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MULTI-DIMENSIONAL TRAVELLING-WAVE SOLUTIONS OF A FLAME PROPAGATION MODEL

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MULTI-DIMENSIONAL TRAVELLING-WAVE SOLUTIONS OF A FLAME PROPAGATION MODEL (*)

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Abstract: We show the existence of a travelling wave solution to the equation

$$\frac{\partial u}{\partial t} + \alpha(y)\frac{\partial u}{\partial x_1} = \Delta u + g(u)$$

in the infinite cylindrical domain $\Sigma = \{(x_1, y) \in \mathbb{R} \times \omega\}$, where ω is a bounded domain in $\mathbb{R}^{N-1}$. This equation describes the propagation of a curved premixed flame in the infinite tube Σ, in the framework of the constant-density approximation and for a unit Lewis number. The existence proof mainly uses elliptic a priori estimates, a topological degree argument and monotonicity arguments. Moreover, we describe a somewhat non classical physical behaviour: we construct travelling wave solutions of this flame propagation model which have an "inversion of the velocity field".

(*) This work was supported in part by DRET (Direction des Recherches, Etudes et Techniques) under contract 87-209.

PAPIER RÉCUPÉRÉ ET RECYCLÉ

SOLUTIONS D'ONDE SIMPLE D'UN MODELE DE PROPAGATION DE FLAMME (*)

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Résumé: Nous montrons l'existence d'une solution d'onde simple de l'équation

$$\frac{\partial u}{\partial t} + \alpha(y)\frac{\partial u}{\partial x_1} = \Delta u + g(u)$$

dans le domaine cylindrique infini $\Sigma = \{(x_1, y) \in I\!R \times \omega\}$, où ω est un domaine borné de $I\!R^{N-1}$. Cette équation décrit la propagation d'une flamme plissée prémélangée dans le tube infini Σ, dans le cadre de l'approximation de densité constante et pour un nombre de Lewis égal à 1. La démonstration de l'existence d'une solution d'onde simple utilise essentiellement des estimations a priori pour une équation elliptique, des arguments de monotonie et un argument de degré topologique. De plus, nous décrivons un comportement physique non classique: nous construisons des solutions stationnaires exhibant une "inversion du champ de vitesse".

(*) Cette étude a été partiellement financée par la DRET (Direction des Recherches, Etudes et Techniques; contrat 87-209).

1. INTRODUCTION

In this paper, we show the existence of a travelling wave solution to the equation:

$$\frac{\partial u}{\partial t}+\alpha(y)\frac{\partial u}{\partial x_1}=\Delta u+g(u)\ , \tag{1.1}$$

set in the infinite cylindrical domain $\Sigma=\{(x_1,y)\in I\!R\times\omega\}$, where ω is a bounded and smooth open domain in $I\!R^{N-1}$. This equation arises in combustion theory: it describes the propagation of a curved premixed flame in the infinite tube Σ, in the framework of the classical thermo-diffusive model, under the assumption that the Lewis number is equal to unity.

Referring to e.g. [3], [4], [12], [14] for more details, we simply recall here the equations of the thermo-diffusive model, which is derived in the framework of the well-known constant-density approximation. We consider a curved flame propagating in the infinite cylindrical tube $\Sigma=I\!R\times\omega\subset I\!R^N$. For $x\in\Sigma$, we write $x=(x_1,y)$ with $x_1\in I\!R$ and $y\in\omega$. With the assumption of a single one-step chemical reaction $\mathcal{R}\to\mathcal{P}$, the equations of the thermo-diffusive model read:

$$\begin{cases} u_t+\alpha(y)u_{x_1}=\Delta u+f(u)v\ , \\ \\ v_t+\alpha(y)v_{x_1}=\dfrac{\Delta v}{Le}-f(u)v \quad \text{in } \Sigma\ ; \end{cases} \tag{1.2}$$

Here, u is the normalized temperature and v is the mass fraction of the reactant. Moreover, $\alpha(y)$ is the x_1-component of the velocity field $\overrightarrow{V}=(\alpha(y),0)$ which is given, parallel to the tube walls $\partial\Sigma$ and divergence free. Lastly, the terms Δu, $\dfrac{\Delta v}{Le}$ and $f(u)v$ correspond to the thermal diffusion, the molecular diffusion (the non-dimensional positive parameter Le is the Lewis number of the reactant $\mathcal{R}$), and the chemical reaction respectively.

The travelling-wave solutions $u(x_1+ct,y)$, $v(x_1+ct,y)$ of (1.2) satisfy:

$$\begin{cases} (c+\alpha(y))u_{x_1}=\Delta u+f(u)v\ , \\ \\ (c+\alpha(y))v_{x_1}=\dfrac{\Delta v}{Le}-f(u)v \quad \text{in } \Sigma\ , \end{cases} \tag{1.3}$$

and the following boundary conditions (which are classical in combustion theory):

$$\frac{\partial u}{\partial \nu}=0\ ,\quad \frac{\partial v}{\partial \nu}=0 \quad \text{on } \partial\Sigma\ , \tag{1.4}$$

$$\begin{cases} u(-\infty,y)=0\ , & v(-\infty,y)=1\ , \\ u(+\infty,y)=1\ , & v(+\infty,y)=0 \quad \text{for } y\in\omega\ . \end{cases} \tag{1.5}$$

In (1.5), ν is the outward unit normal on $\partial\Sigma$. When $Le = 1$, (1.3)-(1.5) obviously imply $u + v = 1$. The travelling-wave solutions are then given by a single elliptic equation [we set $g(u) = (1-u)f(u)$]:

$$(c+\alpha(y))u_{x_1} = \Delta u + g(u) \quad \text{in } \Sigma\ , \tag{1.6}$$

associated with the boundary conditions:

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Sigma\ , \tag{1.7}$$

$$u(-\infty,y)=0\ ,\quad u(+\infty,y)=1 \quad \text{for } y\in\omega\ . \tag{1.8}$$

This is the problem which we investigate in this paper. In (1.6)-(1.8), both $u\in\mathcal{C}^2(\bar\Sigma)$ and $c\in I\!\!R$ are unknown. We will use the following hypotheses [(1.10) corresponds to an ignition temperature assumption]:

$$\alpha \text{ is a continuous function from } \bar\omega \text{ into } I\!\!R\ , \tag{1.9}$$

$$g \text{ is a Lipschitz}-\text{continuous function from } [0,1] \text{ into } I\!\!R_+\ ,\quad g(1)=0\ , \tag{1.10}$$

$$\exists\theta\in(0,1)\ ,\quad \text{such that } g\equiv 0 \text{ on } [0,\theta] \text{ and } g>0 \text{ on } (\theta,1)\ . \tag{1.11}$$

Denoting $<\alpha>= \dfrac{1}{\mid\omega\mid}\displaystyle\int_\omega \alpha(y)dy$, we can now state our main result:

Theorem 1.1:

For any functions α and g satisfying (1.9)-(1.11), there exists a solution (u,c) of (1.6)-(1.8). This solution satisfies:

$$0<u<1 \quad \text{in } \bar\Sigma\ , \tag{1.12}$$

$$u_{x_1}>0 \quad \text{in } \bar\Sigma\ , \tag{1.13}$$

$$<\alpha> + c > 0\ .\bullet \tag{1.14}$$

Remark 1.2: If (u,c) is a solution of (1.6)-(1.8), it is clear from the classical elliptic estimates that $u\in W^{2,p}_{loc}(\Sigma)$ for all $p\in(1,+\infty)$ (see Agmon, Douglis, Nirenberg [1]). Moreover, from classical Schauder's estimates, $u\in\mathcal{C}^2(\Sigma)$ and therefore satisfies (1.6) in the usual sense as soon as the function $\alpha(y)$ is Hölder-continuous. •

Remark 1.3: Uniqueness of the solution of (1.6)-(1.8) has been proved by Berestycki and Nirenberg [7]. Assuming that g is a Lipschitz-continuous function in [0,1], is $\mathcal{C}^2$ in a neighbourhood of 1 and satisfies $g'(1) < 0$, it is shown in [7] that if (u, c) and (u', c') are two solutions of (1.6)-(1.8), then $c = c'$ and $u(x_1, y) = u'(x_1 + \hat{x}_1, y)$ for all $(x_1, y) \in \bar{\Sigma}$, for some $\hat{x}_1 \in I\!R$. Furthermore, that any solution u of (1.6)-(1.8) (with arbitrary $c \in I\!R$) is monotone has also been established by Berestycki and Nirenberg [6], [7] under the assumptions (1.9)-(1.11). This last monotonicity result has been extended to a somewhat more general class of nonlinearities by C. M. Li [13]. •

The existence of a solution (u, c) of (1.6)-(1.8) for any functions α and g satisfying (1.9)-(1.11) is proved in Section 2 below.

Problem (1.6)-(1.8) had been previously investigated by the first two authors in [3], under the additional assumption essentially that α is not too far from being constant. More precisely, under the assumption that:

$$\left(\max_{y\in\bar{\omega}} \alpha(y) - \min_{y\in\bar{\omega}} \alpha(y)\right) \left(<\alpha> - \min_{y\in\bar{\omega}} \alpha(y)\right) < 2\int_0^1 g(s)ds \ , \tag{1.15}$$

it was shown in [3] that there exists a solution (u, c) to (1.6)-(1.8), and that this solution satisfies:

$$c + \min_{y\in\bar{\omega}} \alpha(y) > 0 \ . \tag{1.16}$$

This inequality, which is derived from the additional assumption (1.15), is crucially used in [3] to derive some a priori estimates. In contrast with this situation, for the solutions we construct here, $c + \alpha(y)$ may in general change sign in the domain ω. This phenomenon may be interpreted as an "inversion of the velocity field". Indeed, (1.2)-(1.3) show that $c + \alpha(y)$ is the mixture velocity in the reference frame R_f in which the solution is stationary (R_f is a reference frame attached to the flame and moves with the velocity $-c$ with respect to the original reference frame R_0). Then, (1.16) means that, at every point, the velocity in the reference frame R_f points from left to right, i.e. from the fresh mixture towards the burnt gases, a physically natural situation. But, for solutions satisfying:

$$c + \min_{y\in\bar{\omega}} \alpha(y) < 0 \ , \tag{1.17}$$

there are regions of the tube (where $c + \alpha(y) < 0$) where the velocity is directed in the opposite way, from the burnt gases towards the fresh mixture! It is important to realize here that this non classical situation is by no means non physical: it has indeed been known for a long time (not for a flame in a tube, but in other geometrical configurations, such as for a counterflow diffusion flame; see Williams [15, p. 418]) that the *mixture* velocity in the neighbourhood of the flame may be pointing from the burnt gases towards the fresh mixture. This simply means that the convective effects are locally dominated by the diffusive effects. Moreover, in these conditions, (1.14) says that the average velocity in the reference frame R_f is necessarily positive.

Several results in this direction are shown in Section 3. We prove there that, the function g corresponding to the reaction term being given, one can choose the function α (far enough from a constant) so that the corresponding solution (u, c) of (1.6)-(1.8) actually satisfies (1.17) (such travelling wave solutions have been numerically computed in [2]). Moreover, we show that condition (1.15) is, in some sense, optimal to insure that the inversion of the velocity field does not occur (i.e. that property (1.16) is fulfilled).

2. PROOF OF THE EXISTENCE

This section is devoted to the proof of Theorem 1.1. As in [3], this proof consists in studying an analogous problem posed on the bounded cylindrical domain $R_a = (-a, a)\times\omega$ for $a \in I\!R$ and then in examining the passage to the limit as $a \to +\infty$. Solving the problem in R_a is essentially the same here as in [3]. However [because the solution here does not necessarily satisfy (1.16)], the arguments in [3] would fail to yield the limit as $a \to +\infty$. Hence, we need here another approach to the derivation of the a priori estimates and to the limiting procedure.

We first consider the problem:

$$(c+\alpha(y))u_{x_1} = \Delta u + g(u) \quad \text{in } R_a \ , \tag{2.1}$$

with the mixed boundary conditions:

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } (-a,a)\times\partial\omega \ , \tag{2.2}$$

$$u(-a,y) = 0 \ , \quad u(+a,y) = 1 \quad \text{for } y \in \omega \ . \tag{2.3}$$

To this system we add the following *normalization condition* which we trade in against the freedom to choose c:

$$\max_{(x_1,y)\in \bar{R}_a,\ x_1\le 0} u(x_1,y) = \theta \ . \tag{2.4}$$

The role of this condition (and also analogous normalization conditions) is discussed in Berestycki and Larrouturou [3] and in Berestycki, Nicolaenko and Scheurer [5].

On problem (2.1)-(2.4), we are going to prove:

Proposition 2.1:

Under the assumptions (1.9)-(1.11) and for any $a > 0$, there exists a solution $(u, c) = (u_a, c_a)$ of problem (2.1)-(2.4). •

The proof follows the steps of the one given in [3] with a few minor modifications. For the sake of completeness, we repeat it here. It rests on the following a priori estimates (in what follows we always assume that the definition of g is extended to all of $I\!R$ by setting $g(s) \equiv 0$ for $s \notin [0, 1]$):

Lemma 2.2:

Suppose $g \leq M$ on [0,1] and $\alpha_0 \leq \alpha(y) \leq \alpha_1$ for all $y \in \bar{\omega}$. Then there exists a constant K depending only on a, α_0, α_1 and M such that any solution (u, c) of (2.1)-(2.4) satisfies:

$$| c | \leq K \ , \tag{2.5}$$

and:

$$\|u\|_{C^1(\bar{R}_a)} \leq K \ . \bullet \tag{2.6}$$

PROOF of Lemma 2.2: Since $g(s) = 0$ outside the interval [0,1], it follows from the maximum principle that $0 < u < 1$ in R_a. Hence, by a result of Berestycki and Nirenberg [6, Theorem 4.1], we know that $u_{x_1} > 0$ in $(-a, a) \times \bar{\omega}$ (actually, it is assumed in [6, Theorem 4.1] that u satisfies Dirichlet data on the whole boundary of R_a; but of course the same result holds under the present conditions). Using this information, we infer some inequalities from the maximum principle. Let α_0 and α_1 be such that $\alpha_0 \leq \alpha(y) \leq \alpha_1$ for all $y \in \bar{\omega}$, and denote z_0, z_1 the solution of the following ordinary differential equations:

$$-z_0'' + (c + \alpha_1) z_0' = 0 \quad \text{in } (-a, a) \ , \tag{2.7}$$

$$-z_1'' + (c + \alpha_0) z_1' = M \quad \text{in } (-a, a) \ , \tag{2.8}$$

together with the boundary conditions:

$$z_i(-a) = 0 \ , \quad z_i(+a) = 1 \quad \text{for } i = 0, 1 \ . \tag{2.9}$$

The maximum principle then yields:

$$z_0(x_1) \leq u(x_1, y) \leq z_1(x_1) \quad \text{in } R_a; \tag{2.10}$$

for instance, the first inequality in (2.10) follows from the relation:

$$-\Delta(u - z_0) + (c + \alpha_1)(u - z_0)_{x_1} = g(u) + (\alpha_1 - \alpha(y)) u_{x_1} \geq 0 \ . \tag{2.11}$$

Now, since, for fixed $a > 0$:

$$\lim_{c \to -\infty} z_0(0) = 1 \quad \text{and} \quad \lim_{c \to +\infty} z_1(0) = 0 \ , \tag{2.12}$$

the condition $z_0(0) \leq \theta \leq z_1(0)$ which follows from (2.4) gives the a priori estimate (2.5) on c, with a constant K only depending on a, α_0, α_1 and M. Then, since $0 < u < 1$ and c is bounded, we immediately derive by the elliptic estimates a C^1 bound on u: $\|u\|_{C^1(\bar{R}_a)} \leq K$. •

PROOF of Proposition 2.1: Consider the space $E = C^1(\bar{R}_a)$ and $\mathcal{E} = E \times \mathbb{R}$. For $(v, c) \in \mathcal{E}$ and for $\tau \in [0,1]$, let $u = \phi_\tau(v, c)$ be the unique solution of:

$$-\Delta u + (c + \alpha(y))u_{x_1} = \tau g(v) \quad \text{in } R_a \ , \tag{2.13}$$

with u satisfying (2.2) and (2.3). Hence problem (2.1)-(2.3) has been translated into the equation $u = \phi_1(u, c)$. Next, let:

$$h_\tau(v, c) = \max_{(x_1, y) \in \bar{R}_a, \ x_1 \leq 0} \phi_\tau(v, c) \ . \tag{2.14}$$

Thus our problem (2.1)-(2.4) with unknowns u and c is now written as a fixed point equation in the space $\mathcal{E}$:

$$\begin{cases} u = \phi_1(u, c) \ , \\ h_1(u, c) = \theta \ . \end{cases} \tag{2.15}$$

This problem is of the form $(u, c) - \mathcal{F}_1(u, c) = 0$ where, for $\tau \in [0, 1]$:

$$\mathcal{F}_\tau(u, c) = (\phi_\tau(u, c), c - h_\tau(u, c) + \theta) \ . \tag{2.16}$$

Clearly the mapping $(\tau, (u, c)) \to \mathcal{F}_\tau(u, c)$ from $[0, 1] \times \mathcal{E}$ into $\mathcal{E}$ is continuous and compact. Now, let the reals α_0 and α_1 be given by:

$$\alpha_0 = \min\left(\min_{y \in \bar{\omega}} \alpha(y), 0\right) \ , \quad \alpha_1 = \max\left(\max_{y \in \bar{\omega}} \alpha(y), 0\right) \ , \tag{2.17}$$

so that:

$$\alpha_0 \leq \tau\alpha(y) \leq \alpha_1 \tag{2.18}$$

for all $y \in \bar{\omega}$ and $\tau \in [0, 1]$. Let K be the constant introduced in Lemma 2.2; K depends on a, α_0, α_1 and M. Since $\tau g(s) \leq M$, we know by the estimates in Lemma 2.2 that $(u, c) - \mathcal{F}_\tau(u, c) \neq 0$ for all $(u, c) \in \partial\Omega$ and all $\tau \in [0, 1]$, where:

$$\Omega = \left\{(u, c) \in \mathcal{E} \ , \ \ \|u\|_{C^1(\bar{R}_a)} \leq K \ , \ \ | \, c \, | \leq K\right\} \ . \tag{2.19}$$

Hence, the Leray-Schauder degree $d(I-\mathcal{F}_1,\Omega,0)$ is well defined and:

$$d(I-\mathcal{F}_1,\Omega,0)=d(I-\mathcal{F}_0,\Omega,0) \tag{2.20}$$

by homotopy invariance. Now for $\tau=0$, $\phi_0(u,c)$ is independent of u. We perform a new homotopy by substituting $\tau\alpha(y)$ for $\alpha(y)$ in the definition (2.13) of ϕ_τ and subsequently in the definition (2.14) of h_τ. Continuity and compactness are left unchanged as are our estimates in Lemma 2.2 by the choice we have made of α_0 and α_1. Therefore, by homotopy invariance:

$$d(I-\mathcal{F}_1,\Omega,0)=d(I-\mathcal{F}_1^*,\Omega,0)\ , \tag{2.21}$$

where $\mathcal{F}_1^*(u,c)=\left(\psi_c,c-\hat{h}(c)+\theta\right)$, with $\psi_c=\psi_c(x_1)$ the solution of:

$$\begin{cases} -\psi_c''+c\psi_c'=0 \quad \text{in } (-a,a)\ , \\ \psi_c(-a)=0\ ,\quad \psi_c(+a)=1\ , \end{cases} \tag{2.22}$$

and:

$$\hat{h}(c)=\psi_c(0)\ . \tag{2.23}$$

But $\hat{h}(c)=\dfrac{1-e^{-ca}}{e^{ca}-e^{-ca}}$ and the equation $(u,c)=\mathcal{F}_1^*(u,c)$ uniquely determines c and consequently u ($c=c^*\in(-K,K)$ and $u=\psi_{c^*}$). Then by homotopy invariance, the degree (2.21) is the same as the degree of the mapping:

$$(u,c)\to\left(u-\psi_{c^*},\hat{h}(c)-\theta\right)\ . \tag{2.24}$$

By using the product property of the degree we obtain (letting B_K stand for the ball of radius K in E):

$$d(I-\mathcal{F}_1,\Omega,0)=d(I-\psi_{c^*},B_K,0)\,.\,d\left(\hat{h}(c)-\theta,(-K,K),0\right)\ : \tag{2.25}$$

Since $\psi_{c^*}\in B_K$ and $\hat{h}$ is decreasing with $\hat{h}(-K)>0$, $\hat{h}(K)<0$, we finally get:

$$d(I-\mathcal{F}_1,\Omega,0)=-1\ . \tag{2.26}$$

Hence there exists a solution (u,c) of problem (2.1)-(2.4), which completes the proof of Proposition 2.1. •

Remark 2.3: Under the same assumptions as in Remark 1.3, uniqueness of the solution (u,c) of (2.1)-(2.4) has been proved by Berestycki and Nirenberg [7]. •

Now comes the newer part in the proof of Theorem 1.1. We will now derive the limit of the solution (u_a,c_a) of (2.1)-(2.4) when $a\to+\infty$.

PROOF of Theorem 1.1: We divide it into several steps:

Step 1: A priori estimate on c_a.

Lemma 2.4:

There exists a constant $K > 0$ *independent of* a *such that for any* $a \geq 1$ *the solution* (u_a, c_a) *of (2.1)-(2.4) satisfies:*

$$| c_a | \leq K \ . \bullet \tag{2.27}$$

PROOF: We recall from the proof of Lemma 2.2 that the function:

$$z_0(x_1) = \frac{e^{(c_a+\alpha_1)x_1} - e^{-(c_a+\alpha_1)a}}{2\sinh((c_a+\alpha_1)a)} \tag{2.28}$$

satisfies $z_0(x_1) \leq u_a(x_1, y)$ on $\bar{R}_a$. Hence $z_0(0) \leq \theta$, which implies $(c_a + \alpha_1)a \geq -K'$ for some constant $K' > 0$, or:

$$c_a \geq -\alpha_1 - \frac{K'}{a} \ . \tag{2.29}$$

Let us now derive an upper bound for c_a. To this end, we use the solution $z(x_1)$ of:

$$\begin{cases} -z'' + (c_a + \alpha_0)z' = MH(x_1) & \text{in } (-a, a) \ , \\ z(-a) = 0 \ , \quad z(+a) = 1 \ , \end{cases} \tag{2.30}$$

with M defined as before and $H(x_1) = 1$ if $x_1 > 0$ and 0 otherwise. Clearly $z(x_1) \geq u_a(x_1, y)$ on $\bar{R}_a$ and therefore $z(0) \geq \theta$. A direct computation then yields the estimate $c_a \leq K$ for all $a \geq 1$. $\bullet$

Step 2: Existence of a limit.

From the estimate (2.27) on c_a we see by the classical elliptic estimates (see Agmon, Douglis Nirenberg [1]) that, for any $p > 1$, u_a is bounded in the $W^{2,p}$ norm in any rectangle $(x_1, x_1 + 1) \times \omega$ contained in $\bar{R}_a$, and this holds independently of a and x_1. In particular, there exists $K > 0$ independent of a such that:

$$\|u_a\|_{C^1(\bar{R}_a)} \leq K \tag{2.31}$$

for all $a \geq 1$.

Moreover, we can find a sequence $a_n \to +\infty$ such that $c_{a_n} \to c$ in $\mathbb{R}$ and $u_{a_n} \to u$ locally in C^1 norm. Obviously we obtain a solution of the equation in Σ:

$$(c + \alpha(y))u_{x_1} = \Delta u + g(u) \ , \tag{1.6}$$

which satisfies:

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Sigma \; . \tag{1.7}$$

Furthermore, since $\frac{\partial u_a}{\partial x_1} > 0$ and $0 \leq u_a \leq 1$ for all $a > 0$, we obtain $u_{x_1} \geq 0$ and $0 \leq u \leq 1$ in $\bar{\Sigma}$. •

From now on in this section, (u, c) will always denote the limit of (u_{a_n}, c_{a_n}).

It now remains to study the limits of $u(x_1, y)$ as $x_1 \to \pm\infty$. In particular, we want to avoid u_a converging locally to some constant $\rho \in [0, \theta] \cup \{1\}$.

Step 3: Energy estimates

Lemma 2.5:

The following integral are bounded:

$$\int_\Sigma g(u) < +\infty \; , \quad \int_\Sigma | \nabla u |^2 < +\infty \; . \; \bullet \tag{2.32}$$

Here and thereafter the measure $dx_1 dy$ is understood for integrals taken over Σ or parts of Σ.

PROOF: It is exactly the same as in [3]. For the sake of completeness we repeat it here.

On $I\!R_- \times \omega$, we have $g(u) \equiv 0$ since $u \leq \theta$. Besides, for $z > 0$, let $Q_z =]0, z[\times \omega$ and $U(z) = \int_\omega u(z, y) dy$. Integrating equation (1.6) on Q_z, we have:

$$\int_{Q_z} g(u) = A(z) - A(0) \; , \tag{2.33}$$

with:

$$A(z) = c \int_\omega \alpha(y) u(z, y) dy - U'(z) \; . \tag{2.34}$$

If $\int_{Q_z} g(u) \to +\infty$ when $z \to +\infty$, then $U'(+\infty) = -\infty$, which is impossible since U is bounded. For the second integral in (2.32), we multiply (1.6) by u before integrating on Q_z, and we conclude in the same manner. •

Step 4: Existence of limits as $x_1 \to \pm\infty$

Since $u_{x_1} \geq 0$, we know that the limits $\lim_{x_1 \to \pm\infty} u(x_1, y) = \beta_\pm(y)$ exist. By considering the sequence of functions:

$$u_j^\pm(x_1, y) = u(x_1 \pm j, y) \tag{2.35}$$

for (x_1, y) in the fixed domain R_1, we derive from the classical elliptic estimates that $u_j^\pm \to \beta_\pm$ in the C^1 sense. Since $\int_\Sigma |\nabla u|^2 < +\infty$, we find that, necessarily, $\nabla\beta_\pm = 0$, i.e. β_+ and β_- are constants. Moreover, since u is monotone in the x_1-direction, we have:

$$0 \le \beta_- \le \theta = \max_{y \in \bar{\omega}} u(0, y) \le \beta_+ \le 1 \; . \tag{2.36}$$

Using again the uniform convergence of u_j^+ to β_+ and the finiteness of $\int_\Sigma g(u)$, we infer that $g(\beta_+) = 0$; thus, either $\beta_+ = \theta$, or $\beta_+ = 1$. •

Step 5: The case $\beta_+ = \theta$.

Lemma 2.6:

If $\beta_+ = \theta$, then $\beta_- = \theta$ and $u \equiv \theta$. •

PROOF: If $\beta_+ = \theta$, then $u \le \theta$ and $g(u) \equiv 0$. Integrating the equation (1.6) over the domain R_m and letting $m \to +\infty$, we find (because $\lim_{x_1 \to \pm\infty} |\nabla u(x_1, y)| = 0$ uniformly for $y \in \omega$ from Step 4):

$$(\beta_+ - \beta_-)(c + <\alpha>) = 0 \; , \tag{2.37}$$

where $<\alpha> = \dfrac{1}{|\omega|} \displaystyle\int_\omega \alpha(y) dy$. Similarly, multiplying the equation by u, we find:

$$-\int_\Sigma |\nabla u|^2 - \frac{|\omega|}{2}(\beta_+^2 - \beta_-^2)(c + <\alpha>) = 0 \; . \tag{2.38}$$

Hence, using (2.37), it follows that:

$$\int_\Sigma |\nabla u|^2 = 0 \; , \tag{2.39}$$

and we have proved the claim: u must be constant, and $\beta_- = \theta$. •

Step 6: An estimate for $c + <\alpha>$.

We will now show that $c + <\alpha>$ is bounded away from 0 by a positive number. We first need the next result on the solution (u_a, c_a) of (2.1)-(2.4):

Lemma 2.7:

There exists a constant $\delta > 0$ independent of a such that for any $a \ge 1$ the solution (u_a, c_a) of (2.1)-(2.4) satisfies:

$$\int_{R_a} g(u_a) \ge \delta > 0 \; . \; \bullet \tag{2.40}$$

PROOF: Choose a number λ with $\theta < \lambda < 1$. Let $a \geq 1$. For some $x_0 \in (0,a)$ and $y_0 \in \omega$ we have $u_a(x_0, y_0) = \lambda$. Since $|\nabla u_a| \leq K$ in $\bar{R}_a$ with K independent of a, we see that $a - x_0 \geq \frac{1-\lambda}{K}$, $x_0 \geq \frac{\lambda - \theta}{K}$. Then we can find $\eta > 0$, $\sigma > 0$ and $\epsilon > 0$ independent of a such that:

$$g(u(x_1, y)) \geq \sigma \quad \text{for all } (x_1, y) \in \Sigma \cap B_\eta(x_0, y_0) \ , \tag{2.41}$$

and:

$$|\Sigma \cap B_\eta(x_0, y_0)| \geq \epsilon \ . \tag{2.42}$$

We thus obtain:

$$\int_{R_a} g(u_a) \geq \epsilon\sigma \ , \tag{2.43}$$

which completes the proof. •

A consequence of Lemma 2.7 is the following result:

Lemma 2.8:

Let $\delta > 0$ be the constant defined in Lemma 2.7. Then, c satisfies:

$$<\alpha> + c > \frac{\delta}{|\omega|} \ . \bullet \tag{2.44}$$

PROOF: We will separately consider both cases $\beta_+ = \theta$ and $\beta_+ = 1$.

a) Assume first that $\beta_+ = \theta$, and integrate the equation (2.1) satisfied by u_{a_n} on the domain $R^+_{a_n} = (0, a_n) \times \omega$. We get:

$$\begin{aligned}\int_{R^+_{a_n}} g(u_{a_n}) = (c_{a_n} + <\alpha>) |\omega| - \int_\omega (c_{a_n} + \alpha(y))\, u_{a_n}(0,y)dy \\ - \int_\omega \frac{\partial u_{a_n}}{\partial x_1}(a_n, y)dy + \int_\omega \frac{\partial u_{a_n}}{\partial x_1}(0,y)dy \ .\end{aligned} \tag{2.45}$$

But we know from Lemma 2.6 that u_{a_n} converges in the C^1 sense to θ on any compact subset of $\bar{\Sigma}$. Then the third and fifth terms in (2.45) converge to $(c + <\alpha>)\theta |\omega|$ and 0 respectively. Moreover, using the fact that $\frac{\partial u_{a_n}}{\partial x_1} > 0$ and Lemma 2.7, we get:

$$(1 - \theta) |\omega| (c + <\alpha>) \geq \delta \ , \tag{2.46}$$

whence (2.44).

b) Assuming now that $\beta_+ = 1$, we can argue as in the proof of Lemma 2.7 to show that u (and not u_a) also satisfies:

$$\int_\Sigma g(u) \geq \delta > 0 \ . \tag{2.47}$$

Integrating now equation (1.6) on all of Σ (as in the proof of Lemma 2.6), we get:

$$|\omega|(\beta_+ - \beta_-)(c + \langle\alpha\rangle) = \int_\Sigma g(u) \,, \tag{2.48}$$

from which (2.44) again follows. •

Step 7: Conclusion.

We can now conclude the proof of Theorem 1.1, using the following result:

Lemma 2.9:

Under the assumption $\langle\alpha\rangle + c > 0$, there exists a unique $\lambda > 0$ and a corresponding "eigenfunction" $\Psi = \Psi(y)$ (which is strictly positive in $\bar{\omega}$) solution of:

$$\begin{cases} -\Delta_y \Psi + \lambda(c + \alpha(y))\Psi = \lambda^2 \Psi \quad \text{in } \omega \,, \\ \dfrac{\partial \Psi}{\partial \nu} = 0 \quad \text{on } \partial\omega \,. \bullet \end{cases} \tag{2.49}$$

This result is a particular case of Theorem 3.4 in Berestycki and Nirenberg [7]. We refer the reader to [7] for the complete proof.

Here we use Lemma 2.9 to define a barrier function for u. Since Ψ is defined up to a multiplicative constant, we may as well assume that $\Psi(y) \geq \theta$ on $\bar{\omega}$. Then, the function Φ defined by:

$$\Phi(x_1, y) = e^{\lambda x_1} \Psi(y) \tag{2.50}$$

satisfies:

$$-\Delta(\Phi - u_a) + (c + \alpha(y))(\Phi - u_a)_{x_1} = 0 \quad \text{in } (-a, 0) \times \omega \,, \tag{2.51}$$

together with the boundary inequalities:

$$(\Phi - u_a)(-a, y) \geq 0 \,, \quad (\Phi - u_a)(0, y) \geq 0 \quad \text{for } y \in \omega \,, \tag{2.52}$$

and:

$$\frac{\partial(\Phi - u_a)}{\partial \nu} = 0 \quad \text{on } (-a, 0) \times \partial\omega \,. \tag{2.53}$$

The maximum principle then yields:

$$u_a(x_1, y) \leq \Phi(x_1, y) \tag{2.54}$$

for all $-a \leq x_1 \leq 0$ and $y \in \omega$. Hence:

$$u(x_1, y) \leq e^{\lambda x_1} \Psi(y) \tag{2.55}$$

for all $x_1 \leq 0$ and $y \in \omega$. Since $\lambda > 0$, this shows that $\beta_- = 0$; Step 5 then shows that β_+ cannot be equal to θ. Thus $\beta_+ = 1$ and the proof of Theorem 1.1 is complete. •

3. INVERSION OF THE VELOCITY FIELD

As we said in the introduction, we consider in this section the question of the inversion of the velocity field $c + \alpha(y)$ in the reference frame R_f attached to the flame; that is, we examine the sign of $c + \min_{y\in\bar\omega} \alpha(y)$, (u, c) being the solution of (1.6)-(1.8).

The existence of solutions with $c + \min_{y\in\bar\omega} \alpha(y) < 0$ rests on the following observation concerning a sequence (u_n, c_n) of solutions to:

$$(c_n + \alpha_n(y))\frac{\partial u_n}{\partial x_1} = \Delta u_n + g(u_n) \quad \text{in } \Sigma \ , \tag{3.1}$$

$$\frac{\partial u_n}{\partial \nu} = 0 \quad \text{on } \partial\Sigma \ , \tag{3.2}$$

$$u_n(-\infty, y) = 0 \ , \quad u_n(+\infty, y) = 1 \quad \text{for } y \in \omega \ , \tag{3.3}$$

$$\max_{(x_1,y)\in\bar\Sigma,\ x_1\leq 0} u_n(x_1, y) = \theta \ . \tag{3.4}$$

Proposition 3.1:

Let α_0 and α_1 be two real numbers with $\alpha_0 < \alpha_1$. For $n \geq 2$, let α_n be a continuous function on $\bar\omega$, such that $\lim_{n\to+\infty} \alpha_n(y)$ exists for almost all y in ω:

$$\lim_{n\to+\infty} \alpha_n(y) = \alpha(y) \quad \text{a.e. in } \omega \ , \tag{3.5}$$

and that, for all $n \geq 2$:

$$\alpha_0 \leq \alpha_n(y) \leq \alpha_1 \quad \text{in } \bar\omega \ . \tag{3.6}$$

Let g satisfy the hypotheses (1.10)-(1.11), and let (u_n, c_n) be the solution of (3.1)-(3.4). Then (u_n, c_n) converges in $C^1_{loc}(\bar\Sigma) \times I\!R$ towards (u, c) satisfying (1.6)-(1.8). •

PROOF: From Section 2, the estimates on u_n in $W^{2,p}(R_b)$ for any positive b, on c_n in $I\!R$ and the estimate $<\alpha_n> + c_n > \frac{\delta}{|\omega|}$ only depend on α_0, α_1 and g (and not on n). We can therefore extract a subsequence (u_{n_k}, c_{n_k}) which converges in $C^1_{loc}(\bar\Sigma) \times I\!R$ to (u, c). Then (u, c) satisfies the boundary condition (1.7) on $\partial\Sigma$ and:

$$\max_{(x_1,y)\in\bar\Sigma,\ x_1\leq 0} u(x_1, y) = \theta \ , \quad u_{x_1} \geq 0 \ . \tag{3.7}$$

Moreover (using Lebesgue's bounded convergence theorem), it is easy to see that (u,c) satisfies (1.6) in the sense of the distributions in Σ. Lastly, we have $u(+\infty, y) = \beta_+ \in \{\theta, 1\}$ and $u(-\infty, y) = \beta_- \in [0, \theta]$ for all $y \in \omega$ as in Steps 3 and 4 above. But we also know that:

$$<\alpha> + c = \lim_{n \to +\infty} (<\alpha_{n_k}> + c_{n_k}) \geq \frac{\delta}{|\omega|} > 0 . \tag{3.8}$$

Arguing as in Step 7 above, this shows that $\beta_- = 0$, $\beta_+ = 1$ and it completes the proof: using the uniqueness result of Remark 1.3 above, it is classical to show that the whole sequence (u_n, c_n) converges to (u, c). •

It easily follows from Proposition 3.1 that there exist solutions (u, c) of (1.6)-(1.8) having an inversion of the velocity field, i.e. such that $c + \alpha(y)$ changes sign in ω, or equivalently:

$$c + \min_{y \in \bar{\omega}} \alpha(y) < 0 . \tag{1.17}$$

Indeed, starting with a solution (u,c) for some α, one can modify α about some point $y_0 \in \omega$ without changing c much. Hence one obtains a solution with $c + \alpha(y_0) < 0$. This procedure is detailed below:

Proposition 3.2:

Let g satisfy (1.10)-(1.11). Then there exists a continuous function $\alpha(y)$ such that the corresponding solution (u, c) of (1.6)-(1.8) satisfies (1.17). •

PROOF: Let α_1 be a positive real, to be chosen later. Let $y_0 \in \omega$. For any integer n we define the function α_n by:

$$\alpha_n(y) = \begin{cases} \alpha_1 & \text{if } |y - y_0| > \frac{1}{n} , \\ n\alpha_1 |y - y_0| & \text{if } |y - y_0| \leq \frac{1}{n} . \end{cases} \tag{3.9}$$

For all n, α_n is continuous on $\bar{\omega}$; furthermore, $\min_{y \in \bar{\omega}} \alpha_n(y) = 0$, $\max_{y \in \bar{\omega}} \alpha_n(y) = \alpha_1$, and:

$$\forall y \in \bar{\omega} - \{y_0\} , \quad \lim_{n \to +\infty} \alpha_n(y) = \alpha_1 . \tag{3.10}$$

Then, we know from Proposition 3.1 that the solution (u_n, c_n) of (3.1)-(3.4) converges as $n \to +\infty$ to (u, c) satisfying (1.7)-(1.8) and:

$$(c + \alpha_1) u_{x_1} = \Delta u + g(u) . \tag{3.11}$$

Besides, it is known (see Berestycki, Nicolaenko and Scheurer [5], Johnson [8], Johnson and Nachbar [9], and Kanel' [10], [11]) that the one-dimensional problem:

$$\begin{cases} -\hat{u}_{x_1x_1} + \hat{c}\hat{u}_{x_1} = g(\hat{u}) \quad \text{in } \mathbb{R} , \\ \hat{u}(-\infty) = 0 , \quad \hat{u}(0) = \theta , \quad \hat{u}(+\infty) = 1 , \end{cases} \tag{3.12}$$

(where $\hat{u} = \hat{u}(x_1)$ and $\hat{c}$ are unknown) has a unique solution $(\hat{u}, \hat{c})$. The uniqueness statement recalled in Remark 1.3 then implies:

$$u(x_1, y) = \hat{u}(x_1) \quad \text{for all } (x_1, y) \in \Sigma , \tag{3.13}$$

and:

$$c + \alpha_1 = \hat{c} . \tag{3.14}$$

Observe that $\hat{c}$ only depends on the function g. If we choose from the outstart to take $\alpha_1 > \hat{c}$, then (3.14) says that $c < 0$. For n large enough, we have $c_n < 0$, whence:

$$c_n + \min_{y \in \bar{\omega}} \alpha_n(y) = c_n < 0 , \tag{3.15}$$

which completes the proof: the choice $\alpha(y) = \alpha_n(y)$ for n large enough gives a solution of (1.6)-(1.8) which satisfies (1.17), i.e. which exhibits the inversion of the velocity field. •

We conclude this section by deriving conditions for the inversion of the velocity field to occur. First we recall from [3] that this inversion never occurs when the condition:

$$\left(\max_{y \in \bar{\omega}} \alpha(y) - \min_{y \in \bar{\omega}} \alpha(y) \right) \left(<\alpha> - \min_{y \in \bar{\omega}} \alpha(y) \right) < 2 \int_0^1 g(s) ds \tag{1.15}$$

is fulfilled.

Proposition 3.3 [3]:

Let α and g satisfy (1.10)-(1.11) and (1.15). Then the solution (u, c) of (1.6)-(1.8) satisfies:

$$c + \alpha(y) > 0 \quad \forall\, y \in \bar{\omega} . \bullet \tag{3.16}$$

PROOF: We give here a more direct proof of this property than in [3]. Multiplying (1.6) by $1 - u > 0$ and integrating in Σ, we get (we recall from Section 2 that $\lim_{x_1 \to \pm\infty} | \nabla u(x_1, y) | = 0$ uniformly for $y \in \omega$):

$$\frac{| \omega |}{2} (c + <\alpha>) - \int_\Sigma | \nabla u |^2 \geq 0 . \tag{3.17}$$

Notice that this inequality again shows that $<\alpha> + c > 0$. Besides, multiplying (1.6) by u_{x_1} and integrating in Σ yields:

$$| \omega | \int_0^1 g(s)ds = \int_\Sigma (c + \alpha(y))u_{x_1}^2 \ . \tag{3.18}$$

This implies that $| \omega | \int_0^1 g(s)ds \leq \left(c + \max_{y\in\bar{\omega}} \alpha(y)\right) \int_\Sigma | \nabla u |^2$, whence, using (3.17):

$$2\int_0^1 g(s)ds \leq \left(c + \max_{y\in\bar{\omega}} \alpha(y)\right) . (c + <\alpha>) \ . \tag{3.19}$$

When (1.15) holds, this inequality shows that $c > -\min_{y\in\bar{\omega}} \alpha(y)$, which concludes the proof. •

Our last result says that condition (1.15) is optimal, in the following sense:

Proposition 3.4:

Let $\epsilon > 0$ be given. One can find two functions α and g satisfying the assumptions (1.9)-(1.11) and the inequalities:

$$2\int_0^1 g(s)ds < \left(\max_{y\in\bar{\omega}} \alpha(y) - \min_{y\in\bar{\omega}} \alpha(y)\right) \left(<\alpha> - \min_{y\in\bar{\omega}} \alpha(y)\right) < 2\int_0^1 g(s)ds + \epsilon \ , \tag{3.20}$$

such that the corresponding solution (u, c) of (1.6)-(1.8) has the inversion of the velocity field (i.e. satisfies (1.17)). •

PROOF: The proof is mainly analogous to the previous one, and uses the asymptotic analysis for high activation energies of the one-dimensional problem (3.12). Referring to [5] for the details, we just recall here that the solution $(\hat{u}, \hat{c})$ of (3.12) always satisfies:

$$2\int_0^1 g(s)ds < \hat{c}^2 \ . \tag{3.21}$$

This can also be inferred from the proof of Proposition 3.3. Moreover, it is shown in [5] that one can choose g (corresponding to a high activation energy) such that:

$$2\int_0^1 g(s)ds < \hat{c}^2 < 2\int_0^1 g(s)ds + \frac{\epsilon}{2} \ . \tag{3.22}$$

Having chosen g such that (3.22) holds, we set:

$$\alpha_1 = \sqrt{2\int_0^1 g(s)ds + \epsilon} \ , \tag{3.23}$$

and define the sequence of functions α_n as in (3.9). Again the corresponding solution (u_n, c_n) of (3.1)-(3.4) converges to $(\hat{u}, \hat{c} - \alpha_1)$. For n large enough, α_n satisfies (3.20), and c_n is strictly negative since $\hat{c} - \alpha_1 < 0$ from (3.22)-(3.23); we can then conclude as in the preceding proof. •

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