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HOMOGÉNÉISATION D' UN MODÈLE D' ÉCOULEMENTS MISCIBLES COMPRESSIBLES EN MILIEU POREUX

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Abstract

We consider a 1-D model of flow of two miscible fluids with same compressibility factor in an heteregeneous porous media. We study the existence and homogenization of the parabolic-hyperbolic system governing this flow and show the stability of the derivation of the macroscopic level .

Résumé

On considère un modèle 1-D d' écoulements de deux fluides miscibles ayant même facteur de compressibilité. On étudie l' existence et l' homogénéisation du système parabolic-hyperbolique gouvernant de tels écoulements, et on montre que le problème homogénéisé est de même type.

Key-words

Characteristics, Homogenization, Hyperbolic, Oscillations, Parabolic, Porous Media.

Mots-Clés

Caractéristiques, Homogénéisation, Hyperbolique, Milieu Poreux,
Oscillations, Parabolique.

0. Introduction

With the work of DARCY [10] starts the theory of fluid flow in porous media. From the description he gave of the relationship between pressure and flux of a fluid in a homogeneous porous media, several attempts to generalize the concept to multiphase flow have been made, we refer to BEAR [5], HOUPEURT [13], MATHERON [19], MARLES [18], MUSKAT [21], SCHEIDEGGER [24] and SLATTERY [26].

The objective to give a clear formulation and mathematical description is not achieved considering the complexity of phenomena occuring in a media in presence of a mixture of several fluids and solids.

To derive a macroscopic model for different flows in a heteregeneous porous media various tools have been used,such as the averaging methods and theory of homogenization which are developed in AURIAULT-SANCHEZ PALENCIA [3], BAKHVALOV-PANASENKO [4], MARCHENKO-HROUSLOV [16] and SANCHEZ PALENCIA [23]. Another approach using stochastic analysis and averaging using Markov processes lead to some results in MATHERON [19], CAFLISCH-RUBINSTEIN [8], RUBINSTEIN [22], FIGARI-PAPANICOLAOU-RUBINSTEIN [12] and SHVIDLER [25].

This paper deals with a simpler model in one dimension of the flow of two miscible fluids in a heteregeneous porous media. The aim is to derive a set of effective equation from the microscopic level.

One 1-D non linear model for incompressible miscible displacement has been considered in [1], we have proved the stability of the Darcy law with respect to the oscillations of some parameters.

The homogenization of the first order, linear hyperbolic equation studied in [2] induces for the limiting equation satisfied by the concentration of the fluid some memory effects due to dispersion phenomena. In what follows, we consider two miscible fluids. The composition of the mixture is given by the mass concentration. We neglect the molecular diffusion and dispersion and omit the gravitational terms. The viscosity is assumed constant and the fluids have the same constant compressibility factor. The porosity ϕ and permeability k of the rock are oscillating functions of the spatial variable x and depend on a parameter $\epsilon > 0$ connected with the microstructure.

If the functions p, q and u stand for pressure, rate of flow and concentration of mass respectively,the flow is governed by the following system of parabolic-hyperbolic type : (see CHAVENT-JAFFRE [9], DOUGLAS-ROBERTS [11])

$\Omega =]0,1[$, T is a fixed positive number and $\Omega_T =]0,1[\times\Omega$

$(p^\epsilon, q^\epsilon, u^\epsilon)$ satisfies :

$$(P_\epsilon)\begin{cases} \phi^\epsilon(x)\dfrac{\partial}{\partial t}p^\epsilon(t,x) + \dfrac{\partial}{\partial x}q^\epsilon(t,x) = 0 \\ q^\epsilon(t,x) = -k^\epsilon(x)\dfrac{\partial}{\partial x}p^\epsilon(t,x) \quad (t,x)\in\Omega_T \\ q^\epsilon(t,0) = q_1^\epsilon(t), \quad q^\epsilon(t,1) = q_2^\epsilon(t) \quad t\in]0,T[\\ p^\epsilon(0,x) = p_0^\epsilon(x) \quad x\in\Omega \end{cases}$$

$$(H_\epsilon)\begin{cases} \phi^\epsilon(x)\dfrac{\partial}{\partial t}u^\epsilon(t,x) + q^\epsilon(t,x)\dfrac{\partial}{\partial x}u^\epsilon(t,x) = 0 & (t,x)\in\Omega_T \\ 0\le u^\epsilon(t,x)\le 1 \\ u^\epsilon(t,0) = u_1^\epsilon(t), \quad 0\le u_1^\epsilon(t)\le 1 & t\in]0,T[\\ u^\epsilon(0,x) = u_0^\epsilon(x), \quad 0\le u_0^\epsilon(x)\le 1 & x\in\Omega \end{cases}$$

The boundary data $q_1^\epsilon, q_2^\epsilon, u_1^\epsilon$ and the initial data p_0^ϵ and u_0^ϵ are taken dependent on ϵ, this dependence will be specified further.The boundary condition for u^ϵ is given for $x = 0$ because of the positivity of q^ϵ as will be seen.

We want to analyse the influence of heteregeneities - caracterized by the parameter ϵ - on the effective behavior of the pressure, flux and mass concentration.

The summary of the paper is the following :

In section 1 we consider the case of a periodic microstructure and derive formally the equations obtained asymptotically for the pressure and concentration. In section 2 we recall some properties concerning a parabolic equation with discontinuous coefficients and study the behavior of the flux q^ϵ and derive the homogenized equation for q. The third section is devoted to the existence of a weak solution for the problem (H^ϵ) and in section 4 we look for the limiting equation for u^ϵ.

In a forthcoming paper we will discuss the more realistic model where the two fluids have different compressibility factors.

1. Asymptotic study of media with periodic microstructure

In this section we use the asymptotic methods introduced by BENSOUSSAN- LIONS-PAPANICOLAOU [6], LIONS [15] and SANCHEZ PALENCIA [23] to derive the limiting equations in the case of a periodic structure, namely we assume that the coefficients ϕ^ϵ and k^ϵ are given by :

$$\begin{cases} \phi^\epsilon(x) = \phi(x, \dfrac{x}{\epsilon}), \\ k^\epsilon(x) = k(x, \dfrac{x}{\epsilon}), \end{cases} \tag{1.1}$$

where the functions $\phi(x,y)$ and $k(x,y)$ are 1-periodic with respect to the y variable .

We look for the pressure $p^\epsilon(t,x)$ in the form of an asymptotic expansion :

$$p^\epsilon(t,x) = p^0(t,x,y) + \epsilon p^1(t,x,y) + \epsilon^2 p^2(t,x,y) + \dots$$

where $y = \dfrac{x}{\epsilon}$, and $p^i(t,x,y)$ is a periodic functions of the variables y for $i = 0,1,\dots$.

Using the equation :

$$\phi^\epsilon(x)\frac{\partial}{\partial t}p^\epsilon(t,x) - \frac{\partial}{\partial x}(k^\epsilon(x)\frac{\partial}{\partial x}p^\epsilon(t,x)) = 0 \tag{1.2}$$

and expanding we obtain the following relations :

(1.3)
$$\begin{cases} \frac{\partial}{\partial y}(k(x,y)\frac{\partial}{\partial y}p^0) = 0, \\ \frac{\partial}{\partial y}(k(x,y)\frac{\partial}{\partial y}p^1) + \frac{\partial}{\partial y}(k(x,y)\frac{\partial}{\partial x}p^0) + \frac{\partial}{\partial x}(k(x,y)\frac{\partial}{\partial y}p^0) = 0, \\ \phi(x,y)\frac{\partial}{\partial t}p^0 - \frac{\partial}{\partial y}(k(x,y)\frac{\partial}{\partial y}p^2) \\ = \{\frac{\partial}{\partial y}(k(x,y)\frac{\partial}{\partial x}p^0) + \frac{\partial}{\partial y}(k(x,y)\frac{\partial}{\partial x}p^1) + \frac{\partial}{\partial x}(k(x,y)\frac{\partial}{\partial y}p^1)\}, \\ etc\ldots \end{cases}$$

The first one implies that p^0 doesn't depend on y so $p^0 = p^0(t,x)$. The second one implies that p^1 is well known :

$$p^1(t,x,y) = \chi(x,y)\frac{\partial}{\partial x}p^0(t,x) + \tilde{p}^1(t,x),$$

where χ is the unique solution of :

(1.4)
$$\begin{cases} -\frac{\partial}{\partial y}(k(x,y)\frac{\partial}{\partial y}\chi) = \frac{\partial}{\partial y}k(x,y), \\ \chi \text{ 1- periodic on } y \quad \text{and } \int_0^1 \chi(x,y)\,dy = 0. \end{cases}$$

The periodicity of p^2 leads to the following homogenized equation :

(1.5)
$$\tilde{\phi}(x)\frac{\partial}{\partial t}p^0 - \frac{\partial}{\partial x}\{\int_0^1 k(x,y)(1+\frac{\partial}{\partial y}\chi(x,y))\,dy\frac{\partial}{\partial x}p^0\} = 0.$$

where $\tilde{\phi}(x) = \int_0^1 \phi(x,y)\,dy$. If we assume that the data are independent of ϵ then p^0 is well defined and also $p^l(t,x,y)$ $l \geq 1$.

Therefore the flux q^ϵ is given by :

(1.6)
$$q^\epsilon(t,x) = q^0(t,x,y) + \epsilon q^1(t,x,y) + \epsilon^2 q^2(t,x,y) + \ldots$$

where $q^l(t,x,y) = -k(x,y)[\frac{\partial}{\partial x}p^l + \frac{\partial}{\partial y}p^{l+1}]$ $l \geq 0$.

As for the pressure we assume that the concentration u^ϵ also admits the following asymptotic expansion in ϵ :

(1.7)
$$u^\epsilon(t,x) = u^0(t,x,y) + \epsilon u^1(t,x,y) + \epsilon^2 u^2(t,x,y) + \ldots$$

Plugging (1.6) and (1.7) into the transport equation :

(1.8)
$$\phi^\epsilon(x)\frac{\partial}{\partial t}u^\epsilon + q^\epsilon(t,x)\frac{\partial}{\partial x}u^\epsilon = 0.$$

we get the following system of equations

$$
(1.9)\qquad \begin{cases} q^0(t,x,y)\dfrac{\partial}{\partial y}u^0 = 0, \\ \phi(x,y)\dfrac{\partial}{\partial t}u^0 + q^0(t,x,y)[\dfrac{\partial}{\partial x}u^0 + \dfrac{\partial}{\partial y}u^1] = 0 \\ etc\ldots \end{cases}
$$

As $q^0 \neq 0$ the first equation implies $u^0 = u^0(t,x)$,the second one can be written as

$$
(1.10)\qquad \frac{\phi(x,y)}{q^0(t,x,y)}\frac{\partial}{\partial t}u^0 + \frac{\partial}{\partial x}u^0 = -\frac{\partial}{\partial y}u^1.
$$

Using the periodicity of u^1 in y and integrating over the interval (0,1) we find :

$$
(1.11)\qquad \int_0^1 \frac{\phi(x,y)}{q^0(t,x,y)}\,dy\,\frac{\partial}{\partial t}u^0 + \frac{\partial}{\partial x}u^0 = 0
$$

Taking into account the value of q^0 in terms of p^0

$$
q^0(t,x,y) = -k(x,y)[1 + \frac{\partial}{\partial y}\chi]\,\frac{\partial}{\partial x}p^0
$$

we get the macroscopic equation for the concentration :

$$
(1.12)\qquad \int_0^1 \frac{\phi(x,y)}{k(x,y)[1+\frac{\partial}{\partial y}\chi]}\,dy\,\frac{\partial}{\partial t}u^0 + \frac{\partial}{\partial x}p^0\,\frac{\partial}{\partial x}u^0 = 0.
$$

So we have proved, in this setting that the homogenized equations (1.5) and (1.12) have the same form as the microscopic one (1.2),(1.8).

Remark 1.1

If the data $p^\epsilon(x,0) = p_0^\epsilon(x)$ satisfies :

$$
(1.13)\qquad -\frac{\partial}{\partial x}(k^\epsilon\frac{\partial}{\partial x}p_0^\epsilon) = g \quad \text{with} \quad g \in L^2(\Omega),
$$

then we will see later in §2 that the pressure p^ϵ is such that

$$
\frac{\partial}{\partial t}p^\epsilon \in L^2(\Omega_T),
$$

and therefore we obtain :

$$
q^\epsilon(t,x) \to \tilde{q}^0(t,x) = -\int_0^1 k(x,y)[1 + \frac{\partial}{\partial y}\chi]\,dy\,\frac{\partial}{\partial x}p^0 \qquad \text{in } L^2(\Omega_T).
$$

(1.6) is then replaced by

(1.6′)
$$q^\epsilon(t,x) = \tilde{q}^0(t,x) + \epsilon q^1(t,x,y) + \epsilon^2 q^2(t,x,y) + \dots$$

hence (1.12) by

(1.12′)
$$\tilde{\phi}(x)\frac{\partial}{\partial t}u^0 + \tilde{q}^0(t,x)\frac{\partial}{\partial x}u^0 = 0.$$

The case where the initial data doesn't fulfill such a matching condition leads to a boundary layer (see [23]).

2. Homogenization of a problem of parabolic type

In the sequel we consider the problem :

(2.1)
$$\begin{cases} \phi^\epsilon(x)\frac{\partial}{\partial t}p^\epsilon(t,x) + \frac{\partial}{\partial x}q^\epsilon(t,x) = 0, \\ q^\epsilon(t,x) = -k^\epsilon(x)\frac{\partial}{\partial x}p^\epsilon(t,x) \quad (t,x) \in \Omega_T, \\ q^\epsilon(0,t) = q_1^\epsilon(t),\ q^\epsilon(1,t) = q_2^\epsilon(t) \quad t \in]0,T[, \\ p^\epsilon(x,0) = p_0^\epsilon(x) \quad x \in \Omega, \end{cases}$$

with the asumptions :

(2.2)
$$\begin{cases} \phi^\epsilon,\ k^\epsilon \ \in\ L^\infty(\Omega) \\ 0 < \phi_- \le \phi^\epsilon \le \phi^+, \quad 0 < k_- \le k^\epsilon \le k^+. \end{cases}$$

(2.3)
$$\begin{cases} \text{There exist a constant } \ C > 0 \text{ such that for all } \epsilon > 0 \\ \|q_i^\epsilon\|_{W^{1,\infty}(0,T)} \le C \quad i = 1,2 \quad \|p_0^\epsilon\|_{H^1(\Omega)} \le C \end{cases}$$

The homogenization of probleme of type (2.1) has been treated by SPAGNOLO [27] in the framework of G-convergence and also by MARKOV-OLEINIK [17] in a more general setting with the same motivation to investigate the properties of disperse media.

As the flux q^ϵ appears as a coefficient in the transport equation it is important to point out more precisely its properties for the hypothesis (2.2),(2.3).

Since
$$\frac{\partial}{\partial x}p^\epsilon = -\frac{1}{k^\epsilon}q^\epsilon \quad \text{and} \quad \frac{\partial}{\partial t}p^\epsilon = -\frac{1}{\phi^\epsilon}\frac{\partial}{\partial x}q^\epsilon$$

we can formulate a problem satisfied by q^ϵ :

(2.4)
$$\begin{cases} \frac{1}{k^\epsilon}\frac{\partial}{\partial t}q^\epsilon - \frac{\partial}{\partial x}[\frac{1}{\phi^\epsilon(x)}\frac{\partial}{\partial x}q^\epsilon] = 0 \quad in\ \Omega_T, \\ q^\epsilon(t,0) = q_1^\epsilon(t), \quad q^\epsilon(t,1) = q_2^\epsilon(t), \quad t \in]0,T[, \\ q^\epsilon(0,x) = -k^\epsilon(x)\frac{\partial}{\partial x}p_0^\epsilon(x) \equiv q_0^\epsilon(x) \quad x \in \Omega. \end{cases}$$

We begin to render the boundary condition homogeneous by introducing for $T > 0$ the elliptic problem :

$$\begin{cases} -\frac{\partial}{\partial x}\left(\frac{1}{\phi^\epsilon}\frac{\partial}{\partial x}Q^\epsilon\right) = 0 \quad in\ \Omega \\ Q^\epsilon(0,t) = q_1^\epsilon(t),\ Q^\epsilon(1,t) = q_2^\epsilon(t), \quad t \in]0,T[. \end{cases} \tag{2.5}$$

The solution is given by :

$$q^\epsilon(t,x) = q_1^\epsilon(t) + \frac{\Psi^\epsilon(x)}{\Psi^\epsilon(1)}[q_2^\epsilon(t) - q_1^\epsilon(t)] \tag{2.6}$$

where $\Psi^\epsilon(x) = \int_0^x \phi^\epsilon(\xi)\, d\xi$ Because of (2.2) and (2.3), we have :

$$Q^\epsilon \text{ belongs to a bounded subset of } W^{1,\infty}(\Omega_T). \tag{2.7}$$

The function v^ϵ defined by :

$$v^\epsilon = q^\epsilon - Q^\epsilon \tag{2.8}$$

is such that :

$$\begin{cases} \frac{1}{k^\epsilon(x)}\frac{\partial}{\partial t}v^\epsilon - \frac{\partial}{\partial x}\left[\frac{1}{\phi^\epsilon(x)}\frac{\partial}{\partial x}v^\epsilon\right] = f^\epsilon \equiv -\frac{1}{k^\epsilon}\frac{\partial}{\partial t}Q^\epsilon \\ v^\epsilon(0,t) = v^\epsilon(1,t) = 0 \quad t \in]0,T[\\ v^\epsilon(x,0) = v_0^\epsilon(x) \equiv q_0^\epsilon(x) - Q^\epsilon(x,0) \quad x \in \Omega \end{cases} \tag{2.9}$$

v^ϵ is a solution of a non-degenerate equation of parabolic type with source term f^ϵ uniformely bounded in $L^\infty(\Omega_T)$ and initial data bounded in $L^2(\Omega)$.

The standard theory of linear parabolic equations (cf LADYZENSKAJA, SOLONNIKOV, URAL'CEVA [14]) shows that for all $T > 0$, v^ϵ exists and is unique in the space $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$.

The classical energy estimate

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{1}{k^\epsilon(x)}|v^\epsilon(t,x)|^2dx + \int_\Omega \frac{1}{\phi^\epsilon(x)}\left|\frac{\partial}{\partial x}v^\epsilon(t,x)\right|^2dx = \int_\Omega f^\epsilon(t,x)v^\epsilon(t,x)dx$$

shows that thanks to (2.2),(2.3) and (2.7),the sequence $\{v^\epsilon\}_{\epsilon>0}$ is uniformly bounded with respect to ϵ in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$.

The solution v^ϵ enjoys the following property

Lemma 2.1

Under the hypothesis (2.2),(2.3) the solution v^ϵ of (2.9) has the regularity :

$$t\frac{\partial}{\partial t}v^\epsilon \text{ belongs to a bounded subset of } L^2(\Omega_T) \tag{2.10}$$

To this end we multiply the equation (2.9) by $t^2 \frac{\partial}{\partial t} v^\epsilon$ and integrate over Ω therefore we have

$$\int_\Omega \frac{t^2}{k^\epsilon(x)} |\frac{\partial}{\partial t} v^\epsilon(t,x)|^2 dx + \frac{1}{2}\int_\Omega t^2 \frac{1}{\phi^\epsilon(x)} \frac{\partial}{\partial t} |\frac{\partial}{\partial x} v^\epsilon(t,x)|^2 dx = \int_\Omega t^2 f^\epsilon(t,x) \frac{\partial}{\partial t} v^\epsilon(t,x) dx$$

so

$$t^2 \frac{1}{\phi^\epsilon(x)} \frac{\partial}{\partial t} |\frac{\partial}{\partial x} v^\epsilon(t,x)|^2 = \frac{\partial}{\partial t}\{t^2 \frac{1}{\phi^\epsilon(x)} |\frac{\partial}{\partial x} v^\epsilon(t,x)|^2\} - 2t \frac{1}{\phi^\epsilon(x)} |\frac{\partial}{\partial x} v^\epsilon(t,x)|^2$$

Integration in time on $(0,T)$ yields :

$$\int_{\Omega_T} \frac{t^2}{\phi^\epsilon(x)} |\frac{\partial}{\partial t} v^\epsilon(t,x)|^2 dx dt + \frac{T^2}{2}\int_\Omega \frac{1}{\phi^\epsilon(x)} |\frac{\partial}{\partial x} v^\epsilon(t,x)|^2 dx$$
$$= \int_{\Omega_T} t^2 f^\epsilon(t,x) \frac{\partial}{\partial t} v^\epsilon(t,x) dx dt + \int_{\Omega_T} \frac{t}{\phi^\epsilon(x)} |\frac{\partial}{\partial x} v^\epsilon(t,x)|^2 dx dt,$$

and $\forall \eta > 0$, $\exists C_\eta > 0$ such that the left hand side is less than

$$T\int_{\Omega_T} \frac{1}{\phi^\epsilon(x)} |\frac{\partial}{\partial t} v^\epsilon(t,x)|^2 dx dt + \eta \int_{\Omega_T} t^2 |\frac{\partial}{\partial t} v^\epsilon(t,x)|^2 dx dt + C_\eta \int_{\Omega_T} t^2 |f^\epsilon(t,x)|^2 dx dt$$

which implies with (2.2) and (2.3) that the sequence $\{v^\epsilon\}_{\epsilon>0}$ satisfies :

$$\|t \frac{\partial}{\partial t} v^\epsilon\|_{L^2(\Omega_T)} \leq C$$

where C is a constant independent of ϵ.

Remark 2.2

From (2.9) and (2.10) we deduce that for all $\delta > 0$:

$$\frac{\partial}{\partial x}[\frac{1}{\phi^\epsilon} \frac{\partial}{\partial x} v^\epsilon] \in L^2(]\delta, T[\times\Omega),$$

and then

$$[\frac{1}{\phi^\epsilon} \frac{\partial}{\partial x} v^\epsilon] \in L^2(]\delta, T[, H_0^1(\Omega)),$$

so $\frac{\partial}{\partial x} v^\epsilon$ belongs to a bounded set of $L^2(\delta, T; L^\infty(\Omega))$, which with (2.7) implies that :

$$\|\frac{\partial}{\partial x} q^\epsilon\|_{L^2(\delta,T;L^\infty(\Omega))} \leq C, \tag{2.11}$$

where C is a constant independent of ϵ.

To obtain an estimate of $\frac{\partial}{\partial t}v^\epsilon$ in $L^2(\Omega_T)$, we actually need an additional condition on the data p_0^ϵ. Indeed, from (2.9) we get, after multiplication by $\frac{\partial}{\partial t}v^\epsilon$ and integration over Ω and from 0 to t,

$$\int_\Omega \frac{1}{k^\epsilon(x)}|\frac{\partial}{\partial t}v^\epsilon(t,x)|^2dx + \frac{1}{2}\frac{d}{dt}\int_\Omega \frac{1}{\phi^\epsilon(x)}|\frac{\partial}{\partial x}v^\epsilon(t,x)|^2dx = \int_\Omega f^\epsilon(t,x)\frac{\partial}{\partial t}v^\epsilon(t,x)dx$$

$$\begin{aligned}&\int_0^t\int_\Omega \frac{1}{k^\epsilon(x)}|\frac{\partial}{\partial t}v^\epsilon(\sigma,x)|^2dxd\sigma + \frac{1}{2}\int_\Omega \frac{1}{\phi^\epsilon(x)}|\frac{\partial}{\partial x}v^\epsilon(t,x)|^2dx\\ &=\int_\Omega f^\epsilon(\sigma,x)\frac{\partial}{\partial t}v^\epsilon(\sigma,x)dxd\sigma + \frac{1}{2}\int_\Omega \frac{1}{\phi^\epsilon(x)}|\frac{\partial}{\partial x}v^\epsilon(0,x)|^2dx\end{aligned}$$

Making use of (2.4) (2.9) (2.2) and (2.3) we have :

$$\frac{\partial}{\partial x}v^\epsilon(0,x) = -\frac{\partial}{\partial x}(k^\epsilon(x)\frac{\partial}{\partial x}p_0^\epsilon) - \frac{\partial}{\partial x}Q^\epsilon(0,x).$$

In order that the term

$$\int_\Omega \frac{1}{\phi^\epsilon(x)}|\frac{\partial}{\partial x}v^\epsilon(0,x)|^2dx$$

be uniformly bounded with respect to ϵ it is sufficient that p_0^ϵ satisfies an elliptic problem

$$-\frac{\partial}{\partial x}(k^\epsilon(x)\frac{\partial}{\partial x}p_0^\epsilon) = g^\epsilon \tag{2.12}$$

for any sequence with $\|g^\epsilon\|_{L^2(\Omega_T)} \leq C$. Consequently we obtain :

$$\|\frac{\partial}{\partial t}v^\epsilon\|_{L^2(\Omega_T)} \leq C, \tag{2.13}$$

where C is a constant independent of ϵ.

Let

$$\gamma^\epsilon = min\{\min_{t\in]0,T[} q_1^\epsilon(t), \min_{t\in]0,T[} q_2^\epsilon(t), \min_{x\in\Omega} q_0^\epsilon(x)\}$$

$$\Gamma^\epsilon = max\{\max_{t\in]0,T[} q_1^\epsilon(t), \max_{t\in]0,T[} q_2^\epsilon(t), \max_{x\in\Omega} q_0^\epsilon(x)\}$$

Using the maximum principle (cf [14]) we have :

Lemma 2.3

The solution q^ϵ of (2.4) verifies the following property :

$$\min\{\gamma^\epsilon, 0\} \leq q^\epsilon(t,x) \leq \max\{0, \Gamma^\epsilon\} \tag{2.14}$$

for almost all $(t,x) \in \Omega_T$

If $\gamma^\epsilon \geq \gamma > 0$ then $q^\epsilon(t,x) > 0$ which implies that the hyperbolic problem (H_ϵ) is well posed for the boundary condition at $x = 0$.

Remark 2.4

The hypotheses (2.2) (2.3) imply that we have the energy estimate for pressure p^ϵ :

$$\|\frac{\partial}{\partial t} p^\epsilon\|_{L^2(\Omega_T)} \leq C \tag{2.15}$$

where C is a constant independent of ϵ.

we have also :

$$\|\frac{\partial^2}{\partial t^2} p^\epsilon\|_{L^2(\Omega_T^\delta)} \leq C, \qquad \|\frac{\partial^2}{\partial t \partial x} p^\epsilon\|_{L^2(\Omega_T^\delta)} \leq C, \tag{2.16}$$

where C is a constant independent of ϵ and $\Omega_T^\delta =]\delta, T[\times \Omega$.

Assume that

$$\phi^\epsilon \to \phi, \quad \frac{1}{k^\epsilon} \to \frac{1}{k} \quad in\ L^\infty(\Omega)\ weak * . \tag{2.17}$$

From (2.3) we deduce :

$$q_i^\epsilon \to q_i \quad in\ L^\infty(O,T) \quad i = 1,2,$$

$$q_0^\epsilon \rightharpoonup q_0 \quad in\ L^2(\Omega) \quad weakly.$$

Using the techniques introduced by MURAT [20], TARTAR [28] and the result of MARKOV-OLEINIK [17], we obtain the :

Theorem 2.5

The sequence $\{v^\epsilon\}_{\epsilon>0}$ of solutions of (2.4) is such that :

$$\begin{cases} q^\epsilon \rightharpoonup q & in\ L^2(0,T; L^2(\Omega)) \text{ weakly}, \\ q^\epsilon \to q & \text{uniformly in } [\delta, T] \times \Omega \quad \forall \delta > 0 \end{cases} \tag{2.18}$$

where q is the unique solution of :

$$(2.19)\qquad \begin{cases} \frac{1}{k(x)}\frac{\partial}{\partial t}q - \frac{\partial}{\partial x}(\frac{1}{\phi(x)}\frac{\partial}{\partial x}q) = 0 \quad in \quad \Omega_T, \\ q(t,0) = q_1(t), \quad q(t,1) = q_2(t) \quad t \in (0,T), \\ q(0,x) = q_0(x) \quad x \in \Omega. \end{cases}$$

Corollary 2.6

The solution q of (2.19) belongs to $C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ and satisfies

$$\|t\frac{\partial}{\partial t}v^\epsilon\|_{L^2(\Omega_T)} \leq C,$$
$$\gamma \leq q(t,x) \leq \Gamma \quad a.e.\,(t,x) \in \Omega_T,$$
$$\text{where} \quad 0 < \gamma \leq \gamma^\epsilon \leq \Gamma^\epsilon \leq \Gamma.$$

Remark 2.7

If the sequence $\{p_0^\epsilon\}_{\epsilon>0}$ satisfies (2.12) and

$$g^\epsilon \rightharpoonup g \quad in\ L^2(\Omega) \quad \text{weakly}$$

then :

$$q_0^\epsilon \rightharpoonup q_0 = -k\frac{\partial}{\partial x}p_0 \quad in\ L^2(\Omega) \quad \text{weakly}$$

where k is defined by (2.17) and p_0 is the solution of the homogenized elliptic problem (2.13) described by :

$$-\frac{\partial}{\partial x}(k\frac{\partial}{\partial x}p_0) = g \quad in\ L^2(\Omega).$$

3. Study of a hyperbolic first order equation in a non-divergence form

The function q^ϵ being defined by (2.4), we want to investigate weak solutions for the problem :

$$(3.1)\qquad \begin{cases} \phi^\epsilon(x)\frac{\partial}{\partial t}u^\epsilon(t,x) + q^\epsilon(t,x)\frac{\partial}{\partial x}u^\epsilon(t,x) = 0\, in \quad \Omega_T, \\ u^\epsilon(t,0) = u_1^\epsilon(t), \quad t \in]0,T[, \\ u^\epsilon(0,x) = u_0^\epsilon(x)\ x \in \Omega, \\ 0 \leq u^\epsilon(t,x) \leq 1 \quad (t,x) \in \Omega_T. \end{cases}$$

As in [1], we introduce the following notion of a weak solution :

Definition 3.1

A function u^ϵ belonging to $L^\infty(\Omega_T)$ is a weak solution of (3.1) if u^ϵ has its values in [0,1] and satisfies :

$$\begin{aligned} &\int_0^T \int_\Omega \{\phi^\epsilon \frac{\partial}{\partial t}\varphi + q^\epsilon(t,x)\frac{\partial}{\partial x}\varphi + \frac{\partial}{\partial x}q^\epsilon(t,x)\varphi\}u^\epsilon(t,x)\,dxdt \\ &= -\int_\Omega \phi^\epsilon(x)u_0^\epsilon(x)\varphi(0,x)\,dx - \int_0^T q_1^\epsilon(t)u_1^\epsilon(t)\varphi(t,0)\,dt \end{aligned} \tag{3.2}$$

for any function $\varphi \in C_0^1([0,T[\times[0,1[)$.

To define such a solution we begin by the study of characteristic curves X^ϵ associated with the first order differential operator :

$$\phi^\epsilon(x)\frac{\partial}{\partial t} + q^\epsilon(t,x)\frac{\partial}{\partial x}. \tag{3.3}$$

For $\epsilon > 0$, we consider the equation of the characteristic curve associated to (3.3)

$$\begin{cases} \dfrac{\partial}{\partial t}X^\epsilon = W^\epsilon(t,x) \equiv \dfrac{q^\epsilon(t,X^\epsilon)}{\phi^\epsilon(X^\epsilon)} \\ X^\epsilon(s) = x, \quad s \in [0,T], \quad x \in \Omega. \end{cases} \tag{3.4}$$

The function $W^\epsilon(t,x)$ doesn't satisfy the Caratheodory conditions for the existence of a solution X^ϵ, we proceed by regularization of coefficients ϕ^ϵ and q^ϵ of (3.3).

For the remainder of this section we drop the subcript ϵ.

Let ϕ^n and q^n two sequences of regular functions such that :

$$\begin{cases} \phi^n \to \phi \quad in\ L^2(\Omega), \\ q^n \to q \quad in\ L^2(\Omega_T), \\ \phi^n \in C^1(\Omega) \quad \phi_- \le \phi^n(x) \le \phi^+, \\ 0 < \gamma \le q^n(t,x) \le \Gamma \quad (t,x) \in \Omega_T, \\ \|\dfrac{\partial}{\partial x}q^n(t,.)\|_{L^\infty(\Omega)} \le a(t) \quad a \in L^2(0,T) \quad \text{independent on } n. \end{cases} \tag{3.5}$$

For each integer n we introduce X^n uniquely defined by

$$\begin{cases} \dfrac{\partial}{\partial t}X^n = \dfrac{q^n(t,X^n)}{\phi^n(X^n)}, \\ X^n(s) = x, \quad s \in [0,T], \quad x \in \Omega. \end{cases} \tag{3.6}$$

If Ψ^n is the antiderivative of ϕ^n given by

$$\Psi^n(x) = \int_0^x \phi^n(\xi)\,d\xi, \tag{3.7}$$

the equation (3.6) can also be written

$$\frac{\partial}{\partial t}\Psi^n(X^n) = q^n(t, X^n), \tag{3.8}$$

which gives after integration the following *weak formulation* of the characteristic curve (3.6)

$$\Psi^n(X^n) = \int_s^t q^n(\sigma, X^n(\sigma; x, s))\,d\sigma + \Psi^n(x). \tag{3.9}$$

Now, we want to derive some properties of the sequence $\{X^n\}_{n\geq 0}$.

First the estimate :

$$\|\frac{\partial}{\partial t}X^n\|_{L^\infty(\Omega_T)} \quad \text{is valid.} \tag{3.10}$$

Differentiate (3.9) with respect to x, to obtain the following relation :

$$\phi^n(X^n)\frac{\partial}{\partial x}X^n = \int_s^t \frac{\partial}{\partial x}q^n(\sigma, X^n(\sigma; x, s))\,d\sigma + \phi^n(x). \tag{3.11}$$

We set

$$w^n(t) = \sup_{x\in\Omega} |\frac{\partial}{\partial x}X^n(t; x, s)|.$$

From the hypothesis (3.5) we deduce the inequality :

$$\phi_- w^n(t) \leq \int_s^t a(\sigma)\,w^n(\sigma)\,d\sigma + \phi^+, \tag{3.12}$$

whence by the Gronwall lemma :

$$w^n(t) \leq b(t) \quad \text{with} \quad b \in L^2(0,T), \tag{3.13}$$

i.e.

$$\|\frac{\partial}{\partial x}X^n\|_{L^2(0,T;L^\infty(\Omega))} \leq C. \tag{3.14}$$

Also we define implicitely $S^n(t, x)$ by :

$$X^n(S^n(t,x);x,t) = 0,\ (t,x) \in \Omega_T, \tag{3.15}$$

which implies

$$\phi^n(X^n)\frac{\partial}{\partial x}X^n = \int_t^{S^n(t,x)} q^n(\sigma, X^n(\sigma;x,s))\,d\sigma + \Psi^n(x) \tag{3.16}$$

From (3.15), we have using (3.14) that

$$\|\frac{\partial}{\partial x}S^n\|_{L^2(0,T;L^\infty(\Omega))} \le C, \quad \|\frac{\partial}{\partial t}X^n\|_{L^\infty(\Omega_T)} \le C. \tag{3.17}$$

Hence, there exist two subsequences, also denoted by X^n, S^n such that

$$X^n \to X, \quad S^n \to X \quad \text{strongly in} \quad L^2(\Omega_T). \tag{3.18}$$

Given the regularity properties (3.5) of the coefficients, the function u^n defined by

$$u^n(t,x) = \begin{cases} u_0^\epsilon(X^n(0;x,t)), & \text{if } x > X^n(t;0,0) \\ u_1^\epsilon(S^n(t,x)), & \text{otherwise,} \end{cases} \tag{3.19}$$

is the classical solution of the problem

$$\begin{cases} \phi^n\frac{\partial}{\partial t}u^n(t,x) + q^n(t,x)\frac{\partial}{\partial x}u^n(t,x) = 0 \quad \text{in} \quad \Omega_T, \\ u^n(t,0) = u_1^\epsilon(t) \quad t \in]0,T[, \\ u^n(0,x) = u_0^\epsilon(x) \quad x \in \Omega. \end{cases} \tag{3.20}$$

Clearly

$$0 \le u^n(t,x) \le 1, \quad (t,x) \in \Omega_T$$

if the data satisfy

$$0 \le u_0^\epsilon,\, u_1^\epsilon \le 1.$$

For the limiting functions X and S, we establish some necessary a-priori estimates independent of ϵ in order to pass to the limit as $\epsilon \to 0$ in the next section

Lemma 3.2

$\forall\epsilon > 0$, X and S satisfy the equations

$$\begin{cases} \Psi(X) = \int_s^t q(\sigma, X(\sigma;x,s))\,d\sigma + \Psi(x), \\ X(S(t,x);x,t) = 0 \quad \text{a.e. in} \quad \Omega_T. \end{cases} \tag{3.21}$$

Lemma 3.3

X and S belong to V_δ for all $\delta > 0$ where

$$V_\delta = \{v \in L^\infty(\Omega_T), \frac{\partial}{\partial t}v \in L^\infty(\Omega_T), \frac{\partial}{\partial x}v \in L^2(\delta, T; L^\infty(\Omega))\}. \tag{3.22}$$

Proof. It is based on the above procedure and on the property (2.11) of q^ϵ and hypothesis (2.2).

Let us define the function u by

$$u(t,x) = \begin{cases} u_0^\epsilon(X(0;x,t)), & \text{if } x > X(t;0,0); \\ u_1^\epsilon(S(t,x)), & \text{otherwise.} \end{cases} \tag{3.23}$$

We have the following result :

Proposition 3.4

The sequence $\{u^n\}_{n\geq 0}$ converges strongly in $L^1(\Omega_T)$ to the weak solution of problem (3.1) given by (3.23) .

Proof. We proceed as in [1], let Ω_0^n,Ω_1^n,Ω_0,Ω_1 be the sets defined by

$$\begin{aligned} \Omega_0^n &= \{(t,x) \in \Omega_T; \quad x > X^n(t;0,0)\}, \\ \Omega_1^n &= \{(t,x) \in \Omega_T; \quad x < X^n(t;0,0)\}, \\ \Omega_0 &= \{(t,x) \in \Omega_T; \quad x > X(t;0,0)\}, \\ \Omega_1 &= \{(t,x) \in \Omega_T; \quad x < X(t;0,0)\}. \end{aligned}$$

and their respective characteristic functions χ_0^n,χ_1^n,χ_0 and χ_1.

By virtue of (3.19),(3.21) we have :

$$\begin{aligned} u^n(t,x) - u(t,x) = {} & (u_0^\epsilon(X^n) - u_0^\epsilon(X))\chi_0^n \\ & + (u_1^\epsilon(S^n) - u_1^\epsilon(S))\chi_1^n \\ & + u_0^\epsilon(X)(\chi_0^n - \chi_0) \\ & + u_1^\epsilon(S)(\chi_1^n - \chi_1) \end{aligned}$$

Note that

$$\chi_i^n \to \chi_i \quad \text{in} \quad L^1(\Omega_T) \quad i = 1,2\,. \tag{3.24}$$

This follows from (3.18) and the fact that

$$\int_{\Omega_T} |\chi_i^n - \chi_i|\, dx\, dt \leq \int_0^T |\chi_i^n(t;0,0) - \chi_i(t;0,0)|\, dx\, dt.$$

It is clear that if u_0^ϵ and u_1^ϵ are regular functions then $u^n \to u$ in $L^2(\Omega_T)$. Otherwise we proceed by regularization of the data u_0^ϵ and u_1^ϵ by considering two sequences $u_{0,m}^\epsilon$ and $u_{1,m}^\epsilon$ such that

$$\begin{cases} u_{0,m}^\epsilon \to u_0^\epsilon & in\, L^1(\Omega), \\ u_{1,m}^\epsilon \to u_1^\epsilon & in\, L^1(0,T) \end{cases}. \tag{3.25}$$

$$\int_\Omega |u_0^\epsilon(X^n) - u_0^\epsilon(X)|\, dx \le \int_\Omega |u_0^\epsilon(X^n) - u_{0,m}^\epsilon(X^n)|\, dx$$
$$+\int_\Omega |u_{0,m}^\epsilon(X^n) - u_{0,m}^\epsilon(X)|\, dx + \int_\Omega |u_{0,m}^\epsilon(X^n) - u_0^\epsilon(X)|\, dx$$

The second term of the left hand side tends to zero as $m \to +\infty$ by virtue of the regularity of u_0^ϵ and (3.24).

To evaluate the first term, the change of variable

$$y = X^n(0; x, t), \quad x = X^n(t; y, 0)$$

leads to the inequality :

$$\int_\Omega |u_0^\epsilon(X^n) - u_{0,m}^\epsilon(X^n)|\, dx \le \int_\Omega |u_0^\epsilon(y) - u_{0,m}^\epsilon(y)|\, |\frac{\partial}{\partial y} X^n(t; y, 0)|\, dy.$$

The a-priori estimate (3.14),Lemma 3.3 and (3.25) imply that

$$\int_\Omega |u_0^\epsilon(X^n) - u_{0,m}^\epsilon(X^n)|\, dx \to 0 \quad m \to +\infty,$$

and

$$\int_\Omega |u_0^\epsilon(X) - u_{0,m}^\epsilon(X)|\, dx \to 0 \quad m \to +\infty,$$

hence

$$\int_\Omega |u_0^\epsilon(X^n) - u_0^\epsilon(X)|\, dx \to 0 \quad n \to +\infty.$$

Following the same pattern and using the properties of S^n and S we deduce the convergence of u^n to u in $L^1(\Omega_T)$.

4. The passage to the limit $\epsilon \searrow 0$ in problem $\mathbf{H}_\epsilon$

The results of section 3 imply that the characteristic curves X^ϵ and functions S^ϵ defined by (3.21) lie in a bounded set of the space V_δ for each $\delta > 0$. Thus two subsequences also denoted X^ϵ, S^ϵ converging in $L^1(\delta, T; L^\infty(\Omega))$ can be extracted :

$$X^\epsilon \to X, \quad S^\epsilon \to S. \tag{4.1}$$

Assume that

$$u_i^\epsilon \to u_i \quad i = 0, 1 \tag{4.2}$$

in a sense that we shall make precise in the sequel.

Let us define a function u by

$$u(t,x) = \begin{cases} u_0(X(0;x,t)), & \text{if } x > X(t;0,0); \\ u_1(S(t,x)), & \text{otherwise.} \end{cases} \tag{4.3}$$

Our objective is to establish the link between u^ϵ and u.

First we have the

Lemma 4.1

The functions X and S given by (4.1). satisfy

$$\begin{cases} \Psi(X) = \int_s^t q(\sigma, X(\sigma;x,s))\,d\sigma + \Psi(x) \\ X(S(t,x);x,t) = 0 \quad a.e. in\, \Omega_T \end{cases} \tag{4.4}$$

Proof. From properties (2.2) and (2.16),we have

$$\Psi^\epsilon(x) = \int_0^x \phi^\epsilon(\xi)\,d\xi \to \Psi(x) = \int_0^x \phi(\xi)\,d\xi \quad a.e. \quad x \in \Omega$$

as $|\Psi^\epsilon(x) - \Psi^\epsilon(y)| \le \phi^+|x-y|$ for $x, y \in \Omega$.

then (4.1) implies that

$$\Psi^\epsilon(X^\epsilon) \to \Psi(X) \quad a.e. \quad (t,x) \in \Omega_T. \tag{4.5}$$

From (2.11) the property

$$|q^\epsilon(t,x) - q^\epsilon(t,y)| \le a(t)|x-y| \quad \text{for } x, y \in \Omega \quad \text{holds.} \tag{4.6}$$

This allow us to pass to the limit in the term $\int_s^t q^\epsilon(\sigma, X^\epsilon(\sigma;x,s))\,d\sigma$.
Indeed

$$R^\epsilon = \int_s^t q^\epsilon(\sigma, X^\epsilon(\sigma;x,s)) - q(\sigma, X(\sigma;x,s))\,d\sigma,$$

$$|R^\epsilon| \le \int_s^t |q^\epsilon(\sigma, X^\epsilon(\sigma;x,s)) - q^\epsilon(\sigma, X(\sigma;x,s))|\,d\sigma$$
$$+ \int_s^t |q^\epsilon(\sigma, X(\sigma;x,s)) - q(\sigma, X(\sigma;x,s))|\,d\sigma$$

as $q^\epsilon(t,x) \to q(t,x)$ almost everywhere in Ω_T.

As shown in (2.18) the second term tends to zero, the first one is bounded by

$$\int_s^t a(\sigma)|X^\epsilon(\sigma;x,s) - X(\sigma;x,s)|\,d\sigma$$

thus the convergence of R^ϵ as $\epsilon \searrow 0$ is assured and the validity of the first part of (4.4) is proved.

The second identity is deduced from (4.1) in a straightforward manner.

Now assume that

$$\text{(4.7)} \qquad \begin{cases} u_0^\epsilon \to u_0 & in\, L^1(\Omega), \\ u_1^\epsilon \to u_1 & in\, L^1(0,T), \\ 0 \le u_0\,, u_1 \le 1. \end{cases}$$

Then the following convergence result is obtained

Theorem 4.2

The sequence $\{u^\epsilon\}_{\epsilon>0}$ of solutions of problem (3.1) defined by (3.23) converges almost everywhere in Ω_T to the function u given by (4.3).

Proof. Note that the sequence considered is that associated with the sequences X^ϵ, S^ϵ defined by (4.1).

Let $I_\delta = \int_\delta^T \int_\Omega |u^\epsilon(t,x) - u(t,x)|\, dxdt$.

First we have the decomposition

$$\chi_0^\epsilon u_0^\epsilon(X^\epsilon) - \chi_0 u_0(X) = \chi_0^\epsilon u_0^\epsilon(X^\epsilon) - \chi_0 u_0^\epsilon(X^\epsilon)$$
$$+\chi_0 u_0^\epsilon(X^\epsilon) - \chi_0 u_0(X^\epsilon) + \chi_0 u_0(X\epsilon) - \chi_0 u_0(X),$$

$$\chi_1^\epsilon u_1^\epsilon(S^\epsilon) - \chi_1 u_1(S) = \chi_1^\epsilon u_1^\epsilon(S^\epsilon) - \chi_1 u_1^\epsilon(S^\epsilon)$$
$$+\chi_1 u_1^\epsilon(S^\epsilon) - \chi_1 u_1(S^\epsilon) + \chi_1 u_1(S^\epsilon) - \chi_1 u_1(S),$$

so as in Lemma 3.3, Proposition 3.4 with properties (4.1) and (4.7) imply the different terms of I_δ tend to zero in $L^1(\delta, T; L^1(\Omega))$ when $\epsilon \searrow 0$.

In the more general case we are looking for equations satisfied by the limit as $\epsilon \searrow 0$ of the sequences $\{u^\epsilon\}_{\epsilon>0}$.

The sequence $\{u^\epsilon\}_{\epsilon>0}$ defined by (3.23) is uniformly bounded in $L^\infty(\Omega_T)$, so we can substract a sequence that converges to u in $L^\infty(\Omega_T)$ weak*.

We assume that the hypothesis (2.12) holds. We have

$$\text{(4.8)} \qquad \begin{cases} \dfrac{\partial}{\partial t}(\phi^\epsilon p^\epsilon) + \dfrac{\partial}{\partial x} q^\epsilon = 0, \\ \dfrac{\partial}{\partial t}(\phi^\epsilon u^\epsilon) + \dfrac{\partial}{\partial x}(q^\epsilon u^\epsilon) + \phi^\epsilon u^\epsilon \dfrac{\partial}{\partial t} p^\epsilon = 0. \end{cases}$$

From (2.2),(2.15) and (3.23) it follows that $F^\epsilon = \phi^\epsilon u^\epsilon \dfrac{\partial}{\partial t} p^\epsilon$ belongs to a bounded set of $L^2(\Omega_T)$. If we denote by

$$A^\epsilon = (\phi^\epsilon u^\epsilon, q^\epsilon u^\epsilon), \quad B^\epsilon = (-q^\epsilon, \phi^\epsilon p^\epsilon)$$

we have

$$\text{curl}B^\epsilon = F^\epsilon, \qquad \text{div}A^\epsilon = 0$$

Using the elegant method of compensated compactness initiated by MURAT [20] and TARTAR [28] we have that

$< A^\epsilon, B^\epsilon >$ converges to $< A, B >$ in the sense of distributions.

The following assertions are valid

$$< A^\epsilon, B^\epsilon > = (q^\epsilon u^\epsilon)(\phi^\epsilon p^\epsilon) - (q^\epsilon)(\phi^\epsilon u^\epsilon),$$

$$q^\epsilon \to q \quad \text{a.e. in} \quad \Omega_T,$$

$$p^\epsilon \to p \quad \text{a.e. in} \quad \Omega_T,$$

$$(\phi^\epsilon u^\epsilon) \rightharpoonup \widetilde{\phi u} \quad \text{in} \quad L^\infty(\Omega_T) \quad \text{weak}\star,$$

$$(\phi^\epsilon q^\epsilon) \rightharpoonup \phi q \quad \text{in} \quad L^\infty(\Omega_T) \quad \text{weak}\star,$$

$$u^\epsilon \rightharpoonup u \quad \text{in} \quad L^\infty(\Omega_T) \quad \text{weak}\star$$

which implies

$$< A^\epsilon, B^\epsilon > \rightharpoonup \widetilde{\phi u}\, q - \widetilde{qu}\, \widetilde{\phi p} \quad \text{in the distributional sense .}$$

We have also

$$< A^\epsilon, B^\epsilon > \rightharpoonup (\phi q)\, u - \widetilde{qu}\, \widetilde{\phi p},$$

where the uniqueness of the limit in $\mathcal{D}'(\Omega_T)$ implies that

$$\widetilde{\phi u} = \phi u.$$

Considering (2.16), we have

$$\frac{\partial}{\partial t} p^\epsilon \to \frac{\partial}{\partial t} p \quad \text{a.e. in} \quad \Omega_T,$$

where p is the solution of the homogenized problem (2.1) related to q by

$$\phi(x)\frac{\partial}{\partial t}p + \frac{\partial}{\partial x}q = 0, \quad q = -k\frac{\partial}{\partial x}p$$

$$F^\epsilon \rightharpoonup \phi u \frac{\partial}{\partial t}p \quad \text{in} \quad L^2(\Omega_T) \quad \text{weak .}$$

In a distributional sense u satisfies

$$\phi(x)\frac{\partial}{\partial t}p + q(t,x)\frac{\partial}{\partial x}u = 0 \quad in \quad \Omega_T,$$

which completes the proof.

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