Error analysis of some Galerkin-least-squares methods for the elasticity equations

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Rapports de Recherche

N° 1054

Programme 7

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Juillet 1989

ERROR ANALYSIS OF SOME GALERKIN-LEAST-SQUARES METHODS FOR THE ELASTICITY EQUATIONS

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Abstract :

We consider the recent technique of stabilizing mixed finite element methods by augmenting the Galerkin formulation with least squares terms calculated separately on each element. The error analysis is performed in a unified manner yielding improved results for some methods introduced earlier. In addition, a new formulation is introduced and analyzed.

ANALYSE D'ERREUR DE CERTAINES METHODES DE GALERKIN-MOINDRES CARRES POUR LES EQUATIONS D'ELASTICITE

Résumé :

Nous considérons la technique récente des méthodes stabilisantes d'éléments finis mixtes obtenues en ajoutant à la formulation de Galerkin des termes de moindres carrés calculés séparément sur chacun des éléments. L'analyse d'erreur est réalisée d'une manière unifiée et conduit à des améliorations de résultats antérieurs. En outre, une nouvelle formulation est introduite et analysée.

(*) Professeur invité (Projet MODULEF) du 1 Juin au 30 Septembre 1989.

(**) Professeur invité (Projet MODULEF) du 1 Septembre 1987 au 1 Mars 1989.

1. Introduction

For problems in elasticity the term mixed finite element method is used within two different contexts.

The first of them is in connection with the plain strain or three-dimensional problems near the incompressible limit for which the Poisson ratio equals one half. It is well known that for these problems a standard lower-order displacement finite element method will break down. By now, it is also well known how a working method can be obtained; a new unknown, the "pressure", is introduced and discretized with a finite element space different from that used for the displacements. The finite element method obtained falls into the class of saddle-point problems for which there is a general theory developed by Brezzi and Babuška [4,6]. The application of this theory shows that the method converges optimally if the discrete spaces for the displacement and pressure satisfy a stability condition. For a particular choice of displacement and pressure approximations it can be a difficult task to verify this "Babuška-Brezzi" or "inf-sup" condition. During the last decade this problem has been intensively studied and the performances of many combinations are now known; cf. [17] and [21,24] for some recent extensions. In addition there exists a general technique, utilizing "bubble functions", by which a stable method can be designed [11,17].

The other main class of mixed methods consists of methods where the stresses are introduced as new unknowns in addition to the displacements. The main idea behind this is very appealing: one tries to directly approximate the physically interesting variable, the stress tensor. However, this approach is connected with serious difficulties. The resulting finite element method also has the saddle-point structure. In this case the main compatibility, or stability condition, is between the stress tensor and the displacement. It is trivial to design a stable method by simply choosing discontinuos approximations for the stresses, but this merely gives back a displacement method. Instead, one usually tries to impose some continuity requirements on the stress tensor and then the problem with the stability is far from trivial. Also for this problem the technique of bubble functions can be applied. In addition, if the method should also work in the incompressible limit, another condition (analog to the condition discussed above for the "displacement-pressure" formulation) between the displacement and the "pressure" part of the stress tensor, i.e. the trace of the stress tensor, should be satisfied. Also these problems have been studied in detail, and recently many new methods, converging also in the incompressible limit, have been introduced; cf. [1,2,22,23]. From a practical point of view the stabilization technique for mixed methods (displacement-pressure or stress-displacement) and the new families of [1,2,22,23] have one shortcoming: in general they do not utilize standard isoparametric shape functions and they are (claimed to be) difficult to implement in standard finite element software.

Recently some new techniques of deriving mixed approximations have received considerable attention [7,8,13,14,15,16,18,19]. From these works a simple and general technique has emerged. The idea consists of combining the traditional formulation with least-squares forms of the differential equations. The least-squares terms are usually evaluated separately on each element. This technique has already been applied to compressible and incompressible elasticity (or equivalently Stokes flow), beams, arches, plates and shells (see [15] and references therein). An advantage of this new approach is that the classes of finite element spaces that can be used are considerably enlarged. In particular, standard shape functions can be used, and hence the methods are easily incorporated into existing finite element codes. These new methods fall naturally into two categories. One of them has the familiar saddle-point structure and hence the convergence analysis can be performed using the Babuška-Brezzi theory. These methods have been baptized SBB methods (Satisfying Babuška-Brezzi). For the other class, CBB methods (for Circumventing BB), the stability has to be proven directly, without recourse to the Babuška-Brezzi condition necessary for the Galerkin method.

The purpose of this paper is to give a complete error analysis of methods of this type for compressible and incompressible elasticity. We analyze some methods introduced in [13,15,18] improving and simplifying earlier results. In addition a new alternative formulation is introduced.

2. Preliminaries

Let Ω be a bounded domain in $\mathbf{R}^N$, $N = 2,3$, with a polygonal or polyhedral boundary Γ. The problem to be approximated is: Find the displacement $\mathbf{u} = (u_1, \ldots, u_N)$ such that

$$\begin{aligned} 2\mu\{\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{u}) + \frac{\nu}{1-2\nu}\nabla\nabla\cdot\mathbf{u}\} + \mathbf{f} = \mathbf{0} \quad & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} \quad & \text{on } \Gamma_1, \\ 2\mu\{\boldsymbol{\varepsilon}(\mathbf{u}) + \frac{\nu}{1-2\nu}\nabla\cdot\mathbf{u}\,\mathbf{I}\}\cdot\mathbf{n} = \mathbf{g} \quad & \text{on } \Gamma_2, \end{aligned} \tag{2.1}$$

with $\overline{\Gamma}_1 \cup \overline{\Gamma}_2 = \Gamma$, $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\Gamma_1 \neq \emptyset$. As usual $\boldsymbol{\varepsilon}(\mathbf{u})$ denotes the strain tensor, $\mathbf{f}$ the body force and $\mathbf{I}$ the identity tensor. ν denotes Poisson's ratio and μ is the shear modulus given by

$$\mu = \frac{E}{2(1+\nu)},$$

where E is Young's modulus. We restrict Poisson's ratio to $0 \le \nu < 1/2$ where the upper limit corresponds to an incompressible material. The stress tensor is obtained from the solution of (2.1) through

$$\boldsymbol{\sigma} = 2\mu\{\boldsymbol{\varepsilon}(\mathbf{u}) + \frac{\nu}{1-2\nu}\nabla\cdot\mathbf{u}\,\mathbf{I}\}.$$

Assuming e.g. $\mathbf{f} \in L^2(\Omega)^N$ and $\mathbf{g} \in L^2(\Gamma_2)^N$ the problem has a unique solution $\mathbf{u} \in \mathbf{V}$,

$$\mathbf{V} = \{\,\mathbf{v} \in H^1(\Omega)^N \,|\, \mathbf{v}_{|\Gamma_2} = \mathbf{0}\,\}.$$

Furthermore, one can prove (cf. e.g. [3]) that the problem is well-posed with respect to the Poisson ratio: There exists a positive constant C not depending on ν such that

$$\|\mathbf{u}\|_1 + \|\boldsymbol{\sigma}\|_0 \le C(\|\mathbf{f}\|_0 + \|\mathbf{g}\|_{0,\Gamma_2}).$$

The problem is to design a good approximating method for which the rate of covergence is independent of ν. For standard lower-order displacement methods this is not achieved; for values of ν near $1/2$ the convergence is very slow.

All the methods analyzed in this paper will be shown to converge with a rate independent of ν.

The notation to be used is standard [9,17]. In particular we use $C, C_I, C_1, C_2, \ldots,$ for various positive constants which are all independent of ν, or equivalently ϵ, a positive parameter we introduce as

$$\epsilon = \frac{1-2\nu}{\nu}.$$

For simplicity the usual regular partitioning $\mathcal{C}_h$ of $\overline{\Omega}$ into elements is assumed to consist of triangles (tetrahedrons in $\mathbf{R}^3$) or convex quadrilaterals (hexahedrons). For the two-dimensional case, a mixing of triangles and quadrilaterals is not excluded. The quasiuniformity of $\mathcal{C}_h$ is not assumed. For convenience we adopt the following notation

$$R_m(K) = \begin{cases} P_m(K) & \text{if } K \text{ is a triangle or tetrahedron,} \\ Q_m(K) & \text{if } K \text{ is a quadrilateral or hexahedron.} \end{cases}$$

For deriving the optimal L^2-estimates we use the classical Aubin-Nitsche trick and for this we need the regularity assumption

$$\|\mathbf{u}\|_2 + \|\boldsymbol{\sigma}\|_1 \le C\|\mathbf{f}\|_0 \qquad (2.2)$$

for the solution of (2.1) with $\mathbf{g}=\mathbf{0}$. A well known situation when this is valid is a two-dimensional convex domain Ω with $\Gamma_2 = \emptyset$. For more general situations the assumption (2.2) is unfortunately not very realistic.

Without loss of generality we will analyze the methods assuming $2\mu = 1$.

REMARK. With regards to an implementation of the methods discussed herein we note that the assumption $2\mu = 1$ corresponds to a replacement of $\mathbf{f}$ and $\mathbf{g}$ by $\mathbf{f}/(2\mu)$ and $\mathbf{g}/(2\mu)$, respectively, in the variational formulations (3.4), (3.17), (4.3) and (5.2). ∎

3. A family based on the displacement-pressure formulation

In this formulation the "pressure"

$$p = -\frac{1}{\epsilon}\,\nabla\cdot\mathbf{u} \quad \left(= -\frac{\nu}{1-2\nu}\,\nabla\cdot\mathbf{u}\right)$$

is introduced as a new unknown. We remark that this gives the hydrostatic pressure only in the incompressible limit. With our assumption that $2\mu = 1$, the system obtained is

$$\begin{aligned} -\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{u}) + \nabla p &= \mathbf{f} && \text{in } \Omega,\\ \epsilon p + \nabla\cdot\mathbf{u} &= 0 && \text{in } \Omega,\\ \mathbf{u} &= 0 && \text{on } \Gamma_1,\\ \{\boldsymbol{\varepsilon}(\mathbf{u}) - p\,\mathbf{I}\}\cdot\mathbf{n} &= \mathbf{g} && \text{on } \Gamma_2. \end{aligned} \tag{3.1}$$

We approximate the displacement with a standard conforming approximation

$$\mathbf{V}_h = \{\mathbf{v}\in\mathbf{V}\,|\,\mathbf{v}_{|K}\in R_k(K)^N,\ K\in C_h\}, \tag{3.2}$$

with $k \geq 1$. The pressure is approximated either continuously or discontinuously:

$$P_h = \{p\in L^2(\Omega)\,|\,p_{|K}\in R_l(K),\ K\in C_h\} \tag{3.3a}$$

or

$$P_h = \{p\in C^0(\Omega)\,|\,p_{|K}\in R_l(K),\ K\in C_h\}, \tag{3.3b}$$

where $C^0(\Omega)$ denotes the space of continuous functions on Ω. Clearly, for (3.3a) we can choose any $l \geq 0$, whereas (3.3b) requires that $l \geq 1$.

The natural extension of the method by Hughes and Franca, introduced in connection with Stokes flow [18], is defined as follows: Find $\mathbf{u}_h \in \mathbf{V}_h$ and $p_h \in P_h$ such that

$$B(\mathbf{u}_h, p_h; \mathbf{v}, q) = F(\mathbf{v}, q), \qquad (\mathbf{v}, q)\in\mathbf{V}_h\times P_h, \tag{3.4}$$

with

$$\begin{aligned} B(\mathbf{u},p;\mathbf{v},q) =& (\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}(\mathbf{v})) - (\nabla\cdot\mathbf{v},p) - (\nabla\cdot\mathbf{u},q) - \epsilon(p,q)\\ &- \alpha\sum_{K\in C_h} h_K^2(-\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{u}) + \nabla p, -\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{v}) + \nabla q)_K - \beta\sum_{T\in\Gamma_h} h_T\langle[p],[q]\rangle_T \end{aligned}$$

and

$$F(\mathbf{v},q) = (\mathbf{f},\mathbf{v}) + \langle \mathbf{g},\mathbf{v}\rangle_{\Gamma_2} - \alpha \sum_{K\in C_h} h_K^2(\mathbf{f}, -\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{v}) + \nabla q)_K,$$

where Γ_h stands for the collection of element edges (faces in $\mathbf{R}^3$) in the interior of Ω, and $\langle\cdot,\cdot\rangle_T$ and $\langle\cdot,\cdot\rangle_{\Gamma_2}$ denotes the L^2-inner products on T and Γ_2, respectively. $([p])_{|T}$ denotes the jump in p along T.

In [18] it is shown that the method converges provided $\Gamma_2 = \emptyset$, $\beta > 0$ and $0 < \alpha < C_I$ where C_I is the constant in the inverse inequality

$$C_I \sum_{K\in C_h} h_K^2 \|\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{v})\|_{0,K}^2 \le \|\boldsymbol{\varepsilon}(\mathbf{v})\|_0^2, \qquad \mathbf{v}\in\mathbf{V}_h. \tag{3.5}$$

In the above inequality, a constant C_I independent of C_h follows from standard inverse estimates, since the elements of C_h are assumed to be regular; cf. [9].

In [18] an optimal error estimate was derived for the displacement in the energy norm. The estimate obtained for the pressure was, however, only in a mesh dependent H^1-type norm.

The purpose of this section is to extend the analysis of [18]. We treat the general case $\Gamma_2 \neq \emptyset$ and show that one can choose $\beta = 0$ except for the lowest order methods. We would like to point out that this is of considerable practical significance, since the method with $\beta > 0$ leads a nonstandard assembly process. For discontinuos pressure approximations and for $\beta = 0$ one can take advantage of the standard condensation procedure to eliminate pressure at the element level, resulting in a positive definite algebraic system for displacements only.

We also derive optimal L^2-estimates for the pressure and displacement.

Our results are summarized as follows.

THEOREM 3.1. *Suppose that the solution to (3.1) satisfies* $\mathbf{u}\in H^{k+1}(\Omega)^N$ *and* $p\in H^{l+1}(\Omega)$, *and that one of the following conditions is satisfied*

I. $k \ge N$,

II. $P_h \subset C^0(\Omega)$,

III $\beta > 0$.

Then for $0 < \alpha < C_I$ *(3.4) has a unique solution satisfying*

$$\|\mathbf{u}-\mathbf{u}_h\|_1 + \|p - p_h\|_0 \le C(h^k|\mathbf{u}|_{k+1} + h^{l+1}|p|_{l+1}).$$

If in addition the regularity estimate (2.2) holds, then we have

$$\|\mathbf{u}-\mathbf{u}_h\|_0 \le C(h^{k+1}|\mathbf{u}|_{k+1} + h^{l+2}|p|_{l+1}). \qquad \blacksquare$$

We recall that N is the space dimension and k is the degree of the piecewise polynomials used for displacement.

Before proving the theorem it will be convenient to introduce some new notation and prove intermediate results.

First, let us introduce the abbreviation $|\cdot|_h$ for the seminorm

$$|q|_h = (\alpha \sum_{K\in C_h} h_K^2 \|\nabla q\|_{0,K}^2 + \beta \sum_{T\in \Gamma_h} h_T \int_T \|[q]\|^2 ds)^{1/2}.$$

By standard scaling arguments we get the following estimates

$$|q|_h \leq C\|q\|_0, \qquad q \in P_h, \tag{3.6}$$

$$\inf_{q\in P_h} |p - q|_h \leq Ch^{l+1}|p|_{l+1}, \qquad p \in H^{l+1}(\Omega). \tag{3.7}$$

We note that the interpolant above can be taken as the L^2-projection. This will be done below.

Next, we note the boundness of the bilinear form B.

LEMMA 3.1. *There is a positive constant C such that for all $(\mathbf{u}, p) \in \mathbf{V}_h \times P_h$ and $(\mathbf{v}, q) \in \mathbf{V}_h \times P_h$ we have*

$$B(\mathbf{u}, p; \mathbf{v}, q) \leq C(\|\mathbf{u}\|_1^2 + (\epsilon + 1)\|p\|_0^2)^{1/2} \cdot (\|\mathbf{v}\|_1^2 + (\epsilon + 1)\|q\|_0^2)^{1/2}.$$

Proof: The Schwarz inequality gives

$$B(\mathbf{u}, p; \mathbf{v}, q) \leq C(\|\mathbf{u}\|_1^2 + \alpha \sum_{K\in C_h} h_K^2 \|\nabla \cdot \boldsymbol{\varepsilon}(\mathbf{u})\|_{0,K}^2 + (\epsilon + 1)\|p\|_0^2 + |p|_h^2)^{1/2}$$

$$\cdot (\|\mathbf{v}\|_1^2 + \alpha \sum_{K\in C_h} h_K^2 \|\nabla \cdot \boldsymbol{\varepsilon}(\mathbf{v})\|_{0,K}^2 + (\epsilon + 1)\|q\|_0^2 + |q|_h^2)^{1/2}.$$

The assertion now follows from (3.5) and (3.6). ∎

Hence, the following stability inequality is natural.

LEMMA 3.2. *Let either one of the assumptions I -III of Theorem 3.1 be valid. Then there is a positive constant C such that for $(\mathbf{u}, p) \in \mathbf{V}_h \times P_h$ we have*

$$\sup_{\substack{(\mathbf{v},q)\in \mathbf{V}_h\times P_h \\ (\mathbf{v},q)\neq(0,0)}} \frac{B(\mathbf{u}, p; \mathbf{v}, q)}{(\|\mathbf{v}\|_1^2 + (\epsilon + 1)\|q\|_0^2)^{1/2}} \geq C(\|\mathbf{u}\|_1^2 + (\epsilon + 1)\|p\|_0^2)^{1/2}. \qquad ∎$$

To prove Lemma 3.2 we will use the following result, the essential part of which was proven already in [24] (using an argument by Verfürth [25]). For completeness we give the full proof.

LEMMA 3.3. *Under any of the assumptions I-III of Theorem 3.1 there exist constants* C_1, C_2, *such that*

$$\sup_{0\neq \mathbf{v}\in \mathbf{V}_h} \frac{(\nabla\cdot\mathbf{v},p)}{\|\mathbf{v}\|_1} \geq C_1\|p\|_0 - C_2|p|_h, \qquad p\in P_h.$$

Proof: Consider first the case I. Denote by Π the L^2-projection onto the space of piecewise constants, i.e.

$$\Pi q_{|K} = \frac{1}{\text{area}(K)}\int_K q\,dx, \qquad K\in C_h, \qquad q\in P_h.$$

It is a classical result [11,17] that the pair $(\mathbf{V}_h, \Pi P_h)$ is stable, i.e. there is a constant C_1 such that for every $p\in P_h$ there is a $\mathbf{v}\in\mathbf{V}_h$, with $\|\mathbf{v}\|_1 = 1$, such that

$$(\nabla\cdot\mathbf{v}, \Pi p) \geq C_1\|\Pi p\|_0.$$

Hence, using the orthogonality of Πp and $(I-\Pi)p$, and the interpolation estimate

$$\|(I-\Pi)p\|_{0,K} \leq C_3 h_K\|\nabla p\|_{0,K}, \qquad K\in C_h,$$

we get

$$\begin{aligned}(\nabla\cdot\mathbf{v},p) &= (\nabla\cdot\mathbf{v},\Pi p) + (\nabla\cdot\mathbf{v},(I-\Pi)p)\\ &\geq C_1\|\Pi p\|_0 - \|\mathbf{v}\|_1\|(I-\Pi)p\|_0\\ &\geq C_1\|\Pi p\|_0 - \|(I-\Pi)p\|_0\\ &= C_1\|p\|_0 - (1+C_1)\|(I-\Pi)p\|_0\\ &\geq C_1\|p\|_0 - (1+C_1)C_3\Big(\sum_{K\in C_h} h_K^2\|\nabla p\|_{0,K}^2\Big)^{1/2}.\end{aligned}$$

Recalling the definition of the seminorm $|\cdot|_h$ we obtain the asserted estimate for case I.

Next, let us consider cases II and III.

We first split $p = \bar{p} + \tilde{p}$ with

$$\bar{p} = \frac{1}{\text{area}(\Omega)}\int_\Omega p\,dx.$$

Since $\tilde{p}\in L_0^2(\Omega)$, then there is a non-trivial $\mathbf{w}\in H_0^1(\Omega)^N$ such that

$$(\nabla\cdot\mathbf{w},p) = (\nabla\cdot\mathbf{w},\tilde{p}) \geq C_3\|\tilde{p}\|_0\|\mathbf{w}\|_1.$$

Further one can show (cf. [5], [10] and [17, pp.109-111]) that there is an interpolant $\tilde{\mathbf{w}}\in\mathbf{V}_h\cap H_0^1(\Omega)^N$ to $\mathbf{w}$ such that

$$\Big(\sum_{K\in C_h} h_K^{-2}\|\mathbf{w}-\tilde{\mathbf{w}}\|_{0,K}^2 + \sum_{T\in\Gamma_h} h_T^{-1}\int_T|\mathbf{w}-\tilde{\mathbf{w}}|^2\,ds\Big)^{1/2} \leq C_4\|\mathbf{w}\|_1$$

and

$$\|\tilde{\mathbf{w}}\|_1 \le C_5 \|\mathbf{w}\|_1 .$$

Integrating by parts on each $K \in C_h$, and using the above interpolation estimates we get

$$\begin{aligned}
(\nabla\cdot\tilde{\mathbf{w}}, p) &= (\nabla\cdot\tilde{\mathbf{w}}, \tilde{p}) \\
&= (\nabla\cdot(\tilde{\mathbf{w}}-\mathbf{w}), \tilde{p}) + (\nabla\cdot\mathbf{w}, \tilde{p}) \\
&\ge (\nabla\cdot(\tilde{\mathbf{w}}-\mathbf{w}), \tilde{p}) + C_3\|\mathbf{w}\|_1\|\tilde{p}\|_0 \\
&= \sum_{K\in C_h} (\mathbf{w}-\tilde{\mathbf{w}}, \nabla\tilde{p}) + \sum_{T\in\Gamma_h}\int_T ((\tilde{\mathbf{w}}-\mathbf{w})\cdot\mathbf{n})[\tilde{p}]\,ds + C_3\|\mathbf{w}\|_1\|\tilde{p}\|_0 \\
&\ge -(\sum_{K\in C_h} h_K^{-2}\|\mathbf{w}-\tilde{\mathbf{w}}\|_{0,K}^2 + \sum_{T\in\Gamma_h} h_T^{-1}\int_T |\mathbf{w}-\tilde{\mathbf{w}}|^2\,ds)^{1/2}\cdot \\
&\qquad (\sum_{K\in C_h} h_K^2\|\nabla\tilde{p}\|_{0,K}^2 + \sum_{T\in\Gamma_h} h_T\int_T \|[\tilde{p}]|^2 ds)^{1/2} + C_3\|\mathbf{w}\|_1\|\tilde{p}\|_0 \\
&\ge \left\{-C_4(\sum_{K\in C_h} h_K^2\|\nabla\tilde{p}\|_{0,K}^2 + \sum_{T\in\Gamma_h} h_T\int_T \|[\tilde{p}]|^2 ds)^{1/2} + C_3\|\tilde{p}\|_0\right\}\|\mathbf{w}\|_1 \\
&\ge \left\{-\frac{C_4}{C_5}(\sum_{K\in C_h} h_K^2\|\nabla\tilde{p}\|_{0,K}^2 + \sum_{T\in\Gamma_h} h_T\int_T \|[\tilde{p}]|^2 ds)^{1/2} + \frac{C_3}{C_5}\|\tilde{p}\|_0\right\}\|\tilde{\mathbf{w}}\|_1
\end{aligned}$$

Since either one of the assumptions II or III are valid, we have

$$(\sum_{K\in C_h} h_K^2\|\nabla\tilde{p}\|_{0,K}^2 + \sum_{T\in\Gamma_h} h_T\int_T \|[\tilde{p}]|^2 ds)^{1/2} \le C_6|\tilde{p}|_h = C_6|p|_h .$$

Combining the two estimates above, gives

$$\frac{(\nabla\cdot\tilde{\mathbf{w}}, p)}{\|\tilde{\mathbf{w}}\|_1} \ge C_7\|\tilde{p}\|_0 - C_8|p|_h ,$$

Finally, it is clear that (for any reasonable mesh) there is a $\mathbf{z} \in \mathbf{V}_h$ such that

$$\frac{(\nabla\cdot\mathbf{z}, \bar{p})}{\|\mathbf{z}\|_1} \ge C_9\|\bar{p}\|_0 .$$

Letting now $\mathbf{v} = \|\tilde{\mathbf{w}}\|_1^{-1}\tilde{\mathbf{w}} + \delta\|\mathbf{z}\|_1^{-1}\mathbf{z}$, $\delta > 0$, we get

$$\begin{aligned}
(\nabla\cdot\mathbf{v}, p) &= \frac{(\nabla\cdot\tilde{\mathbf{w}}, p)}{\|\tilde{\mathbf{w}}\|_1} + \delta\frac{(\nabla\cdot\mathbf{z}, p)}{\|\mathbf{z}\|_1} \\
&= \frac{(\nabla\cdot\tilde{\mathbf{w}}, p)}{\|\tilde{\mathbf{w}}\|_1} + \delta\frac{(\nabla\cdot\mathbf{z}, \bar{p})}{\|\mathbf{z}\|_1} + \delta\frac{(\nabla\cdot\mathbf{z}, \tilde{p})}{\|\mathbf{z}\|_1} \\
&\ge C_7\|\tilde{p}\|_0 - C_8|p|_h + \delta C_9\|\bar{p}\|_0 - \delta\|\tilde{p}\|_0 \\
&\ge C_{10}\|p\|_0 - C_{11}|p|_h
\end{aligned}$$

if $\delta < C_7$. Since $\|\mathbf{v}\|_1 \le 1 + \delta$ the assertion is thus proved. ■

We are now ready to prove the stability of the method.

Proof of LEMMA 3.2. First, we note that (3.5) and Korn's inequality yield

$$\begin{aligned} B(\mathbf{u},p;\mathbf{u},-p) &= \|\boldsymbol{\varepsilon}(\mathbf{u})\|_0^2 + \epsilon\|p\|_0^2 + |p|_h^2 - \alpha \sum_{K\in C_h} h_K^2 \|\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{u})\|_{0,K}^2 \\ &\ge (1-\alpha C_I^{-1})\|\boldsymbol{\varepsilon}(\mathbf{u})\|_0^2 + \epsilon\|p\|_0^2 + |p|_h^2 \\ &\ge C_3\|\mathbf{u}\|_1^2 + \epsilon\|p\|_0^2 + |p|_h^2, \end{aligned} \tag{3.8}$$

since we assume $0 < \alpha < C_I$.

Next, let $\mathbf{w} \in \mathbf{V}_h$ be a function for which the supremum of Lemma 3.3 is obtained and assume $\|\mathbf{w}\|_1 = \|p\|_0$. Then, using the bilinearity of B, Lemmas 3.1 and 3.3, and the arithmetic-geometric-mean inequality, we get

$$\begin{aligned} B(\mathbf{u},p;-\mathbf{w},0) &= B(\mathbf{u},0;-\mathbf{w},0) + B(0,p;-\mathbf{w},0) \\ &\ge -C_4\|\mathbf{u}\|_1\|\mathbf{w}\|_1 + (\nabla\cdot\mathbf{w},p) - \alpha\sum_{K\in C_h} h_K^2(\nabla p, \nabla\cdot\boldsymbol{\varepsilon}(\mathbf{w}))_K \\ &\ge -C_4\|\mathbf{u}\|_1\|\mathbf{w}\|_1 + (\nabla\cdot\mathbf{w},p) \\ &\quad -\alpha\Big(\sum_{K\in C_h} h_K^2\|\nabla p\|_{0,K}^2\Big)^{1/2}\Big(\sum_{K\in C_h} h_K^2\|\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{w})\|_{0,K}^2\Big)^{1/2} \\ &\ge -C_4\|\mathbf{u}\|_1\|\mathbf{w}\|_1 + C_1\|p\|_0^2 - C_2|p|_h\|p\|_0 - C_5|p|_h\|\mathbf{w}\|_1 \\ &= -C_4\|\mathbf{u}\|_1\|p\|_0 + C_1\|p\|_0^2 - (C_2+C_5)|p|_h\|p\|_0 \\ &\ge -\frac{C_4}{2\gamma_1}\|\mathbf{u}\|_1^2 + \Big(C_1 - \frac{C_4\gamma_1}{2} - \frac{(C_2+C_5)\gamma_2}{2}\Big)\|p\|_0^2 - \frac{(C_2+C_5)}{2\gamma_2}|p|_h^2 \\ &\ge -C_6\|\mathbf{u}\|_1^2 + C_7\|p\|_0^2 - C_8|p|_h^2 \end{aligned} \tag{3.9}$$

if the positive constants γ_1 and γ_2 are chosen small enough.

Denote $(\mathbf{v},q) = (\mathbf{u}-\delta\mathbf{w},-p)$. Combining (3.8) and (3.9) gives

$$\begin{aligned} B(\mathbf{u},p;\mathbf{v},q) &= B(\mathbf{u},p;\mathbf{u}-\delta\mathbf{w},-p) = B(\mathbf{u},p;\mathbf{u},-p) + \delta B(\mathbf{u},p;-\mathbf{w},0) \\ &\ge (C_3 - \delta C_6)\|\mathbf{u}\|_1^2 + (\epsilon + \delta C_7)\|p\|_0^2 + (1-\delta C_8)|p|_h^2 \\ &\ge C(\|\mathbf{u}\|_1^2 + (\epsilon+1)\|p\|_0^2) \end{aligned} \tag{3.10}$$

when choosing $0 < \delta < \min\{C_3C_6^{-1}, C_8^{-1}\}$.

On the other hand we have

$$\begin{aligned} \|\mathbf{v}\|_1^2 + (\epsilon+1)\|q\|_0^2 &\le 2\|\mathbf{u}\|_1^2 + 2\delta^2\|\mathbf{w}\|_0^2 + (\epsilon+1)\|p\|_0^2 = 2\|\mathbf{u}\|_1^2 + (1+\epsilon+2\delta^2)\|p\|_0^2 \\ &\le C(\|\mathbf{u}\|_1^2 + (\epsilon+1)\|p\|_0^2) \end{aligned}$$

which combined with (3.10) proves the stability estimate. ∎

The error estimates are now obtained in the usual manner.

Proof of THEOREM 3.1. Let $\tilde{\mathbf{u}} \in \mathbf{V}_h$ be the interpolant to $\mathbf{u}$ and let $\tilde{p} \in P_h$ be the L^2-projection of p. The stability now implies the existence of $(\mathbf{v}, q) \in \mathbf{V}_h \times P_h$ such that

$$\|\mathbf{v}\|_1^2 + (\epsilon + 1)\|q\|_0^2 \leq C \tag{3.11}$$

and

$$\|\tilde{\mathbf{u}} - \mathbf{u}_h\|_1 + (\epsilon + 1)^{1/2}\|\tilde{p} - p_h\|_0 \leq B(\mathbf{u}_h - \tilde{\mathbf{u}}, p_h - \tilde{p}; \mathbf{v}, q). \tag{3.12}$$

Since we assume $\mathbf{u} \in H^2(\Omega)^N$ and $p \in H^1(\Omega)$, the scheme is consistent, i.e.

$$B(\mathbf{u}_h - \tilde{\mathbf{u}}, p_h - \tilde{p}; \mathbf{v}, q) = B(\mathbf{u} - \tilde{\mathbf{u}}, p - \tilde{p}; \mathbf{v}, q). \tag{3.13}$$

Here we used the trace theorem to conclude that

$$\sum_{T \in \Gamma_h} \int_T |[p]|^2 ds = 0.$$

For the right hand side we now get

$$\begin{aligned} &B(\mathbf{u} - \tilde{\mathbf{u}}, p - \tilde{p}; \mathbf{v}, q) \\ &\leq C(\|\mathbf{u} - \tilde{\mathbf{u}}\|_1^2 + \sum_{K \in C_h} h_K^2 \|\nabla \cdot \boldsymbol{\varepsilon}(\mathbf{u} - \tilde{\mathbf{u}})\|_{0,K}^2 + \|p - \tilde{p}\|_0^2 + |p - \tilde{p}|_h^2)^{1/2} \\ &\quad \cdot (\|\mathbf{v}\|_1^2 + \sum_{K \in C_h} h_K^2 \|\nabla \cdot \boldsymbol{\varepsilon}(\mathbf{v})\|_{0,K}^2 + \|q\|_0^2 + |q|_h^2)^{1/2} - \epsilon(p - \tilde{p}, q). \end{aligned} \tag{3.14}$$

Since $\tilde{p}$ is defined as the L^2-projection of p, the last term above vanishes. Hence using (3.5),(3.6),(3.11)-(3.13) we get

$$\|\tilde{\mathbf{u}} - \mathbf{u}_h\|_1 + (\epsilon+1)^{1/2}\|\tilde{p} - p_h\|_0 \leq C(\|\mathbf{u} - \tilde{\mathbf{u}}\|_1 + (\sum_{K \in C_h} h_K^2 \|\nabla \cdot \boldsymbol{\varepsilon}(\mathbf{u} - \tilde{\mathbf{u}})\|_{0,K}^2)^{1/2} + \|p - \tilde{p}\|_0 + |p - \tilde{p}|_h).$$

The first estimate of Theorem 3.1. now follows from above using standard interpolation estimates and the triangle inequality.

The L^2-estimate for the displacement is obtained by modifying a well known argument. Assume that (2.2) holds so that the solution $(\mathbf{z}, q) \in \mathbf{V} \times L^2(\Omega)$ to

$$\begin{aligned} -\nabla \cdot \boldsymbol{\varepsilon}(\mathbf{z}) + \nabla q &= \mathbf{u} - \mathbf{u}_h \quad && \text{in } \Omega, \\ \epsilon q + \nabla \cdot \mathbf{z} &= 0 && \text{in } \Omega, \\ \mathbf{z} &= 0 && \text{on } \Gamma_1, \\ \{\boldsymbol{\varepsilon}(\mathbf{z}) - q\,\mathbf{I}\} \cdot \mathbf{n} &= 0 && \text{on } \Gamma_2. \end{aligned} \tag{3.15}$$

satisfies

$$\|\mathbf{z}\|_2 + (\epsilon + 1)\|q\|_1 \le C\|\mathbf{u} - \mathbf{u}_h\|_0. \tag{3.16}$$

Using (3.1), (3.4) and (3.15) we get

$$\|\mathbf{u}-\mathbf{u}_h\|_0^2 = B(\mathbf{u}-\mathbf{u}_h, p-p_h; \mathbf{z}-\tilde{\mathbf{z}}, q-\tilde{q}) + \alpha \sum_{K\in C_h} h_K^2(-\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h)+\nabla(p-p_h), \mathbf{u}-\mathbf{u}_h)_K.$$

Here we used

$$\sum_{T\in\Gamma_h} \langle [p-p_h], [q]\rangle_T = 0,$$

which again is a consequence of the trace theorem. From above we now get

$$\begin{aligned}
&\|\mathbf{u}-\mathbf{u}_h\|_0^2 \\
&\le C(\|\mathbf{u}-\mathbf{u}_h\|_1 + (\sum_{K\in C_h} h_K^2\|\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h)\|_{0,K}^2)^{1/2} + \|p-p_h\|_0 + |p-p_h|_h) \\
&\cdot (\|\mathbf{z}-\tilde{\mathbf{z}}\|_1 + (\sum_{K\in C_h} h_K^2\|\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{z}-\tilde{\mathbf{z}})\|_{0,K}^2)^{1/2} + (\epsilon+1)\|q-\tilde{q}\|_0 + |q-\tilde{q}|_h + h\|\mathbf{u}-\mathbf{u}_h\|_0) \\
&\le C(\|\mathbf{u}-\mathbf{u}_h\|_1 + (\sum_{K\in C_h} h_K^2\|\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h)\|_{0,K}^2)^{1/2} + \|p-p_h\|_0 + |p-p_h|_h)\cdot \\
&\cdot h(\|\mathbf{z}\|_2 + (\epsilon+1)\|q\|_1 + \|\mathbf{u}-\mathbf{u}_h\|_0),
\end{aligned}$$

and the L^2-estimate now follows from (3.16) and the estimate already proven. ∎

Let us close this section by giving a

REMARK *on a method introduced by Douglas and Wang* [12].

The method is defined by

$$B(\mathbf{u}_h, p_h; \mathbf{v}, q) = F(\mathbf{v}, q), \qquad (\mathbf{v}, q) \in \mathbf{V}_h \times P_h, \tag{3.17}$$

with

$$\begin{aligned}
B(\mathbf{u}, p; \mathbf{v}, q) &= (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v})) - (\nabla\cdot\mathbf{v}, p) - (\nabla\cdot\mathbf{u}, q) - \epsilon(p, q) \\
&- \alpha \sum_{K\in C_h} h_K^2(-\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{u}) + \nabla p, \nabla\cdot\boldsymbol{\varepsilon}(v) + \nabla q)_K - \beta \sum_{T\in\Gamma_h} h_T\langle [p], [q]\rangle_T
\end{aligned}$$

and

$$F(\mathbf{v}, q) = (\mathbf{f}, \mathbf{v}) + \langle \mathbf{g}, \mathbf{v}\rangle_{\Gamma_2} - \alpha \sum_{K\in C_h} h_K^2(\mathbf{f}, \nabla\cdot\boldsymbol{\varepsilon}(v) + \nabla q)_K.$$

The finite element spaces $\mathbf{V}_h$ and P_h are defined as before. In [12] the convergence of this method was proved for any positive values of the parameters α and β. In particular, no upper bound on α, as in Theorem 3.1., was imposed. However, the bilinear form B for this method is no longer symmetric.

The analysis technique of [12] is different from that utilized in this paper. Here we would like to point out that our technique gives improved results:

THEOREM 3.2. *Suppose that the solution to (3.1) satisfies* $\mathbf{u} \in H^{k+1}(\Omega)^N$ *and* $p \in H^{l+1}(\Omega)$, *and that one of the following conditions are satisfied*

I. $k \geq N$,

II. $P_h \subset C^0(\Omega)$,

III. $\beta > 0$.

Then for $\alpha > 0$ *(3.17) has a unique solution satisfying*

$$\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0 \leq C(h^k|\mathbf{u}|_{k+1} + h^{l+1}|p|_{l+1}).$$

If in addition the regularity estimate (2.2) holds, then we have

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \leq C(h^{k+1}|\mathbf{u}|_{k+1} + h^{l+2}|p|_{l+1}).$$

Proof: Comparing with the proof of Theorem 3.1 we note that the only difference is to show the stability for any $\alpha > 0$.

To this end we let $\gamma > 1$ and use (3.5) to obtain the following estimate corresponding to (3.8) in the proof of Lemma 3.2

$$\begin{aligned}
B(\mathbf{u}, p; \mathbf{u}, -p) &= \|\boldsymbol{\varepsilon}(\mathbf{u})\|_0^2 + \epsilon\|p\|_0^2 + \alpha \sum_{K\in C_h} h_K^2 \|-\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{u}) + \nabla p\|_{0,K}^2 \\
&\quad + \beta \sum_{T\in\Gamma_h} \|[p]\|_{0,T}^2 \\
&= \|\boldsymbol{\varepsilon}(\mathbf{u})\|_0^2 + \epsilon\|p\|_0^2 + \alpha \sum_{K\in C_h} h_K^2 \|\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{u})\|_{0,K}^2 \\
&\quad + 2\alpha \sum_{K\in C_h} h_K^2(-\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{u}), \nabla p)_{0,K} + |p|_h^2 \\
&\geq \|\boldsymbol{\varepsilon}(\mathbf{u})\|_0^2 + \epsilon\|p\|_0^2 + \alpha(1-\gamma) \sum_{K\in C_h} h_K^2 \|\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{u})\|_{0,K}^2 \\
&\quad + (1 - \frac{1}{\gamma})|p|_h^2 \\
&\geq (1 + \frac{\alpha(1-\gamma)}{C_I})\|\boldsymbol{\varepsilon}(\mathbf{u})\|_0^2 + \epsilon\|p\|_0^2 + (1 - \frac{1}{\gamma})|p|_h^2 \\
&\geq C(\|\mathbf{u}\|_1^2 + \epsilon\|p\|_0^2 + |p|_h^2)
\end{aligned}$$

if we choose $1 < \gamma < (1 + C_I\alpha^{-1})$.

The estimate (3.9) stays unchanged, and hence the stability is proved. ■

It is worthwhile emphasizing that the analysis shows that the β-term can be dropped for higher order methods, which is a result of practical interest, since the inclusion would lead to a nonstandard assembly process and one would not be able to eliminate the pressure at the element level in the usual manner. ■

4. The displacement-stress formulation

We now consider methods where the displacement and stresses are approximated independently and write the problem in the following form

$$\begin{aligned} \boldsymbol{\sigma} - \frac{1}{N+\epsilon} \operatorname{tr} \boldsymbol{\sigma}\, \mathbf{I} - \boldsymbol{\varepsilon}(\mathbf{u}) = 0 & \quad \text{in } \Omega, \\ \nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = 0 & \quad \text{in } \Omega, \\ \mathbf{u} = 0 & \quad \text{on } \Gamma_1, \\ \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{g} & \quad \text{on } \Gamma_2, \end{aligned} \tag{4.1}$$

where

$$\operatorname{tr} \boldsymbol{\sigma} = \sum_{i=1}^{N} \sigma_{ii}.$$

The displacement we approximate with a finite element subspace $\mathbf{V}_h \subset \mathbf{V}$ satisfying (3.2). The stress $\boldsymbol{\sigma}$ above is in the space

$$\mathbf{W} = \{ \boldsymbol{\tau} \in L^2(\Omega)^{N \times N} \mid \tau_{ij} = \tau_{ji}, \quad i,j = 1, \ldots, N \}$$

and the approximating space is defined as

$$\mathbf{W}_h = \{ \boldsymbol{\tau} \in \mathbf{W} \mid \boldsymbol{\tau}_{|K} \in R_m(K)^{N \times N}, \ K \in \mathcal{C}_h \} \tag{4.2a}$$

or

$$\mathbf{W}_h = \{ \boldsymbol{\tau} \in C^0(\Omega)^{N \times N} \cap \mathbf{W} \mid \boldsymbol{\tau}_{|K} \in R_m(K)^{N \times N}, \ K \in \mathcal{C}_h \} \tag{4.2b}$$

with $m \geq 0$ for the first choice and $m \geq 1$ for the second.

We note that this includes both continuous and discontinuous approximations for the stresses.

The GLS method introduced by Franca and Hughes [13,15] is the following: Find $(\mathbf{u}_h, \boldsymbol{\sigma}_h) \in \mathbf{V}_h \times \mathbf{W}_h$ such that

$$B(\mathbf{u}_h, \boldsymbol{\sigma}_h; \mathbf{v}, \boldsymbol{\tau}) = F(\mathbf{v}, \boldsymbol{\tau}), \qquad (\mathbf{v}, \boldsymbol{\tau}) \in \mathbf{V}_h \times \mathbf{W}_h, \tag{4.3}$$

with

$$B(\mathbf{u},\boldsymbol{\sigma};\mathbf{v},\boldsymbol{\tau}) = (\boldsymbol{\sigma},\boldsymbol{\tau}) - \frac{1}{N+\epsilon}(\operatorname{tr}\boldsymbol{\sigma},\operatorname{tr}\boldsymbol{\tau}) - (\boldsymbol{\epsilon}(\mathbf{u}),\boldsymbol{\tau}) - (\boldsymbol{\sigma},\boldsymbol{\epsilon}(\mathbf{v}))$$

$$- \alpha\Big(\boldsymbol{\sigma} - \frac{1}{N+\epsilon}\operatorname{tr}\boldsymbol{\sigma}\,\mathbf{I} - \boldsymbol{\epsilon}(\mathbf{u}),\boldsymbol{\tau} - \frac{1}{N+\epsilon}\operatorname{tr}\boldsymbol{\tau}\,\mathbf{I} - \boldsymbol{\epsilon}(\mathbf{v})\Big) + \beta\sum_{K\in C_h} h_K^2(\nabla\cdot\boldsymbol{\sigma},\nabla\cdot\boldsymbol{\tau})_K$$

and

$$F(\mathbf{v},\boldsymbol{\tau}) = -(\mathbf{f},\mathbf{v}) - \beta\sum_{K\in C_h} h_K^2(\mathbf{f},\nabla\cdot\boldsymbol{\tau})_K.$$

In [13,15] this method was analyzed, and the convergence was shown in the compressible regime (i.e. for $\epsilon \geq C > 0$) for $\beta \geq 0$ and $0 < \alpha < 1$. Herein we establish the uniform convergence for all values of compressibility provided β is strictly positive:

THEOREM 4.1. *Suppose that the solution to (4.1) satisfies* $\mathbf{u} \in H^{k+1}(\Omega)^N$ *and* $\boldsymbol{\sigma} \in H^{m+1}(\Omega)^{N\times N}$, *and that one of the following conditions is satisfied*

I. $k \geq N$,

II. $\mathbf{W}_h \subset C^0(\Omega)^{N\times N}$.

Then for $0 < \alpha < 1$ *and* $\beta > 0$ *(4.3) has a unique solution satisfying*

$$\|\mathbf{u}-\mathbf{u}_h\|_1 + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0 \leq C(h^k|\mathbf{u}|_{k+1} + h^{m+1}|\boldsymbol{\sigma}|_{m+1}).$$

If in addition the regularity estimate (2.2) holds, then we have

$$\|\mathbf{u}-\mathbf{u}_h\|_0 \leq C(h^{k+1}|\mathbf{u}|_{k+1} + h^{m+2}|\boldsymbol{\sigma}|_{m+1}). \qquad \blacksquare$$

The error estimates are proven very similarly as in the preceeding section. The main concern is to show the uniform stability.

Again, we first note that the inverse inequality

$$\sum_{K\in C_h} h_K^2\|\nabla\cdot\boldsymbol{\tau}\|_{0,K}^2 \leq C_I\|\boldsymbol{\tau}\|_0^2, \qquad \boldsymbol{\tau}\in\mathbf{W}_h, \tag{4.4}$$

implies that the bilinear form is bounded in the subspace $\mathbf{V}_h \times \mathbf{W}_h$,

$$B(\mathbf{u},\boldsymbol{\sigma};\mathbf{v},\boldsymbol{\tau}) \leq C(\|\mathbf{u}\|_1^2 + \|\boldsymbol{\sigma}\|_0^2)^{1/2}\cdot(\|\mathbf{v}\|_1^2 + \|\boldsymbol{\tau}\|_0^2)^{1/2},$$

and we have to verify

LEMMA 4.1. *Let either one of the assumptions I or II of Theorem 4.1 be valid. Then there is a positive constant C such that*

$$\sup_{\substack{(\mathbf{v},\boldsymbol{\tau})\in\mathbf{V}_h\times\mathbf{W}_h\\(\mathbf{v},\boldsymbol{\tau})\neq(0,0)}} \frac{B(\mathbf{u},\boldsymbol{\sigma};\mathbf{v},\boldsymbol{\tau})}{(\|\mathbf{v}\|_1^2+\|\boldsymbol{\tau}\|_0^2)^{1/2}} \geq C(\|\mathbf{u}\|_1^2 + \|\boldsymbol{\sigma}\|_0^2)^{1/2}, \qquad (\mathbf{u},\boldsymbol{\sigma})\in\mathbf{V}_h\times\mathbf{W}_h$$

Proof: Let us decompose the stress $\boldsymbol{\sigma} \in \mathbf{W}_h$ in the deviatoric part and the hydrostatic pressure:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^d - p\,\mathbf{I}$$

with $\operatorname{tr} \boldsymbol{\sigma}^d = 0$ which gives

$$p = -\frac{1}{N} \operatorname{tr} \boldsymbol{\sigma}.$$

We note that this gives

$$\boldsymbol{\sigma} - \frac{1}{N+\epsilon} \operatorname{tr} \boldsymbol{\sigma}\,\mathbf{I} = \boldsymbol{\sigma}^d - \frac{\epsilon}{N+\epsilon}\, p\,\mathbf{I} \qquad \text{and} \qquad (\boldsymbol{\sigma}^d,\ \mathbf{I}) = 0.$$

Next, suppose that $0 < \alpha < 1$ and let $\gamma > 1$ be a positive parameter. We then get

$$\begin{aligned}
&B(\mathbf{u},\boldsymbol{\sigma}; -\mathbf{u},\boldsymbol{\sigma}) \\
&= \|\boldsymbol{\sigma}\|_0^2 - \frac{1}{N+\epsilon}\|\operatorname{tr}\boldsymbol{\sigma}\|_0^2 - \alpha\{\|\boldsymbol{\sigma} - \frac{1}{N+\epsilon}\operatorname{tr}\boldsymbol{\sigma}\,\mathbf{I}\|_0^2 - \|\boldsymbol{\varepsilon}(\mathbf{u})\|_0^2\} \\
&\quad + \beta \sum_{K\in\mathcal{C}_h} h_K^2 \|\nabla\cdot\boldsymbol{\sigma}\|_{0,K}^2 \\
&= \|\boldsymbol{\sigma}^d - p\,\mathbf{I}\|_0^2 - \frac{N^2}{N+\epsilon}\|p\|_0^2 - \alpha\{\|\boldsymbol{\sigma}^d - \frac{\epsilon}{N+\epsilon}\, p\,\mathbf{I}\|_0^2 - \|\boldsymbol{\varepsilon}(\mathbf{u})\|_0^2\} \\
&\quad + \beta \sum_{K\in\mathcal{C}_h} h_K^2 \|\nabla\cdot\boldsymbol{\sigma}^d - \nabla p\|_{0,K}^2 \\
&= (1-\alpha)\|\boldsymbol{\sigma}^d\|_0^2 + \frac{\epsilon N(N+(1-\alpha)\epsilon)}{(N+\epsilon)^2}\|p\|_0^2 + \alpha\|\boldsymbol{\varepsilon}(\mathbf{u})\|_0^2 + \beta \sum_{K\in\mathcal{C}_h} h_K^2\|\nabla\cdot\boldsymbol{\sigma}^d\|_{0,K}^2 \\
&\quad - 2\beta \sum_{K\in\mathcal{C}_h} h_K^2 (\nabla\cdot\boldsymbol{\sigma}^d, \nabla p)_K + \beta \sum_{K\in\mathcal{C}_h} h_K^2\|\nabla p\|_{0,K}^2 \\
&\geq (1-\alpha)\|\boldsymbol{\sigma}^d\|_0^2 + \alpha\|\boldsymbol{\varepsilon}(\mathbf{u})\|_0^2 + \beta(1-\gamma)\sum_{K\in\mathcal{C}_h} h_K^2\|\nabla\cdot\boldsymbol{\sigma}^d\|_{0,K}^2 + \beta(1-\frac{1}{\gamma})\sum_{K\in\mathcal{C}_h} h_K^2\|\nabla p\|_{0,K}^2 \\
&\geq (1-\alpha+\beta(1-\gamma)C_I)\|\boldsymbol{\sigma}^d\|_0^2 + \alpha\|\boldsymbol{\varepsilon}(\mathbf{u})\|_0^2 + \beta(1-\frac{1}{\gamma})\sum_{K\in\mathcal{C}_h} h_K^2\|\nabla p\|_{0,K}^2 \\
&\geq C_1(\|\boldsymbol{\sigma}^d\|_0^2 + \|\mathbf{u}\|_1^2 + \sum_{K\in\mathcal{C}_h} h_K^2\|\nabla p\|_{0,K}^2),
\end{aligned} \tag{4.5}$$

for $\beta > 0$, $0 < \alpha < 1$, when we choose $1 < \gamma < 1 + (1-\alpha)(C_I\beta)^{-1}$.

The rest of the proof follows that of Lemma 3.2: First, in analogy with Lemma 3.3 we can prove that

$$\sup_{0\neq \mathbf{v}\in\mathbf{V}_h} \frac{(\nabla\cdot\mathbf{v}, q)}{\|\mathbf{v}\|_1} \geq C_2\|q\|_0 - C_3(\sum_{K\in\mathcal{C}_h} h_K^2\|\nabla q\|_{0,K}^2))^{1/2} \qquad q = \operatorname{tr}\boldsymbol{\tau},\ \boldsymbol{\tau}\in\mathbf{W}_h\,.$$

Hence, there exists a $\mathbf{z} \in \mathbf{V}_h$, with $\|\mathbf{z}\|_1 \leq \|p\|_0$, such that

$$B(\mathbf{u},\boldsymbol{\sigma};\mathbf{z},0) \geq C_2\|p\|_0^2 - C_3(\|\mathbf{u}\|_1^2 + \|\boldsymbol{\sigma}^d\|_0^2 + \sum_{K\in\mathcal{C}_h} h_K^2\|\nabla p\|_0^2). \tag{4.6}$$

Now, taking $\mathbf{v} = -\mathbf{u} + \delta\mathbf{z}$, $\boldsymbol{\tau} = \boldsymbol{\sigma}$ shows that

$$\frac{B(\mathbf{u},\boldsymbol{\sigma};\mathbf{v},\boldsymbol{\tau})}{(\|\mathbf{v}\|_1^2 + \|\boldsymbol{\tau}\|_0^2)^{1/2}} \geq C(\|\mathbf{u}\|_1^2 + \|\boldsymbol{\sigma}\|_0^2)^{1/2},$$

if $\delta > 0$ is chosen small enough. ■

The proof of Theorem 4.1 now follows that of Theorem 3.1.

REMARK. We note that if the stresses are approximated discontinuously, then they can be eliminated at the element level. Furthermore, if sufficiently high order approximations are used for the stresses, then the α-term can be dropped. This gives the method proposed in [16] and analyzed in [15]. ■

5. A new formulation

In this section we consider a method based on the equations obtained upon choosing the augmented stress $\mathbf{T}$, the pressure p (which again coincides with the hydrostatic pressure only in the incompressible limit) and the displacement $\mathbf{u}$ as unknowns:

$$\begin{aligned} \mathbf{T} - \boldsymbol{\varepsilon}(\mathbf{u}) &= 0 && \text{in } \Omega, \\ \epsilon p + \nabla\cdot\mathbf{u} &= 0 && \text{in } \Omega, \\ \nabla\cdot\mathbf{T} - \nabla p + \mathbf{f} &= 0 && \text{in } \Omega, \\ \mathbf{u} &= 0 && \text{on } \Gamma_1, \\ \{\mathbf{T} - p\,\mathbf{I}\}\cdot\mathbf{n} &= \mathbf{g} && \text{on } \Gamma_2. \end{aligned} \tag{5.1}$$

We now use the finite element spaces $\mathbf{W}_h, P_h$ and $\mathbf{V}_h$, as defined in (4.2), (3.3) and (3.2), for approximating the augmented stress, pressure and displacement, respectively. The GLS–method for this formulation is now defined as: Find $(\mathbf{T}_h, p_h, \mathbf{u}_h) \in \mathbf{W}_h \times P_h \times \mathbf{V}_h$ such that

$$B(\mathbf{T}_h,p_h,\mathbf{u}_h;\mathbf{S},q,\mathbf{v}) = F(\mathbf{S},q,\mathbf{v}), \qquad (\mathbf{S},q,\mathbf{v}) \in \mathbf{W}_h \times P_h \times \mathbf{V}_h, \tag{5.2}$$

with

$$\begin{aligned} B(\mathbf{T},p,\mathbf{u};\mathbf{S},q,\mathbf{v}) = {} & (\mathbf{T},\mathbf{S}) - (\boldsymbol{\varepsilon}(\mathbf{u}),\mathbf{S}) - (\mathbf{T},\boldsymbol{\varepsilon}(\mathbf{v})) \\ & + (p,\nabla\cdot\mathbf{v}) + (\nabla\cdot\mathbf{u},q) + \epsilon(p,q) \\ & - \alpha(\mathbf{T} - \boldsymbol{\varepsilon}(\mathbf{u}),\mathbf{S} - \boldsymbol{\varepsilon}(\mathbf{v})) + \beta\sum_{K\in\mathcal{C}_h} h_K^2(\nabla\cdot\mathbf{T} - \nabla p,\nabla\cdot\mathbf{S} - \nabla q)_K \end{aligned}$$

and

$$F(\mathbf{S},q,\mathbf{v}) = -(\mathbf{f},\mathbf{v}) - \beta \sum_{K\in C_h} h_K^2(\mathbf{f},\nabla\cdot\mathbf{S} - \nabla q)_K.$$

For this method we get the following convergence result.

THEOREM 5.1. *Suppose that the solution to (5.1) satisfies* $\mathbf{T} \in H^{m+1}(\Omega)^{N\times N}, p \in H^{l+1}(\Omega)$ *and* $\mathbf{u} \in H^{k+1}(\Omega)^N$, *and that one of the following two conditions is valid*

I. $k \geq N$,

II. $P_h \subset C^0(\Omega)$.

Then for $0 < \alpha < 1$ *and* $\beta > 0$ *(5.2) has a unique solution satisfying*

$$\|\mathbf{T} - \mathbf{T}_h\|_0 + \|p - p_h\|_0 + \|\mathbf{u} - \mathbf{u}_h\|_1 \leq C(h^{m+1}|\mathbf{T}|_{m+1} + h^{l+1}|p|_{l+1} + h^k|\mathbf{u}|_{k+1}).$$

If additionally (2.2) holds, then we have

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \leq C(h^{m+2}|\mathbf{T}|_{m+1} + h^{l+2}|p|_{l+1} + h^{k+1}|\mathbf{u}|_{k+1}). \quad \blacksquare$$

Proof: The technique used for the proof is exactly the same as in the preceeding section. First, estimating as in (4.5), using the inverse inequality (4.4), we get

$$\begin{aligned} B(\mathbf{T},p,\mathbf{u};\mathbf{T},p,-\mathbf{u}) &= (1-\alpha)\|\mathbf{T}\|_0^2 + \epsilon\|p\|_0^2 + \alpha\|\varepsilon(\mathbf{u})\|_0^2 + \beta \sum_{K\in C_h} h_K^2\|\nabla\cdot\mathbf{T} - \nabla p\|_{0,K}^2 \\ &\geq C_1(\|\mathbf{T}\|_0^2 + \epsilon\|p\|_0^2 + \|\mathbf{u}\|_1^2 + \sum_{K\in C_h} h_K^2\|\nabla p\|_{0,K}^2) \end{aligned}$$

Next, again the arguments of Lemma 3.3 supply us with a $\mathbf{z} \in \mathbf{V}_h, \|\mathbf{z}\|_1 \leq \|p\|_0$ such that

$$B(\mathbf{T},p,\mathbf{u};0,0,\mathbf{z}) \geq C_2\|p\|_0^2 - C_3(\|\mathbf{T}\|_0^2 + \|\mathbf{u}\|_1^2 + \sum_{K\in C_h} h_K^2\|\nabla p\|_{0,K}^2).$$

Finally, choosing $(\mathbf{S},q,\mathbf{v}) = (\mathbf{T},p,-\mathbf{u}+\delta\mathbf{z})$, with δ sufficiently small, gives us the stability:

$$\frac{B(\mathbf{T},p,\mathbf{u};\mathbf{S},q,\mathbf{v})}{(\|\mathbf{S}\|_0^2 + \|q\|_0^2 + \|\mathbf{v}\|_1^2)^{1/2}} \geq C(\|\mathbf{T}\|_0^2 + \|p\|_0^2 + \|\mathbf{u}\|_1^2)^{1/2}.$$

The error estimation then follows as in the proof of Theorem 3.1. $\blacksquare$

We would like to point out that the method (5.2) gives us a wide variety of methods. In particular, it enables us to use continuous approximations for all variables involved. An example where this is desirable is some models of viscoelastic fluids [20]. In these models spatial derivatives of the augmented stress occur, and hence a continuous approximation of this variable is desirable. Let us also note that this leads to complicated elements for the traditional Galerkin method (i.e. (5.2) with $\alpha = 0$, and $\beta = 0$) since the stability conditions required are (this is easily seen by applying the Babuška-Brezzi theory, cf. [14] for a similar formulation)

$$\sup_{0 \neq \mathbf{T} \in \mathbf{W}_h} \frac{(\mathbf{T}, \boldsymbol{\epsilon}(\mathbf{u}))}{\|\mathbf{T}\|_0} \geq C\|u\|_1$$

and

$$\sup_{0 \neq \mathbf{u} \in \mathbf{V}_h} \frac{(\nabla \cdot \mathbf{u}, p)}{\|\mathbf{u}\|_1} \geq C\|p\|_0 .$$

We refer to [20] where some proposals for stable Galerkin methods are given.

As it is the case for the methods in Section 4, it is clear that the α-term in (5.2) can be dropped if the augmented stress is approximated discontinuously and with polynomials of sufficiently high order:

THEOREM 5.2. *Consider the method (5.2) with $\alpha = 0$ and $\mathbf{W}_h$ according to the choice (4.2a). Suppose that we have $m \geq k-1$ for triangular (tetrahedral) elements and $m \geq k$ for quadrilaterals (hexahedrons), and let one of the following two conditions be valid*

I. $k \geq N$,

II. $P_h \subset C^0(\Omega)$.

Then for $\beta > 0$ (5.2) has a unique solution satisfying

$$\|\mathbf{T} - \mathbf{T}_h\|_0 + \|p - p_h\|_0 + \|\mathbf{u} - \mathbf{u}_h\|_1 \leq C(h^{m+1}|\mathbf{T}|_{m+1} + h^{l+1}|p|_{l+1} + h^k|\mathbf{u}|_{k+1}).$$

If additionally (2.2) holds, then we have

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \leq C(h^{m+2}|\mathbf{T}|_{m+1} + h^{l+2}|p|_{l+1} + h^{k+1}|\mathbf{u}|_{k+1}). \qquad \blacksquare$$

Proof: First we estimate as before

$$\begin{aligned} B(\mathbf{T}, p, \mathbf{u}; \mathbf{T}, p, -\mathbf{u}) &= \|\mathbf{T}\|_0^2 + \epsilon\|p\|_0^2 + \beta \sum_{K \in C_h} h_K^2 \|\nabla \cdot \mathbf{T} - \nabla p\|_{0,K}^2 \\ &\geq C_1(\|\mathbf{T}\|_0^2 + \sum_{K \in C_h} h_K^2 \|\nabla p\|_{0,K}^2) \end{aligned}$$

Next, the arguments of Lemma 3.3 imply the existence of a $\mathbf{z} \in \mathbf{V}_h$, $\|\mathbf{z}\|_1 \leq \|p\|_0$, such that

$$B(\mathbf{T}, p, \mathbf{u}; 0, 0, \mathbf{z}) \geq C_2\|p\|_0^2 - C_3(\|\mathbf{T}\|_0^2 + \|\mathbf{u}\|_1^2 + \sum_{K \in C_h} h_K^2 \|\nabla p\|_{0,K}^2) .$$

Finally with our assumption on $\mathbf{W}_h$ we can find $\mathbf{Q} \in \mathbf{W}_h$ such that

$$(\mathbf{Q}, \boldsymbol{\varepsilon}(\mathbf{u})) = \|\boldsymbol{\varepsilon}(\mathbf{u})\|_0^2$$
$$\|\mathbf{Q}\|_0 \le C_4 \|\boldsymbol{\varepsilon}(\mathbf{u})\|_0$$

and the arithmetic-geometric mean and inverse inequalities then give

$$\begin{aligned} B(\mathbf{T}, p, \mathbf{u}; -\mathbf{Q}, 0, 0) &= -(\mathbf{T}, \mathbf{Q}) + \|\boldsymbol{\varepsilon}(\mathbf{u})\|_0^2 - \beta \sum_{K \in \mathcal{C}_h} h_K^2 (\nabla \cdot \mathbf{T} - \nabla p, \nabla \cdot \mathbf{Q})_K \\ &\ge C_5 \|\mathbf{u}\|_1^2 - C_6 \|\mathbf{T}\|_0^2 - C_7 \sum_{K \in \mathcal{C}_h} h_K^2 \|\nabla p\|_{0,K}^2 . \end{aligned}$$

The final stability estimate is now obtained by taking $(\mathbf{S}, q, \mathbf{v}) = (\mathbf{T} - \delta \mathbf{Q}, p, -\mathbf{u} + \delta \mathbf{z})$ with δ positive and small enough.

The rest of the error estimation is done as in the proof of Theorem 3.1. ■

REMARKS.

1. In this case the method has the familiar saddle-point structure (it is a SBB method using the terminology of [15]) and the proof could equivalently be organized following the familiar steps of Brezzi's theory [6].

2. We also note that for this method the augmented stresses can be eliminated locally by condensation. This leads to a method for the displacement and pressure which has exactly the same degrees of freedoms as the method of Section 3. The difference is, however, that now no upper limit has to be imposed on the stability parameter β whereas in general this has to be done for the parameter α in the method (3.4). ■

Acknowledgements

The authors wish to thank Professor Michel Bernadou for his kind invitation and for his warm hospitality at INRIA-Rocquencourt where the bulk of this work was developed. This work was concluded at LNCC-Rio de Janeiro where Rolf Stenberg was a visiting scholar for two months. During the course of this work Leopoldo P. Franca was partially supported by FAPERJ Proc. E-29/170.376/88 and by Ministère Français des Affaires Etrangères and Rolf Stenberg was partially supported by FAPERJ Proc. E-29/170.742/88, CAPES CEF 033.89 and CNPq Proc. 400491/89-3.

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Imprimé en France
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