A realization theory for two-point boundary-value descriptor systems

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Abstract:

In this paper, we develop a realization theory for a class of two–point boundary–value descriptor systems (TPBVDS's). Specifically, we consider the class of systems that are extendible, i.e. systems for which the input–output map (weighting pattern) can be extended outwards indefinitely, and stationary for which the weighting pattern is shift–invariant.

The particular realization problem considered consists of constructing a minimal TPBVDS that realizes a given extendible, stationary weighting pattern. In our study, we develop an original transform method, better adapted than the z–transform method to descriptor dynamics because of the symmetrical way that it handles the zero and the infinite frequencies. Using this new transform method, we are able to determine the dimension of minimal realizations and construct minimal realizations by performing, in general more than one, factorizations of rational matrices.

UNE THEORIE DE REALISATION DETERMINISTE POUR LES SYSTEMES IMPLICITES AUX DEUX BOUTS

Résumé:

Une théorie de la réalisation déterministe est développée pour une classe de systèmes linéaires implicites aux deux bouts. En particulier, on considère la classe des systèmes extensibles (c'est–à–dire des systèmes qui peuvent être étendus indéfiniment vers l'extérieur) et stationnaires.

Le problème de réalisation qu'on considère consiste à construire des systèmes implicites aux deux bouts à partir de leurs fonctions de transfert. Pour cela, on introduit une transformation originale bien adaptée à la nature implicite de la dynamique des systèmes étudiés. En utilisant cette transformation, on montre que le problème de réalisation considéré est équivalent à un problème de factorisations rationnelles. En particulier, les réalisations minimales sont obtenues en factorisant une ou, en général, plusieurs matrices rationnelles.

I–INTRODUCTION

There exists a rich theory of state–space realization for linear, time–invariant, causal systems, i.e. systems that can be characterized by the following type of input–output relationship:

$$y(k) = \sum_{j=-\infty}^{\infty} W(k-j)u(j) \tag{1.1}$$

where W, the impluse–response (weighting pattern) of the system, satisfies

$$W(m) = 0, \quad m \leq 0. \tag{1.2}$$

In many physical systems however, in particular when the independent variable is space rather than time, causality, i.e. condition (1.2) does not necessarily hold. For example in describing the heat distribution in a rod, there is no reason that the temperature at any point of the rod be exclusively a function of the applied heat on one side of that point. Weighting patterns not satisfying condition (1.2) are called acausal. In this paper, we develop a realization theory for the class of acausal weighting patterns. Specifically, we consider the problem of realizing acausal weighting patterns by **two–point boundary–value descriptor systems** (TPBVDS).

Motivated by the works of Krener [6,7], and Gohberg and Kaashoek [11,13], a system theory has been developed for TPBVDS dealing with issues such as reachability, observability and minimality [1,2,16–18]. There, extendibility, i.e. the notion of extending the boundaries of a system outwards without modifying its weighting pattern, has also been studied. In particular, extendibility is shown to be a natural property to consider for stationary systems (systems that have shift–invariant weighting patterns). Weighting patterns associated to extendible, stationary TPBVDS's can be extended outwards indefinitely, thus, we can associate to an extendible system an **extended weighting pattern** W(i) defined for $-\infty<i<+\infty$, such that the weighting patterns of the system and all of its extensions are restrictions of W(i).

The realization problem that we consider in this paper can be formulated as follows: given an infinite sequence of matrices W(i), construct a minimal TPBVDS, over a given interval, that has extended weighting pattern W(i). As in the causal realization theory, where thanks to the z–transform method, the realization problem is shown to be closely related to a factorization problem of proper rational matrices, we will show that the TPBVDS realization problem is also related to a factorization problem of rational matrices, but rational matrices in two variables. This result is based on an original transform method which can be considered as a generalization of the z–transform method having the advantage that the zero and the infinite frequencies are treated in a symmetrical fashion. This transfomation method also allows us to introduce a generalization of the McMillan degree of rational matrices and relate it to the dimension of minimal realizations.

We start this paper by reviewing TPBVDS results derived in [17,18] that we will need later, and by characterizing completely the property of extendibility. In particular, we will show that there exists a matrix, called the **projection matrix**, that contains all the contribution of the boundary conditions to the weighting pattern of the system. The notion of projection matrix greatly simplifies the development of the realization theory.

Next, we find necessary and sufficient conditions for realizability and propose a simple realization method which does not always yield a minimal realization.

Then, we introduce the (s,t)–transform and show that the realization problem is related to, in general, more than one factorizations of rational matrices in two variables s and t, and solve this factorization problem.

Finally, using the (s,t)–transform technique and the factorizations results, we construct a method for determining the dimension of minimal realizations and constructing these realizations.

II–TWO–POINT BOUNDARY–VALUE DESCRIPTOR SYSTEMS

In this section, we present some of the system–theoretical results derived for TPBVDS's. We shall be needing these results later in the develpment of our realization theory.

2.1–Introduction

A TPBVDS is described by the following dynamic equation

$$Ex(k+1) = Ax(k) + Bu(k), \quad 0 \leq k \leq N-1 \tag{2.1}$$

with boundary condition

$$V_i x(0) + V_f x(N) = v \tag{2.2}$$

and output

$$y(k) = Cx(k), \quad k=0,1,\ldots,N. \tag{2.3}$$

Matrices E, A, V_i and V_f are nxn, and, C and B are respectively mxn and nxp constant matrices. Also N is assumed to be larger than 2n. In [1] it is shown that if (2.1)–(2.2) is well–posed (i.e. it yields a well–defined map from {u,v} to x), we can assume, without loss of generality that (2.1)–(2.2) is in normalized form, i.e. that there exist scalars α and β such that

$$\alpha E + \beta A = I \tag{2.4}$$

(this is referred to as the standard form for the pencil {E,A}) and in addition

$$V_i E^N + V_f A^N = I\,. \tag{2.5}$$

Note that (2.4) implies that E and A commute, that E, A and the system have a common set of eigenvectors[5], and also that $\{E^k, A^k\}$ is a regular pencil for all $k \geq 0$ (see[1]). But most importantly (2.4) implies that the space of matrices $A^K E^L$, $K,L \geq 0$, is spanned by the n matrices $\{A^k E^{n-1-k} \mid k=0,..,n-1\}$; this property has been introduced in [1] as the generalized Cayley–Hamilton theorem. We assume throughout this paper that (2.4) and (2.5) hold.

[5] v is an eigenvector of the system if $v \neq 0$ and for some σ, $(\sigma E - A)v = 0$. σ is called an eigenmode of the system; for descriptor systems σ can be ∞ as well.

As derived in [1], the map from {u,v} to x has the following form:

$$x(k) = A^k E^{N-k} v + \sum_{j=0}^{N-1} G(k,j)Bu(j), \tag{2.6}$$

where the Green's function G(k,j) is given by

$$G(k,j) = \begin{cases} A^k(A-E^{N-k}(V_iA+\omega V_fE)E^k)E^{j-k}A^{N-j-1}\Gamma^{-1} & j\geq k \\ E^{N-k}(\omega E-A^k(V_iA+\omega V_fE)A^{N-k})E^j A^{k-j-1}\Gamma^{-1} & j<k \end{cases} \tag{2.7}$$

and where ω is any number such that

$$\Gamma \triangleq \omega E^{N+1} - A^{N+1} \tag{2.8}$$

is invertible.

The map from inputs u to outputs y is given by the weighting pattern W of the system. With v=0, we have that

$$y(k) = \sum_{j=0}^{N-1} W(k,j)u(j) \tag{2.9}$$

where, obviously

$$W(k,j) = CG(k,j)B. \tag{2.10}$$

2.2–Stationarity

Note that in contrast with the causal case where time–invariant systems have time–invariant impluse responses, here W(k,j), in general, is not only a function of the difference k–j. TPBVDS's that do have time–invariant weighting patterns are called stationary [17,18]:

Definition 2.1

The TPBVDS (2.1)–(2.3) is stationary if (with the usual abuse of notation)

$$W(k,j) = W(k-j) \tag{2.11}$$

for $0\leq k\leq N$, $0\leq j\leq N-1$.

Theorem 2.1 [17,18]

The TPBVDS (2.1)–(2.3) is stationary if and only if

$$O_s[V_i,E]R_s = O_s[V_i,A]R_s = 0 \tag{2.13a}$$

$$O_s[V_f,E]R_s = O_s[V_f,A]R_s = 0, \tag{2.13b}$$

where [X,Y] denotes the commutator product of X and Y

$$[X,Y] = XY - YX \tag{2.14}$$

and

$$R_s = [A^{n-1}B|EA^{n-2}B|\ldots|E^{n-1}B] \tag{2.15a}$$

$$O_s = \begin{bmatrix} CA^{n-1} \\ CEA^{n-2} \\ \vdots \\ CE^{n-1} \end{bmatrix}. \tag{2.15b}$$

The matrices R_s and O_s in (2.15) are, respectively, the <u>strong reachability</u> and <u>strong observability</u> matrices of the TPBVDS as discussed in [1]. Thus (2.13) states that V_i and V_f must commute with E and A except for parts that are either in the left nullspace of R_s or the right nullspace of O_s. For example, if R_s and O_s are of full rank – i.e. if the TPBVDS is <u>strongly reachable</u> and <u>strongly observable</u> – V_i and V_f must commute with E and A.

It can easily be verified that the weighting pattern of a stationary TPBVDS is given by

$$W(k) = \begin{cases} CV_iA^{k-1}E^{N-k}B & k>0 \\ -CV_fE^{-k}A^{N+k-1}B & k\leq 0 \end{cases}. \tag{2.16}$$

2.3–Minimality

In developing our realization theory, we need to characterize minimality for the calss of stationary TPBVDS's.

Definition 2.2

A TPBVDS is minimal if x has the lowest dimension among all TPBVDS's having the same weighting pattern.

In the causal case, a system is minimal if and only if it is reachable and observable and all minimal realizations are related by similarity transformations. The situation is more complex for TPBVDS's:

Theorem 2.2 [17,18]

The stationary TPBVDS (2.1)–(2.3) is minimal if and only if

(a) $[V_iR_s | V_fR_s]$ has full row rank, (2.17a)

(b) $\begin{bmatrix} O_sV_i \\ O_sV_f \end{bmatrix}$ has full column rank, (2.17b)

(c) $Ker(O_s) \subset Im(R_s)$. (2.17c)

Corollary

Let $(C_j,V_j^i,V_j^f,E_j,A_j,B_j,N)$, j=1,2, be two minimal, extendible and stationary realizations of the same weighting pattern, where $\{E_j,A_j\}$, j=1,2, are in standard form for the same α and β. Then there exists an invertible matrix T so that

$$B_2 = TB_1 \tag{2.18a}$$

$$C_2 = C_1T^{-1} \tag{2.18b}$$

$$O_s^1(V_1^i - T^{-1}V_2^iT)R_s^1 = 0 \tag{2.19a}$$

$$O_s^1(V_1^f - T^{-1}V_2^fT)R_s^1 = 0 \tag{2.19b}$$

and

$$(A_1 - T^{-1}A_2T)R_s^1 = 0 \tag{2.20a}$$

$$(E_1 - T^{-1}E_2T)R_s^1 = 0 \tag{2.20b}$$

$$O_s^1(A_1 - T^{-1}A_2T) = 0 \tag{2.20c}$$

$$O_s^1(E_1 - T^{-1}E_2T) = 0 \tag{2.20d}$$

where R_s^1 and O_s^1 are the strong reachability and observability matrices for system 1.

2.4–Extendibility

Extendibility has been introduced in [18] for stationary TPBVDS's. The concept has been later extended in [17] to the non–stationary case. In this paper, we need only to consider the stationary case.

Definition 2.3

The stationary TPBVDS (2.1)–(2.3) is extendible (or input–output extendible) if given any interval [K,L] containing [0,N], there exists a stationary TPBVDS over this larger interval (called the extension of (2.1)–(2.3) to the interval [K,L]) with the same dynamics as in (2.1) but with new boundary matrices $V_i(K,L)$, $V_f(K,L)$ such that the weighting pattern $W_N(k)$ of the original system is the restriction of the weighting pattern $W_{L-K}(k)$ of the new extended system, i.e.

$$W_N(k) = W_{L-K}(k), \quad 1-N \le k \le N. \tag{2.21}$$

The notion of extendibility is closely related to the inward/outward decomposition of boundary value problems introduced by Krener [6] and extended to the TPBVDS case in [1,2,17,18].

Theorem 2.3 [17,18]

A stationary TPBVDS is extendible if and only if

$$O_s(V_i - V_iE^DE)R_s = 0 \tag{2.22a}$$

$$O_s(V_f - V_fA^DA)R_s = 0 \tag{2.22b}$$

where $(.)^D$ denotes the Drazin inverse [21].

Theorem 2.4

The extendible stationary TPBVDS (2.1)–(2.3) has **extendible** extensions on any interval. Moreover, weighting patterns of extendible extensions of (2.1)–(2.3) are unique.

Proof

We shall prove Theorem 2.3 by constructing the extendible extensions. Let $(C,\tilde{V}_i,\tilde{V}_f,E,A,B,M)$ be a non–extendible extension of (2.1)–(2.3). What we like to show is that we can modify this extension such that the resulting system is extendible and remains an extension of (2.1)–(2.3). Consider the modified TPBVDS $(C,\tilde{V}_iEE^D,\tilde{V}_fAA^D,E,A,B,M)$. It is straightforward to verify that this system is normalized and extendible. Thus, what we need to show is that this modified system is still an extension of (2.1)–(2.3).

Let $\tilde{W}_M(k)$ denote the weighting pattern of $(C,\tilde{V}_i,\tilde{V}_f,E,A,B,M)$, then from the definition of extendibility, we know that

$$\tilde{W}_M(k) = W_N(k) \quad 1-N \le k \le N, \tag{2.23}$$

i.e.,

$$C\tilde{V}_iA^{k-1}E^{M-k}B = CV_iA^{k-1}E^{N-k}B \quad 1 \le k \le N \tag{2.24a}$$

$$C\tilde{V}_fE^{-k}A^{M+k-1}B = CV_fE^{-k}A^{N+k-1}B \quad 1-N \le k \le 0. \tag{2.24b}$$

Equivalently, since $N \ge n$ and using the generalized Cayley–Hamilton theorem,

$$O_s(\tilde{V}_iE^{M-N})R_s = O_sV_iR_s \tag{2.25a}$$

$$O_s(\tilde{V}_fA^{M-N})R_s = O_sV_fR_s. \tag{2.25b}$$

Now using the fact that $Im(R_s)$ is E–, A–, E^D– and A^D–invariant [18], and (2.22), we can show that

$$O_s(\tilde{V}_iE^{M-N})EE^DR_s = O_sV_iEE^DR_s = O_sV_iR_s \tag{2.26a}$$

$$O_s(\tilde{V}_fA^{M-N})AA^DR_s = O_sV_fAA^DR_s = O_sV_fR_s. \tag{2.26b}$$

And thus

$$C\tilde{V}_i EE^D A^{k-1} E^{M-k} B = CV_i A^{k-1} E^{N-k} B \quad 1 \le k \le N \tag{2.27a}$$

$$C\tilde{V}_f AA^D E^{-k} A^{M+k-1} B = CV_f E^{-k} A^{N+k-1} B \quad 1-N \le k \le 0, \tag{2.27b}$$

which means that

$$W_M(k) = W_N(k) \quad 1-N \le k \le N, \tag{2.28}$$

where $W_M(k)$ denotes the weighting pattern of $(C,\tilde{V}_i EE^D,\tilde{V}_f AA^D,E,A,B,M)$. Thus we have constructed the desired extension.

Now we have to show uniqueness of W_M. Using Theorem 2.2, we can show that this weighting pattern can be expressed as follows

$$W_M(k) = \begin{cases} C(\tilde{V}_i E^M)E^D(AE^D)^{k-1}B & M \ge k \ge 1 \\ -C(\tilde{V}_f A^M)A^D(EA^D)^{-k}B & 1-M \le k \le 0 \end{cases} \tag{2.29}$$

Clearly $W_M(k)$ is uniquely specified on $[1-N,N]$ since on this interval it must coincide with $W_N(k)$. Thus we must show that $W_M(k)$ is uniquely specified on $[N+1,M]$ and $[1-M,1-N]$. But this follows immediately (2.29) using the standard Cayley–Hamilton result (remember that $N \ge n$).

Thanks to Theroem 2.3, we can associate to an extendible system a set of extendible systems defined over any desired interval having compatible weighting patterns. We can also associate a unique infinite sequence of matrices called the **extended weighting pattern** to an extendible stationary TPBVDS such that the weighting patterns of the system and all of its extensions are restriction of this extended weigthing pattern.

It is straightforward to verify that the extended weighting pattern of the extendible stationary TPBVDS (2.1)–(2.3) can be expressed as follows

$$\begin{aligned} W(k) &= \begin{cases} C(E^N V_i)E^D(AE^D)^{k-1}B & k>0 \\ -C(A^N V_f)A^D(EA^D)^{-k}B & k \le 0 \end{cases} \\ &= \begin{cases} C(E^N V_i)E^D(AE^D)^{k-1}B & k>0 \\ C[I-(E^N V_i)]A^D(EA^D)^{-k}B & k \le 0 \end{cases} \end{aligned} \tag{2.30}$$

III–Projection Matrix

The extended weighting pattern W(k) of an extendible stationary TPBVDS is completely determined in terms of matrices C, E, A, B and $E^N V_i$. Matrix $E^N V_i$ contains all of the contributions of the boundary conditions to the weighting pattern of the system.

Let us now consider an extendible stationary system. And let us define the projection matrix P as follows.

Definition 3.1

Let (C,V_i,V_f,E,A,B,N) be a stationary and extendible TPBVDS. Then P is the projection matrix of this system if

$$O_s P R_s = O_s(E^N V_i)R_s. \tag{3.1}$$

The extended weighting pattern W of the extendible stationary TPBVDS (2.1)–(2.3) can be expressed in terms of P as follows

$$W(k) = \begin{cases} CPE^D(AE^D)^{k-1}B & k>0 \\ -C(I-P)A^D(EA^D)^{-k}B & k\leq 0 \end{cases} \tag{3.2}$$

Also, by using (2.13), (2.22), (3.1) and the fact that $Im(R_s)$ and $Ker(O_s)$ are E– and A–invariant, we can show that the projection matrix P satisfies

$$O_s(PA-AP)R_s = O_s(PE-EP)R_s = 0 \tag{3.3a}$$

$$O_s(P-PEE^D)R_s = O_s((I-P)AA^D-(I-P))R_s = 0. \tag{3.3b}$$

Every extendible stationary TPBVDS has a projection matrix P in particular

$$P = E^N V_i. \tag{3.4}$$

The above choice for the projection matrix is not unique in general. It is not difficult to see that if P is a projection matrix then so is P+Q where Q is any matrix such that $O_s Q R_s$ equals zero.

We can see that the extended weighting pattern of an extendible stationary TPBVDS (C,V_i,V_f,E,A,B,N) is completely specified in terms of the 5–tuple (C,P,E,A,B). Our goal in this paper is to develop a method for constructing a minimal TPBVDS (C,V_i,V_f,E,A,B,N) from its extended weighting pattern W(k). We now can break up this problem in two: first find a 5–tuple (C,P,E,A,B) of minimal dimension such that (3.2) and (3.3) are verified, then find appropriate boundary matrices V_i and V_f. The following result guarantees that this approach is a valid one.

Theorem 3.1

Consider the 5–tuple (C,P,E,A,B) such that {E,A} is in standard form and such that (3.3) is verified. Then for any interval length N, there exist matrices V_i and V_f such that the TPBVDS (C,V_i,V_f,E,A,B,N) is normalized, extendible and stationary and its extended weighting pattern is equal to W(k) in (3.2). •

Proof

Let

$$V_i = P(E^D)^N + \sigma X(\sigma E^N + A^N)^{-1} \tag{3.5a}$$

$$V_f = (I-P)(A^D)^N + X(\sigma E^N + A^N)^{-1} \tag{3.5b}$$

where

$$X = I - PEE^D - (I-P)AA^D = (I-P)EE^D + PAA^D - EE^DAA^D \tag{3.6}$$

and σ is any scalar such that

$$\sigma E^N + A^N$$

is invertible. Then with V_i and V_f defined as in (3.5), we have that (3.2) and (2.30) are equal thanks to the following

$$O_s X R_s = 0. \tag{3.7}$$

Equations (3.3a) and (3.3b) imply respectively (2.13) and (2.22). Also by direct calculation we can see that V_i and V_f are normalized. Thus the TPBVDS (C,V_i,V_f,E,A,B,N) satisfies the conditions of the theorem.

Corollary

Let (C,V_i,V_f,E,A,B,N) be an extendible, stationary TPBVDS with extended weighting pattern $W(k)$. Then if a matrix P satisfies (3.2) and (3.3), P is a projection matrix of this TPBVDS.

Thus we have shown that there is complete equivalence between the boundary representation in terms of the boundary matrices V_i and V_f and the representation in terms of the projection matrix P as far as the weighiting pattern is concerned.

Example 3.1

Consider the TPBVDS

$$x(k+1) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(k) + u(k) \tag{3.8a}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(0) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(N) = v \tag{3.8b}$$

$$y(k) = x(k). \tag{3.8c}$$

This TPBVDS is in standard–form, stationary and extendible. The projection matrix for this system is

$$P = 0 \tag{3.9}$$

(in this case P is unique because (3.8) is strongly reachable and observable, i.e. R_s and O_s have full rank). The extended weighting pattern W(k) of (3.8) is given by

$$W(k) = \begin{cases} 0 & k>0 \\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} & k\leq 0 \end{cases} . \tag{3.10}$$

Note that (3.8) is an anti–causal system. This in fact can be seen directly from the projection matrix. In general, if P=0, the system is anti–causal, if P=I, it is causal. If $P\neq 0$ and $P\neq I$, it is not necessarily true that we have an acausal system. For example if C=0, for any P, W=0 which is clearly not an acausal weighting pattern. However, we shall see that for minimal systems, we can deduce acausality if $P\neq 0$

and P≠I.

We have seen that the projection matrix completely specifies the contributions of the boundary matrices V_i and V_f to the weighting pattern of an extendible, stationary TPBVDS. Thus, the natural question to ask at this point is can minimality conditions for extendible, stationary TPBVDS's be expressed in terms of the 5–tuple (C,P,E,A,B). The answer is yes:

Theorem 3.2

The extendible, stationary TPBVDS (2.1)–(2.3) is minimal if and only if

(a) $[sE-tA|PB|B]$ has full row rank for all $(s,t)\neq(0,0)$ (3.11a)

(b) $\begin{bmatrix} sE-tA \\ CP \\ C \end{bmatrix}$ has full column rank for all $(s,t)\neq(0,0)$ (3.11b)

(c) $\mathrm{Ker}(O_s) \subset \mathrm{Im}(R_s)$ (3.11c)

where P denotes the projection matrix of the system.

Proof

What we have to show is that conditions (2.17) and (3.11) are equivalent. Let us suppose that (2.17) is verified but (3.11a) is not, i.e. for some vector v,

$$v'A^kE^l = 0 \tag{3.12a}$$

$$v'E^kA^lPB = 0 \tag{3.12b}$$

for all $k,l\geq 0$. But (3.12a) implies that

$$v'R_s = 0, \tag{3.13}$$

and thus from (2.17c) we can deduce that

$$v' \in \mathrm{Im}(O_s). \tag{3.14}$$

But then thanks to (3.3a) we get that

$$v'E^kA^lPB = v'PE^kA^lB = 0, \tag{3.15}$$

or,

$$v'PR_s = v'V_iE^NR_s = 0. \tag{3.16}$$

By combining (3.16) and (3.13), we obtain

$$v'V_fA^NR_s = 0. \tag{3.17}$$

But since the system is supposed to be extendible, we have that

$$v'V_iEE^DR_s = v'V_iR_s \tag{3.18a}$$

$$v'V_fAA^DR_s = v'V_fR_s. \tag{3.18b}$$

Thus, (3.16) and (3.17) are equivalent to

$$v'V_iR_s = v'V_fR_s = 0 \tag{3.19}$$

which clearly contradicts (2.17a). A similar argument can be used to show that (3.3b) is implied by (2.17).

To show the converse, let us suppose that (3.3) is verified and (2.17a) is not, i.e. for some v,

$$v'V_iR_s = v'V_fR_s = 0. \tag{3.20}$$

Which thanks to E– and A–invariance of R_s implies that

$$v'V_iE^NR_s = v'V_fA^NR_s = 0. \tag{3.21}$$

Which in turn implies that

$$v'R_s = 0. \tag{3.22}$$

And from (3.3c) we get that

$$v' \in Im(O_s). \tag{3.23}$$

Thus,

$$v'V_iE^NR_s = v'PR_s = 0. \tag{3.24}$$

But (3.22) and (3.24) contradict (3.3a). Similarly, we can show that (2.17b) is implied thus proving the theorem.

Corollary

Let P_j denote a projection matrix of extendible, stationary TPBVDS $(C_j,V_i^j,V_f^j,E_j,A_j,B_j,N)$, $n\geq 2n$, j=1,2, where $\{E_j,A_j\}$, j=1,2, are in standard form for the same α and β. Then there exists an invertible matrix T such that (2.18) and (2.20) hold and such that

$$O_s^1(P_1-T^{-1}P_2T)R_s^1=0 \tag{3.25}$$

where R_s^1 and O_s^1 denote the strong reachability and observability matrices of the system 1.

Proof

It is not difficult to see from (2.18) and (2.20) that the strong reachability and observability matrices of system 1 and 2 are related as follows

$$O_s^2=O_s^1T^{-1},\ R_s^2=TR_s^1. \tag{3.26}$$

From (2.19a) and using (3.26) we can deduce that

$$O_s^1V_1^iR_s^1=O_s^2V_2^iR_s^2 \tag{3.27}$$

which implies that

$$O_s^1V_1^iE_1^NR_s^1=O_s^2V_2^iE_2^NR_s^2 \tag{3.28}$$

which of course implies that

$$O_s^1P_1R_s^1=O_s^2P_2R_s^2 \tag{3.29}$$

which clearly implies (3.25).

IV–REALIZATION PROBLEM

To illustrate the difficulties encountered in constructing minimal realizations for acausal weighting patterns, let us start with an example. In particular, consider the following weighting pattern,

$$W(m) = \begin{bmatrix} 1 & m\geq 1 \\ 1/2 & m\leq 0 \end{bmatrix}. \tag{4.1}$$

Now consider the problem of finding a realization for this weighting pattern. For this, let us separate the causal and the anti–causal parts of the system as follows

$$W(m) = W_f(m) + W_b(m) \tag{4.2}$$

where

$$W_f(m) = \begin{bmatrix} W(m) & m\geq 1 \\ 0 & m\leq 0 \end{bmatrix} \tag{4.3a}$$

$$W_b(m) = \begin{bmatrix} 0 & m\geq 1 \\ W(m) & m\leq 0 \end{bmatrix}. \tag{4.3b}$$

Weighting patterns W_f and W_b can now be realized separately using classical methods. W_f can be realized by a system running forward in time:

$$x_f(k+1) = x_f(k) + u(k) \tag{4.4a}$$

$$y_f(k) = x_f(k) \tag{4.4b}$$

$$x_f(n_1) = v_1 \tag{4.4c}$$

and W_b by a backwards running system:

$$x_b(k) = x_b(k+1) + (1/2)u(k) \tag{4.5a}$$

$$y_b(k) = x_b(k) \tag{4.5b}$$

$$x_b(n_2) = v_2. \tag{4.5c}$$

Now by combining these two forward and backwards systems (i.e. letting $x=\begin{bmatrix} x_f \\ x_b \end{bmatrix}$ and $y=y_f+y_b$), we can construct a realization for W over the interval $[n_1,n_2]$:

$$x(k+1) = x(k) + \begin{bmatrix} 1 \\ -(1/2) \end{bmatrix} u(k) \tag{4.6a}$$

$$y(k) = [1 \;\; 1]x(k) \tag{4.6b}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(n_1) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(n_2) = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \tag{4.6c}$$

It is not difficult to see (and we shall see later) that this approach can be used in general to realize any **realizable** weighting pattern. The problem with this approach, however, is that the resulting realization is not necessarily minimal, i.e. the "state" x does not have minimal dimension. For example, eventhough both realizations (4.4) and (4.5) are minimal, (4.6) is not minimal. In fact, it can be verified that the following one–dimensional TPBVDS realizes W as well:

$$x(k+1) = x(k) + u(k) \tag{4.7a}$$

$$y(k) = (1/2)x(k) \tag{4.7b}$$

$$2x(n_1) - x(n_2) = v. \tag{4.7c}$$

In this case, the reason we can realize both the causal and the anti–causal parts of W with just a one–dimensional system is that they both have the same mode, namely 1. In general, when the causal and the anti–causal parts of W share a common mode the realization approach used above may not result in a minimal system, in which case the resulting system must be reduced. In this section, we develop a method for constructing directly the minimal realization.

The realization problem considered in this section can be formulated as follows: given an infinite sequence of matrices W(k), find matrices C, E, A, B and P such that (3.2) and (3.3) hold. Once we have done that, we can use Theorem 3.1 to realize W(k) over any desired interval. We are particularly interested in realizations (C,P,E,A,B) of lowest dimension i.e. those satisfying (3.11). The first problem we consider is under what conditions the sequence W(k) admits a finite dimensional realization.

4.1–Realizability Conditions

In this section we study the conditions under which a given sequence W(k) is realizable as the extended weighting pattern of a finite–dimensional extendible, stationary TPBVDS. At the same time we will propose a method for constructing such a TPBVDS.

Theorem 4.1

A sequence of matrices W(k) is the extended weighting pattern of an extendible, stationary TPBVDS if and only if for some scalars α_i, β_i, n_f and n_b,

$$W(n_f+j) = \sum_{i=1}^{n_f} \alpha_i W(n_f-i+j) \quad \text{for all } j>0, \tag{4.8a}$$

$$W(-n_b+j) = \sum_{i=1}^{n_b} \beta_i W(-n_b+i+j) \quad \text{for all } j\leq 0. \tag{4.8b}$$

Proof

The only if part is deduced easily from (4.1) and the usual Cayley–Hamilton result. To show the if part note that we can decompose W(k) as follows

$$W_f(k) = u(k-1)W(k) \tag{4.9a}$$

$$W_b(k) = u(-k)W(k) \tag{4.9b}$$

where u(k)=1 for k≥0 and u(k)=0 otherwise. Clearly then

$$W(k) = W_f(k) + W_b(k). \tag{4.10}$$

Thanks to (4.8), $W_f(k)$ and $W_b(k)$ can be realized by finite–dimensional causal and anti–causal systems, respectively. Let (C_f,A_f,B_f) and (C_b,A_b,B_b) be such realizations, i.e.

$$W_f(k) = C_f A_f^{k-1} B_f \quad \text{for } k>0 \tag{4.11a}$$

$$W_b(k) = C_b A_b^{-k} B_b \quad \text{for } k\leq 0. \tag{4.11b}$$

Then it is clear that extendible stationary TPBVDS

$$(C,P,E,A,B) = \left([C_f \;\; -C_b], \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & A_b \end{bmatrix}, \begin{bmatrix} A_f & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} B_f \\ B_b \end{bmatrix}\right) \tag{4.12}$$

realizes W(k). This completes the proof of the theorem.

An extendible stationary TPBVDS having a representation of the form (4.12) is called separable. The TPBVDS (4.12) can be realized over any desired interval as follows

$$\begin{bmatrix} I & 0 \\ 0 & A_b \end{bmatrix} x(k+1) = \begin{bmatrix} A_f & 0 \\ 0 & I \end{bmatrix} x(k) + \begin{bmatrix} B_f \\ B_b \end{bmatrix} u(k) \tag{4.13}$$

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} x(0) + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} x(N) = \begin{bmatrix} \xi_f \\ \xi_b \end{bmatrix} \tag{4.14}$$

$$y(k) = [C_f \;\; -C_b]x(k). \tag{4.15}$$

The extendible, stationary and separable TPBVDS (4.13)–(4.15) realizes W(k), restricted to the interval [–N+1,N], for any N.

The method described in the proof of Theorem 4.1 can be used to realize any realizable sequence. The realization obtained is not in general minimal and must be reduced (a reduction procedure has been developed in [18]). The minimal TPBVDS is not in general separable. In the next section, we will propose another method based on a transform theory and a factorization problem which yields directly a minimal realization.

4.2–(s,t)–Transform

One problem with using the z–transform in cases where the dynamic of the system is singular is that the infinite frequencies cannot be handled in the same way as other frequencies, even though in such systems (at least in the discrete case) there should be total symmetry between zero and infinite frequencies. For this reason, we propose the following transform

$$H(s,t) = \sum_{k=-\infty}^{+\infty} H(k)t^{k-1}/s^k. \tag{4.16}$$

Clearly if H(s,t) exists, then it is strictly proper in (s,t) but not necessarily in s and t separately. Strictly proper in (s,t) means that for all s and t for which H(s,t) is defined

$$\lim_{\gamma \to \infty} H(\gamma s,\gamma t) = 0. \tag{4.17}$$

This can be easily seen by noting that

$$H(\gamma s,\gamma t) = (1/\gamma)H(s,t). \tag{4.18}$$

In the case in which we are interested, H(s,t) has rational entries in s and t, and strictly proper in this case implies that the denominators of these entries have higher degrees than their corresponding numerators. Note that the z–transform can be obtained from the (s,t)–transform simply by replacing (s,t) with (z,1), and that the (s,t)–transform is obtained from the z–transform by replacing z with s/t and dividing the result by t, so that all rational matrices in z, proper or not, translate into proper rational matrices in (s,t). Thus the (s,t)–transform is proper for all the cases in which we are interested.

In the causal case, the z–transform has an important role in the realization problem. Specifically, the realization problem is reduced to the following factorization of the z–transform of the weighting pattern (impulse response) H(z) of the system

$$H(z) = K(zI-F)^{-1}G \tag{4.19a}$$

for some matrices K, F and G. Hence the realization problem in the causal case reduces to the factorization of a proper rational matrix. For the boundary value systems that we are considering, the situation is more complex: even though we do need to consider the following factorization of rational matrices (in s and t this time)

$$H(s,t) = K(sD-tF)^{-1}G, \tag{4.19b}$$

the realization problem and the factorization problems are not identical.

Let W_f and W_b represent the causal and anticausal parts of W(k) respectively (as defined in (4.9)), and let W(k) be realized as in (3.2). Then, thanks to (3.3b), we can compute the transforms of $W_f(k)$ and $W_b(k)$ as follows

$$W_f(s,t) = \sum_{k=1}^{\infty} (t^{k-1}/s^k) CPE^D (AE^D)^{k-1} B$$

$$= CPE^D (sI - tAE^D)^{-1} B = CP(sE - tA)^{-1} B \tag{4.20a}$$

$$W_b(s,t) = \sum_{k=-\infty}^{0} -(t^{k-1}/s^k) C(I-P) A^D (EA^D)^k B$$

$$= C(I-P)A^D (sEA^D - tI)^{-1} B = C(I-P)(sE - tA)^{-1} B. \tag{4.20b}$$

Note that in general $W_f(s,t)$ and $W_b(s,t)$ do not have the same regions of convergence. However, we will consider their analytical extensions instead (while using the same notation). In that case

$$W_f(s,t) + W_b(s,t) = C(sE - tA)^{-1} B. \tag{4.21}$$

Also observe that

$$[W_f(s,t) \;\; W_b(s,t)] = C(sE - tA)^{-1} [PB \;\; (I-P)B], \tag{4.22}$$

$$\begin{bmatrix} W_f(s,t) \\ W_b(s,t) \end{bmatrix} = \begin{bmatrix} CP \\ C(I-P) \end{bmatrix} (sE - tA)^{-1} B. \tag{4.23}$$

We shall see that factorizations (4.21), (4.22) and (4.23) are directly tied to the 3 Hankel matrices: $O_s R_s$, $O_s R_w$ and $O_w R_s$, respectively (see Theorem 4.2).

Note that given the sequence W(k) we can compute $W_f(s,t)$ and $W_b(s,t)$, so that it appears (thanks to (4.21)) that, as in the causal case, the realization problem has been reduced to a factorization problem. This is only partly true, however, because the minimal TPBVDS is not necessarily strongly reachable or observable and thus the situation is more complex than in the causal case. In fact we shall see that in general, to construct the realization, we first need to perform the 2 factorizations (4.22) and (4.23).

The factorization problem in the case where E is invertible is simple. Many ways of constructing the minimal factorization exist (see [19]). The dimension of the minimal factorization has also been studied and it is shown (e.g. [20]) that this dimension is equal to the McMillan degree of the rational transfer matrix. We shall see in the next section that similar results can be obtained for the case where E is not necessarily invertible.

4.3–Factorization of Rational Matrices in s and t

From causal realization theory, we know how to construct a minimal factorization of a strictly proper rational matrix H(z), i.e. finding matrices K, F and G with F having smallest possible dimension such that

$$H(z) = K(zI-F)^{-1}G. \tag{4.24}$$

In that case, F, K and G are unique (except for similarity transformations).

The singular factorization problem is more complex: we want to find K, D, F, and G of lowest possible dimension such that a given rational matrix H(s,t) can be expressed as

$$H(s,t) = K(sD-tF)^{-1}G. \tag{4.25}$$

Clearly, even with the assumption that (D,F) is in standard form i.e. for some α and β, $\alpha D+\beta F=I$, D and F are not unique. To insure uniqueness we must also choose α and β a priori. In essence, in the causal case we have done that by forcing D to be equal to I which corresponds to $\alpha=1$ and $\beta=0$. Any pair (α,β) is acceptable as long as $H(\alpha,-\beta)$ is defined.

Theorem 4.2

a) Let H(s,t) be a rational matrix in s and t, then H(s,t) is factorizable if and only if (4.18) holds for all $\gamma\neq 0$ and for all s and t such that H(s,t) is defined.

b) Let H(s,t) be factorizable, and let (α,β) be a pair of scalars such that $H(\alpha,-\beta)$ exists. Then there exists a unique minimal factorization of H(s,t) (except for similarity transformations) such that

$$\alpha D + \beta F = I \tag{4.26}$$

$$H(s,t) = K(sD-tF)^{-1}G. \tag{4.27}$$

Moreover, the dimension μ of this minimal factorization is given by

$$\mu[H(s,t)] = \nu(H(\alpha z,1-\beta z)) \tag{4.28}$$

where $\nu(.)$ denotes the usual McMillan degree, and where $H(\alpha z,1-\beta z)$ is a strictly proper rational matrix in z.

Corollary

The factorization

$$H(s,t) = K(sD-tF)^{-1}G \tag{4.29}$$

is minimal if and only if (D,F,G) is strongly reachable and (K,D,F) is strongly observable. Moreover, the dimension of the minimal factorization is equal to the rank of the Hankel matrix O_sR_s where O_s denotes the strong observability matrix (K,D,F) and R_s the strong reachability matrix (D,F,G).

Proof of Theorem

To show part a), notice that the only if part is clearly implied by (4.25). To show the if part, we need to construct a realization. For this let α and β be such that $H(\alpha,-\beta)$ exists. Now consider the rational matrix $H(\alpha z,1-\beta z)$. This matrix is strictly proper in z because

$$\lim_{z\to\infty} H(\alpha z,1-\beta z) = \lim_{z\to\infty} (1/z)H(\alpha,-\beta) = 0. \tag{4.30}$$

Thus it can be realized as

$$H(\alpha z,1-\beta z) = K(zI-F)^{-1}G. \tag{4.31}$$

Now assume that $\alpha\neq 0$ (otherwise reverse the role of D and F) and let

$$w = \alpha/(\alpha t+\beta s) \tag{4.32a}$$

$$z = s/(\alpha t+\beta s). \tag{4.32b}$$

In this case

$$s = \alpha z/w \tag{4.33a}$$

$$t = (1-\beta z)/w, \tag{4.33b}$$

which implies that

$$H(s,t) = wH(\alpha z,1-\beta z) = wK(zI-F)^{-1}G = K(sD-tF)^{-1}G, \tag{4.34}$$

where

$$D = (1/\alpha)I-(\beta/\alpha)F. \tag{4.35}$$

This is the desired realization, completing the proof of part a).

For part b), we have already done most of the work. Notice simply that the factorizations (4.31) and (4.34) with D defined in (4.35) are different only by a scalar mutiplication so that we can construct one from the other and thus the dimension and uniqueness property of the two must be identical.

Proof of Corollary

Note that factorization (4.31) is minimal if and only if (K,F) is observable and (F,G) is reachable, which since α is assumed to be nonzero, are equivalent to (K,D,F) strongly observable and (D,F,G) strongly reachable, respectively.

Also note that we have shown that, when $\alpha\neq 0$, $\mu(H(s,t))$ is equal to the McMillan degree of $H(\alpha z,1-\beta z)$ as defined in (4.31). From results on causal realization theory (see e.g. Chapter 6 of [19]) we know that this McMillan degree is equal to the rank of the Hankel matrix

$$\hat{H} = \hat{O}.\hat{R} \tag{4.36}$$

where $\hat{O}$ and $\hat{R}$ are the observability matrix (K,F) and the reachability matrix (F,G). But with $\alpha\neq 0$, the nullspace of $\hat{O}$ coincides with that of O_s and the image of $\hat{R}$ with that of R_s. Thus, the rank of $\hat{H}$ must equal the rank of O_sR_s. This completes the proof of the theorem.

In the proof of Theorem 4.2 we have developed a factorization method for the factorizable matrix H(s,t). Namely, first choose α and β for which $H(\alpha,-\beta)$ is defined. Then form $H(\alpha z,1-\beta z)$ which is a strictly proper rational matrix in z. Factorize this matrix in the regular form (4.31) which gives us K, F and G. Finally compute D from (4.35).

The dimension μ of the minimal factorization can also be obtained directly from the matrix H(s,t).

Theorem 4.3

The dimension of the minimal factorization of a factorizable H(s,t) is equal to the degree of the least common multiple of the denominators of all the minors of H(s,t).

Proof

First note that all the polynomials that appear in the numerators and the denominators of the entries and thus the minors of H(s,t) are homogeneous, i.e. they have the following form

$$p(s,t) = \sum_{i=0}^{k} \alpha_i s^{k-i} t^i \tag{4.37}$$

where k is the degree of p. This follows from condition (4.18). Moreover, thanks again to (4.18), the degree of the denominator of each entry of H(s,t) is always one plus the degree of the numerator. Therefore, for the minors of H(s,t) the degree of the denominator is the order of the minor plus the degree of the numerator.

Proceeding with the proof, suppose that $K(sD-tF)^{-1}G$ is a minimal factorization of H(s,t). Without loss of generality we can assume that K, D, F and G have the following form (this can always be achieved by a similarity transformation):

$$K = [K_1 \quad K_2],\ D = \begin{bmatrix} D_1 & 0 \\ 0 & N \end{bmatrix},\ F = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix},\ G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \tag{4.38}$$

where N is nilpotent, and D_1 and F_2 are invertible. Now consider the rational matrix

$$H_1(s,t) = K_1(sD_1-tF_1)^{-1}G_1. \tag{4.39a}$$

Note that

$$H_1(s,t) = (1/t)K_1(zD_1-F_1)^{-1}G_1 = (1/t)\tilde{H}_1(z) \tag{4.39b}$$

where z=s/t. Since $H_1(s,t)$ can be obtained from $\tilde{H}_1(z)$ and vice versa, the dimension of the minimal factorization of $H_1(s,t)$ and $\tilde{H}_1(z)$ must be equal. But $\tilde{H}_1(z)$ is a strictly proper rational matrix in z and thus the dimension of its minimal factorization is equal to its McMillan degree, i.e. the degree of $a_1(z)$, the least common multiple of the denominators of the minors of $\tilde{H}_1(z)$ (see Chapter 6 of [20]). Also note that since D_1 is invertible

$$H_1(s,0) < \infty \tag{4.40}$$

and thus t is not a factor of the denominator of any of the entries and consequently minors of $H_1(s,t)$. Let $p_1(s,t)$ denote the least common multiple of the denominators of the minors of $H_1(s,t)$, then t is not a factor of $p_1(s,t)$ and consequently the degree of $p_1(s,t)$ is just the degree (in z) of $p_1(z,1)$. But

$$p_1(z,1) = a_1(z) \tag{4.41}$$

so that the degree of $p_1(s,t)$ equals the McMillan degree of $\tilde{H}_1(z)$ thus it corresponds to the dimension of D_1 and F_1.

For block 2 we proceed similarly: let

$$H_2(s,t) = K_2(sN-tF_2)^{-1}G_2. \tag{4.42}$$

Then

$$H_2(0,t) < \infty \tag{4.43}$$

because A_2 is invertible. So s is not a factor of the least common multiple of the denominators of the minors of H_2 denoted by $p_2(s,t)$. Thus, the degree of $p_2(s,t)$ is just the degree in t of $p_2(1,t)$ which, by analogy with the previous case, is just the dimension N and F_2. Also note that $H_2(s,t)=\infty$ only at t=0 thanks to nilpotency of N and the fact that N and F_2 are in standard form (which imply that the eigenvalues of $sN-tF_2$ are just $t\lambda_j$ where λ_j is an eigenvalue of F_2). Thus,

$$p_2(s,t) = p_2(1,t) = t^{n_2} \tag{4.44}$$

where n_2 denotes the dimension of N and F_2.

Noting that

$$H(s,t) = H_1(s,t) + H_2(s,t) \tag{4.45}$$

and the fact that $p_1(s,t)$ and $p_2(s,t)$ have no common factors, we can easily deduce that the least common multiple p(s,t) of the denominators of the minors of H satisfies

$$p(s,t) = p_1(s,t).p_2(s,t), \tag{4.46}$$

which proves the theorem.

Example 4.1

Consider the following sequence

$$H(k) = \begin{bmatrix} -1 & k=0 \\ 1 & k=1 \\ 0 & \text{elsewhere} \end{bmatrix} . \tag{4.47}$$

The corresponding (s,t)–transform is

$$H(s,t) = 1/s - 1/t \tag{4.48a}$$

and the z–transform is

$$H(z) = -1 + 1/z. \tag{4.48b}$$

Already we can see the advantage of using the (s,t)–transform; H(s,t) has poles at s=0 and at t=0 which means that the sequence has a zero and an infinite mode whereas H(z) has a pole only at z=0.

Applying Theorem 4.3 we can see that the degree of the minimal factorization must be 2 (= degree of st). To construct the minimal factorization simply choose $\alpha=\beta=1$ and perform the following factorization

$$H(z,1-z) = 1/z - 1/(1-z) = (1 \;\; 1)(zI-\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{4.49}$$

which implies that

$$K = (1 \;\; 1), D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, F = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, G = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{4.50}$$

4.4–Direct Realization Method

In previous sections we have defined the minimal factorization and minimal realization problems. From (4.21)–(4.23) we can see that the minimal realization problem involves factorization – in fact several factorizations – but, unlike for the causal case, the 2 problems are not identical. In this section, we make the relationship between these clear as we use the (s,t)–transform and the factorization method discussed in the previous sections to obtain the degree and construct a minimal realization of the realizable weighting pattern W(k).

Theorem 4.4

The dimension n of a minimal realization of W(k) is given by

$$n = \mu([W_f(s,t) \;\; W_b(s,t)]) + \mu(\begin{bmatrix} W_f(s,t) \\ W_b(s,t) \end{bmatrix}) - \mu(W_f(s,t)+W_b(s,t)) \tag{4.51}$$

where $\mu(.)$ denotes the degree of the minimal factorization.

Proof

Let (C,P,E,A,B) be a minimal realization of W(k) and let ρ, ω and τ be defined as follows

$$\omega = \mu(C(sE-tA)^{-1}[PB \;\; (I-P)B]) = \mu([W_f(s,t) \;\; W_b(s,t)]) \tag{4.52a}$$

$$\rho = = \mu(\begin{bmatrix} CP \\ C(I-P) \end{bmatrix}(sE-tA)^{-1}B) = \mu(\begin{bmatrix} W_f(s,t) \\ W_b(s,t) \end{bmatrix}) \tag{4.52b}$$

$$\tau = \mu(C(sE-tA)^{-1}B) = \mu([W_f(s,t) + W_b(s,t)]). \tag{4.52c}$$

From the corollary of Theorem 4.2, it follows that ρ, ω and τ are just the rank of Hankel matrices O_sR_w, O_wR_s and O_sR_s respectively, where

$$R_w = [E^{n-1}(PB \;\; (I-P)B)|...|A^{n-1}(PB \;\; (I-P)B)] \tag{4.53a}$$

$$O_w = \begin{bmatrix} \begin{bmatrix} CP \\ C(I-P) \end{bmatrix} E^{n-1} \\ \vdots \\ \begin{bmatrix} CP \\ C(I-P) \end{bmatrix} A^{n-1} \end{bmatrix}. \tag{4.53b}$$

Then from the minimality conditions (3.11a)–(3.11b), R_w and O_w have full rank which means that ρ and ω are the ranks of the strong reachability R_s and the strong observability matrices O_s respectively. From condition (3.11c) we can easily deduce that rank of O_sR_s equals rank of O_s plus that of R_s minus n (i.e. $\tau=\omega+\rho-n$) which clearly implies (4.51).

Example 4.2

Consdier the following weighting pattern

$$W(k) = \begin{cases} \alpha^k & k\geq 1 \\ \beta\alpha^k & k<1 \end{cases} \tag{4.54}$$

where α and β are scalar parameters and $\alpha<1$. Using Theorem 4.1, it is straightforward to verify that W(k) is realizable. From Theorem 4.4, we can compute the dimension of minimal realizations of W(k):

$$n = \mu([\alpha/(s-\alpha t) \;\; \alpha\beta/(\alpha t-s)]) + \mu(\begin{bmatrix}\alpha/(s-\alpha t) \\ \alpha\beta/(\alpha t-s)\end{bmatrix}) - \mu[(1-\beta)\alpha/(s-\alpha t)]$$

$$= \begin{cases} 1+1-1=1 & \text{for } \beta\neq 1 \\ 1+1-0=2 & \text{for } \beta=1 \end{cases}. \tag{4.55}$$

When $\beta\neq 1$, a minimal realization (C,P,E,A,B) of W(k) is

$$(\alpha/(1-\beta), 1/(1-\beta), 1, \alpha, 1).$$

The causal part $W_f(s,t)$ and the anticausal part $W_b(s,t)$ of W have the same pole, namely $s/t=\alpha$, that is why we can realize them both with just one eigenmode. The resulting realization is strongly reachable, strongly observable and non–separable. In general, any time a minimal realization is not separable, the causal and anticausal parts of W must share a common pole. On the other hand, if the causal and anticausal parts of W do not share any common pole, then all cooresponding minimal realizations are separable. In particular, this is the case when W(k) is summable which means that the causal part of W has poles inside the unit circle and the anticausal part of W has poles outside the unit circle. We shall further study this case later in this paper.

When $\beta=1$, a minimal realization of W is

$$([\alpha \;\; -\alpha], \begin{bmatrix}1 & 0 \\ 0 & 0\end{bmatrix}, I, \begin{bmatrix}\alpha & 0 \\ 0 & \alpha\end{bmatrix}, \begin{bmatrix}1 \\ 1\end{bmatrix}).$$

This separable realization is not strongly reachable and it is not strongly observable. Notice that in the previous realization (for $\beta\neq 1$), as β approachs 1, the system matrices tend to infinity. Thus, in a sense, $\beta=1$ is a singularity point and we can see indeed that the dimension of minimal realizations of W is 2 only when β is exactly equal to 1.

In the proof of Theorem 4.4 we have seen that in computing n, we obtain the dimension of the strong reachability and observability matrices which allows us to determine whether the minimal realization is strongly reachable or strongly observable. Thus to do the actual realization, we need to consider three different cases:

a–The minimal system is strongly reachable

What this implies is the following. If we have a minimal realization (C,P,E,A,B) of W(k), of dimension $n=\rho$, then

$$\begin{bmatrix} CP \\ C(I-P) \end{bmatrix}(sE-tA)^{-1}B = \begin{bmatrix} W_f(s,t) \\ W_b(s,t) \end{bmatrix} \tag{4.56a}$$

is a minimal factorization of $\begin{bmatrix} W_f(s,t) \\ W_b(s,t) \end{bmatrix}$. Thanks to the corollary of Theorem 4.2, we can conclude that any minimal factorization of $\begin{bmatrix} W_f(s,t) \\ W_b(s,t) \end{bmatrix}$ yields a minimal realization of W(k). Note also that since $\rho=n$, any such minimal realization is strongly reachable. Thus our construction is as follows: for any fixed α and β such that $W_f(\alpha,-\beta)$ and $W_b(\alpha,-\beta)$ are defined construct a minimal factorization

$$\begin{bmatrix} W_f(s,t) \\ W_b(s,t) \end{bmatrix} = \begin{bmatrix} C_f \\ C_b \end{bmatrix}(sE-tA)^{-1}B \tag{4.56b}$$

such that $\alpha E+\beta A=I$. From (4.56a), the matrix C is given by

$$C = C_f + C_b. \tag{4.57a}$$

To find P note from (4.56a) that P must satisfy

$$CP = C_f. \tag{4.57b}$$

Also since R_s has full rank, condition (4.2a) becomes

$$O_s(PE-EP) = O_s(PA-AP) = 0. \tag{4.58}$$

From (4.57b) and (4.58) we can see that P can be chosen to be any solution to

$$O_sP = O_s^f, \tag{4.59}$$

where

$$O_s^f = (C_f,E,A) = \begin{bmatrix} C_f A^{n-1} \\ C_f E A^{n-2} \\ \vdots \\ C_f E^{n-1} \end{bmatrix}. \tag{4.60}$$

Of course we are guaranteed that there exists a P satisfying (4.59). Furthermore, the part of P which is not determined from (4.59) is exactly the degree of freedom that exists in the selection of P (see corollary Theorem 3.2).

<u>b–The minimal system is strongly observable (ω=n)</u>

By analogy with the previous case we construct the factorization

$$[W_f(s,t) \;\; W_b(s,t)] = C(sE-tA)^{-1}[B_f \;\; B_b]. \tag{4.61}$$

Then

$$B = B_f + B_b, \tag{4.62}$$

and P is any matrix satisfying

$$PR_s = R_s^f \tag{4.63}$$

where

$$R_s^f = (E,A,B_f) = [A^{n-1}B_f | EA^{n-2}B_f | \ldots | E^{n-1}B_f]. \tag{4.64}$$

<u>c–The minimal system is neither strongly reachable or strongly observable</u>

This case is slightly more complicated because E and A cannot be directly obtained from a minimal factorization; this can be seen by noting that E and A are not even uniquely determined in this case. The factorizations that we have discussed, in this case only partially characterize the system matrices. To see this, suppose that (C,P,E,A,B) is a minimal realization of W(k) and let us do a 4–part Kalman decomposition of it. Thanks to Theorem 3.2, this realization has no strongly unreachable <u>and</u>

unobservable part. Thus, it can be represented as follows

$$A = \begin{bmatrix} A_1 & A_4 & A_6 \\ 0 & A_2 & A_5 \\ 0 & 0 & A_3 \end{bmatrix} \tag{4.65a}$$

$$E = \begin{bmatrix} E_1 & E_4 & E_6 \\ 0 & E_2 & E_5 \\ 0 & 0 & E_3 \end{bmatrix} \tag{4.65b}$$

$$B = \begin{bmatrix} B_1 \\ B_2 \\ 0 \end{bmatrix} \tag{4.65c}$$

$$C = [0 \;\; C_2 \;\; C_3] \tag{4.65d}$$

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}. \tag{4.66}$$

By direct calculation we can show that

$$\begin{bmatrix} W_f(s,t) \\ W_b(s,t) \end{bmatrix} = \begin{bmatrix} C_2P_{21}+C_3P_{31} & C_2P_{22}+C_3P_{32} \\ -C_2P_{21}-C_3P_{31} & C_2(I-P_{22})-C_3P_{32} \end{bmatrix}(s\begin{bmatrix} E_1 & E_4 \\ 0 & E_2 \end{bmatrix}-t\begin{bmatrix} A_1 & A_4 \\ 0 & A_2 \end{bmatrix})^{-1}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \tag{4.67a}$$

$$[W_f(s,t) \;\; W_b(s,t)] = [C_2 \;\; C_3](s\begin{bmatrix} E_2 & E_5 \\ 0 & E_3 \end{bmatrix}-t\begin{bmatrix} A_2 & A_5 \\ 0 & A_3 \end{bmatrix})^{-1}\begin{bmatrix} P_{21}B_1+P_{22}B_2 & -P_{21}B_1+(I-P_{22})B_2 \\ P_{31}B_1+P_{32}B_2 & -P_{31}B_1-P_{32}B_2 \end{bmatrix}, \tag{4.67b}$$

and

$$W_f(s,t) + W_b(s,t) = C(sE-tA)^{-1}B = C_2(sE_2-tA_2)^{-1}B_2. \tag{4.67c}$$

Factorizations (4.67) are minimal and thus if we perform minimal factorizations

$$[W_f(s,t) \;\; W_b(s,t)] = \tilde{C}(s\tilde{E}-t\tilde{A})^{-1}[\tilde{B}_f \;\; \tilde{B}_b]) \tag{4.68a}$$

$$\begin{bmatrix} W_f(s,t) \\ W_b(s,t) \end{bmatrix} = \begin{bmatrix} \hat{C}_f \\ \hat{C}_b \end{bmatrix}(s\hat{E}-t\hat{A})^{-1}\hat{B} \tag{4.68b}$$

for the same α and β (i.e. $\alpha\hat{E}+\beta\hat{A}=\alpha\tilde{E}+\beta\tilde{A}=I$), thanks to part b) of Theorem 4.2, we must have that matrices $(\tilde{C},\tilde{E},\tilde{A},[\tilde{B}_f\ \ \tilde{B}_b])$ are related to the matrices

$$([C_2\ \ C_3], \begin{bmatrix} E_2 & E_5 \\ 0 & E_3 \end{bmatrix}, \begin{bmatrix} A_2 & A_5 \\ 0 & A_3 \end{bmatrix}, \begin{bmatrix} P_{21}B_1+P_{22}B_2 & -P_{21}B_1+(I-P_{22})B_2 \\ P_{31}B_1+P_{32}B_2 & -P_{31}B_1-P_{32}B_2 \end{bmatrix})$$

by a similarity transformation. Similarly, matrices $(\begin{bmatrix} \hat{C}_f \\ \hat{C}_b \end{bmatrix}, \hat{E},\hat{A},\hat{B})$ are related to

$$(\begin{bmatrix} C_2P_{21}+C_3P_{31} & C_2P_{22}+C_3P_{32} \\ -C_2P_{21}-C_3P_{31} & C_2(I-P_{22})-C_3P_{32} \end{bmatrix}, \begin{bmatrix} E_1 & E_4 \\ 0 & E_2 \end{bmatrix}, \begin{bmatrix} A_1 & A_4 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix})$$

by a similarity transformation. Specifically, there exist invertible matrices V and W such that

$$\tilde{C}V = [C_2\ \ C_3],\ V^{-1}\tilde{E}V = \begin{bmatrix} E_2 & E_5 \\ 0 & E_3 \end{bmatrix},\ V^{-1}\tilde{A}V = \begin{bmatrix} A_2 & A_5 \\ 0 & A_3 \end{bmatrix},$$
$$V^{-1}[\tilde{B}_f\ \ \tilde{B}_b] = \begin{bmatrix} P_{21}B_1+P_{22}B_2 & -P_{21}B_1+(I-P_{22})B_2 \\ P_{31}B_1+P_{32}B_2 & -P_{31}B_1-P_{32}B_2 \end{bmatrix}), \tag{4.69a}$$

and

$$\begin{bmatrix} \hat{C}_f \\ \hat{C}_b \end{bmatrix} W = \begin{bmatrix} C_2P_{21}+C_3P_{31} & C_2P_{22}+C_3P_{32} \\ -C_2P_{21}-C_3P_{31} & C_2(I-P_{22})-C_3P_{32} \end{bmatrix},$$
$$W^{-1}\hat{E}W = \begin{bmatrix} E_1 & E_4 \\ 0 & E_2 \end{bmatrix},\ W^{-1}\hat{A}W = \begin{bmatrix} A_1 & A_4 \\ 0 & A_2 \end{bmatrix},\ W^{-1}\hat{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}). \tag{4.69b}$$

Now let

$$\tilde{B} = \tilde{B}_f + \tilde{B}_b \tag{4.70a}$$
$$\hat{C} = \hat{C}_f + \hat{C}_b \tag{4.70b}$$

then it can be seen that $(\tilde{C},\tilde{E},\tilde{A},\tilde{B})$ and $(\hat{C},\hat{E},\hat{A},\hat{B})$ are related respectively to

$$([C_2\ \ C_3], \begin{bmatrix} E_2 & E_5 \\ 0 & E_3 \end{bmatrix}, \begin{bmatrix} A_2 & A_5 \\ 0 & A_3 \end{bmatrix}, \begin{bmatrix} B_2 \\ 0 \end{bmatrix}) \text{ and } ([0\ \ C_2], \begin{bmatrix} E_1 & E_4 \\ 0 & E_2 \end{bmatrix}, \begin{bmatrix} A_1 & A_4 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix})$$

by similarity transformations V and W as well.

Note that factorizations $(\tilde{C},\tilde{E},\tilde{A},\tilde{B})$ and $(\hat{C},\hat{E},\hat{A},\hat{B})$ are strongly observable and strongly reachable respectively. Thus by performing a 4–part Kalman decompositions of $(\tilde{C},\tilde{E},\tilde{A},\tilde{B})$ and $(\hat{C},\hat{E},\hat{A},\hat{B})$, we obtain:

$$\tilde{C} = (\tilde{C}_1 \;\; \tilde{C}_2),\ \tilde{A} = \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ 0 & \tilde{A}_4 \end{bmatrix},\ \tilde{E} = \begin{bmatrix} \tilde{E}_1 & \tilde{E}_2 \\ 0 & \tilde{E}_4 \end{bmatrix},\ \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix} \tag{4.71a}$$

$$\hat{C} = (0 \;\; \hat{C}_2),\ \hat{A} = \begin{bmatrix} \hat{A}_1 & \hat{A}_2 \\ 0 & \hat{A}_4 \end{bmatrix},\ \hat{E} = \begin{bmatrix} \hat{E}_1 & \hat{E}_2 \\ 0 & \hat{E}_4 \end{bmatrix},\ \hat{B} = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}. \tag{4.71b}$$

Note that

$$W_f(s,t) + W_b(s,t) = C(sE-tA)^{-1}B = C_2(sE_2-tA_2)^{-1}B_2 = \tilde{C}(s\tilde{E}-t\tilde{A})^{-1}\tilde{B} = \hat{C}(s\hat{E}-t\hat{A})^{-1}\hat{B} \tag{4.72}$$

which implies that

$$\tilde{C}_1(s\tilde{E}_1-t\tilde{A}_1)^{-1}\tilde{B}_1 = \hat{C}_2(s\hat{E}_4-t\hat{A}_4)^{-1}\hat{B}_2. \tag{4.73}$$

But $(\tilde{C}_1,\tilde{E}_1,\tilde{A}_1,\tilde{B}_1)$ and $(\hat{C}_2,\hat{E}_4,\hat{A}_4,\hat{B}_2)$ are both strongly reachable and observable which implies that they must be related by a similarity transformation, i.e. for some invertible matrix T,

$$\hat{C}_2T^{-1} = \tilde{C}_1,\ T\hat{A}_4T^{-1} = \tilde{A}_1,\ T\hat{E}_4T^{-1} = \tilde{E}_1,\ T\hat{B}_2 = \tilde{B}_1. \tag{4.74a}$$

The matrix T can be computed as follows

$$T = \tilde{R}_s\hat{R}_s'(\hat{R}_s\hat{R}_s')^{-1} \tag{4.74b}$$

where $\hat{R}_s$ and $\tilde{R}_s$ denote, respectively, the strong reachability matrices of $(\hat{E}_4,\hat{A}_4,\hat{B}_2)$ and $(\tilde{E}_1,\tilde{A}_1,\tilde{B}_1)$.

Thus, the C, E, A and B matrices of the minimal realization are given by

$$C=(0\ \tilde{C}_1\ \tilde{C}_2),\ A=\begin{bmatrix} \hat{A}_1 & \hat{A}_2T^{-1} & * \\ 0 & \tilde{A}_1 & \tilde{A}_2 \\ 0 & 0 & \tilde{A}_4 \end{bmatrix},\ E=\begin{bmatrix} \hat{E}_1 & \hat{E}_2T^{-1} & * \\ 0 & \tilde{E}_1 & \tilde{E}_2 \\ 0 & 0 & \tilde{E}_4 \end{bmatrix},\ B=\begin{bmatrix} \hat{B}_1 \\ \tilde{B}_1 \\ 0 \end{bmatrix} \tag{4.75}$$

where * indicates an arbitrary matrix. Finally, to solve for P, let $(C,V_i,V_f,E,A,B,2n-1)$ be a realization of W(k) over an interval of length $2n-1$. Then the boundary matrix V_i satifies

$$O_sPR_s = O_sV_iE^{2n-1}R_s. \tag{4.76}$$

From (2.16) we get that

$$O_sV_iR_s = \left[\ (W_{kj})\ \right], \tag{4.77}$$

where

$$W_{kj} = W_f(2n-1-|k-j|). \tag{4.78}$$

Thus we can first compute a V_i from (4.77). Then P is obtained from (4.76). Note that the nonunicity in the choice of P corresponds exactly to the amount of freedom which is available in choosing P (see corollary of Theorem 3.2) and so any P satisfying (4.76) is a projection matrix. An alternative to (4.76) for solving for P can be obtained as follows. Note that since $Im(R_s)$ is E–invariant, there exists a matrix Z such that

$$ER_s = R_s Z \tag{4.79}$$

and thus

$$E^{2n-1}R_s = R_s Z^{2n-1} \tag{4.80}$$

which along with (4.76) and (4.77) yields

$$O_s P R_s = \left[\; (W_{kj}) \; \right] Z^{2n-1}. \tag{4.81}$$

In summary, to construct a minimal realization in this case we have to proceed as follows. First, perform factorizations (4.68) and use (4.70) to construct $(\tilde{C},\tilde{E},\tilde{A},\tilde{B})$ and $(\hat{C},\hat{E},\hat{A},\hat{B})$. Then perform the decompositions (4.71) and compute T from (4.74b). System matrices C, E, A, and B are then given by (4.75). Finally, compute P from (4.76) or (4.81).

Example 4.3

Let

$$W(k) = \begin{cases} 2 & k=1 \\ 1 & \text{elsewhere} \end{cases} \tag{4.82}$$

then

$$W_f(s,t) = 1/s - 1/(s-t) = -t/[s(s-t)] \tag{4.83a}$$

$$W_b(s,t) = 1/(s-t). \tag{4.83b}$$

Applying Theorem 4.4 gives us the dimension of the minimal realization:

$$n = 2 + 2 - 1 = 3. \tag{4.84}$$

It also tells us that the minimal realization is neither strongly reachable or strongly observable. Thus to obtain a minimal realization we follow procedure c) described above. First we perform the following 2 factorizations

$$[W_f \;\; W_b] = [-t/[s(s-t)] \;\; 1/(s-t)] =$$
$$[1 \;\; 1](sI - t\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix})^{-1}\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix} \qquad (4.85a)$$
$$\begin{bmatrix}W_f\\ W_b\end{bmatrix} = \begin{bmatrix}-t/[s(s-t)]\\ 1/(s-t)\end{bmatrix} = \begin{bmatrix}1 & 1\\ -1 & 0\end{bmatrix}(sI - t\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix})^{-1}\begin{bmatrix}-1\\ 1\end{bmatrix}. \qquad (4.85b)$$

We also find that

$$\tilde{B} = \begin{bmatrix}1\\ 0\end{bmatrix} \qquad (4.86a)$$
$$\hat{C} = [1 \;\; 0]. \qquad (4.86b)$$

In this case we can verify that T can be chosen to be just the identity matrix and the minimal realization is

$$C = [0 \;\; 1 \;\; 1],\; E = \begin{bmatrix}1 & 0 & *\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix},\; A = \begin{bmatrix}1 & 0 & *\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix},$$
$$B = \begin{bmatrix}-1\\ 1\\ 0\end{bmatrix},\; P = \begin{bmatrix}* & * & *\\ 0 & 1 & *\\ 1 & 0 & *\end{bmatrix} \qquad (4.87)$$

where * indicates entries that can be chosen arbitrarily. However, if we want the system (4.87) to be in normalized form we must pick the * in E equal to zero.

The above approach to the construction of a minimal realization is worthwhile only if the resulting realization is not separable. The reason for this is that in the separable case, we can easily perform the realization as was done in the proof of Theorem 4.1. The problem is to find a way of recognizing that the minimal realization is separable before actually constructing this realization. The following result solves this problem:

Theorem 4.5

W(k) has a separable minimal realization if and only if

$$n = \mu(W_f(s,t)) + \mu(W_b(s,t)). \qquad (4.88)$$

Proof

First assume that W has a separable minimal realization, in which case clearly (4.88) holds. On the other hand suppose that (4.88) holds and realize W_f and W_b separately. Then putting the realizations for W_f and W_b in parallel clearly realizes W which must be minimal because n is the degree of the minimal realization.

In the next section, we consider another class of weighting patterns for which the realization procedure is simple. Namely, we consider stable systems, i.e. systems whose impluse response W(k) is summable. As will be shown below, these systems admit separable realizations where the forward and backward subsystems are forward and backward stable.

4.5–The Class of Stable TPBVDS's

In the case where the sequence W(k) is summable, i.e. when

$$\sum_{k=-\infty}^{\infty} |W(k)| < \infty, \tag{4.89}$$

it turns out that the realizability condition, as well as finding the degree of the minimal realization and the realization procedure, are simpler than in the general case.

Theorem 4.6

a) A summable sequence W(k) is realizable if and only if the (s,t)–transform of W(k), W(s,t), is rational in s and t.

b) A summable and realizable sequence W(k) has a minimal realization which consists of a separable TPBVDS where the forward and backward subsystems are forward and backward stable respectively. Moreover, this realization is strongly reachable and observable.

Proof

To show part a) suppose that W(s,t) is rational. Note that

$$W(s,t) = \sum_{k=-\infty}^{\infty} W(k)t^{k-1}/s^k \tag{4.90}$$

is well–defined (i.e. has a region of convergence) thanks to (4.89). Also note that

$$W(s,t) = W_f(s,t) + W_b(s,t). \tag{4.91}$$

Since W(s,t), $W_f(s,t)$ and $W_b(s,t)$ have a common region of convergence (which includes $|t/s|=1$), $W_f(s,t)$ and $W_b(s,t)$ must be rational as well. Note that $W_f(s,t)$ is analytic for $|s|\geq|t|$ and $W_b(s,t)$ for $|t|\geq|s|$, thus $W_f(s,t)$ and $W_b(s,t)$ have the following minimal factorizations

$$W_f(s,t) = C_f(sI-tA_f)^{-1}B_f \tag{4.92a}$$

$$W_b(s,t) = C_b(sA_b-tI)^{-1}B_b \tag{4.92b}$$

where A_f and A_b have eigenvalues inside the unit circle. Now consider the TPBVDS (4.13)–(4.15) with C_f, A_f, B_f, C_b, A_b and B_b as defined in (4.92). It is easy to check that the weighting pattern of this system is just W(k) proving that W(k) is realizable. The only if part is trivial.

To show part b) simply note that the realization constructed above is strongly reachable and observable and thus it is minimal.

Next, let us introduce the notion of stability for extendible stationary TPBVDS's.

Definition 4.1

The extendible stationary TPBVDS (C,P,E,A,B) is called <u>stable</u> if it has a summable weighting pattern.

Essentially a stable system is a separable system where the forward and backward subsystems are forward and backward stable. A stable system has a number of interesting properties:

a– it has a stable minimal realization which is strongly reachable and observable,

b– let (C,V_i,V_f,E,A,B,N), $N\geq 2n$, be any finite interval minimal realization of a stable minimal TPBVDS (C,P,E,A,B) then (C,V_i,V_f,E,A,B,N) is strongly reachable and observable, extendible and separable. If in addition we assume that $\{E,A\}$ has been put in forward–backward stable form,

$$E = \begin{bmatrix} I & 0 \\ 0 & A_b \end{bmatrix}, A = \begin{bmatrix} A_f & 0 \\ 0 & I \end{bmatrix} \tag{4.93}$$

with A_f and A_b having eigenvalues inside the unit circle ($\{E,A\}$ cannot have any eigenmode on the unit circle because $W(s,t)$ has no poles on the unit circle), then the projection matrix P is given by

$$P = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \tag{4.94}$$

and the boundary matrices V_i and V_f are equal to P and I–P, respectively, regardless of the length of the interval N,

c– There exists a realization of the stable TPBVDS (C,P,E,A,B) defined on $[-\infty,+\infty]$. This realization denoted by (C,V_i,V_f,E,A,B,∞) has $W(k)$ for weighting pattern,

d– the projection matrix P of a stable TPBVDS (C,P,E,A,B) is completely determined in terms of the pencil $\{E,A\}$, in fact,

$$P = E^{\infty} \tag{4.95}$$

if the system is in the forward–backward stable form (4.93).

From property c– we can see that the realization procedure for summable sequences just consists in performing the factorization

$$W(s,t) = C(sE-tA)^{-1}B \tag{4.96}$$

and transforming $\{E,A\}$ into the forward–backward stable form (4.93).

V–CONCLUSION

In this paper we have studied the problem of realizing acausal, linear, time–invariant weighting patterns by extendible, stationary TPBVDS's. We have obtained realizability conditions and proposed a method for realizing any realizable weighting pattern with a separable realization. This method however does not always yield a minimal realization. We then proposed a new transform technique, which is well adapted to handling noncausal weighting patterns. This allowed us to obtain a direct method for computing the degree of a minimal realization and for constructing such a realization. This approach generalizes the classical realization theory for causal systems.

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