Quasiconvex sets and size X curvature condition. Application to non linear inversion

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QUASICONVEX SETS AND SIZE × CURVATURE CONDITION,

APPLICATION TO NON LINEAR INVERSION.

ENSEMBLES QUASI CONVEXES ET CONDITION DE TAILLE × COURBURE, APPLICATION A L'INVERSION NON LINEAIRE.

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Summary

We define a family of sets of an Hilbert space ("Quasiconvex Sets") on which a generalization of the usual theory of projection on convex sets can be defined (existence, uniqueness and stability of the projection of all points of some neighborhood of the set). We give then a constructive sufficient condition, called the size × curvature condition, for a set D to be quasiconvex, which involves radii of curvatures of curves lying on the set D. Finally, we use the above result for the study of non-linear least square problems, as they appear in parameter estimation, for which we give sufficient condition ensuring existence, uniqueness and stability.

Résumé

Nous définissons une famille de parties d'un espace de Hilbert (les "ensembles quasiconvexes") auxquels on peut, généraliser la théorie usuelle de la projection sur des convexes (existence, unicité et stabilité de la projection pour tous les points d'un voisinage de l'ensemble). Nous donnons ensuite une condition suffisante constructive pour qu'un ensemble D soit quasiconvexe (la condition de taille × courbure), faisant intervenir les rayons de courbure de courbes situés sur D. Enfin, nous appliquons les résultats précédents à l'étude de problèmes de moindres carrés non linéaires, tels par exemple ceux rencontrés dans l'estimation de paramètres, pour lesquels nous donnons des conditions suffisantes assurant l'existence, l'unicité et la stabilité des solutions.

Keywords

Non-linear least squares, parameter estimation, identification, inverse problems, projection theory, approximation theory.

Mots Clefs

Moindres carrés non linéaires, estimation de paramètres, identification, problèmes inverses, théorie de la projection, théorie de l'approximation.

Content

1 - INTRODUCTION

The initial motivation for this work has been the estimation of distributed parameters in partial differential equations. This problem is most usually stated using an "output least square" formulation :

(1.1) find $\hat{x} \in C$ such that $J(\hat{x}) \leq J(x)$ $\forall x \in C$

with

(1.2) $J(x) = \|\varphi(x)-z\|^2_F$

where x represent one point of the function space E of the parameter to be estimated, C the set of admissible parameters of E, $\varphi(x)$ represents the value of the observation operator applied to the solution of some P.D.E. admitting x among its coefficients, and z is a given point of the data space F.

In this context, C is quite often closed convex and bounded, E a Banach space (sometimes also an Hilbert space), the data space F is an Hilbert space, and the φ mapping, beside being regular (at least C^2) has unfortunately no other interesting properties.

Solving (1.1) (1.2) is also the concern of the approximation theory, but in a somewhat different context : the objective being there to find the best approximation to a given element (usually a function) z of an Hilbert space F using a function $\varphi(x)$ depending on a finite number of parameters $x \in \mathbb{R}^n$. Hence the main difference is that the approximation theory solves the least square problem (1.1) on C = E, or sometimes on C open subset of E, whereas in the parameter estimation problem C is usually closed and bounded. A second difference is that the approximation theory is mainly concerned with finite dimensional parameter spaces E, whereas in parameter estimation one would need to encompass infinite dimensional parameter spaces E.

The two domains have evolved quite independantly : the approximation theory has developped mathematical tools for the study of the fundamental properties of (1.1), as existence and uniqueness of $\hat{x}$, continuity and derivability of the $z \rightarrow \varphi(\hat{x})$ solution, based mainly on a geometrical study of the variety $\varphi(\mathbb{R}^n)$ in term of its radii of curvature (see for example [1],[2]).

The parameter estimation theory has first tackled with the infinite dimensionality of the parameter x, especially concerning the more practical problem of finding efficient ways of calculating the derivative J'(x) without having to solve the state equation or its derivatives infinitely many times : optimal control theory and the adjoint state technique have become the standard now, and allow for efficient numerical implementation of optimization algorithms for the solution of (1.1) on computers. Tools have also been developped ([3]) which allow to check that the finite dimensional approximation of (1.1) (1.2) retain the fundamental properties of (1.1) (1.2) as existence, uniqueness, stability, provided of course that these latter hold. But there existed until now virtually no way of checking wether or not these fundamental properties hold, neither for the continuous probiem (1.1) (1.2) nor for its finite dimensional counterpart. The only available existence results were based on compactness (on C or through regularization), which could give no insight into the uniqueness problem ; stability results were obtained for local minima of (1.1) (1.2) only ([8],[9]).

The technique presented in this paper is the developpement of previous work by the author

([4],[5],[6],[7]). It gives sufficient conditions for existence, uniqueness and stability of global minimum for problem (1.1) (1.2), which take advantage of the boundedness of C, and require neither compactness nor finite dimensionality of E (corollary 4.19 or, more general but more technical, theorem 4.18). This approach is based on a geometrical study of φ(C), using the notion of radii of curvature of curves lying in φ(C).

The two basic ingredients for this are :

- A generalization, to a family of so-called quasiconvex sets, of the usual theory of projection on convex sets in Hilbert spaces.
- A constructive sufficient condition for set-quasiconvexity, called the size × curvature condition (paragraph 3.2).

Then the basic additional assumption which will ensure that (1.1) (1.2) is wellposed is that φ(C) satisfy the size × curvature condition. Of course, when φ is not injective this wellposedness is to be understood in a sense which is precised in paragraph 4.

The potential interest of the technique presented here for parameter estimation problem is that it may allow to check numerically on the discretized finite dimensional counterpart of (1.1) (1.2) for its wellposedness, and/or obtain non linear confidence regions ; this technique will also apply to the regularized problems, as we shall see in a forthcoming paper. Its practical limitation is that the required amount of computation becomes very large when the dimension of E increases. More over, though theory holds for infinite dimensional E, there is yet no such example - except for regularized problems - where it has been shown to hold.

Let us also mention that the tools developped in this paper yield a sufficient condition ("φ(C) satisfies the size × curvature condition") which, added to the hypothesis that φ is a local homeomorphism (i.e. φ'(C) continuously invertible) will imply that φ is a global homeomorphism from C on to φ(C). We refer for this point of view to [11], where a "path-lifting" condition is given to obtain a global homeomorphism over the whole of E.

2 - QUASICONVEX SETS AND PROJECTION THEORY

Let :

F = Hilbert space

(2.0.1) $D \subset F$

$z \in F$

be given. We consider in this paragraph the problem of projecting z on D :

(2.0.2) find $\hat{X} \in D$ such that $J(X) = \|X-z\|^2_F = \min$ over D

We are interested in studying the existence and uniqueness of global minimum $\hat{X}$, the location of possible local minima, and the continuity of the $z \rightarrow \hat{X}$ mapping. We want a theory which reduces, when D is convex, to the usual theory of projection on convex sets.

2.1. Equipping a set with pathes

We begin by equipping the set D on which we want to project with something which will play the role of the segments of a convex set, namely a collection P of pathes $P : [0,1] \to D$.

By similarity to the fact that any [X,Y] segment of F can be naturally parametrized by $P : \bar{v} \in [0,1] \to (1-\bar{v})X + \bar{v}Y$, which satisfy $\|P'(\bar{v})\|_F = \|Y-X\|_F$ = length of segment [X,Y], we shall require that :

$$(2.1.1) \qquad P \in P \Rightarrow \begin{cases} \bar{v} \to P(\bar{v}) \text{ is } C^2 \text{ from } [0,1] \text{ in } D \\ \|P'(\bar{v})\|_F = \text{non zero constant } \forall \bar{v} \in [0,1] \end{cases}$$

which allows us to define :

$$(2.1.2) \qquad \begin{cases} \delta(P) \overset{\Delta}{=} \|P'(\bar{v})\| > 0 = \text{length of path P} \\ \bar{v} \in [0,1] = \text{reduced arc length along the path P.} \end{cases}$$

Also by similarity to the properties of segments of a convex, we shall require that P contains enough pathes so that one can connect any two distinct points of D :

(2.1.3) for any X, Y ∈ D, X≠Y, there exists at least one path $P \in P$ connecting X to Y, i.e. satisfying $P(0) = X$, $P(1) = Y$

and that it is stable with respect to restriction :

(2.1.4) for any $P \in P$ and any $\bar{v}', \bar{v}'' \in [0,1]$, $\bar{v}' < \bar{v}''$, the path $\tilde{P} : \bar{v} \in [0,1] \to P(1-v)v'+vv'')$ belong to P .

As we shall be using collection of pathes P satisfying (2.1.1), (2.1.3) and (2.1.4) through out all this paper, we state the

<u>Definition 2.1.</u> We shall say that :

P a collection of pseudo-segments
if and only if
P satisfies (2.1.1), (2.1.3) and (2.1.4).

We discuss now shortly possible choices for P in a non necessarily convex set D :

A first natural idea, which is the direct generalization of one possible definition of the segments in a convex, is to define P as the collection of all minimum-length pathes connecting any point X of D to any other point Y of D (provided of course that such pathes satisfy (2.1.1) and (2.1.3). This choice, which is undoubtedly the most intrinsic one, would probably give the best estimations in the size × curvature condition of §3, as discussed in [7] ; when D is convex, it resumes to choose for P the collection of all segments of D, in which case the forthcoming projection theory exactly reduces to the usual convex theory. However, this choice is not necessarily the best, even when D is convex, for certain applications as the estimation of the best lipschitz constant of the projection operator ; moreover minimum length pathes an extremely difficult to compute when D is not convex, and others choices can be made.

For example, in the very usual case where D is the image, by the C^2-mapping φ, of a segments of C ; this approach will be developped in paragraph 4 for the solution of non linear least squares problems.

We give now some notations and properties concerning the pathes P of a collection of pseudo-segment P :

We define the velocity $v(\bar{v})$ and the accelleration $a(\bar{v})$ with respect to the reduced arc length $\bar{v}$ by :

(2.1.5) $\quad v(\bar{v}) = P'(\bar{v}) \quad , \quad a(\bar{v}) = P''(\bar{v})$

which of course satisfy :

(2.1.6) $\quad < v(\bar{v}), a(\bar{v}) > = 0 \quad \forall \bar{v} \in [0,1].$

We shall also use the radius of curvature of the path P at $P(\bar{v})$, given by :

(2.1.7) $$\rho(\bar{v}) = \frac{\|v(\bar{v})\|_F^2}{\|a(\bar{v})\|_F} = \frac{\delta(P)^2}{\|a(\bar{v})\|} \in IR \cup \{+\infty\}$$

Not surprisingly, as the pathes P of P play the role of the segments in a convex set, we shall need to measure the "distance in D" of two points X and Y of D by the length of the pathes connecting X to Y :

Definition 2.2. Let D be equipped with a collection of pseudosegments P . Thus, for any X, Y $\in$ D, we call "distance in D of X and Y" the quantity :

(2.1.8) $$\delta(X,Y) = \underset{\substack{P \in P \\ P:X \to Y}}{\mathrm{Sup}} \ \delta(P)$$

with the convention that :

(2.1.9) $\delta(X,Y) = 0$ if there is no path from X to Y.

(Due to hypothesis (2.1.3) the latter case may arise only if Y=X).

Notice of course that :

(2.1.10) $\quad \|X-Y\| \leq \delta(X,Y) \quad \forall X,Y \in D$

and that, under the sole hypothesis that P is a collection of pseudosegments, it may happen that some pathes "make a loop", i.e. that

(2.1.11) $\quad \delta(X,X) > 0$

for some X of D.

We conclude with a remark on the notations : we use $\bar{v}$ to denote the reduced arc length along the path P ; the reason for that is that in paragraphs 3 and 4 we shall use v to denote an other parametrization of the same path P : through out the paper the bar notation will refer to quantities relative to the parametrization by the relative arc length.

2.2. Quasiconvex sets

We recall first elementary results on quasiconvex functions :

Definition 2.3. A function $d : [0,1] \to \mathbb{R}$ is strictly quasi convex if and only if :

$$v_1,v_2 \in [0,1],\ v \in]\, v_1,v_2 [\ \Rightarrow d(v) < \operatorname{Max}\ \{d(v_1),d(v_2)\}$$

Proposition 2.4. A continuous quasiconvex function from [0,1] to $\mathbb{R}$ has :

- one and only one global minimum
- no other local minimum

We turn now to our definition of quasiconvex sets :

Definition 2.5. A set D equipped with a collection of pathes $\mathcal{P}$ (in short a set $(D,\mathcal{P})$) is said to be quasiconvex if and only if :

i) $\mathcal{P}$ is a collection of pseudo-segments,

ii) There exists a neighborhood $\mathcal{V}$ of D in F, and a continuous function $\varepsilon : \mathcal{V} \to (\mathbb{R}^+ - \{0\}) \cup \{+\infty\}$ such that :

$$(2.2.1)\quad \begin{cases} z \in \mathcal{V} \\ 0 < \eta < \varepsilon(z) \\ P \in \mathcal{P}(z,\eta) \end{cases} \Rightarrow\ k(z,P) = \operatorname*{Max}_{\bar{v} \in [0,1]} \frac{< z-P(\bar{v}),a(\bar{v}) >}{\delta(P)^2} \le k(z,\eta) < 1$$

where :

(2.2.2) $\mathcal{P}(z,\eta) = \{P \in \mathcal{P} \mid \|P(j)-z\|_F \le d(z,D) + \eta,\ j = 0,1\}$

$\bar{v}$ = reduced arc length along the path P.

We give first a geometrical interpretation of condition (2.2.1) ; define (cf.figure 2.1) :

(2.2.3) $\qquad d(\bar{v}) = \|P(\bar{v}) - z\|_F$

(2.2.4) $\qquad \alpha(\bar{v})$ = angle between $a(\bar{v})$ and $z-P(\bar{v})$.

Then, we obtain from (2.1.7) that :

$$(2.2.5)\qquad k(z,P) = \operatorname*{Max}_{\bar{v} \in [0,1]} \frac{d(\bar{v})}{\rho(\bar{v})} \cos \alpha(\bar{v})$$

and the condition $k(z,P) < 1$ thus means that the projection H of z on to the normal to the path P at $P(\bar{v})$ stays on the half-line originating at center of curvature C of the path and pointing towards the path. The condition $k(z,\eta) < 1$ means that the projection H of z stays away from center of curvature C uniformly for all pathes in D whose extremities are at a distance of z less than or equal to $d(z,D)+\eta$.

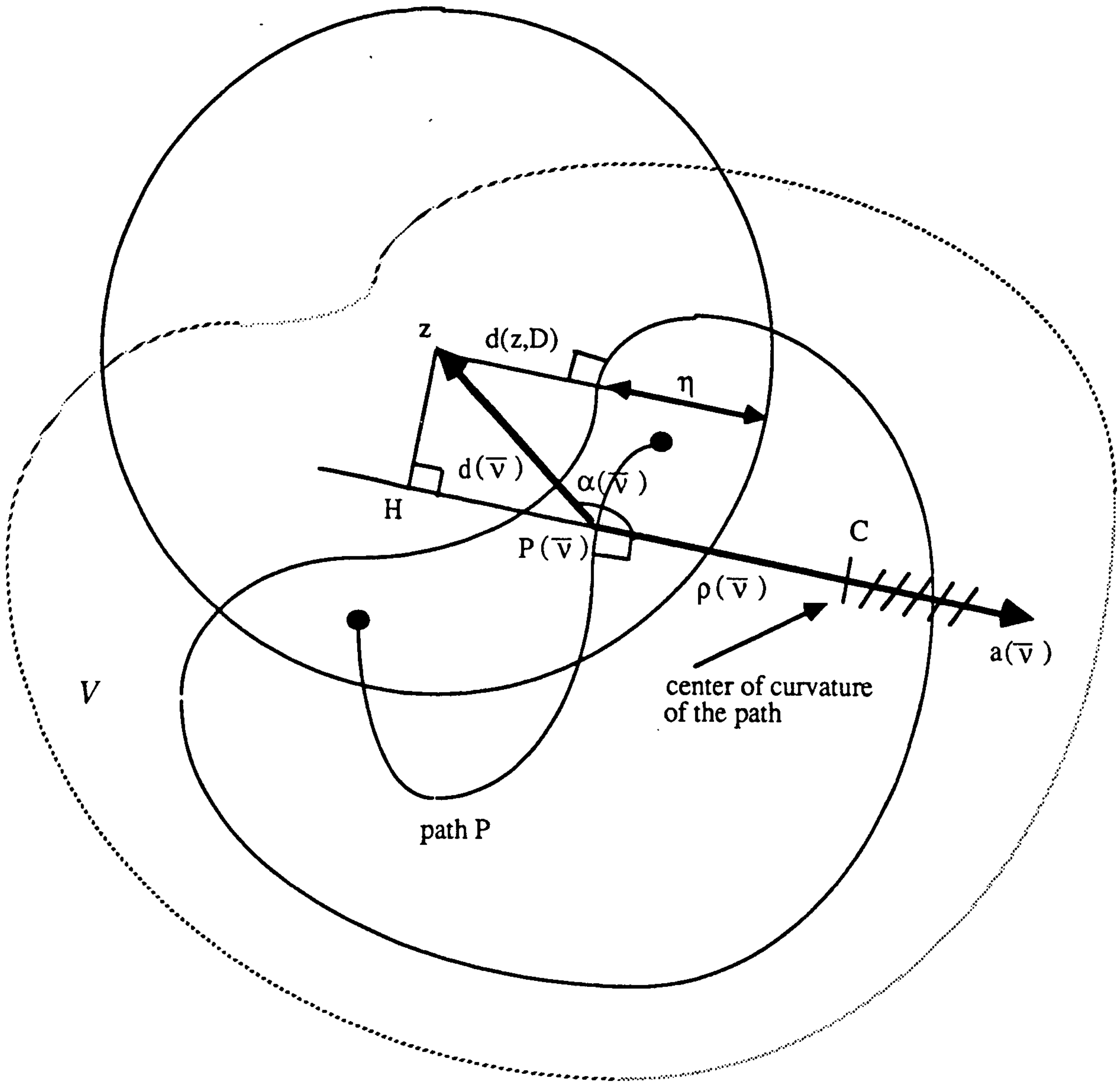


Figure 2.1. : Geometrical interpretation of the quasiconvexity condition for the set D.

It will be convenient to define :

(2.2.6) $$\mathcal{P}(z) = \bigcup_{0<\eta<\varepsilon(z)} \mathcal{P}(z,\eta)$$

$$= \left\{ P \in \mathcal{P} \mid \|P(j)-z\|_F < d(z,D)+\varepsilon(z),\ j=0,1 \right\}$$

so that :

(2.2.7) $$z \in V\ ,\ \ P \in \mathcal{P}(z) \Rightarrow k(z,P) < 1$$

We give now two reasons for calling "quasiconvex" a set satisfying (2.2.1) :

The first reason is its relations with quasiconvex functions :

Proposition 2.6

Let $(D, \mathcal{P})$ be quasiconvex.

Then, for any $z \in \mathcal{V}$ and $P \in \mathcal{P}(z)$,

i) $\bar{v} \to f(\bar{v}) = \| P(\bar{v}) - z \|^2_F$ is strictly convex

ii) $\bar{v} \to d(\bar{v}) = \| P(\bar{v}) - z \|_F$ is strictly quasiconvex.

Proof : An easy calculation yields :

$$(2.2.8) \qquad f''(\bar{v}) = 2\,\delta(P)^2 \left(1 - \frac{< z - P(\bar{v}), a(\bar{v}) >}{\delta(P)^2} \right)$$

so that :

$$(2.2.9) \qquad f''(\bar{v}) \geq 2\delta(P)^2\,(1 - k(z,P)) > 0$$

as P is taken in $\mathcal{P}(z)$ with $z \in \mathcal{V}$. Hence $\bar{v} \to f(\bar{v})$ is strictly convex, and its square root $\bar{v} \to d(\bar{v})$ is strictly quasiconvex. ∎

Remark 2.7

The strict convexity of $\bar{v} \to f(\bar{v})$ is not retained when the curvilinear abscissa $\bar{v}$ is replaced by some other parametrization v, whereas the strict quasiconvexity of $\bar{v} \to d(\bar{v})$ is. ∎

The second reason is that we shall be able in paragraph 2.3 to extend to quasiconvex sets the usual theory of projection on convex sets in Hilbert space ; in particular, a convex set is quasiconvex (take $\mathcal{P}$ = {segments of C with their natural parametrization}, $\mathcal{V} = F$, $\varepsilon(z) = +\infty$, for which (2.2.1) holds with $k(z,\eta) = 0 < 1$).

We conclude this paragraph with a simple property of pathes in a quasiconvex set :

Proposition 2.8

Let $(D, \mathcal{P})$ be quasiconvex.

Then the pathes P of $\mathcal{P}$ cannot "make a loop", i.e., for any $P \in \mathcal{P}$, the $\bar{v} \to P(\bar{v})$ mapping is injective, which is equivalent to say that, for any two points X and Y of D :

$$(2.2.10) \quad \delta(X,Y) = 0 \quad \Leftrightarrow \quad X = Y$$

Proof : We proove that $\bar{v} \to P(\bar{v})$ mapping is injective. If not, there would exist $\bar{v}', \bar{v}'' \in [0,1]$ such that :

$$\begin{cases} P(\bar{v}') = P(\bar{v}'') = \text{some } X \in D \\ \bar{v}' \neq \bar{v}'' \end{cases}$$

Introducing the path $\tilde{P} : \bar{v} \rightarrow P((1-\bar{v})v'+\bar{v}\ v'')$ we have :

. $\tilde{P} \in P$ using (2.1.4)

. $X \in D \subset V$

. $\| \tilde{P}(j) - X \|_F = 0 = d(X,D) < d(X,D) + \varepsilon(X) \Rightarrow \tilde{P} \in P\ (X)$

Hence the function $\bar{v} \rightarrow d(\bar{v}) = \| \tilde{P}(\bar{v}) - X \|_F$ is strictly quasiconvex as we can see from proposition 2.6, but this is contradictory to the fact that obviously $d \geq 0$ with $d(0) = d(1) = 0$. ∎

2.3. Projecting on quasiconvex sets

We give in this paragraph a step-by-step construction of the projection operator on a quasiconvex set (D,P). The projection operator will be defined on the neighborhood V of D. The construction essentially mimics the convex theory. We begin with :

Theorem 2.9 (Uniqueness of the projection)

If :

(2.3.1) (D,P) is quasiconvex

(2.3.2) $z \in V$

Then :

(2.3.4) The "distance to z" function has at most one global minimum over D. Moreover, all possible other local minima yield a value larger than or equal to $d(z,D) + \varepsilon(z)$.

Proof : Let $X_0, X_1 \in D$, $X_0 \neq X_1$, be two distinct local minima on D of the "distance to z" function, and $P \in P$ be a path from X_0 to X_1, which exists because of (2.1.3). Necessarily, the function $\bar{v} \rightarrow d(\bar{v}) = \| P(\bar{v}) - z \|_F$ has two distinct local minima at $\bar{v}=0$ and $\bar{v}=1$, and hence is not strictly quasiconvex. Supposing that $\| P(j)-z \|_F < d(z,D) + \varepsilon(z)$, $j=0,1$ would imply that $P \in P$ (z) and then $\bar{v} \rightarrow d(\bar{v})$ would be strictly quasiconvex by proposition 2.6, which is impossible .So necessarily :

$$\max_{j=0,1} \| P(j)-z \|_F \geq d(z,D) + \varepsilon(z)$$

or :

(2.3.5) $$\max_{j=0,1} \| X_j - z \|_F \geq d(z,D) + \varepsilon(z) > d(z,D)$$

This shows that at least one of the two local minima X_0 and X_1 has to be at a distance of z larger than $d(z,D) + \varepsilon(z)$. Hence there can be at most one global minimum(which by definition is at the distance $d(z,D) < d(z,D) + \varepsilon(z)$ of z !), and all other local minima are necessarily at a distance of z larger than or equal to $d(z,D) + \varepsilon(z)$. ∎

We turn now to :

Proposition 2.10 (Obtuous angle lemma)

Let D be equipped with a collection of pseudo segments P , $z \in F$ and $P \in P$ be given, and define $d(\bar{v}) = \| P(\bar{v}) - z \|_F$.

If :

(2.3.6) the $\bar{v} \to d(\bar{v})$ function has a local minimum at $\bar{v}=0$

Then :

(2.3.7) $d(0)^2 + (1-k)\delta(P)^2 \leq d(1)^2$

where :

(2.3.8) $k = k(z,P)$

(k non necessarily strictly smaller than one !).

Proof : The function :

$$f(\bar{v}) = d(\bar{v})^2 = \| P(\bar{v})-z \|^2_F$$

has also a local minimum at $\bar{v} = 0$, so that :

$$f'(0) \geq 0.$$

But, as is the proof of proposition (2.6), we have :

$$f''(\bar{v}) \geq 2\delta(P)^2(1-k(z,P))$$

(k(z,P) non necessarily strictly smaller than one !).Then the Taylor expansion of f :

$$f(1) = f(0) + f'(0) + 1/2\, f''(\bar{v}_0) \quad \text{for some } \bar{v}_0 \in [0,1]$$

yield the sought result. ∎

Remark 2.11 (geometrical interpretation of the theorem of the obtuous angle).

If $(D,\mathcal{P})$ is quasiconvex, if z is taken in $\mathcal{V}$ and P in $\mathcal{P}(z)$, then $k = k(z,P) < 1$. Then (2.3.7) is the analogous, for the curvilinear triangle (z,P(0),P(1)) (see figure 2.2), of the property that in a triangle, the sum of the squared length of edges adjacent to an obtuous angle is smaller than the squared length of the opposite edge. ∎

Proposition 2.11 (Continuity lemma)

Let D be equipped with a collection of pseudo segments $\mathcal{P}$, and $z_0, z_1 \in F$ be two points of F admitting projections X_0 and X_1 on D.

Then one has, for any path $P \in \mathcal{P}$ from X_0 to X_1 :

(2.3.9) $(1-k(P))\delta(P)) \leq \| z_0-z_1 \|_F$

where :

(2.3.10) $k(P) = (k(z_0,P) + k(z_1,P))/2$

(k(P) non necessarily strictly smaller than one !).

Proof : From (2.1.2) we know that :

$$\delta(P) > 0.$$

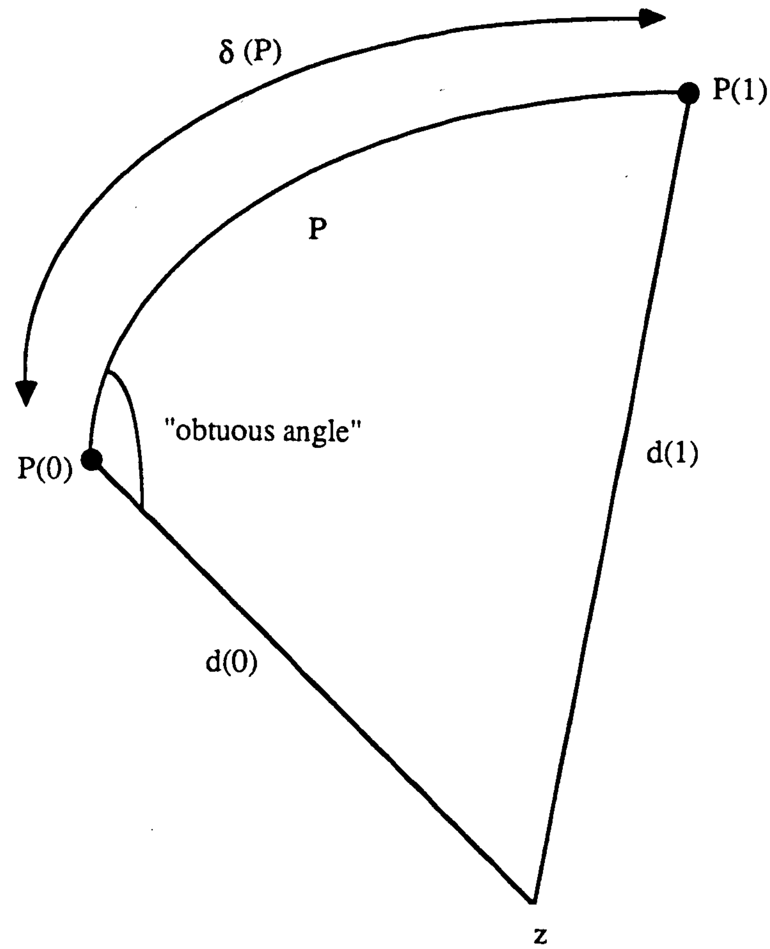


Figure 2.2. : Illustration of the obtuous angle theorem.

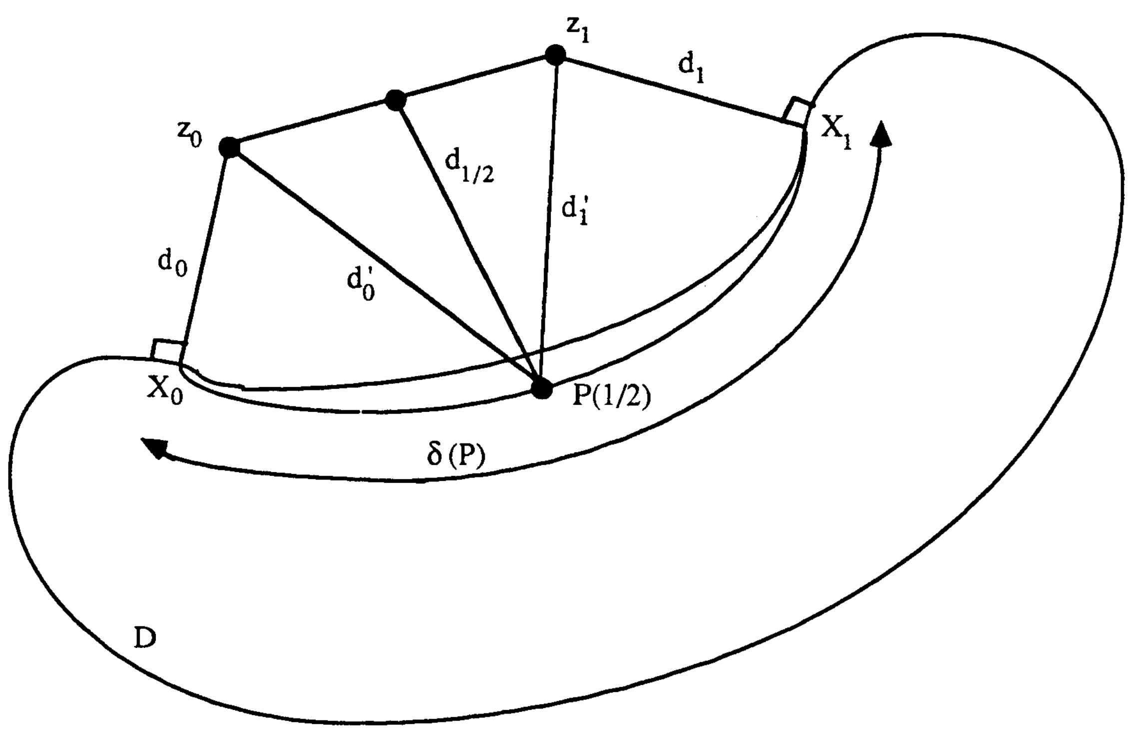


Figure 2.3. : Notations for the continuity lemma.

Defining a function $\bar{v} \to f(\bar{v})$ by :

(2.3.11) $\quad f(\bar{v}) = \| P(\bar{v}) - (1-\bar{v})z_0 - \bar{v} z_1\|^2_F \qquad \forall \bar{v} \in [0,1]$

we obtain :

$$f''(\bar{v}) = 2 \| v(\bar{v}) + z_0 - z_1 \|^2_F + 2 < P(\bar{v}) - (1-\bar{v})z_0 - \bar{v} z_1, a(\bar{v}) >$$

Developping then the square, using the Cauchy-Schwarz inequality and the definition of k(z,P) yields :

(2.3.12) $$f''(\bar{v}) \geq 2\delta(P)^2 \left\{ 1 - 2\frac{\|z_0-z_1\|_F}{\delta(P)} + \frac{\|z_0-z_1\|_F^2}{\delta(P)^2} - (1-\bar{v})k_0 - \bar{v}k_1 \right\}$$

where :

(2.3.13) $$k_j = k(z_j,P) = \operatorname*{Sup}_{\bar{v} \in [0,1]} \frac{< z_j - P(\bar{v}), a(\bar{v}) >}{\delta(P)^2} \qquad j = 0,1$$

From (2.3.12) we see that the function :

$$\bar{v} \to f(\bar{v}) + \bar{v}(1-\bar{v})\delta(P)^2 \left\{ 1 - \frac{2}{3}(k_0+k_1) + \frac{1}{3}((1-\bar{v})k_1+\bar{v}k_0) - 2\frac{\|z_0-z_1\|_F}{\delta(P)} + \frac{\|z_0-z_1\|_F^2}{\delta(P)^2} \right\}$$

is convex, and hence, with the notations of figure 2.3 and of (2.3.10) :

(2.3.14) $$d_{\frac{1}{2}}^2 + (1-k(P))\frac{\delta(P)^2}{4} - \frac{\delta(P)}{2}\|z_0-z_1\|_F + \frac{1}{4}\|z_0-z_1\|_F^2 \leq \frac{1}{2}d_0^2 + \frac{1}{2}d_1^2$$

Using now the obtuous angle lemma in the curvilinear triangles (z_0,X_0,P(1/2)) and (z_1,x_1,P(1/2)) yields, as the length along P from X_0 or X_1 to P(1/2) is δ(P)/2 :

(2.3.15) $$\begin{cases} d_0^2 + (1-k_0)\dfrac{\delta(P)^2}{4} \leq d'^2_0 \\ d_1^2 + (1-k_1)\dfrac{\delta(P)^2}{4} \leq d'^2_1 \end{cases}$$

Plugging these results in the RHS of (2.3.14) and rearranging the terms, we obtain :

(2.3.16) $$(1-k(P))\frac{\delta(P)^2}{2} - \frac{\delta(P)}{2}\|z_0-z_1\|_F \leq \frac{1}{2}d'^2_0 + \frac{1}{2}d'^2_1 - d_{\frac{1}{2}}^2 - \frac{1}{4}\|z_0-z_1\|_F^2$$

But F is an Hilbert space, and from the mediane theorem we see that the R.H.S. of (2.3.16) vanishes ! Dividing then (2.3.16) by $\delta(P)/2 > 0$ yields the sough result (2.3.9). ∎

We can now state the

Theorem 2.12 (Continuity of the projection)

If :

(2.3.17) (D,P) is quasiconvex

(2.3.18) z_0, $z_1 \in V$ admit projections X_0,X_1 on D

z_0,z_1 "are not too far one from the other", precisely if there exists $d \geq 0$ such that

(2.3.19)

$$\|z_0-z_1\|_F + \operatorname*{Max}_{j=0,1} d(z_j,D) < d < \operatorname*{Min}_{j=0,1} \left\{ d(z_j,D)+\varepsilon(z_j) \right\}$$

Then :

(2.3.20) $\|X_0-X_1\|_F \le \delta(X_0,X_1) \le (1-k)^{-1} \|z_0,z_1\|_F$

where :

(2.3.21) $k = (k_0+k_1)/2$

(2.3.22) $k_j = k(z_j,\eta_j) < 1 \qquad j = 0,1$

(2.3.23) $0 < \eta_j = d - d(z_j,D) < \varepsilon(z_j) \qquad j = 0,1$

which proves the continuity of the projection.

We remark first that hypothesis (2.3.19) will actually be satisfied as soon as z_0 and z_1 are close enough : if for example $z_1 \to z_0$, then :

$$\|z_0-z_1\|_F + \underset{j=0,1}{\mathrm{Max}}\, d(z_j,D) \to d(z_0,D)$$

$$\underset{j=0,1}{\mathrm{Min}} \left\{ d(z_j,D)+\varepsilon(z_j) \right\} \to d(z_0,D) + \varepsilon(z_0)$$

because of the continuity of the $z \to \varepsilon(z)$ function. As $\varepsilon(z_0) > 0$, the existence of some d satisfying (2.3.19) will be ensured for z_1 close enough to z_0.

Proof of theorem 2.12

We remark first that (2.3.23) follows immediately from hypothesis (2.3.19), and that (2.3.22) follows from (2.3.18), (2.3.23) and the quasiconvexity of D. Hence $k < 1$ and the coefficient in the RHS of (2.3.20) is strictly positive. We are left with the proof of (2.3.20), which we separate in two cases :

Case 1 : $X_0 = X_1$: Then, from proposition 2.8, we know that necessarily $\delta(X_0,X_1) = 0$, so that (2.3.20) holds trivially.

Case 2 : $X_0 \neq X_1$: Then we know from (2.1.3) that there exists $P \in \mathcal{P}$ going from X_0 to X_1, so that we can apply the continuity lemma (proposition 2.11), which yields :

(2.3.24) $\qquad (1-k(P))\delta(P) \le \|z_0-z_1\|_F$

with :

$$k(P) = (k(z_0,P) + k(z_1,P))/2$$

As X_0 and X_1 are the projections of z_0 and z_1, we have :

$$\|x_j-z_j\|_F = d(z_j,D) < d(z_j,D) + \eta_j$$

But from (2.3.19) and (2.3.23) we see that :

$$\begin{cases} \|X_k-z_j\|_F \le \|z_j-z_k\|_F + \|X_k-z_k\| < d = d(z_j,D) + \eta_j \\ j,k = 0,1 \quad , \quad j \neq k \end{cases}$$

which shows that :

$$P \in \mathcal{P}(z_j,\eta_j) \qquad j = 0,1$$

Hence, using the property (2.2.1) of quasiconvex sets, we find that :

$$k(z_j,P) \le k_j = k(z_j,\eta_j) < 1 \qquad j = 0,1$$

so that :

$$k(P) \le k = (k_1+k_2)/2 < 1$$

Plugging this inequality in the L.H.S. of (2.3.24) and dividing by $1-k > 0$ yields the sought formula (2.3.20).

We conclude the proof by checking that the majoration (2.3.20 thru 23) implies the continuity of the projection (provided of course it exists, which will be proved in theorem 2.15) : Let $z_0 \in V$ be given, and suppose that $z_1 \to z_0$. For z_1 close enough to z_0, hypothesis (2.3.18) ($z_1 \in V$) and (2.3.19) (see comments before the proof) will be satisfied, so that (2.3.20) holds, with the following upper bound for k :

$$k \leq (k_0+1)/2 < 1$$

which is independant from z_1 and hence proves the continuity of the projection. ∎

We give now the :

Proposition 2.13 (Continuity of the injection of (D, $\|\ \|_F$) in (D,δ)).

If :

(2.3.25) (D,P) is quasiconvex

Then :

(2.3.26) $$\begin{cases} X_n,\ X \in D \\ \|X_n - X\|_F \to 0 \end{cases} \Rightarrow \delta(X_n,X) \to 0$$

Proof : This proposition results from the application of proposition 2.12 to $z_j = X_j \in D$, j=0,1, in which case the projection of z_j on D obviously exists ! . ∎

Proposition 2.14 (Mediane Lemma)

Let D be equipped with a collection of pesudo segments P , $z \in F$ and $P \in P$ be given, and define $d(\bar{v}) = \| P(\bar{v})-z \|_F$.

Then the following inequality hold (see figure 2.4) :

(2.3.27) $$d(\tfrac{1}{2})^2 + (1-k)\frac{\delta(P)^2}{4} \leq \frac{1}{2}\, d(0)^2 + \frac{1}{2}\, d(1)^2$$

where :

(2.3.28) $k = k(z,P)$

(k non necessarily strictly smaller than 1 !).

Proof : As in propositions 2.6 and 2.10, we define $f(\bar{v}) = d(\bar{v})^2 = \| P(\bar{v})-z \|^2_F$, and find that :

$$f''(\bar{v}) \geq 2\delta(P)^2(1-k)$$

where k is defined by (2.3.28). Hence the function $\bar{v} \to f(\bar{v}) - \bar{v}(1-\bar{v})\delta(P)^2(1-k)$ is convex, which proves (2.3.27). ∎

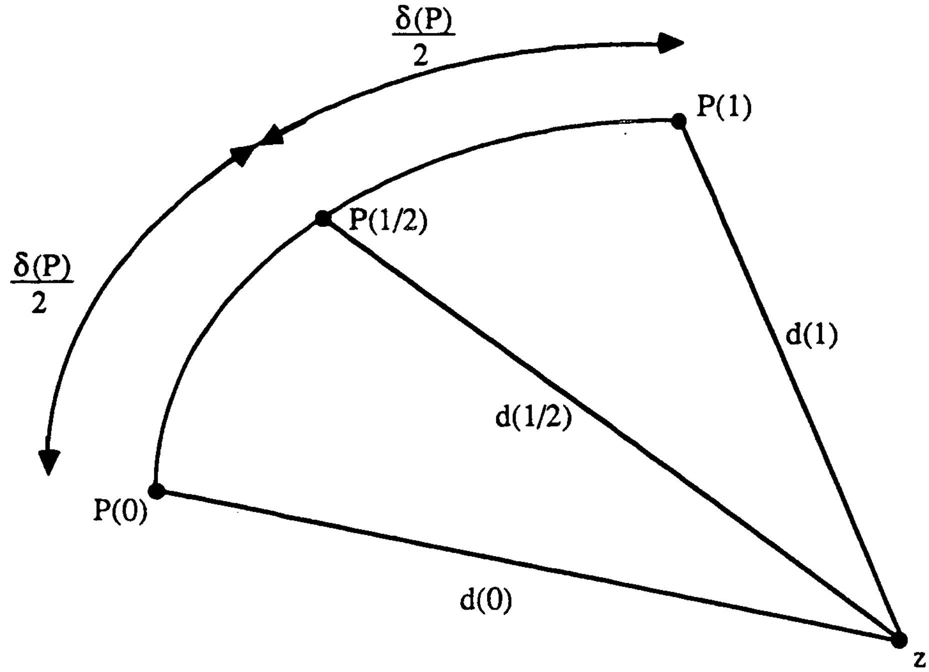


Figure 2.4. : Illustration of the mediane lemma.

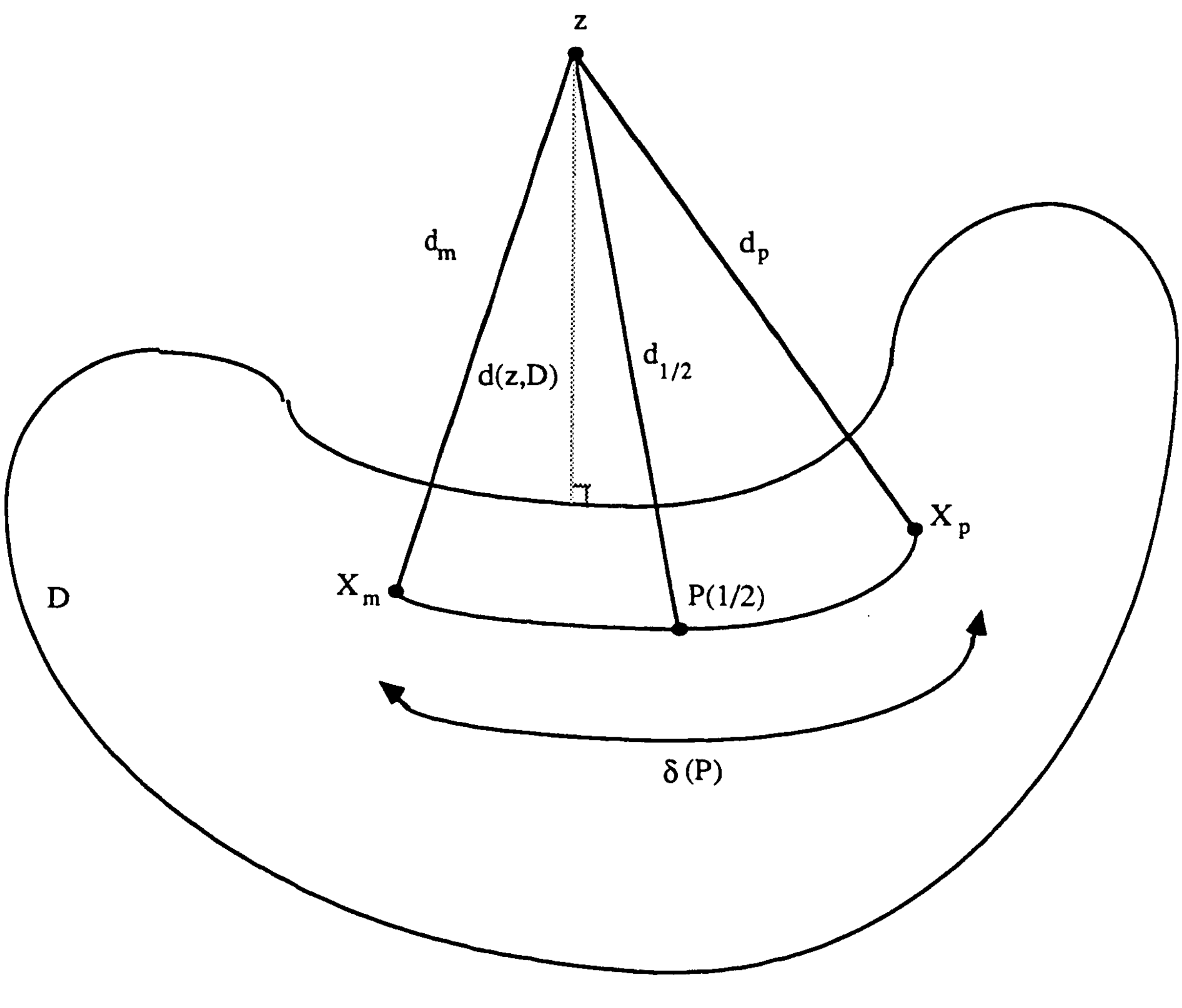


Figure 2.5. : Notations for the theorem of existence of the projection on D.

Theorem 2.15 (Existence of the projection)

If :

(2.3.29) $(D,\mathcal{P})$ is quasiconvex

(2.3.30) D is closed in F

Then :

(2.3.31) any $z \in V$ has a unique projection $\hat{X}$ on D

(2.3.32) any minimizing sequence is converging toward $\hat{X}$ for both the $\|X-Y\|_F$ and the $\delta(X,Y)$ distances.

Proof :

Let $z \in V$ be given, choose d such that :

(2.3.33) $\quad d(z,D) < d < d(z,D) + \varepsilon(z),$

and let $\{X_n \in D, n \in \mathbb{N}\}$ be a minimizing sequence of the "distance to z" function $J(X) = \|X-z\|^2_F$ defined in (2.0.2). Of course, there exists $N \in \mathbb{N}$ such that :

(2.3.34) $\quad d_m = \| X_m-z \|_F \le d , \qquad d_p = \| X_p-z \|_F \le d$

for m,p larger than N.

In order to proove that $\delta(X_m,X_p) \to 0$ when $m,p \to +\infty$, it is sufficient to consider only couple of indices m,p for which $\delta(X_m,X_p) > 0$!

So let $m,p \ge N$ be chosen such that $\delta(X_m,X_p) > 0$

This implies by proposition 2.8 that $X_m \ne X_p$, and hence that there exists a path $P \in \mathcal{P}$ going from X_m to X_p. From (2.3.33-34) we see that :

$$P \in \mathcal{P}(z,\eta)$$

with :

$$o < \eta = d - d(z,D) < \varepsilon(z)$$

so that :

(2.3.35) $\quad k(z,P) \le k(z,\eta) < 1$

Applying now the mediane lemma to the curvilinear triangle (z, X_m, X_p) yields, with the notations of figure 2.5 :

$$d_{\frac{1}{2}}^2 + (1-k)\frac{\delta(P)^2}{4} \le \frac{1}{2}\, d_m^2 + \frac{1}{2}\, d_p^2$$

where $k = k(z,P)$. Using (2.3.35) and the fact that $d_{1/2} > d(z,D)$ yields :

$$(1-k(z,\eta))\frac{\delta(P)^2}{4} \le \frac{1}{2}\, d_m^2 + \frac{1}{2}\, d_p^2 - d^2(z,D)$$

This inequality holds for any path P going from X_m to X_p. Hence :

$$(1-k(z,\eta))\frac{\delta(X_m,X_p)^2}{4} \le \frac{1}{2}\, d_m^2 + \frac{1}{2}\, d_p^2 - d^2(z,D)$$

with $k(z,\eta) < 1$ and independant of m and p. But when $m,p \to +\infty$ we have :

$$d^2_m \to d^2(z,D)$$

$$d^2_p \to d^2(z,D)$$

which shows that :

(2.3.36) $\quad \delta(X_m,X_p) \to 0$ when $m,p \to +\infty$

As $\| X_m, X_p \| \leq \delta(X_m, X_p)$, (2.3.36) implies that $\{X_n\}$ is a Cauchy Sequence, thus converging in F toward some $\hat{X} \in D$, as F is complete (Hilbert space) and D closed :

$$\| X_n - \hat{X} \|_F \to 0 \qquad \text{when } n \to \infty$$

But , as we have seen in proposition 2.13,this implies that:

$$\delta(X_n, X) \to 0 \qquad \text{when } n \to \infty$$

which ends the proof of theorem 2.15. ■

2.4. Obtaining a locally uniform lipschitz constant for the projection operator

We make in this paragraph the additional assumption that the radii of curvature along the pathes stay away from zero locally uniformly, i.e. :

$$(2.4.1) \qquad \begin{cases} \forall z \in V, \quad \forall \eta \text{ s.t. } 0 < \eta \leq \varepsilon(z), \ \forall P \in P(z,\eta) \\ \rho(\bar{v}) \geq \bar{R}(P) \geq \bar{R}(z,\eta) \qquad \forall \bar{v} \in [0,1] \end{cases}$$

This hypothesis will be satisfied in most of the cases where (D,P) will be quasiconvex, as prooving that k(z,P), which is given by (2.2.5), stays away from 1 will usually involve prooving that $\rho(\bar{v})$ stay away from zero.

Then the projection on D becomes locally lipschitz continuous on V :

Theorem 2.16

Let (D,P) be quasiconvex and satisfy (2.4.1).

Then, for any $z \in V$, there exists $\eta > 0$ and $k < 1$ such that :

(2.4.2) $\| X_0, X_1 \|_F \leq \delta(X_0, X_1) \leq (1-k)^{-1} \| z_0 - z_1 \|_F$

for all $z_0, z_1 \in B(z,\eta)$ admitting projections X_0, X_1 on D.

Given $z \in V$, η and k can be selected by the following procedure :

. choose $\eta > 0$ such that :

(2.4.3) $2\eta < \varepsilon(z)$

$$(2.4.4) \qquad k(z,2\eta) + \frac{\eta}{\bar{R}(z,2\eta)} < 1$$

. define $k < 1$ by :

$$(2.4.5) \qquad k = k(z,2\eta) + \frac{\eta}{\bar{R}(z,2\eta)}$$

Proof :

It is a simple corollary of the continuity lemma 2.11. If $X_0 = X_1$, then (2.4.2) holds trivially. If $X_0 \neq X_1$, we see from proposition 2.11 that (2.3.9), (2.3.10) hold for any path $P \in P$ going from X_0 to X_1. But we have now :

$$k(z_j,P) \leq \underset{\bar{v} \in [0,1]}{\text{Max}} \frac{< z-P(\bar{v}), a(\bar{v}) >}{\delta(P)^2} + \underset{\bar{v} \in [0,1]}{\text{Max}} \frac{< z_j - z, a(\bar{v}) >}{\delta(P)^2}$$

hence, using (2.4.1) and the fact that $z_j \in B(z,\eta)$:

$$(2.4.6) \qquad k(z_j,P) \leq k(z,P) + \frac{\eta}{\bar{R}(P)} \qquad \forall j = 0,1$$

But, as $z_j \in B(z,\eta)$:

(2.4.7) $\quad P \in \mathcal{P}(z,2\eta)$

so that (2.4.6) can be majorated by :

(2.4.8) $$k(z_j,P) \leq k(z,2\eta) + \frac{\eta}{\overline{R}(z,2\eta)}$$

i.e. using (2.4.4), (2.4.5) :

(2.4.9) $\quad k(z_j,P) \leq k < 1 \qquad \forall j = 0,1$

which, together with (2.3.10) and the fact that (2.3.9) holds for any path P from X_0 to X_1 completes the proof of 2.4.2. ∎

Remark 2.17

When (D,$\mathcal{P}$) is quasiconvex and satisfy (2.4.1), we may define at every point $z \in V$:

(2.4.10) $$k(z) = \inf_{0<\eta<\varepsilon(z)} k(z,\eta) \in [-\infty, 1[$$

Then we see from (2.4.2), (2.4.5) that :

(2.4.11) $\quad (1-k(z))^{-1}$ = lower bound to the lipschitz constant of the projection on D around z.

When z admits a projection X on D, k(z) can be quite easily computed by the formula :

(2.4.12) $$k(z) = d(z,D) \sup_{P \text{ goes thru } X} \frac{\cos\alpha}{\rho}$$

with obvious definition for α and ρ (see figure 2.6). ∎

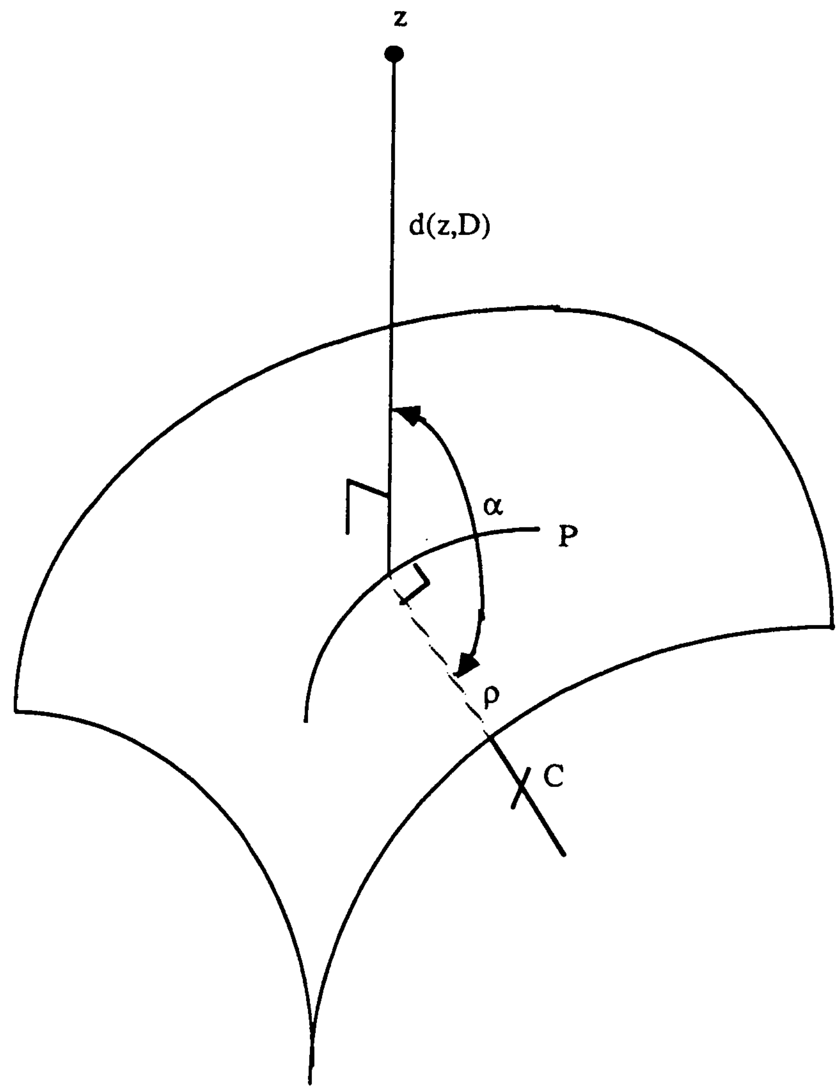


Figure 2.6. : Notation for k(z).

3. A SIZE × CURVATURE SUFFICIENT CONDITION

We want to develop in this paragraph a sufficient condition for the quasiconvexity of a set D of the Hilbert space F, with a cylindrical neighborhood V :

$$V = \{z \in F \mid d(z,D) < \gamma(D)\}$$

where the number $\gamma(D)$ will be computed by considering all pathes P of P. We shall consider in this section pathes P parametrized by some arbitrary parameter $\nu \in [0,1]$, as this will prove useful for the applications.

In order to avoid any confusion, we shall use the bar notation for all quantities related to the parametrization by the arc length ; for example :

P shall denote the $\nu \to P(\nu)$ mapping
$\overline{P}$ shall denote the $\overline{\nu} \to \overline{P}(\overline{\nu})$ mapping
v shall denote the velocity with respect to ν
$\overline{v}$ shall denote the velocity with repect to ν
ρ shall denote the $\nu \to \| v(\nu) \|^2 / \| a(\nu) \|$ mapping
$\overline{\rho}$ shall denote the $\overline{\nu} \to \| \overline{v}(\overline{\nu}) \|^2 / \| \overline{a}(\overline{\nu}) \|$ mapping
etc...

3.1. Not necessarily the reduced arc length.

We equip the set D with a collection *P* of pathes P : [0,1] → D which satisfy (compare with paragraph 2.1) :

(3.1.1) $P \in \mathcal{P} \Rightarrow \nu \to P(\nu)$ is C^2 from [0,1] in D

(3.1.2) for any X,Y ∈ D, X≠Y, there exists at least one path P ∈ *P* connecting X to Y, i.e. satisfying P(0) =X, P(1) = Y

(3.1.3) for any P ∈ *P* and any ν', ν" ∈ [0,1], ν' < ν", the path $\tilde{P} : \nu \in [0,1] \to P((1-\nu)\nu'+\nu\nu'')$ belongs to *P*

and :

$$(3.1.4) \quad \begin{cases} P \in \mathcal{P} \\ \nu_0 \in [0,1] \\ v(\nu_0) = 0 \end{cases} \Rightarrow \begin{cases} v(\nu) \neq 0 \text{ for } \nu \neq \nu_0, |\nu-\nu_0| \text{ small enough} \\ \text{and} \\ \dfrac{v(\nu)}{\|v(\nu)\|} \text{ and } \dfrac{a(\nu)}{\|v(\nu)\|^2} - \dfrac{v(\nu)}{\|v(\nu)\|} < \dfrac{v(\nu)}{\|v(\nu)\|}, \dfrac{a(\nu)}{\|v(\nu)\|^2} >_F \\ \text{have a limit when } \nu \to \nu_0 \end{cases}$$

where v(ν) and a(ν) denote the velocity and accelleration with respect to ν along the path P :

(3.1.5) $v(\nu) = P'(\nu), \qquad a(\nu) = P''(\nu)$

Note that (3.1.4) eliminates parametrizations which "stationate" too much, and in particular constant pathes.

We check now if one can re-parametrize pathes satisfying (3.1.1 thru 4) as C^2 functions of the reduced arc length, which would make of *P* a collection of pseudo-segments.

Using (3.1.1) we can <u>define the length δ(P)</u> of the path P and the <u>reduced arc length $\overline{\nu} \in [0,1]$</u> along this path by :

$$(3.1.6) \qquad \delta(P) = \int_0^1 \| v(\nu) \|_F \, d\nu$$

$$(3.1.7) \qquad \overline{\nu} = s(\nu) = (1/\delta(P)) \int_0^{\nu} \| v(\nu) \|_F \, d\nu$$

From (3.1.4) we obtain that the $s : \nu \to \overline{\nu}$ mapping is strictly monotonous, and hence can be inverted, thus allowing to parametrize the path P by its reduced arc length $\overline{\nu}$ by setting :

(3.1.8) $\overline{P}(\overline{v}) = P(s^{-1}(\overline{v}))$.

Defining then the velocity $\overline{v}(\overline{v})$ and the acceleration $\overline{a}(\overline{v})$ with respect to the arc length $\overline{v}$ by :

(3.1.9) $\overline{v}(\overline{v}) = \overline{P}'(\overline{v}), \qquad \overline{a}(\overline{v}) = \overline{P}''(\overline{v}),$

a simple calculation yields, at points when ≠0 :

(3.1.10) $$\overline{v}(\overline{v}) = \frac{v}{\|v\|_F}\,\delta(P)$$

(3.1.11) $$\overline{a}(\overline{v}) = \left\{ \frac{a}{\|v\|_F^2} - \frac{v}{\|v\|_F} < \frac{v}{\|v\|_F}, \frac{a}{\|v\|_F^2} > \right\} \delta(P)^2$$

Hence we have the :

<u>Proposition 3.1</u>

Let D be equipped with a family of paths $P = \{P : v \in [0,1] \rightarrow P(v) \in D\}$, and define $\overline{P} = \{\overline{P} : \overline{v} \in [0,1] \rightarrow \overline{P}(\overline{v}) \in D\}$ by (3.1.8).

Then :

P satisfies (3.1.1 thru 4) $\Rightarrow \overline{P}$ is a collection of pseudo segments.

<u>Proof</u> : The pathes $\overline{P}$ of $\overline{P}$ satisfy obviously (2.1.3) and (2.1.4). The derivatives $\overline{P}'$ and $\overline{P}''$ are defined by (3.1.10) and (3.1.11) at all points $\overline{v}$ such that $v(s^{-1}(\overline{v})) \neq 0$, and are continuous at these points ; but from (3.1.4) we can also define $\overline{P}'$ and $\overline{P}''$, using a continuity argument, at points $\overline{v}_0$ such that $s^{-1}(\overline{v}_0) = 0$; Hence $\overline{P}'$ and $\overline{P}''$ are defined and continuous all over [0,1], i.e. $\overline{P}$ is a C^2 function, and from (3.1.10) we see that $\| \overline{v}(\overline{v}) \|_F = \delta(P)$ = constant, so that $\overline{P}$ satisfies also (2.1.1). ■

By similarity to the formula (2.1.7) for the radius of curvature of the path, which we rewrite here as :

(3.1.12) $\overline{\rho}(\overline{v}) = \| \overline{v}(\overline{v}) \|^2_F / \| \overline{a}(\overline{v}) \|_F \in \mathbb{R} \cup \{+\infty\}$

we shall define, for pathes parametrized by an arbitrary parameter v, the quantity :

(3.1.13) $\rho(v) = \| v(v) \|^2_F / \| a(v) \|_F \in \mathbb{R} \cup \{+\infty\}$

which has no simple geometrical interpretation;we can call it the <u>cinematic radius or curvature</u>,as it includes the accelleration along the trajectory. In order to compare $\rho(v)$ and $\overline{\rho}(\overline{v})$ we see from (3.1.10), (3.1.11) that :

(3.1.14) $\| \overline{v} \|_F = \delta(P)$

(3.1.15) $$\| \overline{a} \|_F = \frac{\| a \|_F}{\| v \|_F^2} \left\{ 1 - < \frac{v}{\| v \|_F}, \frac{a}{\| a \|_F} >^2 \right\}^{\frac{1}{2}} \delta(P)^2$$

Hence:

(3.1.16) $$\overline{\rho} = \rho \left\{ 1 - < \frac{v}{\| v \|_F}, \frac{a}{\| a \|_F} >^2 \right\}^{-\frac{1}{2}}$$

and

(3.1.17) $\rho(v) \leq \overline{\rho}(\overline{v})$

so that $\rho(v)$ can be seen as a lower bound to the radius of curvature $\overline{\rho}(\overline{v})$. But from (3.1.4) we see that, for any $P \in P$ we have :

(3.1.18) $$R(P) = \operatorname*{Inf}_{v \in [0,1]} \rho(v) > 0$$

and hence from (3.1.17) :

(3.1.19) $$\bar{R}(\bar{P}) = \operatorname*{Inf}_{\bar{v} \in [0,1]} \bar{\rho}(\bar{v}) \geqslant R(P) > 0.$$

This will allow us to proove the :

Proposition 3.2

Let P satisfy (3.1.1 thru 4). Then a path $P \in P$ goes at most a finite number of times through a given point $X \in D$.

Proof : Using proposition 3.1. we can parametrize P by its reduced arc length $\bar{v}$. Let $\bar{v}', \bar{v}''$ be two consecutives values of $\bar{v}$ for which $P(\bar{v}) = X$; the Taylor formula yields:

$$\bar{P}(\bar{v}'') = \bar{P}(\bar{v}') + \bar{P}(\bar{v}')(\bar{v}''-\bar{v}') + \int_{\bar{v}'}^{\bar{v}''} (\bar{v}''-\bar{v})\bar{P}''(\bar{v})\, d\bar{v}$$

Hence, with the notations (3.1.9) :

$$(\bar{v}''-\bar{v}')\bar{v}(\bar{v}') = - \int_{\bar{v}'}^{\bar{v}''} (\bar{v}''-\bar{v})\bar{P}''(\bar{v})\, d\bar{v}$$

and, using (3.1.12), (3.1.14), (3.1.19) :

$$|\bar{v}''-\bar{v}'|\, \delta(P) \leq \frac{\delta(P)^2}{\bar{R}(\bar{P})} \int_{\bar{v}'}^{\bar{v}''} (\bar{v}''-\bar{v})\, d\bar{v}$$

so that :

$$(v''-v')\delta(P) \geq 2\bar{R}\,(\bar{P}) \geq 2R(P) > 0$$

which prooves the proposition. ∎

We shall also need the lower bound of the radii of curvature of all pathes of P ; so we define :

(3.1.20) $$R(D) = \operatorname*{Inf}_{P \in P} R(P)$$

and :

(3.1.21) $$\bar{R}(D) = \operatorname*{Inf}_{\bar{P} \in \bar{P}} \bar{R}(\bar{P}) \geq R(D)$$

where of course $\bar{P}$ is made of the same pathes as P , but reparametrized as function of the reduced arc length.

3.2. How much does a set deviates from a convex ? The size × curvature condition.

Let D be equipped with a collection P of pathes satisfying (3.1.1 thru 4). We want to find one measure of the deviation of D from a convex.

As the role of the segments for a convex set is played for D by the pathes $P \in P$, we first measure how much a path P departs from a segment. For that purpose we define along the path P a (positive concave) function g(v) by the 1-D elliptic problem :

(3.2.1) $\quad - g''(v) = \| a(v) \| \quad , \quad g(0) = g(1) = 0$

and we compare g(v) to the cinematic radius of curvature $\rho(v) = \| v(v) \|^2 / \| a(v) \|$ by defining a number $\gamma(P) \in \mathbb{R} \cup \{+\infty\}$ by :

(3.2.2) $$\gamma(P) = \operatorname*{Inf}_{v \in [0,1]} \left\{ \rho(v) - g(v) \right\} \leq R(P)$$

We do not pretend to give here any motivation for the construction of this number $\gamma(P)$: the reason for this definition of $\gamma(P)$ will only be apparent when going through the proofs of paragraph 3.3.

But we want to show that the above defined number $\gamma(P)$ gives a measure of the deviation of the path P from a segment.

Consider first for that purpose the case where the parameter $v \in [0,1]$ along the path P happens to be the reduced arc length $\bar{v} \in [0,1]$. Then equations (3.2.1), (3.2.2) read, using (3.1.12), (3.1.14) :

(3.2.3) $$\bar{g}''(\bar{v}) = \frac{\delta^2(P)}{\bar{\rho}(v)} \quad , \quad \bar{g}(0) = \bar{g}(1) = 0$$

and :

(3.2.4) $$\bar{\gamma}(P) = \operatorname*{Inf}_{\bar{v} \in [0,1]} \left\{ \bar{\rho}(\bar{v}) - \bar{g}(\bar{v}) \right\}$$

If we denote by $\bar{\rho}_{min}$ and $\bar{\rho}_{max}$ lower and upper bounds of the radius of curvature $\bar{\rho}(\bar{v})$ along the path P :

(3.2.5) $$0 < \bar{\rho}_{min} \leq \bar{\rho}(\bar{v}) \leq \bar{\rho}_{max}$$

then a simple calculation shows that :

(3.2.6) $$\bar{\rho}_{min} - \frac{\delta^2(P)}{8\bar{\rho}_{min}} \leq \bar{\gamma}(P) \leq \bar{\rho}_{max} - \frac{\delta^2(P)}{8\bar{\rho}_{max}}$$

So if we compare various pathes P having the same length $\delta(P)$ we see that :

. $\bar{\gamma}(P) \to +\infty$ when $\bar{\rho}_{min} \to \infty$, i.e. when the path tends to become a segment.

. $\bar{\gamma}(P) \to -\infty$ when $\bar{\rho}_{max} \to 0$, i.e. when the path tends to accumulate into a point, thus departing more and more from a segment.

. $\bar{\gamma}(P) > 0$ as soon as $\delta(P) < 2\sqrt{2}\ \bar{\rho}_{min}$. Hence circular pathes satisfy $\bar{\gamma}(P) > 0$ as long as they turn from an angle strictly smaller than $2\sqrt{2} < \pi$ radians.

If we consider now the general case of a path P parametrized by some $v \in [0,1]$, the above interpretation remains qualitatively valid for $\gamma(P)$ defined by (3.2.1), (3.2.2), provided that the tangential accelleration along the path $v \to P(v)$ remains bounded.

This allows us to give the following <u>interpretation of the sign of $\gamma(P)$ for any path P :</u>

(3.2.7) $\quad \gamma(P) > 0 \quad \Leftrightarrow \quad$ the length of the path P is not "too large" with respect to its radii of curvature,

so that the number $\gamma(P)$ gives us some kind of measurement of the deviation of the path P from a segment. In the case of a path $\tilde{P} \in \mathcal{P}$ which is a part of one other path $P \in \mathcal{P}$, as in (3.1.3), one checks easily (cf.[7]) from the definition of $\gamma(P)$ that :

(3.2.8) $\gamma(P) \leq \gamma(\tilde{P})$

We can now measure the deviation of the set D from a convex by considering the worst deviation of a path $P \in \mathcal{P}$ from a segment ; so we associate to D the number $\gamma(D)$ defined by :

(3.2.9) $$\gamma(D) = \operatorname*{Inf}_{P \in \mathcal{P}} \gamma(P) \leq R(D)$$

Remark 3.3

It is sufficient, in sight of (3.1.3), (3.2.8), to calculate $\gamma(D)$ by taking in (3.2.9) the infimum only over the maximal pathes of D (provided of course that they exist) :

(3.2.10) $$\gamma(D) = \operatorname*{Inf}_{\substack{P \in \mathcal{P} \\ P \text{ maximal}}} \gamma(P)$$

where :

(3.2.11) $P \in \mathcal{P}$ is maximal iff there exists no $P' \in \mathcal{P}$ such P is a subpath of P'. ■

Definition 3.4

The set D equipped with the collection of pathes $\mathcal{P}$ is said to satisfy the size × curvature condition if and only if :

(3.2.12) the collection of pathes $\mathcal{P}$ satisfies (3.1.1. thru 4)

(3.2.13) $\gamma(D) > 0 \Leftrightarrow$ the length of each path P of D is not "too large" with respect to its radii of curvature.

The number $\gamma(D)$ which we use to measure the deviation of the set D from a convex depends on the geometry and the parametrization of the collection of pathes $\mathcal{P}$ used for its definition ; we have discussed in [7] the choice of $\mathcal{P}$ which yields the less restrictive size × curvature condition, i.e. which gives the largest $\gamma(D)$: there are hints that, for a given geometry of the path, the (reduced) arc length $\bar{\nu}$ is a good parametrization (it gives at least the largest value to $\rho(\nu)$ as we have seen in (3.1.17)!), and that among all possible path geometries, minimum length pathes won't do too bad.

3.3. Size × curvature condition and quasi convexity

As in the previous paragraph, we consider here a set D equipped with a collection $\mathcal{P}$ of pathes satisfying (3.1.1 thru 4).

In the convex case, the "distance to z" is, for a given z, a convex function along a segment ; we see now what this becomes when the segment is replaced by a curved path :

Lemma 3.5

Let D equipped with a collection $\mathcal{P}$ of pathes satisfying (3.1.1. thru 4), $z \in F$ and $P \in \mathcal{P}$ be given, and define (cf.figure 3.1) :

(3.3.1) $d(\nu) = \| P(\nu)-z \|_F \qquad \forall \nu \in [0,1]$

Then :

(3.3.2) $d(\nu) \leq (1-\nu)d(0) + \nu d(1) + g(\nu) \qquad \forall \nu \in [0,1]$

where g is the concave function associated to the path P by (3.2.1).

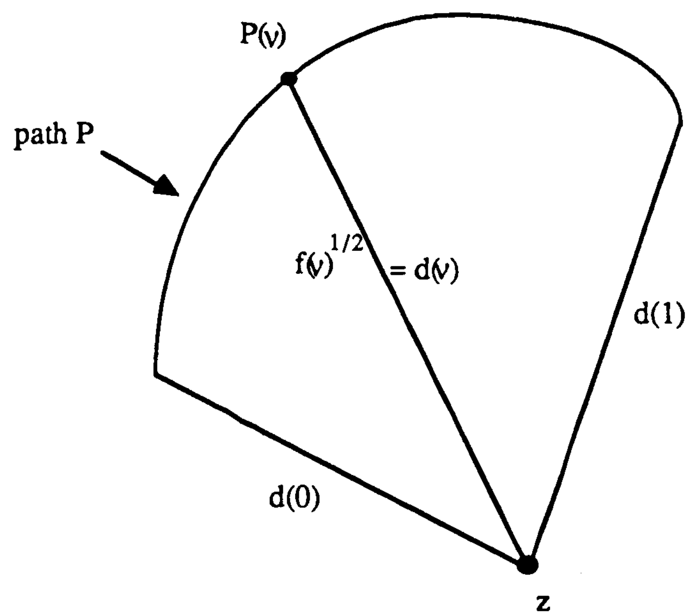


Figure 3.1. : Notations for lemma 3.5.

Proof : Define $f : [0,1] \to IR^+$ by :

(3.3.3) $\quad f(v) = \| P(v)-z \|^2_F = d(v)^2$

Differentiating twice (3.3.3) yields, with notations (3.1.5) :

(3.3.4) $\quad f'(v) = 2 < P(v)-z, v(v) >$

(3.3.5) $\quad f''(v) = 2 \| v(v)-z \|^2_F + 2 < P(v)-z, a(v) >$

From (3.3.4) we get :

(3.3.6) $\quad |f'(v)| \leq 2f(v)^{1/2} \| v(v) \|$

Plugging (3.3.6) into (3.3.5) yields, when $f(v) > 0$:

$$f''(v) \geq \frac{f'(v)^2}{2f(v)} - 2f(v)^{\frac{1}{2}} \| a(v) \|$$

and, using the definition (3.2.1) of g :

$$\frac{f''(v)}{2f(v)^{\frac{1}{2}}} - \frac{f'(v)^2}{4f(v)^{\frac{3}{2}}} - g''(v) \geq 0$$

and hence, as $d(v) = f^{1/2}(v)$:

(3.3.7) $\quad d''(v) - g''(v) \geq 0$

But from proposition 3.2. we know that f may vanish only for a finite number of values of v ; hence (cf.figure 3.2) :

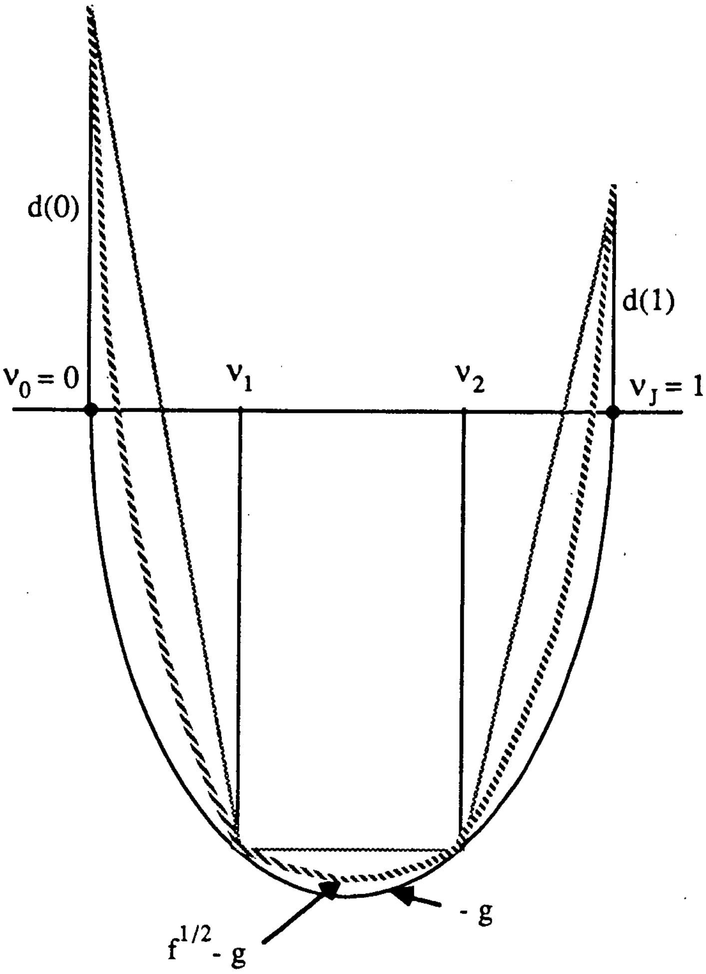


Figure 3.2.

there exists $0 = v_0 < v_1 < ... < v_J = 1$ such that :

(3.3.8) $\quad f(v_0) \geq 0, \quad f(v_j) = 0, j = 1...J\text{-}1, \quad f(v_J) \geq 0$

$\quad f(v) > 0 \quad \forall v \in]v_{j-1}\text{-}v_j[\quad \forall j = 1,2...J.$

From (3.3.7) and (3.3.8) we find that :

(3.3.9) $\quad$ the function $v \rightarrow d(v) - g(v)$ is convex over each $[v_{j-1}\text{-}v_j]$ interval, $j = 1...J$

which, using (3.3.8) and the fact that the function -g is convex with $g(0) = g(1) = 0$, yields the sought result (3.3.2). ■

In the convex case, d(v) defined in (3.3.1) is convex, and hence $f(v) = d(v)^2$ is also convex. In the next lemma, we investigate the meaning, in the non convex case, of the local concavity of f :

Lemma 3.6

Let D equipped with a collection *P* of pathes satisfying (3.1.1. thru 4), $z \in F$ and $P \in$ *P* be given, and define :

(3.3.3) $\quad f(v) = \| P(v)\text{-}z \|^2_F = d(v)^2$

If :

(3.3.10) $\quad f''(v) \leq 0$ over some non void subinterval of [0,1]

Then :

(3.3.11) $\quad \text{Max } \{d(0),d(1)\} \geq \gamma(P)$

Proof : Let $0 \leq v' < v'' \leq 1$ be such that :

$$f''(v) \leq 0 \qquad \forall v \in [v',v'']$$

From (3.3.5) we obtain :

(3.3.12) $\quad 0 \geq f''(v) \geq 2 \| v(v) \|^2_F - 2d(v) \| a(v) \|_F \qquad \forall v \in [v',v'']$

But from (3.1.4), $\| v(v) \|_F$ cannot be identically zero over the $[v',v'']$ interval. Hence we can select a $v_0 \in [v',v'']$ such that :

$$f''(v_0) \leq 0 \text{ and } \| v(v_0) \| > 0$$

Then (3.3.12) shows that $\| a(v_0) \| > 0$, so that we can divide (3.3.12), written at $v = v_0$, by $\|a(v_0) \|$, which yields ; using (3.1.13) :

(3.3.13) $\quad d(v_0) \geq \dfrac{\|v(v_0)\|_F^2}{\|a(v_0)\|_F} = \rho(v_0)$

This, together with lemma 3.5, yields :

$$\rho(v_0) \leq d(v_0) \leq (1-v_0)d(0) + v_0 d(1) + g(v_0)$$

and, using the definition (3.2.2) of $\gamma(P)$:

$$\text{Max } \{d(0),d(1)\} \geq (1-v_0)d(0) + v_0 d(1) \geq \rho(v_0) - g(v_0) \geq \gamma(P)$$

which is the sought result. ■

An immediate consequence is the :

Lemma 3.7

Let (D,P) satisfy the size × curvature condition (definition 3.4), and let $z \in F$ and $P \in P$ be given :

If, using notation (3.3.1) :

(3.3.14) $\text{Max } \{d(0),d(1)\} < \gamma(P)$

Then :

(3.3.15) $v \to f(v) = \| P(v)-z \|^2_F$ is strictly convex over [0,1]

(3.3.16) $v \to d(v) = \| P(v)-z \|_F$ is strictly quasiconvex over [0,1]

Proof : From (3.3.14) and Lemma 3.6 we see that $f''(v)$ cannot be negative on any subinterval of [0,1]. As f'' is a continuous function, this implies that $f''(v) \geq 0$, $\forall v \in [0,1]$ so that f is convex. Moreover, f cannot be affine on any subinterval of [0,1], as this would imply that $f''(v) = 0$ on this subinterval. Hence f is strictly convex on [0,1]. Then $d(v) = f(v)^{1/2}$ is strictly quasiconvex as the square root of a strictly convex function. ■

We prove now the main theorem of this section :

Theorem 3.8

Let D be equipped with a collection P of pathes (parametrized by an arbitrary parameter v). If :

(D,P) satisfies the size × curvature condition (definition 3.4), so that $R(D) \geq \gamma(D) > 0$.

Then :

(D,P) is quasiconvex (definition 2.5), with the following choices for V and $\varepsilon(z)$:

(3.3.17) $\quad V = \{z \in F \mid d(z,D) < \gamma(D)\}$

(3.3.18) $\quad \varepsilon(z) = \gamma(D) - d(z,D)$

and the resulting value for $k(z,\eta)$ is :

$$(3.3.19) \qquad k(z,\eta) = \frac{d(z,D)+\eta}{\bar{R}(D)} < \frac{\gamma(D)}{R(D)} \leq 1, \qquad 0 < \eta < \varepsilon(z)$$

Proof : Let (D,P) satisfy the size × curvature condition of definition 3.4. From (3.2.9), (3.2.13) we get :

(3.3.20) $\quad R(D) \geq \gamma(D) > 0$

This allows us to define V and $\varepsilon(z)$ by (3.3.17) and (3.3.18).

Let us choose now $z \in D$, $0 < \eta < \varepsilon(z)$ and $P \in P(z,\eta)$ as in the left hand side of (2.2.1). First we see from (2.2.2), (3.3.18) and notation (3.3.1) that :

(3.3.21) $\quad \mathrm{Max}\{d(0),d(1)\} \leq d(z,D)+\eta < \gamma(D)$

which, together with (3.2.9) shows that lemma 3.7 applies, hence the $\nu \to d(\nu)$ function is quasiconvex.

In order to proove (2.2.1) we majorate now $k(z,P)$ using (2.2.5) and the bar notation introduced at the beginning of paragraph 3 :

$$k(z,P) \leq \underset{\bar{\nu} \in [0,1]}{\mathrm{Max}} \frac{\bar{d}(\bar{\nu})}{\bar{\rho}(\bar{\nu})}$$

But the $\bar{\nu} \to \bar{d}(\bar{\nu})$ function is quasiconvex (as the $\nu \to d(\nu)$ one is !), which, together with (3.1.19), (3.1.21) yields :

$$k(z,P) \leq \frac{\mathrm{Max}\{d(0),d(1)\}}{\bar{R}(\bar{P})} \leq \frac{\mathrm{Max}\{d(0),d(1)\}}{\bar{R}(D)}$$

and, using (3.3.21), (3.1.21) and (3.2.9) :

$$k(z,P) \leq \frac{d(z,D)+\eta}{\bar{R}(D)} < \frac{\gamma(D)}{R(D)} \leq 1$$

which shows that (k,z,η) define by (3.3.19) satisfies (2.2.1). This ends the proof of theorem 3.8. ∎

We specialize in the next theorem the results on quasiconvex sets to sets satisfying the size × curvature condition :

Theorem 3.9

Let D be equipped with a collection P of pathes satisfying (3.1.1 thru 4).

i) If D,P satisfy the "size × curvature" condition (definition 3.4), so that :

(3.3.22) $\quad \bar{R}(D) \geq R(D) \geq \gamma(D) > 0$

Then :

. The $\nu \to P(\nu)$ mapping are injective, $\forall P \in P$

. The "distance in D" $\delta(X,Y)$ of definition 2.2 and the usual distance in F, $\|X-Y\|_F$, are equivalent in the sense that :

(3.2.23) $\quad \| X-Y \|_F \leq \delta(X,Y) \leq \left(1-\frac{d}{R(D)}\right) \| X-Y \|_F$

as soon as :

(3.3.24) $\quad \| X-Y \|_F \leq d < \gamma(D)$

ii) If moreover :

(3.3.25) $\quad d(z,D) < \gamma(D)$

Then :

. There exists at most one projection $\hat{X}$ of z on D, i.e. the "distance to z" function has at most one global minimum over D.

. All possible local minima are at a distance of z larger than or equal to $\gamma(D)$.

. The "projection on D" mapping, when it exists, is lipschitz continuous : if z_0 and z_1 satisfying (3.3.25) admit projections $\hat{X}_0$ and $\hat{X}_1$ on D, one has :

(3.3.26) $\quad \| \hat{X}_0-\hat{Y}_1 \|_F \leq \delta(\hat{X}_0,\hat{Y}_1) \leq \left(1-\frac{d}{R(D)}\right)^{-1} \| z_0-z_1 \|_F$

as soon as :

(3.3.27) $\quad \| z_0-z_1 \|_F + \underset{\varphi=0,1}{\mathrm{Max}}\ d(z_j,D) \leq d < \gamma(D)$

iii) If moreover :

(3.3.28) D is closed in F.

Then :

z has a unique projection $\hat{X}$ on D, and any minimizing sequence is a Cauchy sequence for both the $\delta(X,Y)$ and $\| X-Y \|_F$ distances, and converges towards $\hat{X}$ for both distances.

Proof : This theorem is just a rewriting of theorems 2.9, 2.12 and 2.15 using theorem 3.8. In fact, theorem 2.12 implies (3.3.23) and (3.3.26) only when the numbers d are chosen such as to satisfy the double strict inequality in (3.3.24) and (3.3.27), but a continuity argument yields immediately the sought result. ■

4 - APPLICATION TO NON-LINEAR LEAST-SQUARES INVERSION

We apply in this paragraph the results on quasiconvex sets and the size × curvature condition to the least-square solution of non-linear equations. Let :

(4.01)
- E = a Banach space, with the norm $\| \ \|_E$,
- $C \subset E$ = a convex subset of E,
- F an Hilbert space, with scalar product $<,>_F$,
- $\varphi : C \to F$ a C^2 mapping
- $z \in F$ = a given data point

We consider now the non-linear least square problem :

(4.02) $\quad$ find $\hat{x} \in C$ such that $J(x) = \| \varphi(x)-z \|^2_F$ = min over C

As we have seen in the introduction, such problem arises in various area, and especially in parameter estimation problems : x represents the parameter to be identified, C the set of admissible parameters, z the recorded data and φ the parameter $\rightarrow$ output mapping, usually defined through the resolution of state equations and the application of an observation operator. In this context, hypothesis (4.01) are quite often satisfied, but as we shall see, they are far from being sufficient for ensuring well-posedness of the optimization problem (4.0.2), i.e. existence and uniqueness of its solution $\hat{x}$ and continuity of the $z \rightarrow \hat{x}$ mapping on some neighborhood of $\varphi(C)$, also called "Output Least-Square Identifialility" in the context of parameter estimation problems.

So we shall investigate problem (4.0.2) under the additional hypothesis that $\varphi(C)$ is quasiconvex. As we want to state only constructive results, we shall consider only the case where $\varphi(C)$ satisfies the sufficient size × curvature condition of paragraph 3.

We first handle in paragraph 4.1 the possible non-injectivity of the φ mapping by setting problem (4.0.2) on the set $C^{\#}$ of connected equivalence classes generated by φ, rather than on the set C itself.

Then we equip in paragraph 4.2 the set C with a collection Π of pathes in such a way that the image pathes by φ satisfy the conditions required for testing the size × curvature condition on $\varphi(C)$.

We equip in paragraph 4.3 the set $C^{\#}$ of connected equivalence classes with two pseudo-distances, one transported on $C^{\#}$ by φ^{-1} from the "distance in $\varphi(C)$", and one defined directly from the norm in E.

Finally we restate in paragraph 4.4 the results of the projection theory of paragraph 3 in the context of the non-linear least square problem (4.0.2), and demonstrate their analogy with the classical results for the linear least square problem :

(4.0.3) find $\hat{x} \in C$ such that $J(x) = \| \Phi.x-z \|^2_F = \min$ over C

where :

(4.0.4) $\Phi \in L(E;F)$,

to which they reduce exactly when φ happens to be linear.

When the quasiconvexity of $\varphi(C)$ cannot established, and/or when one wants to overcome the non-injectivity of φ, the solution since Tychonov [12] is to replace problem (4.0.2) by its regularized version :

(4.0.5) find $\hat{x}_\varepsilon \in C$ such that $J_\varepsilon(x) = J(x) + \varepsilon^2 \| x-x_0 \|^2_E = \min$ over C

where :

(4.0.6)
$x_0 \in E$ is some a-priori estimate of the sought solution
$\varepsilon > 0$ is the regularizing parameter

(in its original work, Tychonov used a $\varepsilon^2 \| x-x_0 \|^2_{\mathcal{E}}$ regularizing term, where the subspace $\mathcal{E}$ was compactly unbedded in E, but one advantage of the theory developped in this paper is that it will not require compactness, which is replaced by an hypothesis on the "shape" of $\varphi(C)$ such as the size × curvature condition).

A detailed study of regularization theory for non-linear least square problems, with the help of quasiconvex sets and size × curvature condition will be presented in a next paper. We refer to references [6] and [10] for preliminary (including numerical) results.

4.1. Handling the non-injecitivy of φ

When the mapping φ to be inverted is not injective, the first idea is that we have to replace the search for one solution x by the search for an equivalence class $\dot{x}$ of the quotient set $\dot{C}$ of C by the equivalence relation "$x \sim y$ iff $\varphi(x) = \varphi(y)$".

But this is not the right notion, as by definition φ is always injective on $\dot{C}$, so that the question of uniqueness seems to disappear ! In fact, the equivalence classes $\dot{x}$ are too large for our purpose. So we split each equivalence class $\dot{x}$ into its connected components $x^\#$: from a "physical" point of view, two solutions inside the same connected equivalence class $x^\#$ cannot be distinguished, as one can pass "continuously" from one to the other, whereas two solutions located in two distinct connected components have to be distinguished as they correspond to completely different physical situations.

Definition 4.1

Let hypothesis (4.0.1) hold. We define :

(4.1.1) $\dot{C} = \{\dot{x} = \{x \in C \mid \varphi(x) = X\} \mid X \in \varphi(C)\}$

(set of all equivalence classes)

(4.1.2) $C^\# = \{x^\# = \text{connected component of } \dot{x} \in \dot{C}\}$

(set of all connected equivalence classes)

As the mapping φ is C^2, the elements $x^\#$ of $C^\#$ are closed, connected, regular manifold of C, with tangent space at $x \in x^\#$ included in the kernel of $\varphi'(x)$.

So we shall consider in the sequal the well posedness on $C^\#$ of the non-linear least square problem (4.0.2). This setting restores the problem of uniqueness, which will be achieved as soon as $C^\# = \dot{C}$, and will require the definition of some "distance" on $C^\#$ (cf.paragraph 4.2).

A practically interesting case is the case where the connected equivalence classes $x^\#$ contain only one point x, which we shall denote with a slight abuse of langage, by writing $C^\# = C$:

Definition 4.2

We shall say that :

(4.1.3) φ is locally injective $\Leftrightarrow C^\# = C$

By opposition, when φ is injective in the usual sense on C, we shall say that :

(4.1.4) φ is globally injecive $\Leftrightarrow \dot{C} = C$

A first sufficient condition for the local injectivity of φ is that $\varphi'(x)$ is injective and continuously invertible :

Proposition 4.3

Let hypothesis (4.0.1) hold. If :

(4.1.5) $\exists \alpha > 0$ s.t. $\| \varphi'(x).y \|_F \geq \alpha \| y \|_E$ $\quad \forall x \in C, \forall y \in E$

Then :

(4.1.6) φ is locally injective, i.e. $C^\# = C$

This proposition is a special case of the next one, so we skip its proof.

But injectivity of $\varphi'(x)$ is not necessarily required for the local injectivity of φ : the graph of φ may have a few "stationary points", as for example in the :

Proposition 4.4

Let hypothesis (4.0.1) hold, and suppose that E is an Hilbert space.

If :

(4.1.7) $\dim \operatorname{Ker} \varphi'(x) < +\infty$ $\quad \forall x \in C$

(4.1.8) $\exists \alpha > 0$ s.t. $\| \varphi'(x).y^\perp \|_F \geq \alpha \| y^\perp \|_E$ $\quad \forall x \in C, \forall y^\perp \in \operatorname{Ker} \varphi'(x)^\perp$

(4.1.9)
$$\begin{cases} x \in C \\ y \in E \\ \varphi'(x).y = 0 \end{cases} \Rightarrow \begin{cases} \varphi''(x)(y,y) = 0 \\ \text{and} \\ \exists \varepsilon > 0 \text{ st } 0 < |v| < \varepsilon \Rightarrow \varphi'(x+vy).y \neq 0 \end{cases}$$

Then :

(4.1.10) φ is locally injective, i.e. $C^\# = C$

The proof is quite technical and given in Appendix.

Example 4.5

On $C = [-1,+1]$ the functions $\varphi(x) = x^m$, $m \in \mathbb{N}$, satisfy the hypothesis of proposition 4.4 for $m \geq 3$, and are (globally) injective for m odd and locally injective for m even. ■

4.2. Equipping C with pathes

In order to be able to use the theory of paragraphs 2 and 3 for the solution of the non-linear least square problem (4.0.2), we need to define a collection P of pathes on the set $D = \varphi(C)$. This can be achieved by equipping the set C with a collection Π of pathes $\Pi : [0,1] \to C$ satisfying (compare with (3.1.1. thru 4)) :

(4.2.1) $\pi \in \Pi \Rightarrow v \to \pi(v)$ is C^2 from $[0,1]$ in C

(4.2.2) for any $x^\#, y^\# \in C^\#$, $x^\# \neq y^\#$, there exists $\pi \in \Pi$ such that $\pi(0) \in x^\#$ and $\pi(1) \in y^\#$

(4.2.3)
$$\begin{cases} \text{for any } \pi \in \Pi \text{ and any } v', v'' \in [0,1],\ v' \leq v'', \text{ the path} \\ \qquad \tilde{\pi} : v \in [0,1] \to \pi((1-v)v' + vv'') \text{ belongs to } \Pi \end{cases}$$

$$
(4.2.4)\quad \begin{cases} \text{for any } \pi \in \Pi \text{, condition (3.1.4) holds with } \pi \in \Pi \text{ instead of } P \in \mathcal{P} \text{ ,} \\ \text{and } v(v) \text{ and } a(v) \text{ defined by :} \\ v(v) = \varphi'(\pi(v)).\pi'(v) \\ a(v) = \varphi''(\pi(v))(\pi'(v),\pi'(v)) + \varphi'(\pi(v)).\pi''(v), \\ \text{which implies that } \varphi o \pi \text{ can be reparametrized as a } C^2 \text{ function of the arc length } \bar{v} . \end{cases}
$$

Then obviously the following proposition holds :

Proposition 4.6

If Π satisfies (4.2.1 thru 4), then $P = \varphi o \Pi$ satisfies (3.1.1 thru 4), and $\overline{P} = \overline{\varphi o \Pi}$ is a collection of pseudo-segments of $\varphi(C)$ (cf.proposition 3.1).

Remark 4.7

The most typical situation is when there is only one path Π going from $x^\#$ to $y^\#$, $y^\# \neq x^\#$. The fact that (4.2.2) allows many such pathes is a technical facility which may be useful in some situations, as for example when pathes are defined as some minimum length arcs, or when φ is not even locally injective and one does not want to get into the structure of the connected equivalence classes $x^\#$, which leads to choose for Π the collection of all segments of C for example. ■

The condition (4.2.4) implies (cf.figure 4.1) that the pathes π of Π are not allowed to remain tangent to a connected equivalence class : among the three pathes π_1, π_2 and π_3 of the figure 4.1 going from $x^\#$ to $y^\#$, π_1 satisfy necessaryly (4.2.4), π_2 may satisfy (4.2.4), and π_3 surely does not satisfy (4.2.4).

We give now two simple examples for the choice of the family Π of pathes of C :

Example 4.8

When φ is linear continuous and E is an Hilbert space :

(4.2.5) $\qquad \varphi(x) = \Phi.x \qquad$ with $\Phi \in \mathcal{L}(E;F)$

then all equivalence classes in C are connected :

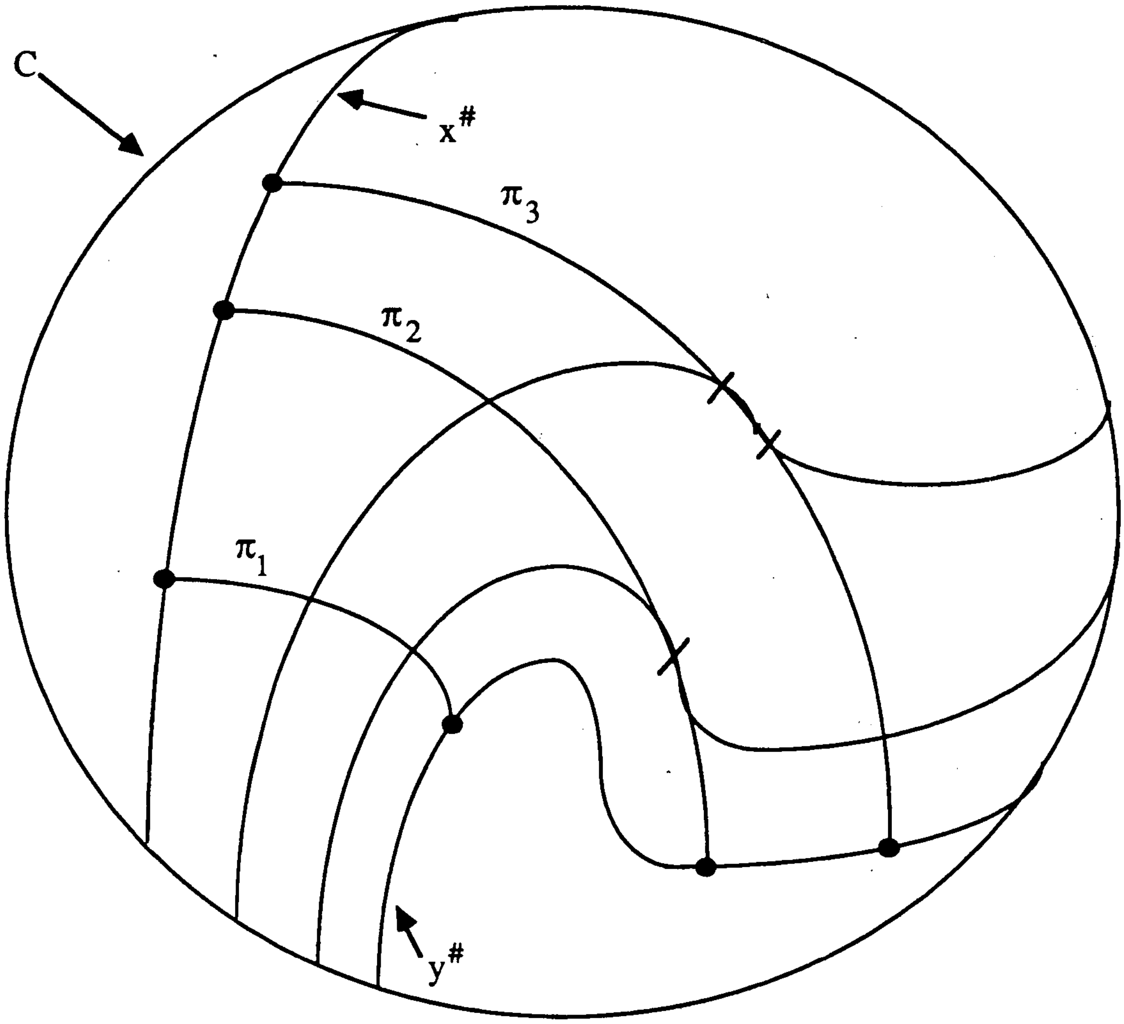


Figure 4.1 : Geometrical illustration of condition (4.2.4).

(4.2.6) $C^{\#} = \dot{C}$

and one can choose for Π the family of all segments of C whose direction is not the kernel of Φ :

(4.2.7) $\Pi = \{ \nu \rightarrow (1-\nu)x + \nu y \text{ for } x,y \in C, x-y \notin \mathrm{Ker}\Phi\}$

If we decide (cf.figure 4.2) to represent each (connected) equivalence class $x^{\#} = \dot{x}$ by its representant $x^{\perp}$ in the supplementary space $\mathrm{Ker}\Phi^{\perp}$, we may define :

(4.2.8) $C^{\perp} = \{x^{\perp} \in \mathrm{Ker}\Phi^{\perp} \text{ st } \dot{x} \cap C \neq \phi\}$

and then take for Π the family of all segments of $C^{\perp}$:

(4.2.9) $\Pi = \{\nu \rightarrow (1-\nu)x^{\perp} + \nu y^{\perp} \text{ for } x^{\perp}, y^{\perp} \in C^{\perp}\}$

Of course, both (4.2.7) and (4.2.9) satisfy hypothesis of proposition 4.5, and yield the same family of pathes in $\varphi(C)$, namely all segments of $\varphi(C)$. ∎

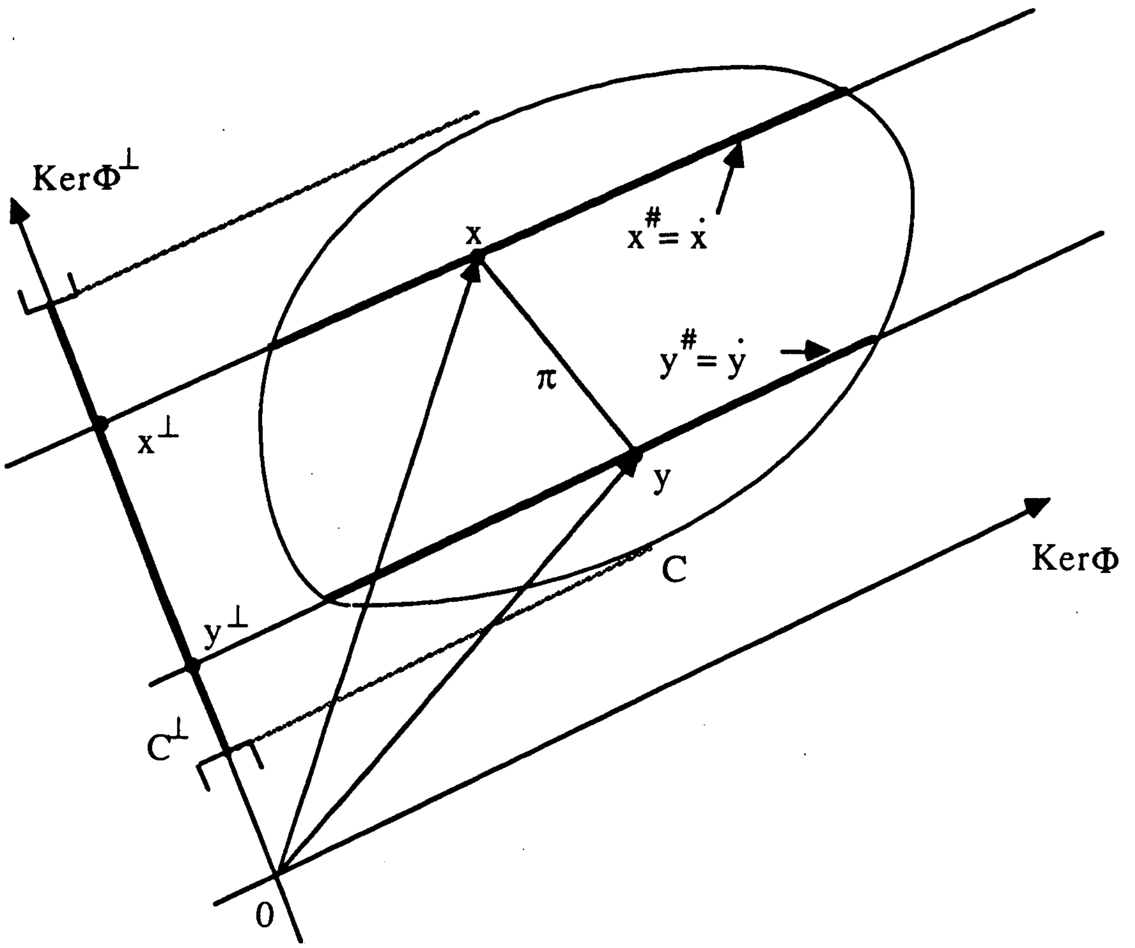


Figure 4.2 : Illustration of the equivalence classes and pathes in the linear case.

Example 4.9

When φ is non-linear, but satisfies :

$$(4.2.10)\qquad \begin{cases} x \in C \\ y \in E \\ \varphi'(x).y = 0 \end{cases} \Rightarrow \begin{cases} v(\nu) \neq 0 \text{ for } |\nu| \text{ small enough} \\ \text{and} \\ \dfrac{v(\nu)}{\| v(\nu) \|} \text{ and } \dfrac{a(\nu)}{\| v(\nu) \|^2} - \dfrac{v(\nu)}{\| v(\nu) \|} < \dfrac{v(\nu)}{\| a(\nu) \|}, \dfrac{a(\nu)}{| v(\nu) \|^2} > \\ \text{have a limit when } \nu \rightarrow 0 \end{cases}$$

where :

$$(4.2.11)\qquad \begin{cases} v(\nu) = \varphi'(x+\nu y).y \\ a(\nu) = \varphi''(x+\nu y)(y,y) \end{cases}$$

then the collection Π of all segments of C :

$$(4.2.12)\qquad \Pi = \{\nu \rightarrow (1-\nu)x + \nu y \text{ for } x,y \in C\}$$

satisfy the hypothesis of proposition 4.6.

Of course, condition (4.2.10) holds as soon as as $\varphi'(x)$ is injective for any x of C. But it allows for "inflexion points" in the graph of φ : for example the functions $\varphi = x^m$ of example 4.5 satisfy (4.2.10) as soon as m is odd ! ■

4.3. Equipping $C^\#$ with pseudo distances

We can now measure the "distance" of two connected equivalence class $x^\#$ and $y^\#$ of $C^\#$ using the pathes Π defined in the previous paragraph.

As the stability result for the projection on $\varphi(C)$ is expressed in term of the length of the pathes in $\varphi(C)$, we begin by the :

Definition 4.10

Let C be equipped with a collection Π of pathes satisfying (4.2.1 thru 4).

Then, for any $x^\#, y^\# \in C^\#$ we call "distance in $\varphi(C)$" of $x^\#$ and $y^\#$ the quantity :

(4.3.1) $\delta(x^\#,y^\#) = \delta(\varphi(x), \varphi(y))$ for any $x \in x^\#$, $y \in y^\#$

where $\delta(X,Y)$ is given in definition 2.2. One checks easily that $\delta(x^\#,y^\#)$ is given by the formula :

(4.3.2) $$\delta(x^\#,y^\#) = \sup_{\pi : x^\# \to y^\#} \int_0^1 \| \varphi'(\pi(v)).\pi'(v) \|_F \, dv$$

or, in case there is no path from $x^\#$ to $y^\#$ (which may happen only if $x^\# = y^\#$) :

(4.3.3) $\delta(x^\#,y^\#) = 0$

We try now to figure out the "shape" of the "ball" :

(4.3.4) $B_\delta(x^\#,\varepsilon) = \{y^\# \in C^\# \mid \delta(x^\#,y^\#) \leq \varepsilon\}$,

as this "ball" will be the non-linear uncertainty domain for the solution of the non-linear least square problem (4.0.2) in presence of some uncertainty on the data z.

In the general case, $B_\delta(x^\#,\varepsilon)$ will be a Π-star shaped domain of $C^\#$ as shown in figure 4.3.a. We specialize now to the case where Π is chosen as in examples 4.8 and 4.9 of the preceeding paragraph :

Example 4.8. revisited (φ linear and E = Hilbert space)

One finds immediately that :

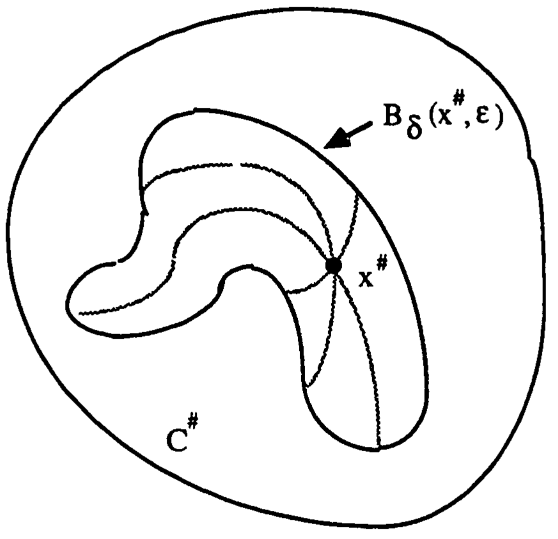
(4.3.5) $$\begin{cases} \delta(x^\#,y^\# = \| \Phi.(x-y) \|_F \\ \forall x \in x^\# = \dot{x}, \forall y \in y^\# = \dot{y} \end{cases}$$

or equivalently :

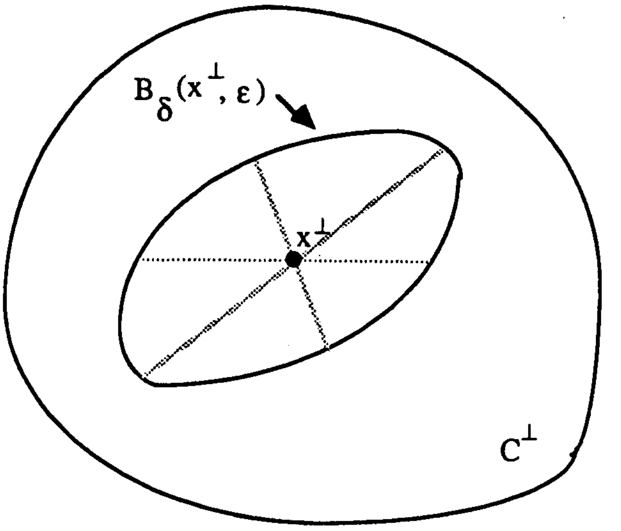
(4.3.6) $$\begin{cases} \delta(x\#,y\#) = \| \Phi.(x^\perp - y^\perp) \|_F \\ \forall x^\perp, y^\perp \in C^\perp, \end{cases}$$

so that $B_\delta(x^\#,\varepsilon)$ is an ellipsoid in $C^\#$, i.e. in $C^\perp$, as shown in figure 4.3.b.

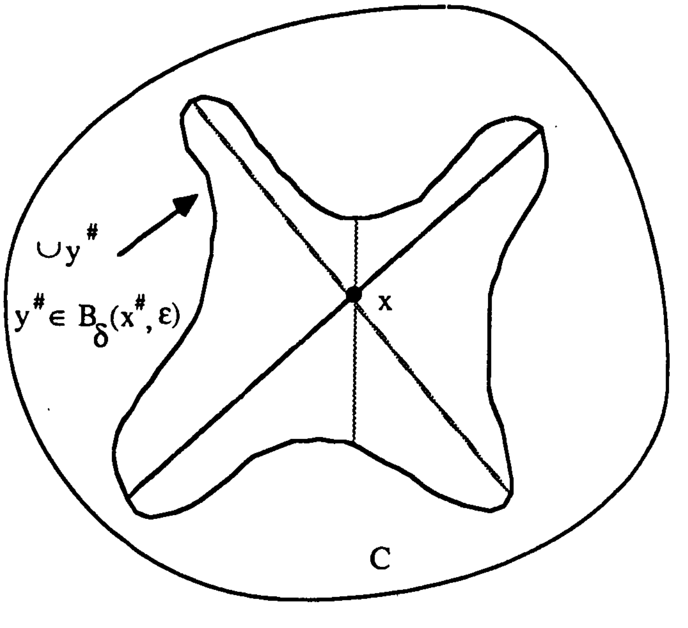
Of course, if Φ is injective, then $B_\delta(x^\#,\varepsilon)$ is an ellipsoid in C itself. ■



a - General case



b - Example 4.8. (linear)



c - Example 4.9. when $x^\# = \{x\}$

Figure 4.3. : Typical shape of $B_\delta(x^\#,\varepsilon)$, (dotted lines represent pathes π).

Example 4.9. revisited

Once again, as the pathes Π are taken to be the segments of C, $B_\delta(x^\#,\varepsilon)$ is expected to be "star-shaped" in $C^\#$, but this is quite weak as we don't know the structure of $C^\#$. But under the additional hypothesis :

(4.3.7) $\qquad x^\# = \{x\}$

then one can see easily from (4.3.2) that the union of all $y^\# \in B_\delta(x^\#,\varepsilon)$ is a star-shaped domain of C, as shown in figure 4.3.c.

Of course, if C and φ satisfy more over the hypothesis of proposition 4.4, then $C^\# = C$, and (4.3.7) automatically holds, and $B_\delta(x^\#,\varepsilon)$ itself is a star-shaped domain of C. ▬

The distance $\delta(x^\#,y^\#)$ on $C^\#$ is transported on C by φ^{-1} from some distance in the data space F. In many applications, one would like to use a distance on $C^\#$ which is defined directly from the distance in E, and which in the good cases reduces to this latter.

So we give the :

Definition 4.1 Let C be equipped with a collection Π of pathes satisfying (4.2.1 thru 4), and E be an Hilbert space.

Then, for any $x^\#, y^\# \in C^\#$, we call "distance in C" of $x^\#$ and $y^\#$ the quantity :

(4.3.8) $$d(x^\#,y^\#) = \underset{\pi : x^\# \to y^\#}{\text{Sup}} \int_0^1 \| \pi'(v)^\perp \|_E dv$$

or, in case there is no path from $x^\#$ to $y^\#$ (which may happen only if $x^\# = y^\#$) :

(4.3.9) $d(x^\#,y^\#) = 0$,

where :

(4.3.10) $\pi'(v)^\perp$ is the projection of $\pi'(v)$ on Ker $\varphi'(\pi(v))^\perp$

We show on figure 4.4 the geometrical interpretation of formula (4.3.8) : for a given path π going from $x \in x^\#$ to $y \in y^\#$, the integral in the R.H.S. of (4.3.8) is the "normal length" of the path π, i.e. its length measured along the normals to the encountered connected classes of equivalence.

We first compare $\delta(x^\#,y^\#)$ and $d(x^\#,y^\#)$ in the linear case :

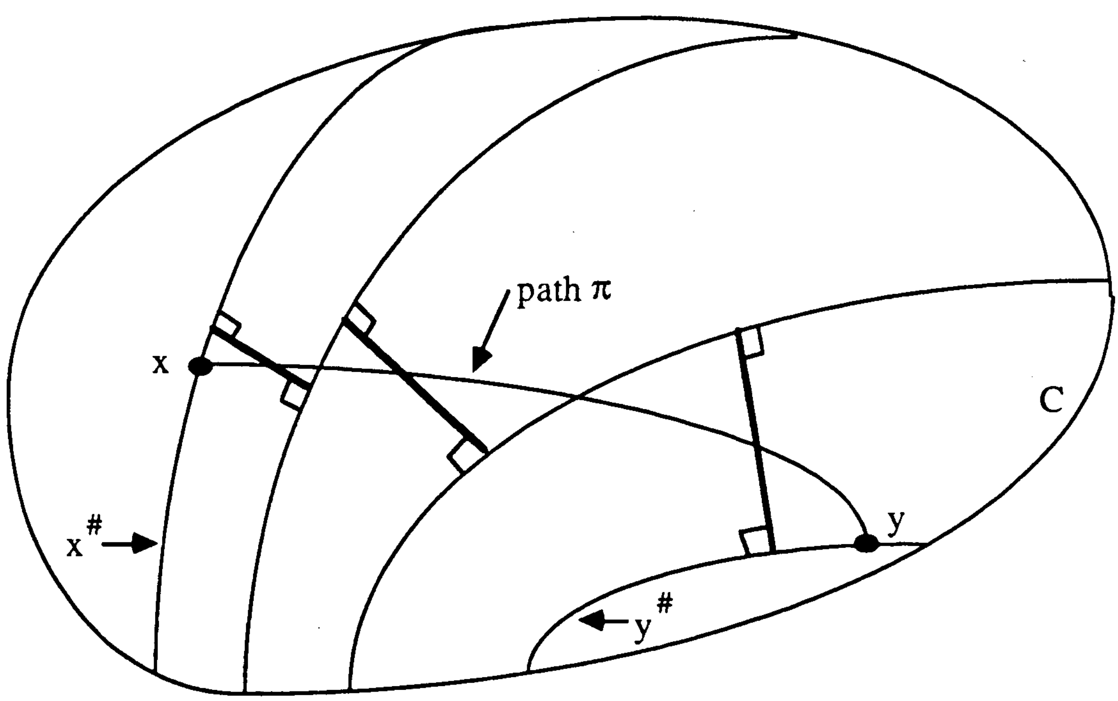


Figure 4.4. : The sum of the length of the reinforced segments is an approximation to $\int_0^1 \| \pi'(v)^\perp \|_E dv$

Proposition 4.12

Let hypothesis and notations of example 4.8 hold : φ is linear, $C^{\#} = \dot{C}$ can be represented by $C^{\perp}$ (see figure 4.2), and Π is made of all segments of $C^{\perp}$.

i) If :

(4.3.11) $\exists \alpha > 0$ s.t. $\| \Phi.y^{\perp} \|_F \geq \alpha \| y^{\perp} \|_E \quad \forall y^{\perp} \in \text{Ker}\Phi^{\perp}$

Then :

(4.3.12) $\delta(x^{\perp},y^{\perp}) \geq \alpha d(x^{\perp},y^{\perp}) \quad \forall x^{\perp},y^{\perp} \in C^{\perp}$

with :

(4.3.13) $d(x^{\perp},y^{\perp}) = \| x^{\perp}-y^{\perp} \|_E$

ii) If moreover :

(4.3.14) Φ is injective

Then $C^{\#} = \dot{C} = C^{\perp} = C$, and

(4.3.14) $d(x,y) = \| x-y \|_E \quad \forall x,y \in C$

Remark 4.13

Hypothesis (4.3.11) + (4.3.14) can be written more simply as :

(4.3.15) $\exists \alpha > 0$ s.t. $\| \Phi.y^{\perp} \|_F \geq \alpha \| y \|_E \quad \forall y \in E.$ ■

We compare now δ and d in one non-linear case :

Proposition 4.14

Let hypothesis and notations of example 4.9 hold : φ is non-linear, with $\varphi'(x)$ injective nearly everywhere, and Π is made of all segments of C.

i) If :

(4.3.16) $\exists \alpha > 0$ s.t. $\| \varphi'(x).y^{\perp} \|_F \geq \alpha \| y^{\perp} \|_E$

$\forall x \in C, \ \forall y \in \text{Ker}\varphi'(x)^{\perp}$

Then :

(4.3.17) $\delta(x^{\#},y^{\#}) \geq \alpha\, d(x^{\#},y^{\#})$

ii) If moreover :

(4.3.18) $\varphi'(x)$ is injective $\quad \forall x \in C$

Then φ is locally injective, i.e. $C^{\#} = C$ (proposition 4.3), and :

(4.3.19) $d(x,y) = \| x-y \|_E \quad \forall x,y \in C$

Remark 4.15

Hypothesis (4.3.16) + (4.3.18) can be written more simply as :

(4.3.20) $\exists \alpha > 0$ s.t. $\| \varphi'(x).y \|_F \geq \alpha \| y \|_E \quad \forall x \in C, \ \forall y \in E.$ ■

4.4. The Non-Linear Least Square Problem

Having now developped in the above paragraphs the necessary tools, we apply now the projection theory on sets satisfying the size × curvature condition to the resolution of the non-linear least square problem (4.0.2).

In addition to :

(4.4.1) E,C,F,φ,z satisfying (4.0.1)

we need to choose a collection Π of pathes of C such that :

(4.4.2) Π satisfies (4.2.1 thru 4).

Remark 4.16

The result we shall obtain on the non-linear least-square problem (4.0.2) will be relative to the choice of Π we have made in (4.4.2). If we change our collection of pathes, we may obtain qualitatively and quantitatively different results ! ▬

Once the choices of E,C,F,φ,z,Π have been made according (4.4.1) and (4.4.2), we can apply the theory of projection on sets satisfying the size × curvature condition, as developped in §3, to the projection on :

(4.4.3) $D = \varphi(C)$

equipped with the pathes :

(4.4.4) $P = \varphi o \Pi$,

as we know from proposition 4.6 that P satisfies the required hypothesis. Some can define, as in (3.1.21) :

(4.4.5) $$\overline{R} \triangleq \overline{R}(\varphi(C)) = \operatorname*{Inf}_{\overline{P} \in \overline{P}} \overline{R}(\overline{P}) \geq R(\varphi(C)) \triangleq R$$

(smallest radius of curvature along the pathes of P), and, as in (3.2.9) :

(4.4.6) $$\gamma \triangleq \gamma(\varphi(C)) = \operatorname*{Inf}_{P \in P} \gamma(P) \leq R$$

(size × curvature product of $\varphi(C)$). The sign of this number γ will tell us if, given the choice we have made for Π, the projection theory of paragraph 3 applies or not to the resolution of the non-linear least square problem (4.0.2). So this number γ contains very useful information on the well posedness of the non-linear least square problem ; its computation requires to calculate the numbers $\gamma(\varphi o \pi)$ for all pathes π going from any connected equivalence class $x^\#$ to any other connected equivalence class $y^\#$.

Remark 4.17

When C happens to be convex and closed and Π happens to be the collection of all segments of C, the amount of computational effort required for the computation of γ is some what lowered :

(4.4.7) $$\gamma = \operatorname*{Inf}_{\substack{\pi = [x,y] \\ x,y \in \partial C}} \gamma(\varphi o p)$$ ▬

We rewrite now the projection theorem 3.9 in the context of non-linear least squares :

Theorem 4.18

Let C,Π,φ satisfy (4.4.1), (4.4.2).

i) If C,Π,φ satisfy the "size × curvature" condition

(4.4.8) $\quad \bar{R} \geq R \geq \gamma \overset{\Delta}{=} \gamma(\varphi(C)) > 0$

Then :

. φ is injective on $C^\#$, i.e. $C^\# = \dot{C}$

. The "distance in $\varphi(C)$" $\delta(x^\#,y^\#)$ of two connected equivalence classes, defined in (4.3.1) satisfies :

(4.4.9) $\quad \| \varphi(x^\#)-\varphi(y^\#) \|_F \leq \delta(x^\#,y^\#) \leq \left(1 - \frac{d}{\bar{R}}\right)^{-1} \| \varphi(x^\#) - \varphi(y^\#) \|_F$

as soon as :

(4.4.10) $\quad \| \varphi(x^\#)-\varphi(y^\#) \|_F \leq d < \gamma$

ii) If moreover $z \in F$ satisfies the "error-size" condition

(4.4.11) $\quad d(z,\varphi(C)) < \gamma$

Then :

. There exists at most one connected equivalence class $\hat{x}^\#$ solution of the non-linear least square problem (4.0.2).

. Connected equivalence classes $y^\#$ corresponding to a local minimum of J yield a value $J(y^\#)$ larger than or equal to γ.

. The $z \rightarrow \hat{x}^\#$ mapping, when defined, is Lipschitz continuous : if z_0,z_1 satisfy (4.4.11) and if the corresponding non-linear least-square problems (4.0.2) admit solutions $\hat{x}^\#_0$, $\hat{x}^\#_1$, one has :

(4.4.12) $\quad \delta(\hat{x}^\#_0,\hat{x}^\#_1) \leq \left(1 - \frac{d}{\bar{R}}\right)^{-1} \| z_0-z_1 \|_F$

as soon as :

(4.4.13) $\quad \| z_0-z_1 \|_F + \underset{j=0,1}{\text{Max}} \ d(z_j,\varphi(C)) \leq d < \gamma$

iii) If moreover

(4.4.14) $\quad \varphi(C)$ is closed in F

Then :

The non-linear least square problem (4.0.2) has a unique solution $\hat{x}^\#$ in the set $C^\#$ of connected equivalence classes, and any minimizing sequence $\{x^\#_n\}$ of J over $C^\#$ is a Cauchy sequence for the "distance in $\varphi(C)$" $\delta(x^\#,y^\#)$, and :

(4.4.15) $\quad \delta(x^\#_n,\hat{x}^\#) \rightarrow 0 \quad$ when $n \rightarrow \infty$

This theorem shows that the size × curvature condition $\gamma > 0$ implies the well posedness of the non-linear least square problem (4.0.2) stated in term of connected equivalence classes $x^\# \in C^\#$. Of course, when $C^\# = C$ (as for example in propositions 4.3 and 4.4 of paragraph 4.2) this yields to the well-posedness of problem (4.0.2) itself.

We compare now the results obtained in theorem 4.18 for the non-linear least square problem (4.0.2) to the well-known stability results for the linear least square problem (4.0.3), (4.0.4), which read :

(4.4.16) $$\begin{cases} \| \Phi(\hat{x}_0-\hat{x}_1) \|_F \leq \| z_0-z_1 \|_F \\ \text{where } \hat{x}_j \text{ is any solution of (4.0.3) for } z = z_j \end{cases}$$

This in turn implies, when :

(4.4.17) $\exists\alpha > 0$ s.t. $\| \Phi y^\perp \|_E \geq \alpha \, \|y^\perp \|_F \qquad \forall y \in \mathrm{Ker}\Phi^\perp$

the following stability estimate in the E norm for the equivalence classes solution of the linear least square problem (4.0.3) :

(4.4.18) $\alpha \| \hat{x}^\perp_0 - \hat{x}^\perp_1 \|_E \leq \| z_0-z_1 \|_F$

where $\hat{x}^\perp_j$ denotes the minimum norm solution of problem (4.0.3) with $z = z_j$.

Finally, when Φ is more over injective, i.e. when :

(4.4.19) $\exists\alpha > 0$ s.t. $\| \Phi y \|_F \geq \alpha \, \|y \|_E \qquad \forall y \in E$

then one has the following stability estimate for the solution of the linear least square problem (4.0.3) :

(4.4.20) $\alpha \| \hat{x}_0 - \hat{x}_1 \|_E \leq \| z_0-z_1 \|_F$.

If we come back now to the results for the non-linear least square problem (4.0.2) stated in theorem 4.18, we see first that the stability estimate (4.4.12), (4.4.13) is exactly the non-linear counter part to (4.4.16), to which it reduces exactly when φ happens to be linear (see example 4.8).

The non-linear counter part to (4.4.18) is obtained from (4.2.12) and proposition 4.14 i) under the hypothesis (4.3.16) (which is to be compared to (4.4.17)), and reads :

(4.4.21) $$\alpha \, d(x^\#,y^\#) \leq \left(1 - \frac{d}{R}\right)^{-1} \| z_0-z_1 \|_F$$

which once again reduces exactly to (4.4.18) when φ happens to be linear (use propostion 4.12 i).

Finally, the counter part to (4.4.20) is obtained from (4.2.12) and proposition 4.14 under the hypothesis (4.3.20) (which is to be compared to (4.4.19)), and reads :

(4.4.22) $$\alpha \| \hat{x}_0 - \hat{x}_1 \|_E \leq \left(1 - \frac{d}{R}\right)^{-1} \| z_0-z_1 \|_F$$

which once again reduces exactly to (4.4.20) when φ happens to be linear (use proposition 4.12 ii)).

To conclude, we summarize the results of theorem 4.18 in the case where (4.3.20) holds :

<u>Corollary 4.19</u>

Let C,φ satisfy (4.0.1), and choose :

(4.4.23) $\Pi = \{$ segments of $C\}$

<u>If</u> :

(4.4.24) φ satisfies (4.3.20)

(4.4.25) C,Π,φ satisfy the size × curvature condition (4.4.8)

(4.4.26) C is closed in E

Then :

. φ is injective on C

. For any $z \in F$ such that :

(4.4.27) $\quad d(z,\varphi(C)) < \gamma \quad$ (error-size condition)

the non-linear least square problem (4.0.2) has a unique solution $\hat{x} \in C$, all possible local minima yield for J a value larger than or equal to γ, and any minimizing sequence $\{x_n\}$ of J is converging to $\hat{x}$ in E.

. The $z \to \hat{x}$ mapping thus defined is Lipschitz-continuous : the solutions $\hat{x}_0$ and $\hat{x}_1$ of (4.0.2) corresponding to the data z_0 and z_1 satisfy :

$$(4.4.28) \qquad \alpha \, \| \hat{x}_0 - \hat{x}_1 \|_E \leq \left(1 - \frac{d}{\bar{R}}\right)^{-1} \| z_0 - z_1 \|_F$$

as soon as :

$$(4.4.29) \qquad \| z_0 - z_1 \|_F + \underset{j=0,1}{\mathrm{Max}} \; d(z_j, \varphi(C)) \leq d < \gamma$$

Proof : We see from (4.4.24) and from example 4.9 that Π defined by (4.4.23) satisfies (4.2.1 thru 4). Hence we can apply theorem 4.18, provided we prove that $\varphi(C)$ is closed in F. So let $\{X_n \in \varphi(C)\}$ be converging in F towards $X \in F$. From proposition 4.3 and part i) of theorem 4.18, we know that φ is injective on C, so that we can define uniquely a sequence $\{x_n \in C\}$ by $x_n = \varphi^{-1}(X_n)$. The converging sequence $\{X_n\}$ is necessarily a Cauchy sequence in F, which implies using (4.4.9) that $\{x_n\}$ is a Cauchy sequence of C for the "distance in $\varphi(C)$" $\delta(x,y)$, which in turn implies, using proposition 4.14 that $\{x_n\}$ is a Cauchy sequence in E. But E is an Hilbert space, hence complete, and C is closed, so that there exists $\hat{x} \in C$ such that $x_n \to \hat{x}$. Hence, as φ is continuous, $X_n = \varphi(x_n) \to \varphi(\hat{x})$, which implies that $X = \varphi(\hat{x}) \in \varphi(C)$, so that $\varphi(C)$ is closed. ∎

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APPENDIX

Proof of Proposition 4.4.

Let $X \in \varphi(C)$ be given, $x^{\#}$ be one connected equivalence class of $\varphi^{-1}(X)$, and x_0 be one point of $x^{\#}$.

We shall use throughout the proof the following decomposition of the Hilbert space E :

$T = \text{Ker}\ \varphi'(x_0)$

$N = T^{\perp}$ = orthogonal supplementary space to T

which are two closed subspaces of E, equipped with the norm induced from E. Moreover T is finite dimensional from hypothesis (4.1.7).

We make now the hypothesis :

(A.1) Suppose that $x^{\#}$ is not reduced to $\{x_0\}$

and take three steps to obtain a contradiction :

Step 1 : T contains at least one non-zero $y \in E$

Define a sequel $\{x_n \in x^{\#}, n=1,2...\}$ by recurrence as follows :

- for n=1 we choose for x_1 any point of $x^{\#}$ distinct from x_0 (such point necessarily exists using hypothesis (A.1)).
- given $x_n \in x^{\#}, x_n \neq x_0$, we define x_{n+1} as any point of $x^{\#}$ distinct from x_0 and lying in the ball centered at x_0 and with radius $1/2\ \|x_n - x_0\| > 0$. If there were no such x_{n+1}, this would imply that the ball $B(x_0, 1/2\ \|x_n - x_0\|)$ contains no other point of $x^{\#}$ than its center x_0, which is contradictory to the fact that x_0 and x_n are in the same connected component $x^{\#}$.

We define then $y_n \in E$ as :

$y_n = (x_n - x_0)/\|x_n - x_0\|$ so that

$\|y_n\| = 1$

and project it on T and N :

$y_n = y_{nT} + y_{nN}$, with

$y_{nT} \in T$, $y_{nN} \in N$.

Using the derivability of φ at x_0 and the fact that $\varphi(x_0) = \varphi(x_n) = X$, we obtain

$\varphi'(x_0).y_n + R\ (x_n - x_0) = 0$ with

$R\ (x_n - x_0) \to 0$ when $n \to \infty$.

Using the decomposition of y_n on T and N and hypothesis (4.1.8) yields :

$\alpha\|y_{nN}\| \leq \|R\ (x_n - x_0)\|$ so that

$\|y_{n,N}\| \to 0$ when $n \to \infty$.

This implies, as $\|y_{n,T}\|^2 + \|y_{n,N}\|^2 = \|y_n\|^2 = 1$, that $\|y_{n,T}\|^2 \to 1$ when $n \to \infty$.
As T is finite dimensional, we can find a subsequence $y_{n,T}$ and element $y \in T$ such that :

$$y_{n,T} \to y \text{ (strongly) in T, hence } \|y\| = 1$$

which prooves that T contains at least the non-zero element y.

Step 2 : Write the variety $x^\#$ of E as the graph of a C^2-mapping g from T to N locally around x_0

Using the decomposition x = t+n of any vector x of E on T and N, and the fact that $x^\#$ is determined, locally around x_0, by the equation $\varphi(x) = X$, we consider the equation :

$$\phi(t,n) = X \qquad \text{where } \phi(t,n) \text{ is defined by}$$

$$\phi(t,n) = \varphi(t+n)$$

The ϕ function is C^2 and satisfies :

$$\left\| \frac{\partial\phi}{\partial n} (t_0,n_0).\delta n \right\|_F \geq \alpha \, \| \delta n \|_E \quad \forall \delta n \in N, \quad \text{where } x_0 = t_0 + n_0$$

Hence we can use the implicit function theorem to express n as a function of t in the neighborhood of t_0,n_0 in the equation $\varphi(t+n) = X$:

there exists an open ball B_T of T centered at t_0,
and a C^2-mapping $g : B_T \to N$, such that :

$$g(t_0) = n_0$$

$$\varphi(t+g(t)) = X \qquad \forall t \in B_T$$

$$\| \varphi'(t+g(t)).\delta n \| \geq \frac{\alpha}{2} \| \delta n \| \qquad \forall t \in B_T \quad \forall \delta n \in N$$

Step 3 : g is an affine function from B_T to N, i.e. $x^\#$ is an affine variety around x_0

For any $t \in B_T$ we define :

$$x(t) = t+g(t) \in x^\#$$

For any $\delta t \in T$ we obtain, deriving twice in the δt direction :

$$\delta x(t) = x'(t).\delta t = \delta t + g'(t).\delta t$$

$$\delta^2 x(t) = x''(t).(\delta t,\delta t) = g''(t)(\delta t,\delta t)$$

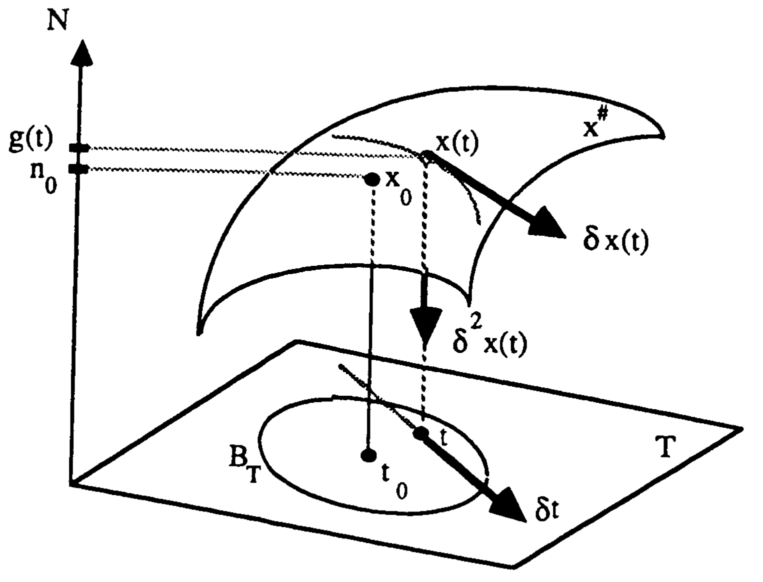
But as $x(t) \in x^\#$ we have :

$$\varphi(x(t)) = 0 \qquad \forall t \in B_T$$

which we derivate twice in the δt direction :

$$\varphi'(x(t)).\delta x(t) = 0$$

$$\varphi''(x(t))(\delta x(t),\delta x(t)) + \varphi'(x(t)).\,\delta^2 x(t) = 0$$



But using hypothesis (4.1.9) we see that $\varphi'(x(t)).\delta x(t) = 0$ implies that $\varphi''(x(t))(\delta x(t),\delta x(t)) = 0$ too, so that the last above equation reduces to :

$$\varphi'(x(t)).\ \delta^2 x(t) = 0, \qquad \text{or}$$

$$\varphi'(x(t)).[g''(t)(\delta t,\delta t)] = 0$$

But $g''(t)(\delta t,\delta t) \in N$, and from the last formula of step 2 we see that :

$$g''(t)(\delta t,\delta t) = 0 \qquad \forall\ \delta t \in T$$

As one has, fro any $\delta t,\ \delta u \in T$:

$$2g''(t)(\delta t,\delta u) = g''(t)(\delta t+\delta u,\delta t+\delta u) - g''(t)(\delta t,\delta t) - g''(t)(\delta u,\delta u)$$

we find that :

$$g''(t)(\delta t,\delta u) = 0 \qquad \forall\ \delta t,\delta u \in T$$

and hence $g''(t) = 0$ for any $t \in B_T$, which proves that g is an affine function on B_T.

Then the results of Step 1 and 3 are obviously in contradiction with the second part of hypothesis (4.1.9) : let y be the vector defined in Step 1, which satisfies :

$$y \in T, \text{ i.e. } \varphi'(x_0).y = 0 \ \text{ and } y \neq 0$$

which, as $x^\#$ is affine around x_0, means that y is in the vector space associated to the affine variety $x^\#$. Hence we have $\varphi'(x_0+\nu y).y = 0$ for ν small enough, which contradicts (4.1.9).

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