X1X2) - \partial_1^h W (X1^2 + 2X1 - 1) \right) \right) \\ U2_{n+1} = \mathcal{P}_{[0,1]} \left(U2_n - \rho_2 \left((-X2 + 1.45X1 - 0.45)W + \partial_2^h W (X2 - 1)^2 + 0.55\partial_1^h W (X1 - 1)^2 \right) \right) \\ U3_{n+1} = \mathcal{P}_{[0,1]} \left(U3_n - \rho_3 \left((X2 - 0.55X1 - 0.45)W - \partial_2^h W (X2 - 1)^2 - 1.45\partial_1^h W (X1 - 1)^2 \right) \right) \end{array} \right)$$

This algorithm converges when the step ρ_1 satisfies:

$$0 < \rho_1 < 2kK^{-2}$$

with

$$k|W|^2 \leq D_U^2 \mathcal{H}^h(W)W \leq K|W|^2$$

6.4 Numerical results

We perform a numerical test with:

number of space discretization points : [24, 24]

precision required for the implicit resolution : 0.01

step for the implicit resolution: 0.004

maximal number of iterations for the implicit resolution: 5000

maximal number of iterations for the optimization of the hamiltonian: 36

precision required for the convergence test: 0.01

The Bellman function V , solution of eq.(1) and the optimal controls $U1, U2, U3$ as well as their contour-lines are displayed on the following figures.

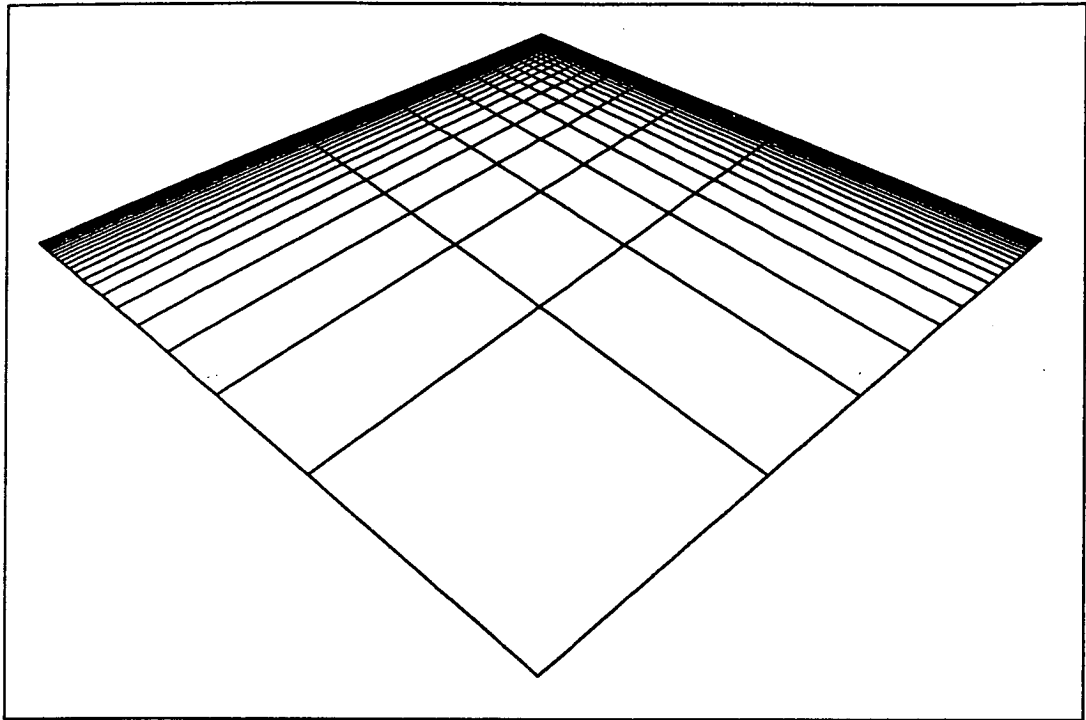


Figure 1: Negative of the maximum of the utility function: V

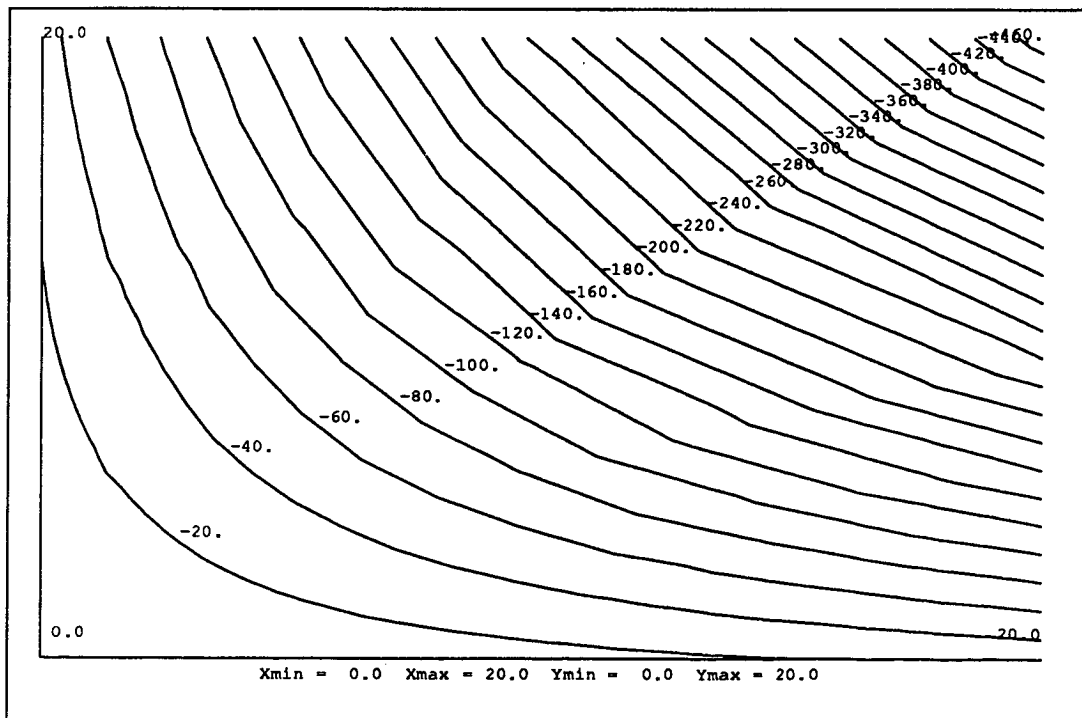


Figure 2: Contour-lines of V

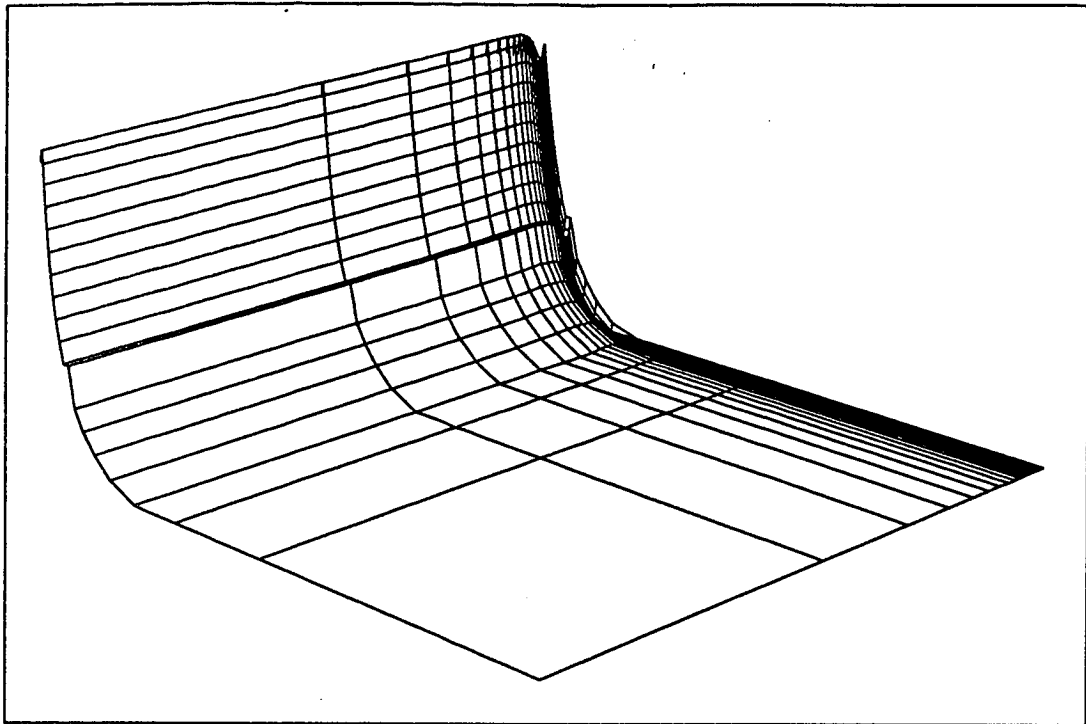


Figure 3: Optimal control U_1 (consumption)

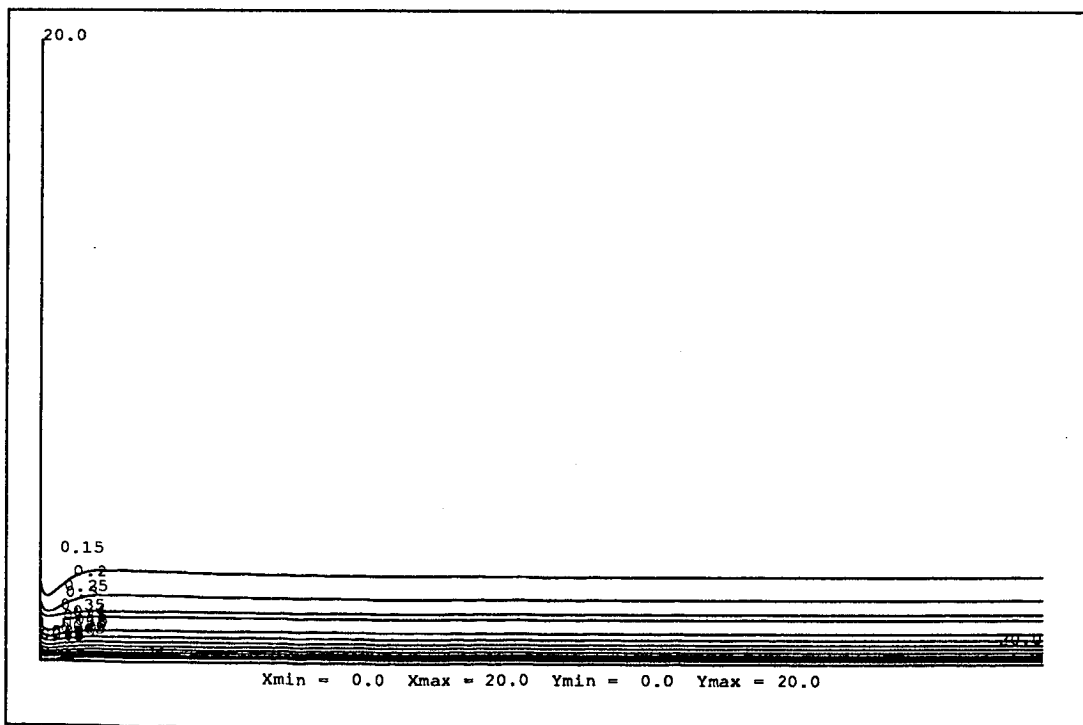


Figure 4: Contour-lines of the optimal control U_1

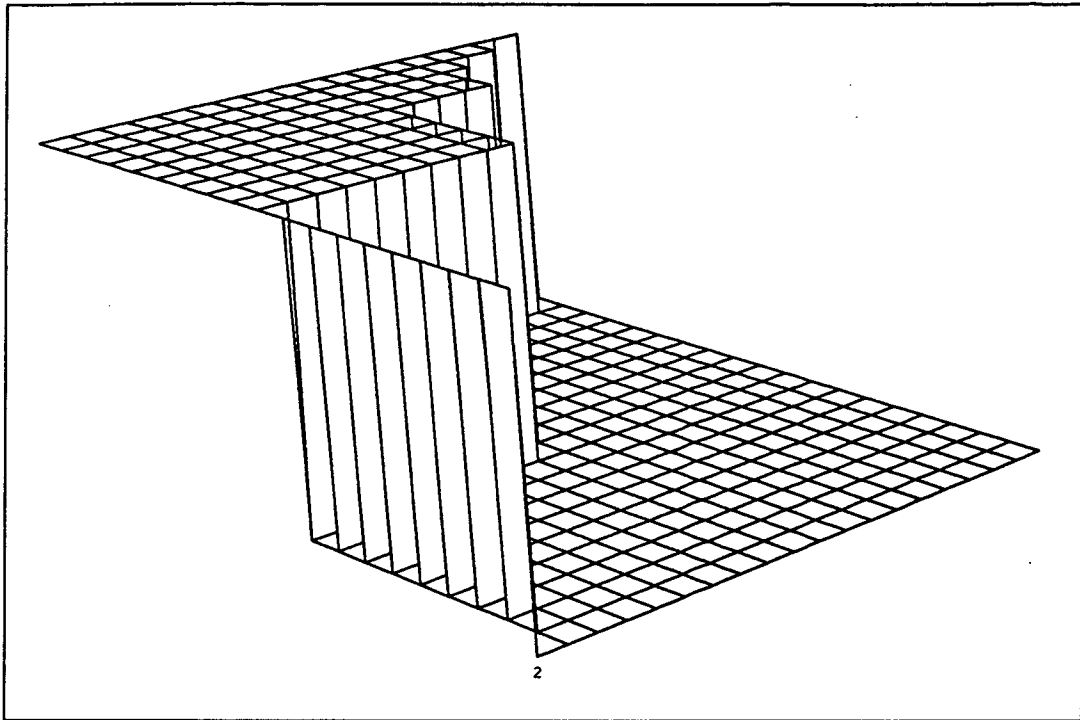


Figure 5: Optimal bang-bang control U_2 (purchase of stock)

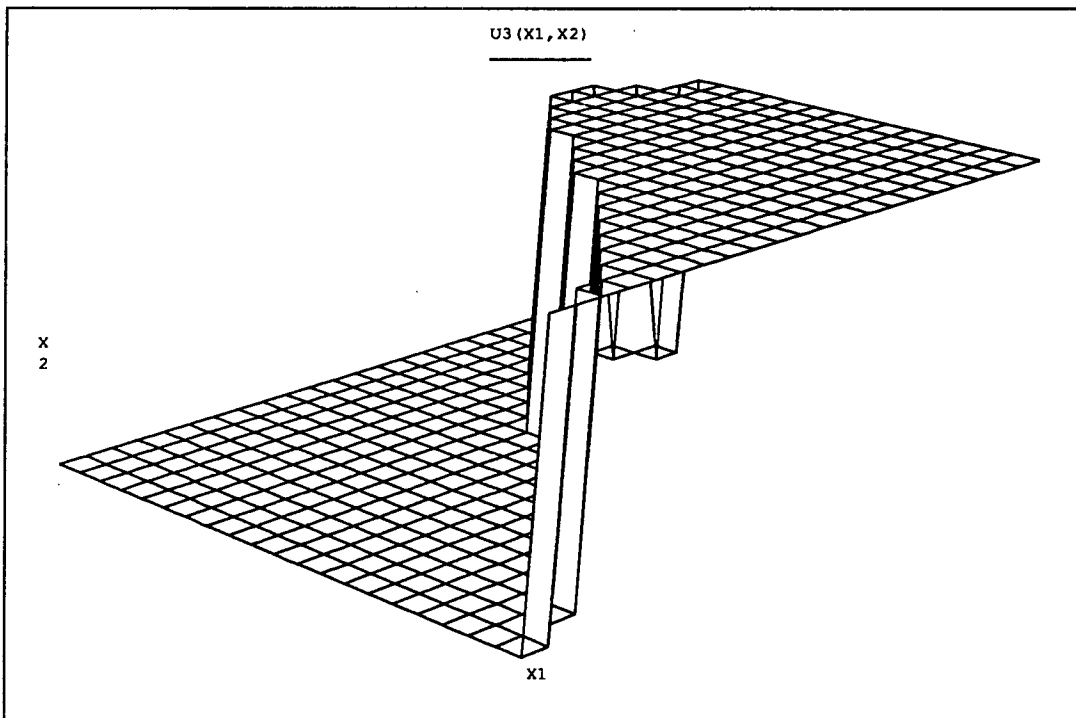


Figure 6: Optimal bang-bang control U_3 (sale of stock)

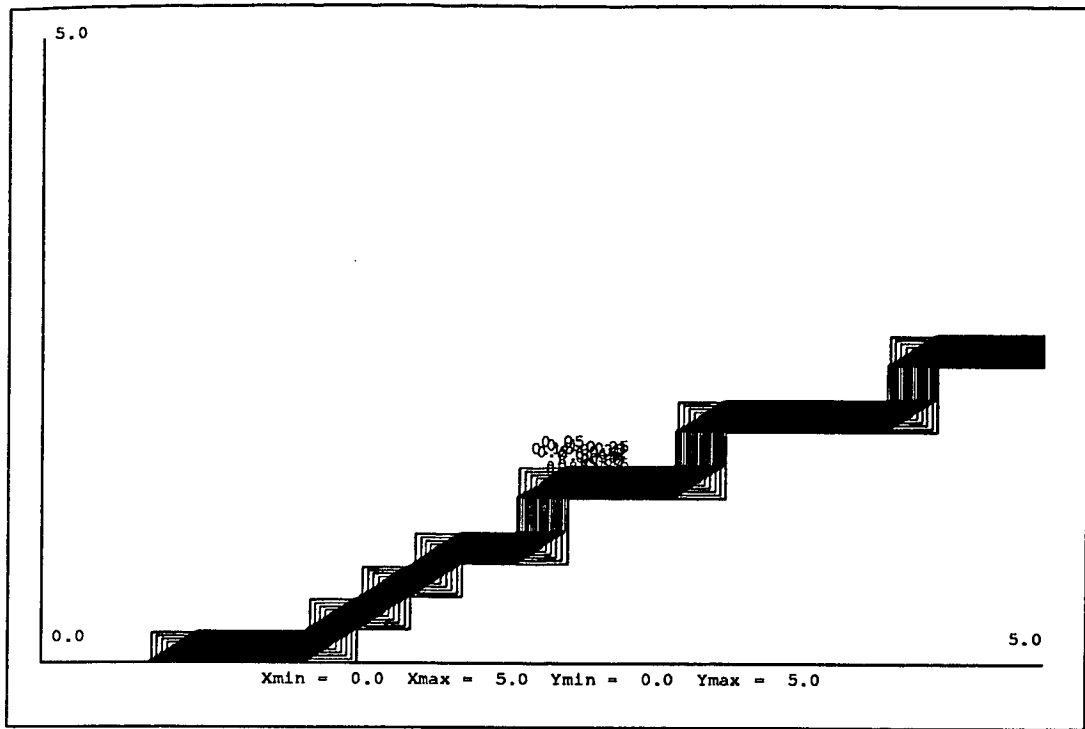


Figure 7: Contour-lines of the optimal control U2. (It is optimal to buy in the region below the curve)

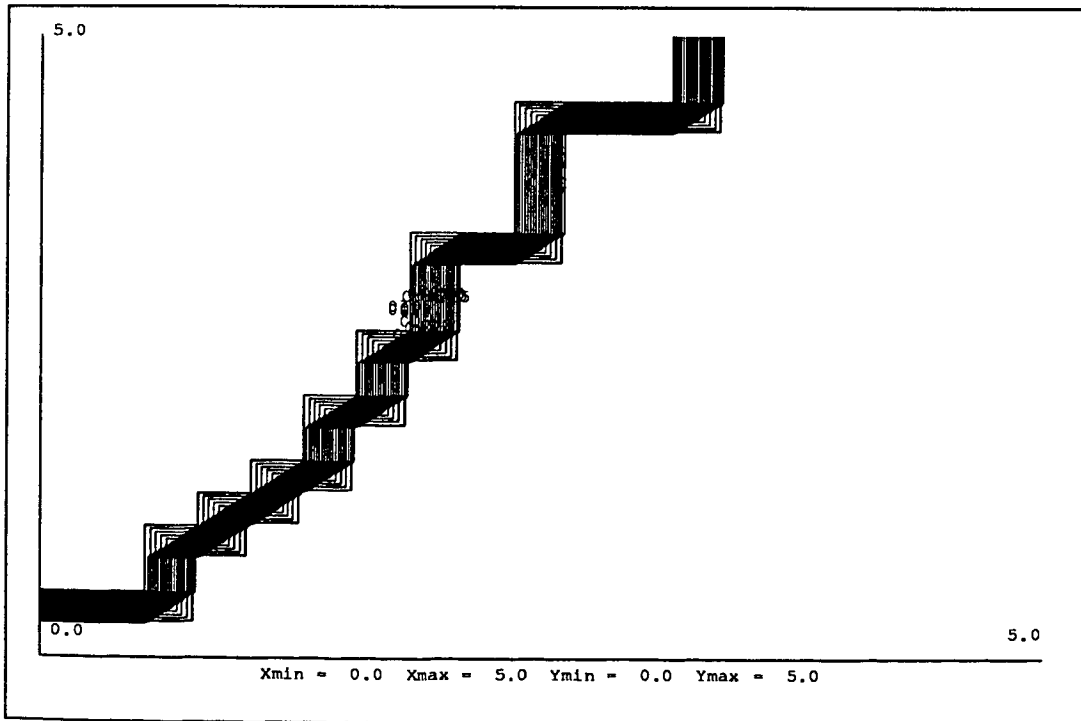


Figure 8: Contour-lines of the optimal control U3. (It is optimal to sell in the region above the curve)

A Appendix: The generated Fortran routines

```

SUBROUTINE PPD(N1,N2,V,EPSIMP,IMPMAX,RO,U,EPS,NMAX,ROG)
DIMENSION V(N1,N2),U(3,N1,N2)
C      Resolution de l equation de Kolmogorov dans le cas ou:
C      Les parametres sont
C      L etats-temps est: X1 X2
C      La dynamique du systeme est decrite par l operateur
C      4 3 2 2
C      plus( 0.5 Q2 X2 - Q2 X2 - 0.060000002 Q2 X2 - 0.5 P2 X2
C      + 0.559999994 Q2 X2 + 0.5 P2 X2 + W (0.5 X2 + 0.2 X1)
C      2
C      - 0.2 P1 X1 + 0.2 P1 X1 + 0.1568 Q2 + 0.1 Q1 , Minu(
C      2 0.3
C      (P2 U2 - 1.0 P2 U3) X2 + (- 3.3333333 U1 X1 + 2.0 P2 U3
C      0.3
C      - 2.0 P2 U2 + 3.3333333 U1 ) X2
C      + W (- U3 (- X2 - 0.55 (1 - X1) + 1)
C      + U2 (- X2 - 1.45 (1 - X1) + 1) - U1 (1 - X1))
C      2
C      + (- 1.45 P1 U3 + 0.55 P1 U2 - 1.0 P1 U1) X1
C      0.3
C      + (2.9 P1 U3 - 1.1 P1 U2 + 2.0 P1 U1 + 3.3333333 U1 ) X1
C      + (- 1.0 P2 - 1.45 P1) U3 + (P2 + 0.55 P1) U2 - 1.0 P1 U1
C      0.3
C      - 3.3333333 U1 ) )
C      ou v(..) et w designe le cout
C      ou pi designe sa derivee premiere par rapport a xi
C      ou qi designe sa derivee seconde par rapport a xi
C      Le probleme est statique
C      Les conditions aux limites sont:
C      X2 = 0 p2 = 0
C      X2 = 1 -p2 = 0
C      X1 = 0 p1 = 0
C      X1 = 1 -p1 = 0
C      Les nombres de points de discretisation sont: N1 N2
C      X2 = 1 correspond a I2 = N2 - 1
C      X2 = 0 correspond a I2 = 2
C      X1 = 1 correspond a I1 = N1 - 1
C      X1 = 0 correspond a I1 = 2
C      Le taux d actualisation vaut: 0.9
C      impmax designe le nbre maxi d iterations du systeme implicite
C      epsimp designe l erreur de convergence du systeme implicite
C      ro designe le pas de la resolution du systeme implicite
C      par une methode iterative
C      P2 est discretise par difference divise symetrique
C      P1 est discretise par difference divise symetrique
C      Minimisation par la methode de gradient avec projection

```

```

C                                     de l'Hamiltonien:
C
C                                     2           0.3
C          (P2 U2 - 1.0 P2 U3) X2  + (- 3.3333333 U1   X1
C                                     0.3
C          + 2.0 P2 U3 - 2.0 P2 U2 + 3.3333333 U1   ) X2
C          + W (- U3 (- X2 - 0.55 (1 - X1) + 1)
C          + U2 (- X2 - 1.45 (1 - X1) + 1) - U1 (1 - X1))
C                                     2
C          + (- 1.45 P1 U3 + 0.55 P1 U2 - 1.0 P1 U1) X1
C                                     0.3
C          + (2.9 P1 U3 - 1.1 P1 U2 + 2.0 P1 U1 + 3.3333333 U1   ) X1
C          + (- 1.0 P2 - 1.45 P1) U3 + (P2 + 0.55 P1) U2 - 1.0 P1 U1
C          0.3
C          - 3.3333333 U1
C          contraintes sur le controle:
C          0 =< U3 =< 1
C          0 =< U2 =< 1
C          0.1 =< U1 =< 10
C          nmax designe le nombre maxi d iteration de la methode de
C                                     gradient avec projection
C          eps designe l erreur de convergence de la methode de
C                                     gradient avec projection
C          rog designe le pas, qui est constant, dans la methode de gradi#
C          ent
C          H2=0.999999/(N2-3)
C          H1=0.999999/(N1-3)
C          U3=U(3,1,1)
C          U2=U(2,1,1)
C          U1=U(1,1,1)
C          HIH2=H2**2
C          HIH1=H1**2
C          H22=2*H2
C          H21=2*H1
C          NM2=N2-1
C          NM1=N1-1
C          ROA=10000*ROG
C          do 1017 I2=1,N2,1
C          do 1017 I1=1,N1,1
C          V(I1,I2)=0.0
1017 CONTINUE
          IMITER=1
1013 CONTINUE
          ERIMP=0
          do 1011 I1=1,N1,1
          X1=H1*(I1-2)
          V(I1,N2)=V(I1,N2-2)
          V(I1,1)=V(I1,2)
1011 CONTINUE

```

```

do 1009 I2=2,NM2,1
X2=H2*(I2-2)
V(N1,I2)=V(N1-2,I2)
V(1,I2)=V(2,I2)
1010 CONTINUE
do 1009 I1=2,NM1,1
X1=H1*(I1-2)
Q2=(V(I1,I2+1)-2*V(I1,I2)+V(I1,I2-1))/HIH2
Q1=(V(I1+1,I2)-2*V(I1,I2)+V(I1-1,I2))/HIH1
P2=(V(I1,I2+1)-V(I1,I2-1))/H22
P1=(V(I1+1,I2)-V(I1-1,I2))/H21
IF (I1 .EQ. 2) P1=(V(I1+1,I2)-V(I1,I2))/H1
IF (I2 .EQ. 2) P2=(V(I1,I2+1)-V(I1,I2))/H2
W=V(I1,I2)
NITER=0
WO=-1.0e20
1001 CONTINUE
NITER=NITER+1
if (NITER-NMAX) 1002,1002,1003
1003 CONTINUE
WRITE(8,1801) I1,I2
1801 FORMAT(' descente n a pas converge',2i3)
GOTO 1004
1002 CONTINUE
UN1=U1**((-7.0)/10.0)*((ROG*X1-ROG)*X2+P1*ROG*U1**((7.0/10.0)*X1**2
1 +(-ROG*U1**((7.0/10.0)*W-2*P1*ROG*U1**((7.0/10.0)-ROG)*X1+ROG*U1*
2 *(7.0/10.0)*W+U1**((17.0/10.0)+P1*ROG*U1**((7.0/10.0)+ROG)
UN2=-((20*P2*ROA*X2**2+(-20*ROA*W-40*P2*ROA)*X2+11*P1*ROA*X1**2+(29
1 *ROA*W-22*P1*ROA)*X1-9*ROA*W-20*U2+(20*P2+11*P1)*ROA)/20.0
UN3=((20*P2*ROA*X2**2+(-20*ROA*W-40*P2*ROA)*X2+29*P1*ROA*X1**2+(11*
1 ROA*W-58*P1*ROA)*X1+9*ROA*W+20*U3+(20*P2+29*P1)*ROA)/20.0
U1=UN1
U2=UN2
U3=UN3
U1=AMAX1(U1,0.1)
U1=AMIN1(U1,10)
U2=AMAX1(U2,0)
U2=AMIN1(U2,1)
U3=AMAX1(U3,0)
U3=AMIN1(U3,1)
WW=((P2*U2-1.0*P2*U3)*X2**2+(-3.3333333*U1**0.3*X1+2.0*P2*U3-2.0*P2
1 *U2+3.3333333*U1**0.3)*X2+W*(-U3*(-X2-0.55*(1-X1)+1)+U2*(-X2-1.
2 45*(1-X1)+1)-U1*(1-X1))+(-1.45*P1*U3+0.55*P1*U2-1.0*P1*U1)*X1**
3 2+(2.9*P1*U3-1.1*P1*U2+2.0*P1*U1+3.3333333*U1**0.3)*X1+(-1.0*P2
4 -1.45*P1)*U3+(P2+0.55*P1)*U2-1.0*P1*U1-3.3333333*U1**0.3
ER=ABS(WW-W0)
if (ER-EPS) 1004,1004,1005
1005 CONTINUE

```

```

      WO=WW
      GOTO 1001
1004 CONTINUE
      U(1,I1,I2)=U1
      U(2,I1,I2)=U2
      U(3,I1,I2)=U3
      WO=WW
      W1=0.5*Q2*X2**4-Q2*X2**3-0.060000002*Q2*X2**2-0.5*P2*X2**2+0.55999
1      994*Q2*X2+0.5*P2*X2+W*(0.5*X2+0.2*X1)-0.2*P1*X1**2+0.2*P1*X1+0.
2      1568*Q2+0.1*Q1
      WO=W1+WO
      WO=WO-0.9*V(I1,I2)
      VNEW=R0*WO+V(I1,I2)
      V(I1,I2)=VNEW
      ERIMP=ABS(WO)+ERIMP
1009 CONTINUE
      WRITE(6,*) ERIMP
      IMITER=IMITER+1
      if (IMITER-IMPMAX) 1016,1015,1015
1016 CONTINUE
      if (EPSIMP-ERIMP) 1013,1012,1012
1015 CONTINUE
      WRITE(8,1807)
1807 FORMAT(' schema implicite n a pas converge')
1012 CONTINUE
      RETURN
      END

```

```

PROGRAM M_PPD
REAL V(24,24),U(3,24,24)
  LISPFUNCTION FLFTNV 'CLIMAX::FORT-MACS' (REAL(24,24),CHARACTER*2
1  0,INTEGER(2))
  INTEGER NFTNV(2)
  CHARACTER*20 FTNVF
  LISPFUNCTION FLFTNU 'CLIMAX::FORT-MACS' (REAL(3,24,24),CHARACTER
1  *20,INTEGER(3))
  INTEGER NFTNU(3)
  CHARACTER*20 FTNUF
  NFTNV(1)=24
  NFTNV(2)=24
  FTNVF= 'FTNV'
  NFTNU(1)=24
  NFTNU(2)=24
  NFTNU(3)=3
  FTNUF= 'FTNU'
c-faire
      DO 1002 K1=1,24
c-faire

```

```

                DO 1001 K2=1,24
                  V(K1,K2)=0.0
c-faire
                DO 1000 J=1,3
                  U(J,K1,K2)=0.1
1000             CONTINUE
c-fin-de-faire.
1001             CONTINUE
c-fin-de-faire.
1002             CONTINUE
c-fin-de-faire.
                CALL PPD(24,24,V,0.01,5000,0.004,U,0.010000001
1                ,36,0.05)
                CALL FLFTNV(V,FTNVF,NFTNV)
                CALL FLFTNU(U,FTNUF,NFTNU)
STOP
END

```

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APPENDIX B

SECOND REPORT AUTOMATICALLY GENERATED BY PANDORE

An homogeneous one-dimensional model of a portfolio selection problem with transaction costs

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Abstract

This paper deals with portfolio selection with transaction costs. The purpose is to optimize a utility function. The optimal cost satisfies a Bellman equation derived from the optimal principle of Dynamic Programming which is solved numerically.

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1 Notation

- State variables : s_0, s_1
 s_0 : bank account
 s_1 : risky asset
- Control variables : c, l, u
 c : rate of consumption
 l : purchase of stock
 u : sale of stock

- Time variable : t
- I-th state variable : x_i
- Derivative operator with respect to x_i : ∂_i

2 Formulation of the problem

2.1 Evolution equation of the system

- Evolution of the bank account

$$(ds_0)_t = (rs_0(t) + l(1 - \mu) - u(\lambda + 1) - c)dt$$

- Evolution of the risky asset

$$(ds_1)_t = (\alpha s_1(t) - l + u)dt + \sigma s_1(t)dW_t$$

where

- $c \in [0.10, 10]$
- $l \in [0, 1]$
- $u \in [0, 1]$

2.2 Value function

The value function is defined as the negative of the maximum of the utility function . The problem is to minimise the expectation of the function:

$$J(S) = \int_0^{+\infty} e^{-\delta t} \left(\frac{-c^\gamma}{\gamma} \right)_t dt$$

in the feedback class , i.e. the applications $S: [s_0, s_1] \mapsto [c, l, u]$

2.3 Optimality conditions

The Bellman function V is defined by:

$$V(x, y) = \min_S (E [J(s) | s_0(0) = x, s_1(0) = y])$$

V satisfies the Dynamic Programming equation [6,3]

$$-\delta V + \min_{c,l,u} (K(c, l, u)) = 0$$

with

$$K(c, l, u) = \frac{\sigma^2}{2} y^2 \partial_2^2 V + \alpha y \partial_2 V + u((1 - \mu) \partial_1 V - \partial_2 V) + (rx - c) \partial_1 V - l((\lambda + 1) \partial_1 V - \partial_2 V) - \frac{c^\gamma}{\gamma}$$

3 Change of variables-Reduction to a one-dimensional problem

We make the change of state variables:

- total wealth

$$\rho = s_1 + s_0$$

- fraction of wealth invested in the risky asset

$$X = \frac{s_1}{(s_1 + s_0)}$$

and we make the following changes of control variables:

- fraction of wealth dedicated to consumption

$$U1 = \frac{c}{(s_1 + s_0)}$$

- fraction of wealth dedicated to purchase of stock

$$U2 = \frac{l}{(s_1 + s_0)}$$

- fraction of wealth dedicated to sale of stock

$$U3 = \frac{u}{(s_1 + s_0)}$$

We have

$$V(x, y) = V(\rho(1 - X), \rho X) \equiv \tilde{V}(\rho, X)$$

The function $\tilde{V}(\rho, X)$ satisfies the Bellman equation

$$-\delta\tilde{V} + \min_{U1, U2, U3} (\tilde{H}(U1, U2, U3)) = 0$$

avec

$$\begin{aligned} \tilde{H}(U1, U2, U3) = & \frac{\sigma^2}{2}(1 - X)^2 X^2 \frac{\partial^2 \tilde{V}}{\partial X^2} + \frac{\sigma^2}{2} \rho^2 X^2 \frac{\partial^2 \tilde{V}}{\partial \rho^2} + \rho \sigma^2 (1 - X) X^2 \frac{\partial^2 \tilde{V}}{\partial \rho \partial X} \\ & + \left(\sigma^2 (X - 1) X^2 + U3(\mu X - 1) + U2(\lambda X + 1) + (\alpha - r)(1 - X)X + U1X \right) \frac{\partial \tilde{V}}{\partial X} \\ & + \rho(-\mu U3 - \lambda U2 + \alpha X + r(1 - X) - U1) \frac{\partial \tilde{V}}{\partial \rho} - \frac{\rho^\gamma U1^\gamma}{\gamma} \end{aligned}$$

the function \tilde{V} has the property

$$\tilde{V}(\rho, X) = \rho^\gamma \tilde{V}(1, X)$$

Let us now state

$$W(X) = \tilde{V}(1, X)$$

This lead to solve an equation in W in one variable X :

$$(\gamma r - \delta)W + \min_{U1, U2, U3} (H(U1, U2, U3)) = 0 \quad (1)$$

with

$$\begin{aligned} H(U1, U2, U3) = & \frac{\sigma^2}{2}(1-X)^2 X^2 \partial_1^2 W \\ & + \left((\gamma - 1)\sigma^2(1-X)X^2 + (\mu U3 + \lambda U2 + U1)X + (\alpha - r)(1-X)X - U3 + U2 \right) \partial_1 W \\ & + \left(\frac{\sigma^2}{2}(\gamma - 1)\gamma X^2 - \gamma(\mu U3 + \lambda U2 + U1) + (\alpha - r)\gamma X \right) W - \frac{U1^\gamma}{\gamma} \end{aligned}$$

under the following well-posedness condition

$$\delta > \gamma \left(\frac{(\alpha - r)^2}{2\sigma^2(1 - \gamma)} + r \right)$$

4 Dynamic Programming Method

For the numerical study of equation (1), a small noise ϵ will be added. This will not distort the solution W but will slightly translate it by ϵ .

We take the following numerical values:

$$\alpha = 0.12$$

$$r = 0.07$$

$$\lambda = 0.03$$

$$\mu = 0.01$$

$$\gamma = 0.5$$

$$\delta = 0.1$$

$$\sigma = 0.4$$

$$\epsilon = 0.3$$

W satisfies

$$\min_{U1, U2, U3} (A(U1, U2, U3)W + c(U1, U2, U3)) - 6.50 \cdot 10^{-2} W = 0 \quad (2)$$

with

$$\begin{aligned} A(U1, U2, U3)W &= e_1 + e_2 + e_3 \\ e_3 &= e_4 \partial_1 W \\ e_4 &= -0.08(1-X)X^2 + 0.05(1-X)X + (0.01U3 + 0.03U2 + U1)X - U3 + U2 \\ e_1 &= -(0.02X^2 - 0.025X + 0.005U3 + 0.015U2 + 0.50U1)W \\ e_2 &= 0.08(1.3 - X)^2 (X + 0.30)^2 \partial_1^2 W \end{aligned}$$

and

$$c(U1, U2, U3) = -2.0U1^{0.50} \quad (3)$$

Limit conditions :

If the little noise ϵ had not been added, the limit conditions would have been natural, i.e. the process would have been singular for $X = 0$ and $X = 1$. As we are very closed to this situation, we put reflections on the boundaries.

$$\partial_1 W(0) = 0$$

$$\partial_1 W(1) = 0$$

Our purpose is to solve equation (2) after discretization. This is possible because the state dimension is small.[14,15,7,9]

4.1 Discretization

We denote:

h : discretization step

We define the following operator:

$S_i : W(x) \mapsto W(x + h)$

$$\partial^h = \frac{(S - S^{-1})}{2h}$$

$$\Delta^h = \frac{(S^{-1} + S - 2)}{h^2}$$

We thus approximate

$\partial_1^2 W$ by $\Delta^h W$

$\partial_1 W$ by $\partial^h W$

The discretized equation is

$$\min_{U1, U2, U3} (c(U1, U2, U3) + A^h(U1, U2, U3)W^h) - 6.50 \cdot 10^{-2} W^h = 0$$

with

$$\begin{aligned} A^h(U1, U2, U3)W &= e_5 + e_6 + e_7 \\ e_5 &= (-0.02X^2 + 0.025X - 0.005U3 - 0.015U2 - 0.5U1 - 0.0243h^{-2})W \\ e_6 &= 1.22 \cdot 10^{-2} h^{-2} S^{-1}W \\ e_7 &= 1.22 \cdot 10^{-2} h^{-2} SW \end{aligned} \tag{4}$$

4.2 Probabilistic interpretation of the discretized equation

The discretization of the Bellman equation :

$$\min_{U1, U2, U3} (A(U1, U2, U3)W + c(U1, U2, U3)) - \lambda W = 0$$

can be interpreted as a control problem of Markov chain with discount factor λk and instantaneous cost kC . The associated cost function is

$$\sum_{n=0}^{\infty} k(\lambda k + 1)^{-1-n} C(X_n, U_n)$$

and the Markov matrix M can be written:

$$M = kA + I$$

where I is the Identity matrix, λ the discount factor and k the inverse of the diagonal of A . M is defined by:

| <i>Init - pt</i> | <i>Final - pt</i> | <i>Transition - probability</i> |
|------------------|-------------------|--|
| [X] | [X] | 0 |
| [X] | [X + h] | $\frac{1.22 \cdot 10^{-2} h^{-2}}{(0.02X^2 - 0.025X + 0.005U3 + 0.015U2 + 0.50U1 + 0.0243h^{-2})}$ |
| [X] | [X - h] | $\frac{1.22 \cdot 10^{-2} h^{-2}}{(0.02X^2 - 0.025X + 0.005U3 + 0.015U2 + 0.50U1 + 0.0243h^{-2})}$ |
| [0] | [0] | 0 |
| [0] | [h] | $\frac{2.43 \cdot 10^{-2} h^{-2}}{(0.005U3 + 0.015U2 + 0.50U1 + 2.43 \cdot 10^{-2} h^{-2})}$ |
| [1] | [1] | 0 |
| [1] | [1 - h] | $\frac{2.43 \cdot 10^{-2} h^{-2}}{(0.005U3 + 0.015U2 + 0.50U1 + 2.43 \cdot 10^{-2} h^{-2} - 0.005)}$ |

Indeed, if the following stability conditions are satisfied

$$h \leq \min_{X, U1, U2, U3} \left(\frac{0.16(1.3 - X)^2(X + 0.30)^2}{|0.08(1 - X)X^2 - 0.05(1 - X)X - (0.01U3 + 0.03U2 + U1)X + U3 - U2|} \right)$$

the matrix coefficients are positive and the sum of the coefficients on a same line is equal to 1. The matrix M is thus a transition matrix of a Markov chain. Moreover the optimal cost obeys

$$(\lambda k + 1)W^h = \min_{U1, U2, U3} (M(U1, U2, U3)W^h + kC(U1, U2, U3))$$

Thus we can use the contraction iteration

$$W_{n+1}^h = \frac{\min_{U1, U2, U3} (M(U1, U2, U3)W_n^h + kC(U1, U2, U3))}{(\lambda k + 1)}$$

4.3 Optimization of the Hamiltonian

The problem is to minimize the control-dependant part \mathcal{H}^h of the discretized hamiltonian with respect to $U = [U1, U2, U3]$ by a projected gradient method.

$$U_{n+1} = \mathcal{P}_{[0.1,10] \otimes [0,1] \otimes [0,1]}(U_n - \rho \frac{d\mathcal{H}^h(U_n)}{dU_n})$$

that is

$$\left(\begin{array}{l} U1_{n+1} = \mathcal{P}_{[0.10,10]}(U1_n - \rho_1(-0.50W + \partial_1^h W X - 1.0U1_n^{-0.50})) \\ U2_{n+1} = \mathcal{P}_{[0,1]}(U2_n - \rho_2(-1.50 \cdot 10^{-2} W + 3.00 \cdot 10^{-2} \partial_1^h W X + \partial_1^h W)) \\ U3_{n+1} = \mathcal{P}_{[0,1]}(U3_n - \rho_3(-5.00 \cdot 10^{-3} W + 1.00 \cdot 10^{-2} \partial_1^h W X - 1.0\partial_1^h W)) \end{array} \right)$$

This algorithm converges when the step ρ_1 satisfies:

$$0 < \rho_1 < 2kK^{-2}$$

with

$$k|W|^2 \leq D_U^2 \mathcal{H}^h(W)W \leq K|W|^2$$

4.4 Numerical results

We perform a numerical test with:

number of space discretization points : [70]

precision required for the implicit resolution : 0.01

step for the implicit resolution: 0.006

maximal number of iterations for the implicit resolution: 6000

maximal number of iterations for the optimization of the hamiltonian: 36

convergence tests:

precision required: 0.01

The solution W of equ. (2) and the optimal controls $U1, U2, U3$ are displayed on the following figures.

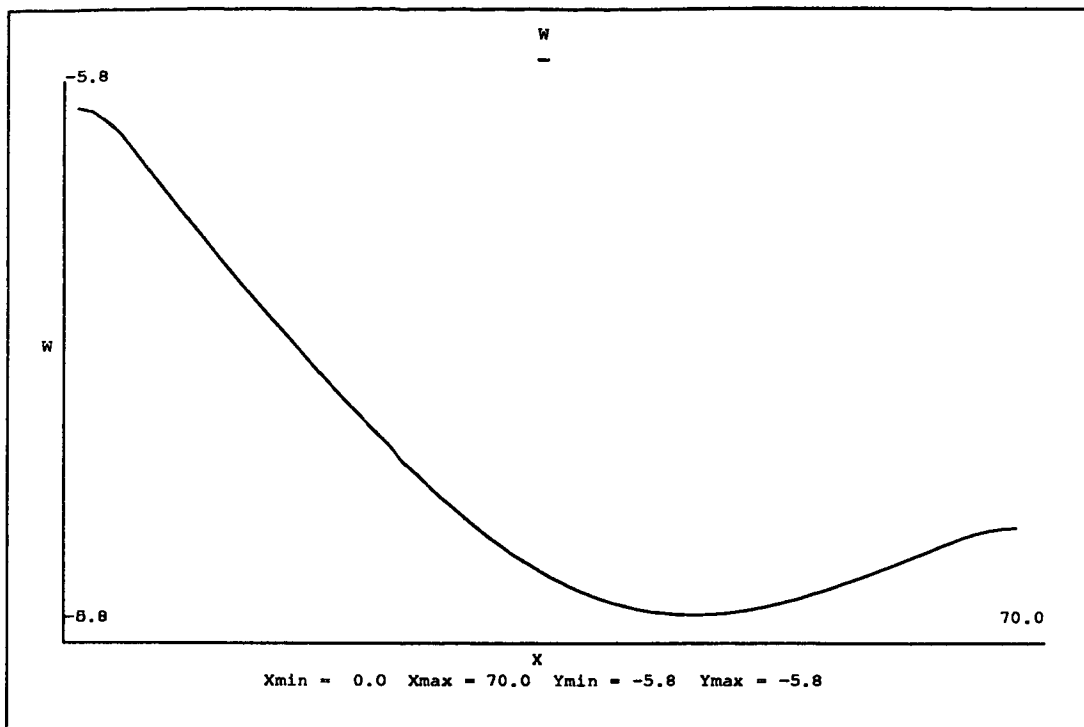


Figure 1: Negative of the maximum of the utility function: W

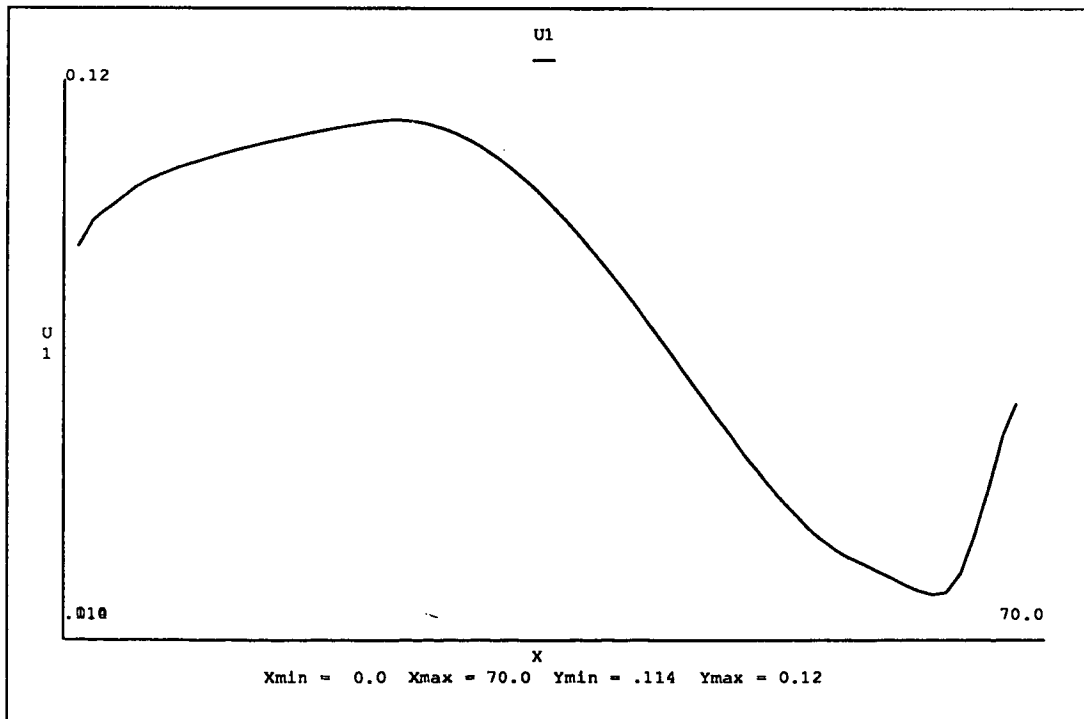


Figure 2: Optimal control U1 (consumption)

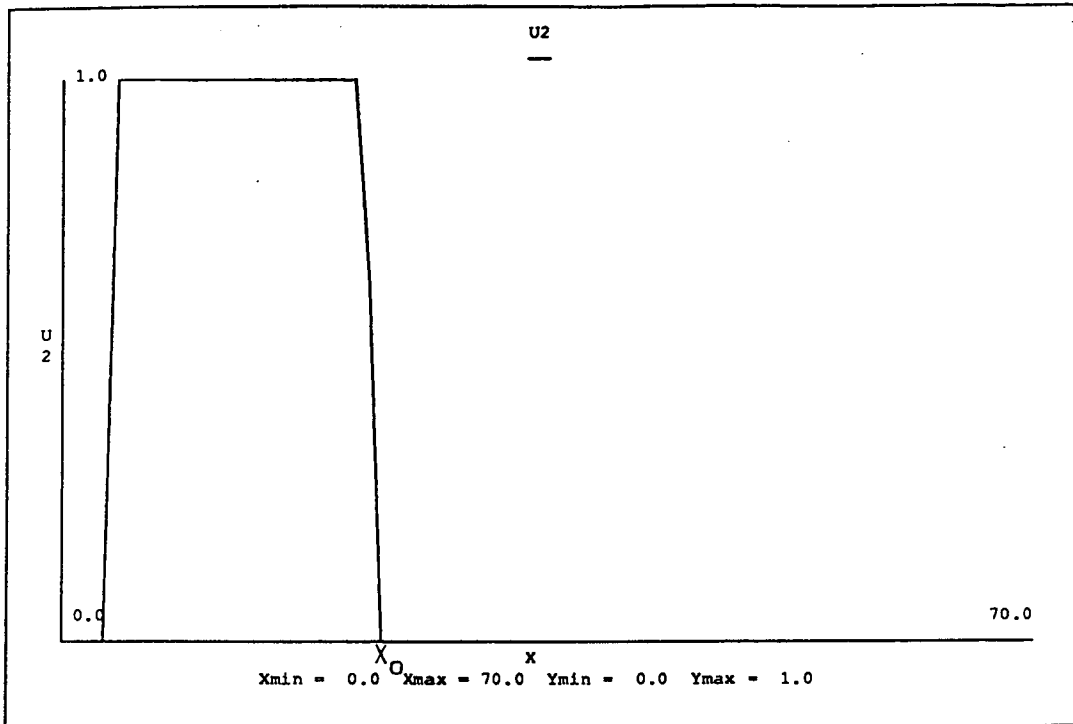


Figure 3: contrôle optimal bang-bang U2. Il est optimal d'acheter des actions si $X \leq X_0$

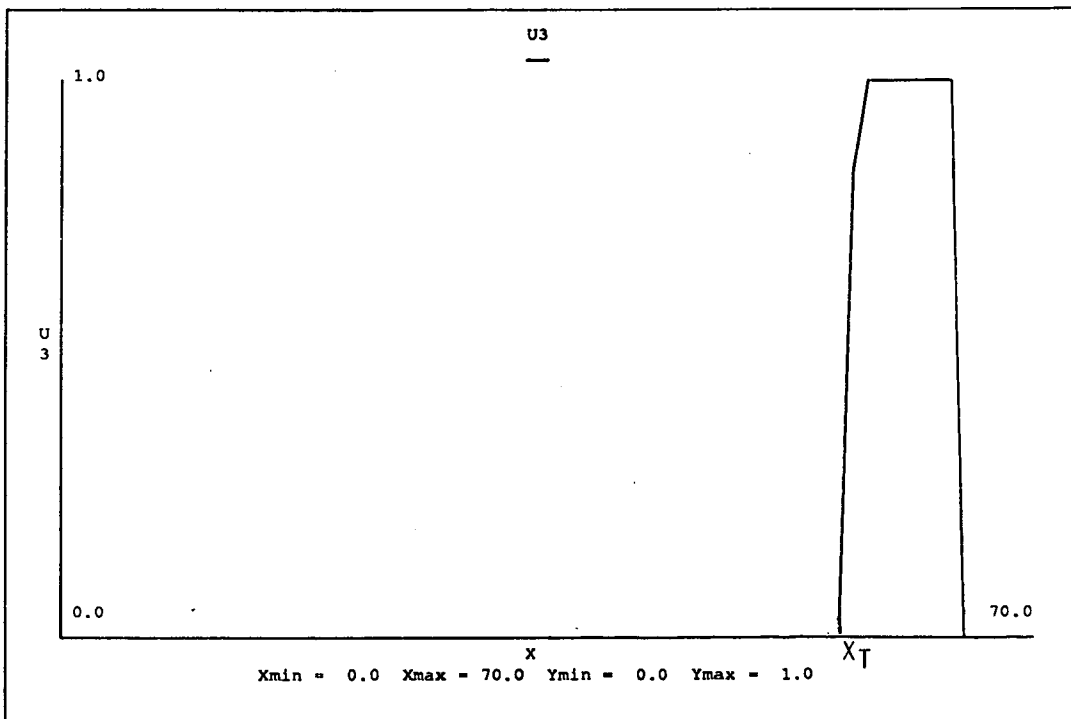


Figure 4: Contrôle optimal bang-bang U3. Il est optimal de vendre des actions si $X \geq X_T$ et de ne pas effectuer de transactions pour $X_0 \leq X \leq X_0$


```

C      contraintes sur le controle:
C          0 =< U3 =< 1
C          0 =< U2 =< 1
C          0.1 =< U1 =< 10
C      nmax designe le nombre maxi d iteration de la methode de
C                                          gradient avec projection
C      eps designe l erreur de convergence de la methode de
C                                          gradient avec projection
C      rog designe le pas, qui est constant, dans la methode de gradi#
C      ent
      H1=0.999999/(N1-3)
      U3=U(3,1)
      U2=U(2,1)
      U1=U(1,1)
      HIH1=H1**2
      H21=2*H1
      NM1=N1-1
      ROA=10000*ROG
      do 1016 I1=1,N1,1
      V(I1)=0.0
1016 CONTINUE
      IMITER=1
1012 CONTINUE
      ERIMP=0
      V(N1)=V(N1-2)
      V(1)=V(3)
1010 CONTINUE
      do 1009 I1=2,NM1,1
      X1=H1*(I1-2)
      Q1=(V(I1+1)-2*V(I1)+V(I1-1))/HIH1
      P1=(V(I1+1)-V(I1-1))/H21
      IF (I1 .EQ. 2) P1=(V(I1+1)-V(I1))/H1
      IF (I1 .EQ. NM1) P1=(V(I1)-V(I1-1))/H1
      W=V(I1)
      NITER=0
      WO=-1.0e20
1001 CONTINUE
      NITER=NITER+1
      if (NITER-NMAX) 1002,1002,1003
1003 CONTINUE
      WRITE(8,1801) I1
1801 FORMAT(' descente n a pas converge',i3)
      GOTO 1004
1002 CONTINUE
      UN1=-(P1*ROG*U1*X1-GAM*ROG*U1*W-ROG*U1**GAM-U1**2)/U1
      UN2=-LAM*P1*ROA*X1+GAM*LAM*ROA*W+U2-P1*ROA
      UN3=-MU*P1*ROA*X1+GAM*MU*ROA*W+U3+P1*ROA
      U1=UN1

```

```

U2=UN2
U3=UN3
U1=AMAX1(U1,0.1)
U1=AMIN1(U1,10)
U2=AMAX1(U2,0)
U2=AMIN1(U2,1)
U3=AMAX1(U3,0)
U3=AMIN1(U3,1)
WW=P1*((MU*U3+LAM*U2+U1)*X1-U3+U2)-GAM*(MU*U3+LAM*U2+U1)*W-U1**GAM
1 /GAM
ER=ABS(WW-WO)
if (ER-EPS) 1004,1004,1005
1005 CONTINUE
WO=WW
GOTO 1001
1004 CONTINUE
U(1,I1)=U1
U(2,I1)=U2
U(3,I1)=U3
WO=WW
W1=P1*((GAM-1)*SIG**2*(1-X1)*X1**2+(ALP-R)*(1-X1)*X1)+W*((GAM-1)*G
1 AM*SIG**2*X1**2/2.0+GAM*(ALP-R)*X1+GAM*R)+Q1*SIG**2*(-X1+EP1+1)
2 **2*(X1+EP1)**2/2.0
WO=W1+WO
WO=WO-ZDEL*V(I1)
VNEW=RO*WO+V(I1)
V(I1)=VNEW
ERIMP=ABS(WO)+ERIMP
1009 CONTINUE
WRITE(6,*) ERIMP
IMITER=IMITER+1
if (IMITER-IMPMAX) 1015,1014,1014
1015 CONTINUE
if (EPSIMP-ERIMP) 1012,1011,1011
1014 CONTINUE
WRITE(8,1807)
1807 FORMAT(' schema implicite n a pas converge')
1011 CONTINUE
RETURN
END

```

```

PROGRAM M_PPD
REAL V(70),U(3,70)
LISPFUNCTION FLFTNV 'CLIMAX::FORT-MACS' (REAL(70),CHARACTER*20,I
1 NTEGER(1))
INTEGER NFTNV(1)
CHARACTER*20 FTNVF
LISPFUNCTION FLFTNU 'CLIMAX::FORT-MACS' (REAL(3,70),CHARACTER*20

```

```

1      ,INTEGER(2))
      INTEGER NFTNU(2)
      CHARACTER*20 FTNUF
      NFTNV(1)=70
      FTNVF= 'FTNV'
      NFTNU(1)=70
      NFTNU(2)=3
      FTNUF= 'FTNU'
c-faire
      DO 1001 K1=1,70
          V(K1)=0.0
c-faire
      DO 1000 J=1,3
          U(J,K1)=0.1
1000      CONTINUE
c-fin-de-faire.
1001      CONTINUE
c-fin-de-faire.
      CALL PPD(70,V,0.01,6000,0.006,U,0.01,36,0
1      .05,0.4,0.3,0.12,0.07,0.5,0.03,0.01,0.1)
      CALL FLFTNV(V,FTNVF,NFTNV)
      CALL FLFTNU(U,FTNUF,NFTNU)
      STOP
      END

```

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