Computation of the asymptotic states for linear half space kinetic problems

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ABSTRACT

A spectral numerical scheme computing the asymptotic states for linear half space problems is described in the case of a simple transport equation and the linearized Bhatnagar-Gross-Krook (BGK) model (see [5]). This method seems very efficient and the results are in good agreement with those obtained by more direct computations and by other authors.

KEYWORDS Spectral method, Boltzmann equation, BGK model, Kinetic boundary layer.

CALCUL DES ETATS ASYMPTOTIQUES DE PROBLEMES DE TRANSPORT LINEAIRE DANS UN DEMI-ESPACE

RESUME

Nous décrivons une méthode numérique spectrale pour calculer l'état asymptotique de problèmes linéaires de demi-espace dans le cas d'une équation simple de transport et du modèle Bhatnagar-Gross-Krook (BGK) linéarisé (voir [5]). Cette méthode semble très performante et les résultats sont en accord avec ceux obtenus à l'aide de simulations plus directes et par d'autres auteurs.

MOTS CLES : Méthode spectrale, Equation de Boltzmann, modèle BGK, couche limite cinétique.

I MOTIVATIONS

The flow field around reentry aircrafts is well described by a density distribution $f(x,v,t)$ governed by the Boltzmann equation (see Cercignani [5], Chapman, Cowling [7], Ferziger, Kaper [14]). Numerical methods used to solve it, for example the Monte-Carlo Simulation (see Babovsky [1], Bird [2], Deshpande [13], Nanbu [22]) or more direct computations (see Chorin [8], Yen [28]) are expensive in computer time. When the mean free path of molecules is small, fluid dynamics (Navier-Stokes or Euler equations) gives good results. Thus, it seems of great interest to be able to match regions in which we compute either the Boltzmann equation or the Navier-Stokes system according to the local value of the mean free path (or some other relevant criteria). A general strategy for this domain decomposition problem is described by Golse [15] and will be developped in [11].

The distribution of particles coming from the Boltzmann cell to the Navier-Stokes cell is computed through the resolution of the Boltzmann equation using for example a Monte-Carlo method. Usually, this distribution is not exactly Maxwellian and we have to proceed some kind of projection to get from the knowledge of this distribution the right boundary conditions for the Navier-Stokes cell. This can be done by introducing a Knudsen layer term, the support of which is of the thickness of some mean free paths. This layer term makes the transition between the kinetic and the fluid descriptions of the flow, and it satisfies the Boltzmann equation. Moreover, near the matching interface, the local Knudsen number (ratio of the mean free path by a characteristic length of variation of the fluid quantities) is assumed to be small, in order to be sure that the Navier-Stokes equations give a good approximation of the flow. Therefore, the kinetic correction satisfies a linear half space problem

$$(\xi_1 + v)\frac{\partial}{\partial x}\chi + L\chi = 0, \quad 0 \le x < +\infty, \tag{I.1.1}$$

$$\chi(0,\xi_1) = \varphi(\xi), \quad \text{for} \quad \xi_1 + v > 0 \tag{I.1.2}$$

where v is the mean velocity normal to this interface and L the Boltzmann operator linearized around the absolute Maxwellian. Existence and uniqueness for problem (I.1.1)-(I.1.2) was studied by Greenberg, Van der Mee [17] for the BGK model and by Coron, Golse, Sulem [10] for the linearized Boltzmann operator.

The theoretical behavior of the solution of (I.1.1)-(I.1.2) is well known. We want to describe a numerical method to compute the asymptotic behavior of χ when x tends to infinity.

Since the problem (I.1.1)-(I.1.2) is to be solve at each cell of the interface to make domain decompositions (see Golse [15]), it should be very easy to perform. The direct computation of the solution of (I.1.1)-(I.1.2) is too expensive from a computational point of view (see section II.4.B.3); thus we propose a spectral method which is much cheaper. In this method, we choose a vector space of finite dimension and we look for exponentially decreasing solution of (I.1.1). This leads to a generalized eigenvalue problem. A decomposition of the incoming flux on these eigenvectors gives an approximation of the asymptotic limit of the solution of (I.1.1)-(I.1.2).

Note that linear half space problems (I.1.1)-(I.1.2) with $v = 0$ is also fundamental to get the coefficients of the slip boundary equations for the Navier-Stokes system (see Coron [9] and Sone, Onishi [25]).

The following sections are organised as follows

in section II, we study the case of a simple transport equation. We present the spectral method and test it by computing the " extrapolation length" and by perfoming a direct computation for (I.1.1)-(I.1.2).

in section III, we extend the method to the linearized BGK model and we test it on the computation of the slip coefficients.

II SIMPLE TRANSPORT PROBLEM

1. Introduction

We first study the simple transport problem

$$(\mu+c)\frac{\partial}{\partial x}u+u-\tilde{u}=0, \quad 0\leq x<+\infty, \quad -1\leq\mu\leq+1 \qquad (II.1.1)$$

$$u(0,\mu)=\varphi(\mu), \quad \text{for} \quad \mu+c>0 \qquad (II.1.2)$$

$c\in[-1,+1]$ is a shift constant, and

$$\tilde{u}(x)=\frac{1}{2}\int_{-1}^{+1}u(x,\mu)d\mu$$

The following proposition can be easily proved (see for example [16])

Proposition II.1.1

For any $c\geq 0$ and φ such that $\int_{0\leq\mu+c\leq 1}\varphi^2 d\mu<+\infty$, there exists a unique solution of (II.1.1)-(II.1.2) in $L^\infty(dx,L^2(d\mu))$. This solution decreases exponentially fast to a constant when x goes to infinity.

We are going to describe a numerical scheme to compute the asymptotic limit of problem (II.1.1)-(II.1.2) when $c\geq 0$.

2. Desciption of the spectral method

We extend the method proposed by Degond, Mas-Gallic [12] for a model Fokker-Planck equation, to this transport problem.

We consider the operators A_c and B defined respectively by

$$A_c u=(\mu+c)u, \quad Bu=u-\tilde{u}$$

Following [12], we remark that if (λ,Φ_λ) is a solution of the generalized eigenvalue problem

$$\lambda A_c\Phi_\lambda(\mu)=B\Phi_\lambda(\mu) \qquad (II.2.1)$$

then

$$\Psi_\lambda(x,\mu)=e^{-\lambda x}\Phi_\lambda(\mu)$$

satisfies the equation (II.1.1).

Because we are looking for bounded solution of (II.1.1)-(II.1.2), we are interested in eigenfunctions Φ_λ of (II.2.1) with $\lambda\geq 0$.

In the case $c=0$, it is well known that the function φ can be decomposed on the eigenfunctions Φ_λ associated with positive or null eigenvalues

$$\varphi=\alpha_0\Phi_0+\sum_{\lambda>0}\alpha_\lambda\Phi_\lambda \qquad (II.2.2)$$

This property is known as "half range completeness" (see [12]).

When x goes to infinity, $\Psi_\lambda(x,\mu)$ (with $\lambda>0$) tends to zero. Thus the asymptotic limit of the solution of (II.1.1)-(II.1.2) is equal to α_0.

Now, we consider the space spanned by the n first Legendre polynomials and write down the operator B and a truncation of A_c in this vector space. We are going to solve the "discrete" eigenvalue problem corresponding to (II.2.1) by performing an approximate decomposition similar to (II.2.2). We shall obtain numerically the asymptotic limit of the solution of (II.1.1)-(II.1.2).

A) the discrete problem

Let P_i be the normalized Legendre polynomial

$$P_i(\mu) = \sqrt{\frac{2i+1}{2}}\frac{1}{2^i i!}\frac{d^i}{d\mu^i}((\mu^2-1)^i)$$

The (P_i) are normalized

$$\int_{-1}^{+1} P_i(\mu)P_j(\mu)d\mu = \delta_{i,j}$$

and form a complete system of $L^2(d\mu)$.
We recall the induction relations

$$\mu P_i = m_i P_{i-1} + m_{i+1}P_{i+1}, \quad 1 \leq i \leq n-2$$

with

$$m_i = \frac{i}{\sqrt{(2i-1)(2i+1)}} \quad \text{and} \quad \mu P_0 = \frac{\mu}{\sqrt{2}} = \frac{1}{\sqrt{3}}P_1 = m_1 P_1$$

The operator A_{cn}, troncation of A_c on $Span(P_0, ...P_{n-1}) = E_n$ is defined by the matrix

$$A_{cn} = \begin{pmatrix} c & m_1 & 0 & \dots & \dots & \dots & 0 \\ m_1 & c & m_2 & \ddots & & & \vdots \\ 0 & m_2 & c & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & c & m_{n-1} \\ 0 & \dots & \dots & \dots & 0 & m_{n-1} & c \end{pmatrix}$$

The restriction of B to E_n, referred to as B_n, is defined by the matrix

$$B_n = \begin{pmatrix} 0 & 0 & \dots & \dots & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}$$

These particular forms of operators A_{cn}, B_n are due to the choice of the Legendre polynomials.

Taking the discrete version of (II.2.1), we are looking for solutions of the following generalized eigenvalue problem

$$\lambda_i A_{cn}\Phi_i = B_n \Phi_i \qquad (II.2.3)$$

As in the continuous case, for $\lambda_i = 0$, the space of solutions of (II.2.3) is equal to $Span(P_0)$.
We then decompose the incoming flux data, $\varphi(\mu)$ (given for $\mu + c \geq 0$), on the solutions Φ_i of the equation (II.2.3) with $\lambda_i \geq 0$

$$\varphi(\mu) \sim \sum_{\lambda_i>0} \alpha_i \Phi_i \quad + \alpha_0 P_0 \qquad (II.2.4)$$

The asymptotic limit calculated by this spectral method is

$$\alpha_0 P_0 = \frac{\alpha_0}{\sqrt{2}}$$

The main step in this method is the way to perform the projection (II.2.4). In order to do this, following [12], we remark that we have in the continuous case

$$\varphi(\mu) = \sum_{\mu_0+c>0} \varphi(\mu_0)\delta_{\mu_0}(\mu), \quad \mu + c > 0 \tag{II.2.5}$$

where δ_{μ_0} is the Dirac distribution at point μ_0 and is also the generalized eigenfunction of the operator A_c associated to the eigenvalue $\mu_0 + c$

$$A_c\delta_{\mu_0} = (\mu_0 + c)\delta_{\mu_0}$$

The decomposition (II.2.5) can be viewed as the projection of φ on the eigenfunction of A_c associated with positive eigenvalues.

We derived from this interpretation the way to perform the projection (II.2.4) in the discrete case.

We consider the space E^+_{cn} spanned by the eigenfunctions of A_{cn} with positive eigenvalue and we denote by Π^+_{cn} the L^2 orthogonal projection on E^+_{cn}.

Let φ_n be an approximation of φ in E_n; we compute α_0, α_i such that

$$\Pi^+_{cn}(\varphi_n) = \sum_{\lambda_i>0} \alpha_i \Pi^+_{cn}(\Phi_i) \quad + \alpha_0 \Pi^+_{cn}(P_0) \tag{II.2.6}$$

We prove in the section (II.3.D) that $\left((\Pi^+_{cn}(\Phi_i))_{\lambda_i>0}, \Pi^+_{cn}(P_0)\right)$ is a basis of E^+_{cn}, so that there exists a unique sequence (α_0, α_i) satisfying (II.2.6).

Note that (II.2.6) is a system of n equations with $Card\{\lambda_i/\lambda_i > 0\} + 1$ unknown. Thus, from a computational point of view, we have used a least square algorithm to solve (II.2.6).

The decomposition (II.2.4) gives also an approximate solution of the problem (II.1.1)-(II.1.2) and in particular, a prediction of $u(x = 0, \mu)$ for $\mu + c < 0$ for any given φ.

3. Properties of the eigenvalue problems

A) Eigenvalue of A_{cn}

We have

$$A_{cn} = A_{0n} + cI_n$$

where I_n is the $n \times n$ identity matrix.

A_{cn} and A_{0n} have the same eigenvectors and their eigenvalues differ by c.

Let us study the case $c = 0$

Since A_{0n} is a symmetric matrix, it has n real eigenvalues and n eigenvectors. Moreover, if $\Phi_i = \sum_{k=0}^{n-1} c_{i,k}P_k$ satisfies $\lambda_i\Phi_i = A_{0n}\Phi_i$ then one can compute $c_{i,k}$ in terms of $c_{i,0}$ using the following induction formula

$$c_{i,1} = \frac{\lambda_i}{m_1}c_{i,0} \tag{II.3.1}$$

$$c_{i,k} = \frac{\lambda_i}{m_k}c_{i,k-1} - \frac{m_{k-1}}{m_k}c_{i,k-2}, \quad \text{for} \quad 2 \le k \le n-1 \tag{II.3.2}$$

The eigenvalues of A_{0n} are therefore simple.

The determinant of A_{0n} is equal to zero if n is odd and to $(-1)^{n/2}(m_1)^2(m_3)^2...(m_{n-1})^2$ if n is even.

From now on, we consider the case where n is even.

We also remark that if

$$A_{0n}\left(\sum_{k=0}^{n-1} c_{i,k}P_k\right) = \lambda_i\left(\sum_{k=0}^{n-1} c_{i,k}P_k\right)$$

then

$$A_{0n}\left(\sum_{k=0}^{n-1} c_{i,k}(-1)^k P_k\right) = -\lambda_i\left(\sum_{k=0}^{n-1} c_{i,k}(-1)^k P_k\right)$$

Thus, for $c = 0$, the set of eigenvalues is symmetric with respect to 0 and the number of positive eigenvalue is $n/2$.

For $\Phi \in E_n$, we have

$$|(A_{0n}\Phi, \Phi)| = |\int_{-1}^{+1} \mu\Phi^2 d\mu| \leq \int_{-1}^{+1} \Phi^2 d\mu \qquad (II.3.3)$$

the inequality beeing strict if $\Phi \neq 0$. Thus the eigenvalues of A_{0n} are smaller than 1.

B) Generalized eigenvalue problem (II.2.3) $\lambda_i A_{cn}\Phi_i = B_n\Phi_i$

As mentionned previously, the space of solution of (II.2.3) with $\lambda_i = 0$ is equal to the space spanned by P_0. Let $\Phi_i = \sum_{k=0}^{n-1} \Phi_{i,k}P_k$ be a solution of (II.2.3), we obtain

$$c\Phi_{i,0} + m_1\Phi_{i,1} = 0 \qquad (II.3.4)$$

Let us study the generic case $c \neq 0$

If $\lambda_i \neq 0$, (II.2.3) is equivalent to

$$\Phi_{i,0} = -\frac{m_1}{c}\Phi_{i,1} \qquad (II.3.5)$$

and

$$\tilde{A}_{cn}\begin{pmatrix}\Phi_{i,1}\\ \vdots \\ \Phi_{i,n-1}\end{pmatrix} = \frac{1}{\lambda_i}\begin{pmatrix}\Phi_{i,1}\\ \vdots \\ \Phi_{i,n-1}\end{pmatrix} \qquad (II.3.6)$$

where $\tilde{A}_{cn}$ is the symmetric matrix

$$\tilde{A}_{cn} = \begin{pmatrix} c-\frac{m_1^2}{c} & m_2 & 0 & \dots & \dots & 0 \\ m_2 & c & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & c & m_{n-1} \\ 0 & \dots & \dots & 0 & m_{n-1} & c \end{pmatrix}$$

This matrix has $n-1$ simple eigenvalues (the proof is similar to that given for A_{0n}). The determinant of $\tilde{A}_{cn}$ is a rational function of c with at most a finite number of zeros. Except for these critical values of c, (II.2.3) gives $(n-1)$ solutions.

We thus have found n solutions of the generalized eigenvalue problem (II.2.3).

The particular case $c = 0$.

For $\lambda_i \neq 0$, (II.2.3) is equivalent to the system

$$\Phi_{i,0} = -\frac{m_2}{m_1}\Phi_{i,2} \qquad (II.3.7)$$

$$\Phi_{i,1} = 0 \qquad (II.3.8)$$

$$\begin{pmatrix} 0 & m_3 & 0 & \dots & \dots & 0 \\ m_3 & 0 & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 0 & m_{n-1} \\ 0 & \dots & \dots & 0 & m_{n-1} & 0 \end{pmatrix} \begin{pmatrix} \Phi_{i,2} \\ \vdots \\ \Phi_{i,n-1} \end{pmatrix} = \frac{1}{\lambda_i} \begin{pmatrix} \Phi_{i,2} \\ \vdots \\ \Phi_{i,n-1} \end{pmatrix} \qquad (II.3.9)$$

System (II.3.9) defines $n-2$ simple eigenvalues (n is even, so the symmetric square matrix in (II.3.9) does not have 0 as an eigenvalue). We have $n-1$ solutions for the generalized eigenvalue problem (II.2.3).

Remark. According to (II.3.8), solutions of (II.2.3) have no component on P_1. In fact $P_1(\mu) = \sqrt{3/2}\mu$ corresponds to the unique polynomial unbounded solution of (II.1.1) with $c = 0$

$$u(x, \mu) = x - \mu$$

Such a solution does not exist for $c \neq 0$.

C) Number of positive eigenvalues

Proposition II.3.1 *For $c \geq 0$, the number of positive or null eigenvalues of A_{0n} and of problem (II.2.3) are equal.*

Proof

When $c = 1$, there exist n positive eigenvalues of A_{cn} because $E_{cn}^{+} = E_n$ (see (II.3.3)). Similarly, if (λ_i, Φ_i) satisfy (II.2.3) with $c = 1$ then $\lambda_i \geq 0$.

Let us study the case where $c \neq 0$ and $c \neq 1$.

The eigenvalues of $\tilde{A}_{cn}$ vary continuously when $c \in]0, 1]$. Therefore, according to (II.3.6), the number of eigenvectors corresponding to positive or null eigenvalue of problem (II.2.3) is constant between the critical values of c where the determinant of $\tilde{A}_{cn}$ in null. The eigenvectors are given by (II.3.5)-(II.3.6) and thus vary continuously when c describes $]0, 1]$.

We remark that λ_i becomes infinite if and only if Φ_i satisfies $A_{cn}\Phi_i = 0$. This relation implies that c is minus an eigenvalue of A_{0n} and Φ_i a corresponding eigenvector.

Let us prove that the sign of λ_i changes when c crosses one of these critical values.

We denote by $(\Phi_{ic}, 1/\lambda_{ic})$ the continuous solution of (II.3.5)-(II.3.6) corresponding to different values of c around the critical value c_0. Let

$$c = c_0 + \delta c, \quad A_{cn} = A_{c_0 n} + \delta A_{cn}$$

$$\Phi_{ic} = \Phi_{ic_0} + \delta\Phi_{ic}, \quad \frac{1}{\lambda_{ic}} = \frac{1}{\lambda_{ic_0}} + \delta(\frac{1}{\lambda_{ic}}) = \delta(\frac{1}{\lambda_{ic}})$$

we have

$$A_{c_0 n}\Phi_{ic_0} = 0$$

and

$$A_{cn}\Phi_{ic} = A_{c_0 n}\delta\Phi_{ic} + \delta A_{cn}\Phi_{ic_0} + \delta A_{cn}\delta\Phi_{ic} = \delta(\frac{1}{\lambda_{ic}})B_n\Phi_{ic_0}$$

The image of A_{c_0n} is the orthogonal space to Φ_{ic_0}. We thus get

$$\delta(\frac{1}{\lambda_{ic}})(B_n\Phi_{ic_0},\Phi_{ic_0}) = (\delta A_{cn}\Phi_{ic_0} + \delta A_{cn}\delta\Phi_{ic},\Phi_{ic_0}) \qquad (II.3.10)$$

where $(.,.)$ denotes the scalar product. We have $\delta A_{cn} = (\delta c)I_n$. Equation (II.3.10) proves that

$$\delta(\frac{1}{\lambda_{ic}}) \quad \text{and} \quad \delta c$$

have the same sign. This result proves that the sign of λ_{ic} changes when c crosses the value c_0 which is an eigenvalue of A_{0n}.

The numbers of positive or null eigenvalue of A_{cn} and of the problem (II.2.3) are both equal to n when $c = 1$; the two numbers decrease when c decreases and change by one when c crosses an eigenvalue of A_{0n}. They are thus equal for any $c \in]0,1]$.

When $c = 0$, the number of positive or null eigenvalue of A_{0n} and of problem (II.2.3) are both equal to $n/2$ (see section II.3.A, II.3.B).

D) The decomposition

Let $(\Phi_i,\ i \in I_c)$ be the solutions of the generalized eigenvalue problem (II.2.3) with positive, null or infinite (that is to say $A_{cn}\Phi_i = 0$) eigenvalue.

Proposition II.3.2 *The vectors $(\Pi^+_{cn}(\Phi_i),\ i \in I_c)$ are linearly independent and thus constitute a basis of E^+_{cn}.*

Proof

We denote by $\Phi_0 = 1/\sqrt{2}$, the eigenvector associated to $\lambda_0 = 0$. If

$$\sum \alpha_i \Pi^+_{cn}(\Phi_i) = 0 \qquad (II.3.11)$$

then

$$\sum \alpha_i \Phi_i \in (E^+_{cn})^\perp$$

and we have

$$0 \geq (A_{cn}(\sum \alpha_i\Phi_i), \sum \alpha_i\Phi_i) = \sum_i \alpha_i^2 (A_{cn}\Phi_i,\Phi_i) + 2\sum_{i,j} \alpha_i\alpha_j (A_{cn}\Phi_i,\Phi_j) \qquad (II.3.12)$$

The second term in the right hand side of (II.3.12) is equal to zero because, for $0 \leq i < j$ and $\lambda_j \neq 0$

$$(A_{cn}\Phi_i,\Phi_j) = (\Phi_i, A_{cn}\Phi_j) = \frac{1}{\lambda_j}(\Phi_i, B_n\Phi_j) = \frac{1}{\lambda_j}(B_n\Phi_i, B_n\Phi_j) = 0$$

(see section II.3.B for this identity). If $\lambda_j = 0$ then $A_{cn}\Phi_j = 0$ which gives the same result.
Inequality (II.3.12) becomes

$$0 \geq \alpha_0 (A_{cn}\Phi_0,\Phi_0) + \sum_{i\neq 0} \frac{\alpha_i^2}{\lambda_i}(B_n\Phi_i,\Phi_i) \qquad (II.3.13)$$

for $i \neq 0$, $(B_n\Phi_i,\Phi_i) = (B_n\Phi_i, B_n\Phi_i) > 0$ because $B_n\Phi_i \neq 0$ (see section (II.3.b)).
for $i = 0$, $(A_{cn}\Phi_0,\Phi_0) = c/2$.
a) In the generic case where c is different from 0 and the eigenvalue of A_{0n}, we get $\alpha_i = 0$ for all i.
b) If $c = 0$, then the inequality (II.3.12) gives $\alpha_i = 0$ for $i \neq 0$ and using (II.3.11), we obtain

$$\alpha_0 \Pi^+_{0n}(\Phi_0) = 0 \qquad (II.3.14)$$

but the induction relations (II.3.1)-(II.3.2) prove that the component of the eigenvector of A_{0n} on Φ_0 are never zero; thus $\alpha_0 = 0$.

c) If c is equal to an eigenvalue of A_{0n}, then there exists a unique j such that $A_{cn}\Phi_j = 0$. Inequality (II.3.12) proves that $\alpha_i = 0$ for $i \neq j$. Using (II.3.11), we get $\alpha_j \Pi^+_{cn}(\Phi_j) = 0$ but $\Phi_j \in E^+_{cn}$ (see section II.3.B), so $\alpha_j = 0$.

We have proved that $\Pi^+_{cn}(\Phi_i)$ form a basis of E^+_{cn}, which allows the decomposition (II.2.6).

4. Numerical results

From a computational point of view, the spectral method is easy to implement. First, the eigenvectors and eigenvalues (Φ_i, λ_i), $1 \le i \le n$ of problem (II.2.3) are computed. This computation can be done with a high efficiency because the matrix A_{cn}, B_n are symmetric and tridiagonal. The eigenvectors of A_{cn} associated with positive eigenvalues are also computed and we thus obtain the projection operator Π^+_{cn}. The vectors $\Pi^+_{cn}(\varphi)$ and $\Pi^+_{cn}(\Phi_i)$ for $\lambda_i \ge 0$ are computed and decomposition (II.2.6) is performed using a least square algorithm. This decomposition gives the spectral solution for problem (II.1.1)-(II.1.2).

We now discuss the results obtained with this spectral method for the computation of the asymptotic limit and the outgoing flux of the solution of problem (II.1.1)-(II.1.2) for $c = 0$ and $c > 0$.

A) The case $c = 0$

1) Computation of the extrapolation length

We first test this method on the problem of the computation of the extrapolation length. We consider the case $c = 0$, $\varphi(\mu) = \mu$. The bounded solution of (II.1.1)-(II.1.2) converges to a constant l when x goes to infinity. This constant is called the extrapolation length and its numerical value is $l = 0.71044609$ (see Williams [27]). The results obtained by the spectral method are the following

N	l_N	$(l - l_N)/l$
4	0.69402480	2.31 %
10	0.70823854	0.31 %
20	0.70991539	0.075 %
40	0.71031562	0.018 %
50	0.71036285	0.012 %
60	0.71038841	0.008 %
70	0.71040377	0.006 %

As in [12], we found that this spectral method gives very accurate results even for a small value of N.

Note that the Marshak approximation (see the following section) gives a value of l equal to $l_{Mar} = 2/3$, this leads to an error of 6.16 % .

A variational method (see Loyalka [21], Golse[16]) gives $L_{Var} = 0.7083$ the error is of 0.3 %.

Remark We have tested another way to perform the decomposition (II.2.4). Instead of computing α_0, α_i by the relation (II.2.6), we define α_0, α_i by

$$(\varphi, P_0)_+ = (\sum_{\lambda_i > 0} \alpha_i \Phi_i + \alpha_0 P_0, P_0)_+ \qquad (II.4.1)$$

and

$$(\varphi, \Phi_i)_+ = (\sum_{\lambda_i > 0} \alpha_i \Phi_i + \alpha_0 P_0, \Phi_i)_+ \quad \text{for all} \quad \lambda_i > 0 \qquad (II.4.2)$$

where

$$(f, g)_+ = \int_0^1 f(\mu) g(\mu) d\mu \qquad (II.4.3)$$

The results are not as good as those obtained by the previous decomposition.

N	l_N	$(l-l_N)/l$
4	0.66175	6.85 %
10	0.69505	2.17 %
20	0.70364	0.96 %

In the next sections we use decomposition (II.2.6).

2) The Albedo operator

As it was noticed in the section II.2, the spectral method gives not only the asymptotic limit of the solution of (II.1.1),(II.1.2) but also the outgoing flux at $x = 0$ according to the decomposition (II.2.4)

$$u(0,\mu) = \sum_{\lambda_i > 0} \alpha_i \Phi_i(\mu) \quad + \alpha_0 P_0(\mu), \quad \mu + c < 0 \qquad (II.4.4)$$

For this simple transport problem (II.1.1)-(II.1.2) with $c = 0$, Chandrasekhar's calculus gives

$$u(0,-\mu) = \frac{1}{2} H(\mu) \int_0^1 \frac{\mu' H(\mu')\varphi(\mu')}{\mu + \mu'} d\mu', \quad \mu > 0 \qquad (II.4.5)$$

where the function H is the solution of some integral equation (see Chandrasekhar [6]).
We have tested the spectral method for the Albedo problem for $\varphi(\mu) = \mu$, for $0 < \mu < 1$.
The equation (II.4.5) gives

$$u(0,-\mu) = \frac{H(\mu)}{\sqrt{3}} - \mu \qquad (II.4.6)$$

On figure 1, the first curve represents the graph of the solution u at $x = 0$

$$u(0,\mu) = \mu, \quad \text{for } \mu > 0 \qquad (II.4.7)$$

$$u(0,\mu) \quad \text{given by (II.4.6)}, \quad \text{for } \mu < 0 \qquad (II.4.8)$$

the second curve $\Pi^+_{cn}(\mu)$ (for $c = 0$ and $n = 20$), the third curve the approximation of the solution given by the decomposition (II.2.4) at $x = 0$, the fourth one is the prediction (constant with respect to μ) of the Albedo given by Marshak's method.

As expected, $\Pi^+_{0n}(\mu)$ is very close to 0 for $\mu < 0$ and to μ for $\mu > 0$. The oscillations in the approximation of the solution given by the spectral method are due to the fact that the exact solution is discontinuous at $x = 0$ (Gibbs phenomenon). The spectral method seems to give a good approximation of the solution in the Legendre space.

B) The case $c \neq 0$

1) The Marshak approximation

Description of Marshak's method.
For u solution of (II.1.1), the quantity

$$\int_{-1}^{+1} (\mu + c) u(x,\mu) d\mu \qquad (II.4.9)$$

is independent of x. In the Marshak's approximation (see [16]), one assumes that the half flux

$$\int_{\substack{0 \leq \mu + c \\ -1 \leq \mu \leq +1}} (\mu + c) u(x,\mu) d\mu \qquad (II.4.10)$$

is conserved.
From (II.1.2), this quantity at $x = 0$, is equal to

$$\int_{\substack{0 \leq \mu + c \\ -1 \leq \mu \leq +1}} (\mu + c)\varphi(x,\mu)d\mu$$

The limit of u at infinity, u_∞, is given by

$$u_\infty \int_{\substack{0 \leq \mu + c \\ -1 \leq \mu \leq +1}} (\mu + c)d\mu = \int_{\substack{0 \leq \mu + c \\ -1 \leq \mu \leq +1}} (\mu + c)\varphi(x,\mu)d\mu \qquad (II.4.11)$$

For $\varphi(\mu) = \mu$ and $c \geq 0$, one obtains

$$u_\infty = \frac{2}{3} - \frac{c}{3} \qquad (II.4.12)$$

For $c = 0$, this method gives for the extrapolation length the value 2/3.

For $c = 1$, we remark that the half flux defined by (II.4.10) is equal to the total flux (II.4.9) and thus is rigorously conserved. The asymptotic limit given by (II.4.12) is therefore exact ($u_\infty = 1/3$).

2) The spectral method

We compared the prediction of the asymptotic values u_∞ given by the Marshak approximation (formula (II.4.12)) to those given by the spectral method for $N = 4$, $N = 10$, $N = 20$ (figure 2), for different values of c.

We notice that the result for $N = 10$ and $N = 20$ are very close to each other. We see on the curve of the spectral method some bumps. They correspond to the parameters c being eigenvalues of A_{0n}. At these critical values, the dimension of E^+_{cn} changes. In section (II.3.c), we have obtained that if we denote by

$$-1 < -\lambda_{n/2} < -\lambda_{n/2\,-1} < \dots < -\lambda_1 < 0 < \lambda_1 < \dots < \lambda_{n/2\,-1} < \lambda_{n/2}$$

the eigenvalue of A_{0n}, then
if $0 \leq c < \lambda_1$ then $\dim(E^+_{cn}) = n/2$,
if $\lambda_i \leq c < \lambda_{i+1}$ then $\dim(E^+_{cn}) = (n/2) + i$,
if $\lambda_{n/2} \leq c < 1$ then $\dim(E^+_{cn}) = n$.
For $N = 4$, we have $\lambda_1 \simeq 0.34$, $\lambda_2 \simeq 0.86$
For $N = 10$, $\lambda_1 \simeq 0.149$, $\lambda_2 \simeq 0.433$, $\lambda_3 \simeq 0.680$, $\lambda_4 \simeq 0.865$, $\lambda_5 \simeq 0.974$

Remark. For $\lambda_{n/2} < c \leq 1$, $\dim(E^+_{cn}) = n$, the generalized eigenvectors Φ_i form a basis of E_n and the decomposition (II.2.4) is exact.

$$\varphi(0,\mu) = \mu = \sum_{\lambda_i > 0} \alpha_i \Phi_i \quad + \alpha_0 P_0 \qquad (II.4.13)$$

Thus

$$(A_{cn}P_0, \mu) = (P_0, A_{cn}\mu) = \sum_{\lambda_i > 0} (P_0, \frac{\alpha_i}{\lambda_i} B_n \Phi_i) \quad + \alpha_0 (P_0, A_{cn}P_0) = \alpha_0 (P_0, A_{cn}\dot{P}_0) = \alpha_0 c$$

and we get

$$\lim_{x \to +\infty} u(x,\mu) = \alpha_0 P_0 = \frac{1}{3c} \qquad (II.4.14)$$

It is not very surprising to notice that Marshak's formula (II.4.12) and the spectral method for c close to 1 (relation (II.4.14)) give the same value for u_∞ and the same value for du_∞/dc at $c = 1$ (this result is apparent on figure 2).

Proposition II.4.1

The asymptotic limit given by the spectral method is continuous with respect to c.

Proof The continuity of the spectral result with respect to c is obvious at any point different from an eigenvalue of A_{0n}. Let us prove that it is also continuous at these critical points.

Let c_0 be an eigenvalue of A_{0n} and Φ_j such that $A_{c_0n}\Phi_j = 0$. We denote with subscript c_0^+ (respectively c_0^-) the limit of expressions dependent of c when $c_0 \geq c \to c_0$ (respectively $c_0 < c \to c_0$). We get the following natural relations

$$E^+_{c_0^+n} = Span\big((\Psi)_{i\neq j}, \Phi_j\big) \qquad (II.4.15)$$

$$E^+_{c_0^-n} = Span\big((\Psi)_{i\neq j}\big) \qquad (II.4.16)$$

where $A_{c_0n}\Psi_i = \mu_i\Psi_i, \quad \mu_i > 0$ and $\Psi_j = \Phi_j$.
Taking the limit $c \to c_0^+$, the projection formula (II.2.6) gives

$$\Pi^+_{c_0^+n}(\mu) = \sum_{i\neq j}\beta_i\Psi_i \quad + \beta_j\Phi_j = \sum_{i\neq j}\alpha_i^+\Pi^+_{c_0^+n}(\Phi_i) \quad + \alpha_j^+\Pi^+_{c_0^+n}(\Phi_j) \qquad (II.4.17)$$

where
$\lambda_i A_{c_0n}\Phi_i = B_n\Phi_i, \quad \lambda_i > 0.$
When $c \to c_0^-$ (II.2.6) gives

$$\Pi^+_{c_0^-n}(\mu) = \sum_{i\neq j}\beta_i\Psi_i = \sum_{i\neq j}\alpha_i^-\Pi^+_{c_0^-n}(\Phi_i) \qquad (II.4.18)$$

(we have used $(\Psi_i, \Phi_j) = 0$ for $i \neq j$ to obtain the same β_i in expression (II.4.17)-(II.4.18). From (II.4.15)-(II.4.16)

$$\Pi^+_{c_0^+n}(\Phi_j) = (\Phi_j), \quad \Pi^+_{c_0^-n}(\Phi_j) = 0$$

and we have

$$\Pi^+_{c_0^+n}(\Phi_i) = \Pi^+_{c_0^-n}(\Phi_i) + (\Phi_j, \Phi_i)\Phi_j$$

Thus $\alpha_i^+ = \alpha_i^-$ and $\alpha_j^+ = \beta_j - \sum\alpha_i^+(\Phi_j, \Phi_i)$. In particular, for index i corresponding to $\Phi_i = P_0$, the equality $\alpha_i^+ = \alpha_i^-$ proves that, using the spectral method, we obtain the same asymptotic value for problem (II.1.1)-(II.1.2), when $c \to c_0^+$ and $c \to c_0^-$. This concludes the proof of the above proposition.

3) Comparison with a direct computation

Description of the algorithm

Problem (II.1.1)-(II.1.2) is studied in the slab $[0, L]$ instead of the half space $[0, +\infty[$. At $x = L$, we prescribe one of the two different types of boundary conditions

- incoming flux

$$u(L, \mu) = f(\mu) \quad \text{given for} \quad -1 < \mu < -c \qquad (II.4.19)$$

- reflexion

$$u(L, \mu) = u(L, -2c - \mu), \quad -1 < \mu < -c \qquad (II.4.20)$$

Note that, for $c > 0$, the solution in the slab does not depend on f except in a boundary layer near $x = L$. The boundary condition (II.4.20) prevents the apparition of this boundary layer (see figure 3) and allows us to take a smaller value of L.

We introduce the velocity and space discretization

$$\mu_m = -1 + m\,\Delta\mu, \quad 0 \le m \le M, \quad \mu_M = 1$$

$$x_p = p\,\Delta x, \quad 0 \le p \le P, \quad x_P = L$$

We use the following iterative algorithm
Starting from $\tilde{u}_p^0 = 0$, for $0 \le p \le P$, and knowing $\tilde{u}_p^n$ solve
- for any integer m such that $\mu_m + c \ge 0$

$$(\mu_m + c)\frac{u_{p+1,m}^{n+1} - u_{p,m}^{n+1}}{\Delta x} + \frac{u_{p+1,m}^{n+1} + u_{p,m}^{n+1}}{2} - \frac{\tilde{u}_{p+1}^{n} + \tilde{u}_{p}^{n}}{2} = 0, \quad 0 \le p \le P-1 \qquad (II.4.21)$$

$$u_{0,m}^{n+1} = \varphi(\mu_m) \qquad (II.4.22)$$

- then from the knowledge of $u_{p=P,m}^{n+1}$ for any m such that $\mu_m + c \ge 0$ and the boundary condition (II.4.19) or (II.4.20), we get $u_{p=P,m}^{n+1}$ for integer m such that $\mu_m + c \le 0$. We then solve equation (II.4.21) for these values of m with this boundary condition instead of (II.4.22).
- from $u_{p,m}^{n+1}$, we compute the value of $\tilde{u}_p^{n+1}$ by

$$\tilde{u}_p^{n+1} = \frac{1}{\Delta\mu}\left(u_{p,0}^{n+1} + 2\sum_{m=1}^{m=M-1} u_{p,m}^{n+1} + u_{p,M}^{n+1}\right)$$

For $c = 0.5$, $L = 8$, $M = 50$, $P = 100$, using 80 iterations, we have represented on figure 3 the value of $\tilde{u}(x)$ given by this method using the two different conditions (II.4.19) and (II.4.20), ($\varphi(\mu) = \mu$ and $f(\mu) = 1$).

We have thus obtain the asymptotic value of the solution $u_\infty = 0.51369$. For $N = 4, 10, 40$ the spectral method gives respectively $u_\infty = 0.51167, 0.51337, 0.51370$ and the Marshak's approximation $u_\infty = 0.5$.

On figure 4, the first curve represents the solution $u(0,\mu)$ computed by this direct method (with the same numerical parameters) and the other curves, the results given by the spectral method for $N = 20$ (the second curve is $\Pi_{cn}^{+}(\mu)$, the third curve is the approximate solution given by the spectral method at $x = 0$). For the Albedo problem, we are interested in the distribution of particles with velocity μ less than $-c$. Instead of taking the restriction of the approximate solution for those values of μ, we use the orthogonal projection Π_{cn}^{-} on the eigenvectors of A_{cn} associated with negative eigenvalues (cf the restriction for $\mu > -c$ and the introduction of Π_{cn}^{+}). The fourth curve is the projection Π_{cn}^{-} of this approximation and gives a better solution of the Albedo problem (we have less oscillations).

Comparison of the computational cost

In the spectral method, we solve two eigenvalues problem for nxn symmetric and tridiagional matrix and we use a least square algorithm to compute the solution of the system (II.2.6).

In the direct method, the computational cost is proportional to the product of M by P and by the number of iterations (which gives 40 000 in our case). In this direct method, more iterations are needed for small values of c.

The spectral method is in any case less time consuming and gives accurate results.

Notice also that if we want to solve problem (II.2.1)-(II.2.2) for different functions φ, the computation of the two eigenvalue problems is only needed to be performed once. Moreover, if we compute the inverse of the decomposition operator, we get the matrix of the linear operator which gives the coefficients of the decomposition (II.2.6) (and therefore the solution of the initial problem (II.2.1)-(II.2.2)) in terms of φ. Then, using this matrix, we get the solution of this problem for any φ in a very efficient way.

Remark. In the spectral method, one can use a large value of N and this gives a good result in the Legendre space. However, when, for example, we want to compute the outgoing flux for different values of μ, we have to face a problem of precision because for $|\mu|$ close to 1, it is difficult to evaluate a Legendre polynomial of order $N \ge 40$ at point μ. Therefore, to use the spectral method with $N \ge 40$ for the Albedo problem, we have introduce a filter. When we have the solution in the Legendre space, we then take only into account the 20 first polynomials to plot the solution in terms of μ. However, the quality of the solution is close to the results obtained with the spectral method with $N = 20$; to get better results, it seems that we have to improve our filter.

III THE LINEARISED B.G.K. EQUATION

The B.G.K. equation is simpler than the Boltzmann equation and for this reason was intensively studied as a model for rarefied gas dynamics. In this model, the collision operator is replaced by a relaxation towards the local Maxwellian state (see [5]).

We now derive the spectral method for the linearised B.G.K. equation.

1. Introduction

We linearise the B.G.K. relaxation term around a given Maxwellian

$$M_0(\xi) = \frac{1}{(2\pi T_0)^{3/2}} exp\left(-\frac{|\xi|^2}{2T_0}\right) \qquad (III.1.1)$$

(the presence of a mean velocity in $M_0(\xi)$ is already taken into account by the drift v in (I.1.1)).
The linearised B.G.K. operator is written as

$$Lf = f - \pi(f) \qquad (III.1.2)$$

where $\pi(f)$ is the orthogonal projection in $L^2(d\xi)$ of f on $Span(\psi_\alpha)$, $\alpha = 0,1,2,3,4$

$$\pi(f) = \sum_{\alpha=0}^{\alpha=4} < \psi_\alpha, f > \psi_\alpha \qquad (III.1.3)$$

$$< f, g >= \int f(\xi)g(\xi)d\xi$$

$$\psi_0(\xi) = M_0^{1/2}; \quad \psi_i(\xi) = \frac{\xi_i}{\sqrt{T_0}} M_0^{1/2}, \ i = 1,2,3; \quad \psi_4(\xi) = \frac{|\xi|^2 - 3T_0}{\sqrt{6}T_0} M_0^{1/2}$$

In order to use Hermite polynomials, we take $T_0 = 1/2$ (this covers in fact the general case by a change of velocities ξ and v in (I.1.1)).

We have the following result

Proposition III.1
- If $0 \le v < \sqrt{5T_0/3} = \sqrt{5/6}$, for any $\varphi \in L^2((1+|\xi|)d\xi)$, there exists a unique solution $\chi \in L^\infty(dx, L^2((1+|\xi|)d\xi)$ of (I.1.1)-(I.1.2) such that

$$\int \xi_1 \chi(x,\xi) d\xi = 0 \qquad (III.1.4)$$

- if $v > \sqrt{5/6}$ (respectively $-\sqrt{5/6} < v < 0$; $v < -\sqrt{5/6}$), we have existence and uniqueness of a solution of (I.1.1)-(I.1.2) without the additional condition (III.1.4) (respectively with 4; 5 additional conditions).

(see Greenberg and Van der Mee [17] for a proof, and also [10]).

We use the Hermite polynomials multiplied by $(\pi)^{-1/4} exp(-x^2/2)$

$$P_0(x) = \frac{1}{\pi^{1/4}} exp(-\frac{x^2}{2}), \quad P_1(x) = x\frac{\sqrt{2}}{\pi^{1/4}} exp(-\frac{x^2}{2}), \quad P_2(x) = \frac{2x^2-1}{\sqrt{2}\pi^{1/4}} exp(-\frac{x^2}{2}), \ldots$$

We thus obtain

$$\psi_0(\xi) = P_0(\xi_1)P_0(\xi_2)P_0(\xi_3), \quad \psi_1(\xi) = P_1(\xi_1)P_0(\xi_2)P_0(\xi_3), \text{ (idem for } \psi_2, \psi_3),$$

$$\psi_4(\xi) = \frac{1}{\sqrt{3}}(P_2(\xi_1)P_0(\xi_2)P_0(\xi_3) + P_0(\xi_1)P_2(\xi_2)P_0(\xi_3) + P_0(\xi_1)P_0(\xi_2)P_2(\xi_3)) \qquad (III.1.5)$$

As in the previous section, we assume that $\varphi(\xi)$ is decomposed on $P_i(\xi_1)P_j(\xi_2)P_k(\xi_3)$

$$\varphi(\xi) = \sum_{i,j,k} c_{i,j,k} P_i(\xi_1)P_j(\xi_2)P_k(\xi_3)$$

We thus have to study the case $\varphi(\xi) = P_i(\xi_1)P_j(\xi_2)P_k(\xi_3)$.

Remark III.1 If $j > 2$ or $k > 2$ then

$$\chi(x,\xi) = 0, \quad \text{for } \xi_1 + c < 0$$
$$\chi(x,\xi) = exp(-\frac{x}{\xi_1 + c})P_i(\xi_1)P_j(\xi_2)P_k(\xi_3), \quad \text{for } \xi_1 + c > 0$$

is a solution of (I.1.1)-(I.1.2) (we have $\pi(\chi) = 0$).

Moreover, if φ is odd (respectively even) with respect to ξ_2, there exists a solution χ of (I.1.1) odd (respectively even) with respect to ξ_2. We also notice that the variables ξ_2 and ξ_3 play similar roles.

We thus have to study the two cases

$$\varphi(\xi) = P_i(\xi_1)P_1(\xi_2)P_0(\xi_3)$$
$$\varphi(\xi) = P_i(\xi_1)P_0(\xi_2)P_0(\xi_3) \quad \text{or} \quad P_i(\xi_1)P_2(\xi_2)P_0(\xi_3)$$

2. The case $\varphi(\xi) = P_i(\xi_1)P_1(\xi_2)P_0(\xi_3)$

A) Reduction to a one dimensional velocity problem

We are looking for a solution χ of (I.1.1)-(I.1.2) of the form

$$\chi(x,\xi) = \chi'(x,\xi_1)P_1(\xi_2)P_0(\xi_3) \qquad (III.2.1)$$

From (I.1.1)-(I.1.2), χ' must satisfy

$$(\xi_1 + v)\partial_x\chi' + \chi' - \tilde{\chi}' = 0 \qquad (III.2.2)$$
$$\chi'(0,\xi_1) = P_i(\xi_1), \quad \xi_1 + v > 0 \qquad (III.2.3)$$

with

$$\tilde{\chi}'(x,\xi_1) = P_0(\xi_1)\left(\int \chi'(x,\xi_1')P_0(\xi_1')d\xi_1'\right) \qquad (III.2.4)$$

Problem (III.2.2)-(III.2.4) is quite similar to the simple transport equation studied in section II (the velocity variable belongs to a one dimensional space) except that the velocity space is not bounded.

Using Hermite polynomials instead of Legendre polynomials, we derive the same kind of spectral method.

B) Numerical results

1) The case $v = 0$

When $v = 0$ and $\varphi(\xi) = P_1(\xi_1)P_1(\xi_2)P_0(\xi_3)$, there exists a constant l such that

$$\lim_{x\to+\infty} \chi(x,\xi) = l\ P_0(\xi_1)P_1(\xi_2)P_0(\xi_3) \qquad (III.2.5)$$

This constant is related to the coefficient of the slip boundary condition for the tangential velocity at the Navier-Stokes level (see Coron [9]). This problem have been intensively studied for the B.G.K. model (but also for the linearised Boltzmann equation). Direct computations of the Couette flow have been performed, comparisons with experiments were made by Reynolds, Smolderen and Wendt [23] and variational methods have been proposed (Cercignani [5], Golse [16], Loyalka [21]).

The Marshak's approximation gives $l = 1.253$.

The variational method of Loyalka [21] gives $l \simeq 1.4245$.

The true value is $l \simeq 1.437$ (see Cercignani [4], Loyalka [21], Sone and Onishi [25]).
The spectral method gives

N	l_N	N	l_N
4	1.344	40	1.432
10	1.411	50	1.433
20	1.426	60	1.434
30	1.430	70	1.434

2) The case $v \neq 0$

The Marshak's approximation presented in section II.4.B.1 leads to

$$\int_{\xi_1+v>0} (\xi_1+v)P_1(\xi_1)P_0(\xi_1)d\xi_1 = l\int_{\xi_1+v>0} (\xi_1+v)P_0(\xi_1)P_0(\xi_1)d\xi_1$$

Thus

$$l = \frac{\sqrt{2}\int_{-v}^{+\infty} exp(-\xi_1^2)d\xi_1}{2v\int_{-v}^{+\infty} exp(-\xi_1^2)d\xi_1 + exp(-v^2)} \qquad (III.2.6)$$

In figure 5, we have represented the value of l predicted by the Marshak method, formula (III.2.6) (curve 1), and the results of the spectral method with $N = 4, 10, 20$ (curves 2,3,4) for $v \in [0,2]$.

As expected, the results are similar to those obtained for the simple transport equation.

3. The case $\varphi(\xi) = P_i(\xi_1)P_0(\xi_2)P_0(\xi_3)$ or $P_i(\xi_1)P_2(\xi_2)P_0(\xi_3)$

A) Derivation of the spectral method

We are looking for a solution χ of (I.1.1)-(I.1.2) when

$$\varphi(\xi) = P_i(\xi_1)P_0(\xi_2)P_0(\xi_3) \quad \text{or} \quad P_i(\xi_1)P_2(\xi_2)P_0(\xi_3)$$

of the following form

$$\chi(x,\xi) = \chi_1(x,\xi)P_0(\xi_2)P_0(\xi_3) + \chi_2(x,\xi)P_2(\xi_2)P_0(\xi_3) + \chi_3(x,\xi)P_0(\xi_2)P_2(\xi_3) \qquad (III.3.1)$$

where χ_1, χ_2, χ_3 satisfy linear half space coupled equations. To solve this problem with the spectral method, we consider the first n Hermite polynomials $P_0, P_1, ...P_{n-1}$ and we define by E_{3n} the space of dimension $3n$ spanned by $P_i(\xi_1)P_0(\xi_2)P_0(\xi_3), P_i(\xi_1)P_2(\xi_2)P_0(\xi_3), P_i(\xi_1)P_0(\xi_2)P_2(\xi_3)$ for $0 \leq i \leq n-1$.

As in section II, A_{v3n} is the truncation on E_{3n} of the operator of multiplication by $\xi_1 + v$ and B_{3n} the operator on E_{3n} corresponding to $Lf = f - \pi(f)$.

The matrix of A_{v3n} is formed by three identical $n \times n$ blocks (the multiplication by $\xi_1 + v$ acts only on the dependence with respect to ξ_1).

For the spectral method, we consider the generalized eigenvalue problem

$$\lambda_i A_{v3n}\Phi_i = B_{3n}\Phi_i \qquad (III.3.2)$$

The space of solutions of (III.3.2) with $\lambda_i = 0$ is spanned by ψ_0, ψ_1, ψ_4.

As in section II, let E^+_{v3n} be the space spanned by the eigenvectors of A_{v3n} associated with positive eigenvalues and Π^+_{v3n}, the orthogonal projection on E^+_{v3n}. The decomposition of φ

$$\Pi^+_{v3n}(\varphi) = \Pi^+_{v3n}\Big(l_0\psi_0 + l_1\psi_1 + l_4\psi_4 + \sum_{\lambda_i>0} c_i\Phi_i\Big) \qquad (III.3.3)$$

gives in particular the asymptotic limit of the solution χ

$$\chi_\infty = l_0\psi_0 + l_1\psi_1 + l_4\psi_4 \qquad (III.3.4)$$

For any value of n, we obtain that , for $\sqrt{5/6} < v$ (respectively $0 < v < \sqrt{5/6}$), the number of positive or null generalised eigenvalues λ_i of the equation (III.3.2) is equal to the dimension of E^+_{v3n} (respectively $\dim E^+_{v3n}+1$).

For $0 \le v < \sqrt{5/6}$, this corresponds to the fact that we can prescribe the additional condition (III.1.4) for the problem (I.1.1)-(I.1.2). For the spectral method, this additional condition corresponds to the constraint $l_1 = 0$ in (III.3.3).

B) Numerical results

Notice that the Marshak's approximation leads to the following system for the unknown components l_0, l_1, l_4 of χ_∞ on ψ_0, ψ_1, ψ_4 (see (III.3.4))

$$\int_{\xi_1+v>0} (\xi_1 + v)\varphi\psi_\alpha d\xi = \sum_{i=0,1,4} l_i \int_{\xi_1+v>0} (\xi_1 + v)\psi_i\psi_\alpha d\xi \quad \text{for} \quad \alpha = 0,1,4 \qquad (III.3.5)$$

Thus (III.3.5) is a system of 3 equations. For $v > \sqrt{5/6}$, there are 3 unknown l_0, l_1, l_4 but for $0 < v < \sqrt{5/6}$, we are looking for a solution χ_∞ with $l_1 = 0$; there are only two unknowns.

We have studied the case

$$\varphi(\xi) = (\sqrt{3}/2)P_3(\xi_1)P_0(\xi_2)P_0(\xi_3) + (P_1(\xi_1)P_2(\xi_2)P_0(\xi_3) + P_1(\xi_1)P_0(\xi_2)P_2(\xi_3))/2$$

For $v = 0$ using the Marshak's method with equations (III.3.5) with $\alpha = 0,4$ only, we obtain $l_0 = -5\sqrt{\pi}/16 \simeq -0.5539$, $l_4 = 5\sqrt{6\pi}/16 \simeq 1.35675$ whereas a least square method to solve (II.3.5) gives $l_0 \simeq -0.547$, $l_4 \simeq 1.355$.

Note that l_4 is related to the temperature jump at the wall due to the slip boundary conditions for the Navier-Stokes system (see [9]).

A variational method (Loyalka [21]) gives $l_4 \simeq 1.5767$ whereas a direct computation performed by Larini and Brun [20] gives $l_4 \simeq 1.6$ and Sone, Onishi [25] find $l_4 \simeq 1.596$.

The spectral method gives

N	l_0	l_4
4	-0.37156	1.25127
10	-0.7026	1.5638
16	-0.7230	1.5792

In the subsonic case, $0 \le v < \sqrt{5/6}$, figures 6 and 7 are devoted to the variation, with respect to v, of the components l_0, l_4 of the asymptotic state of the solution χ with the mass flux l_1 equal zero. The first curve represents the Marshak's approximation using equations (II.3.5) with $\alpha = 0,4$. The three other curves represent the results of the spectral method for $N = 4, 10, 16$.

In the supersonic case ($\sqrt{5/6} < v$), the components l_0, l_1, l_4 are represented on figures 8, 9, 10. The first curve represents the Marshak's approximation, the three other ones, the results of the spectral method ($N = 4, 10, 16$).

For $n = 4$, the spectral method is not very accurate whereas for N greater or equal to 10 the results obtained seems to be correct.

For $v >> 1$, the Marshak method gives the asymptotics

$$l_0 \simeq -\frac{5}{12}v^{-3}, \quad l_1 \simeq -\frac{5}{6\sqrt{2}}v^{-2}, \quad l_4 \simeq \frac{5}{2\sqrt{6}}v^{-1}$$

These results are in agreement with the spectral method for $n = 4, 10, 16$.

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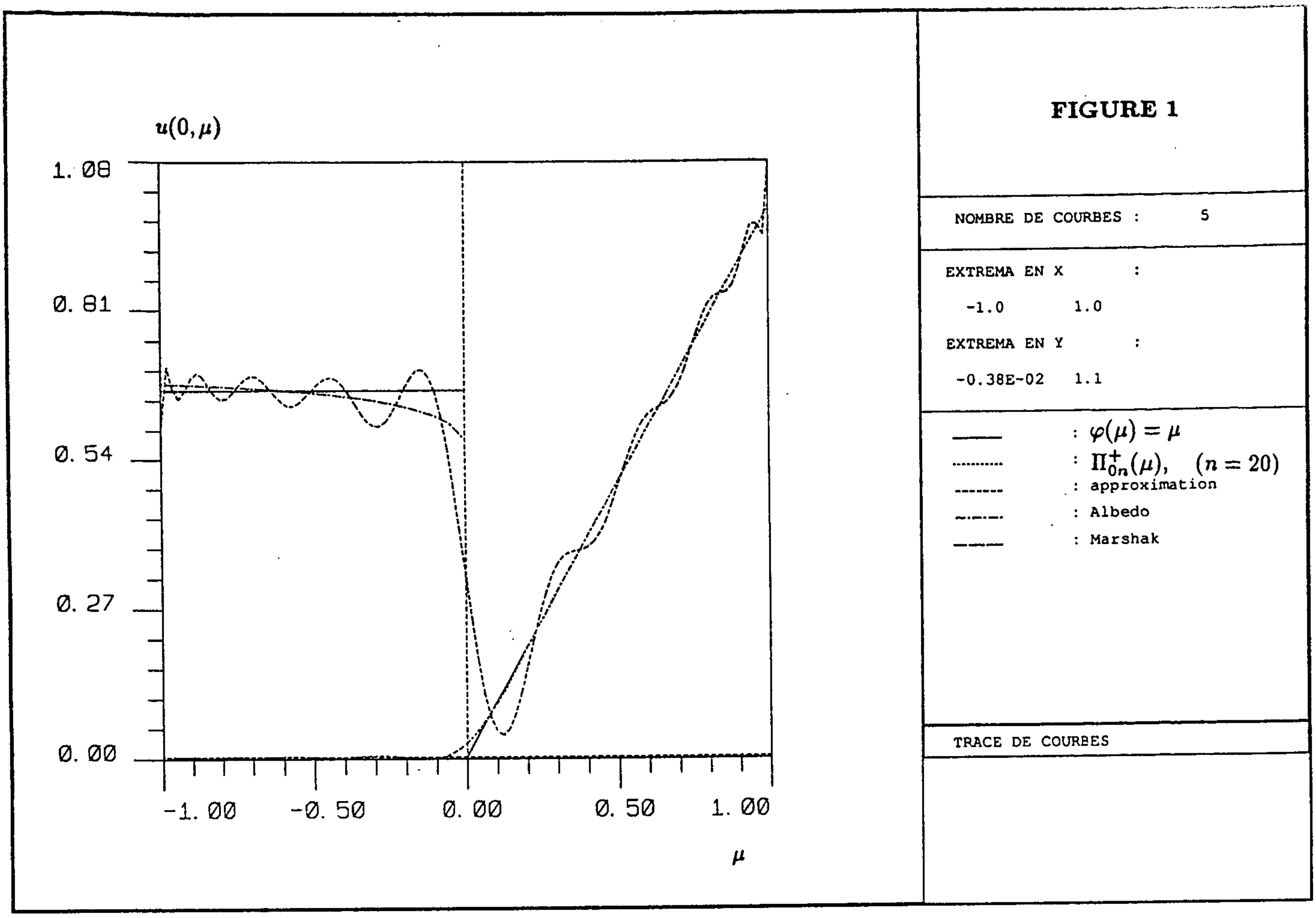

FIGURE 1
u(0,μ)
1.08
0.81
0.54
0.27
0.00
-1.00
-0.50
0.00
0.50
1.00
μ
NOMBRE DE COURBES : 5
EXTREMA EN X :
-1.0 1.0
EXTREMA EN Y :
-0.38E-02 1.1
: φ(μ) = μ
: Π⁺₀ₙ(μ), (n = 20)
: approximation
: Albedo
: Marshak
TRACE DE COURBES

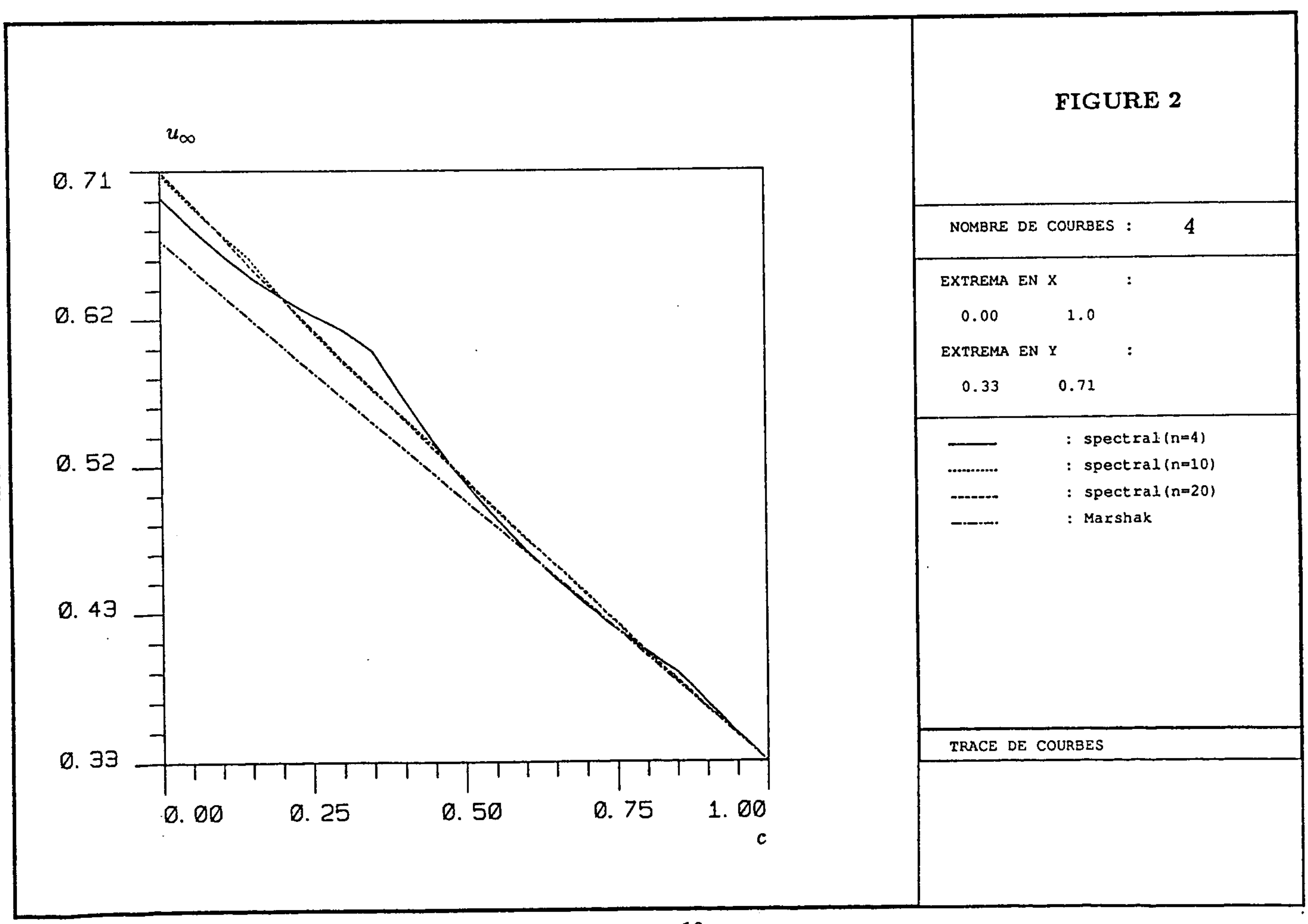

FIGURE 2
u∞
0.71
0.62
0.52
0.43
0.33
0.00
0.25
0.50
0.75
1.00
c
NOMBRE DE COURBES : 4
EXTREMA EN X :
0.00 1.0
EXTREMA EN Y :
0.33 0.71
: spectral(n=4)
: spectral(n=10)
: spectral(n=20)
: Marshak
TRACE DE COURBES

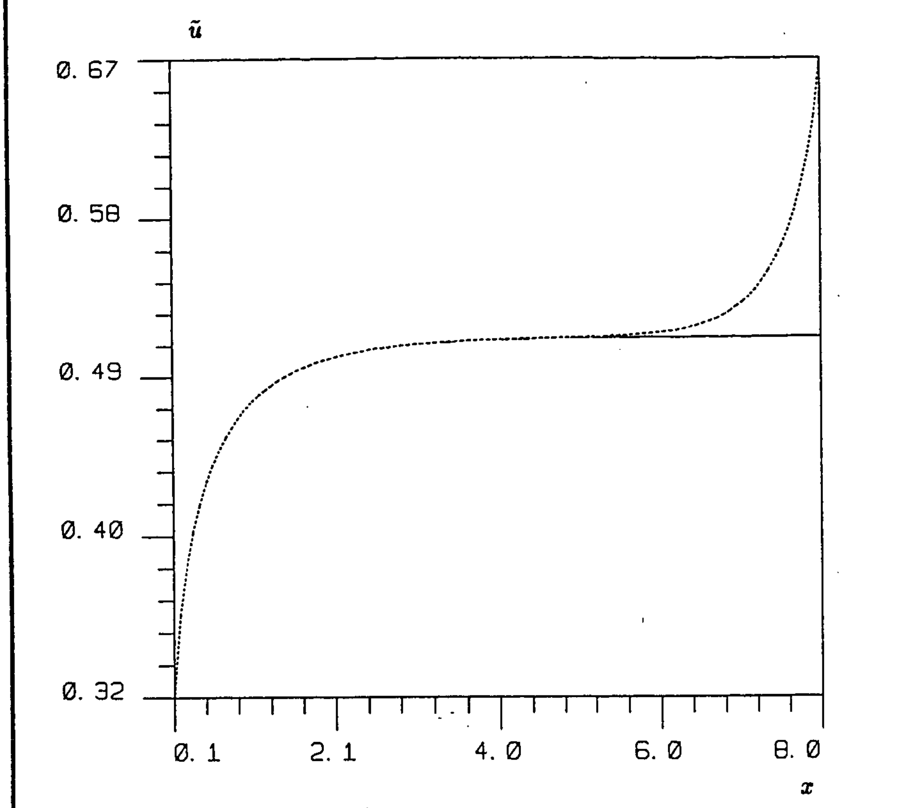

FIGURE 3

NOMBRE DE COURBES : 2

EXTREMA EN X :

0.80E-01 8.0

EXTREMA EN Y :

0.32 0.67

——— : reflexion (cf. II.4.20)

.... : incoming-flux (cf. II.4.19)

TRACE DE COURBES

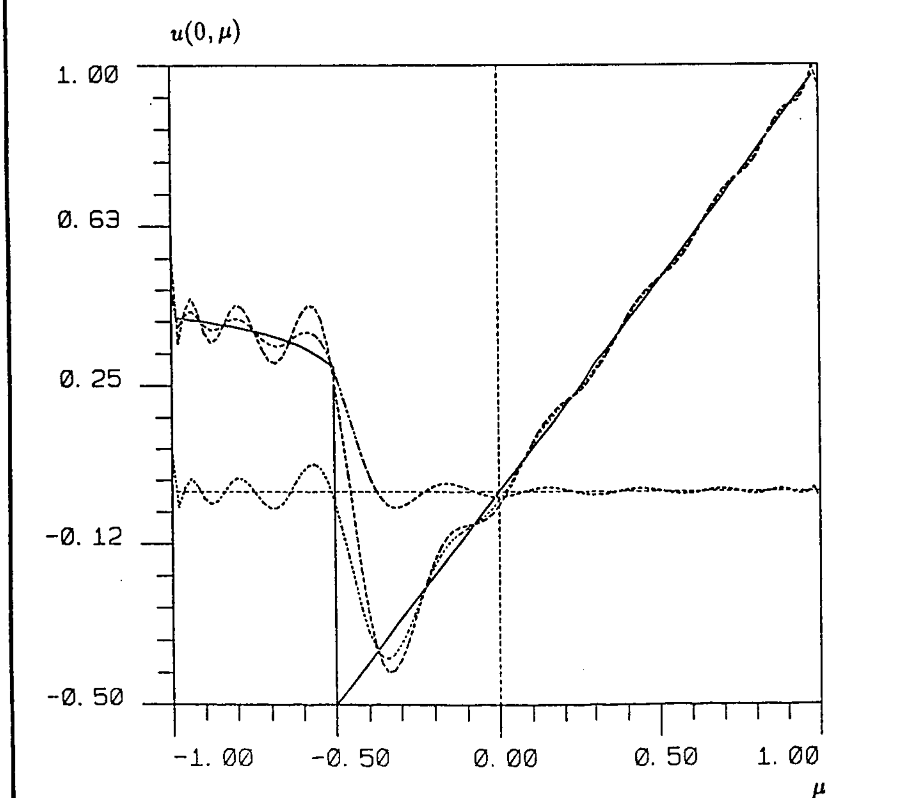

FIGURE 4

NOMBRE DE COURBES : 4

EXTREMA EN X :

-1.0 1.0

EXTREMA EN Y :

-0.50 1.0

——— : direct

.... : $\Pi^+_{cn}(\mu)$

------- : spectral method

-·-·-·- : $\Pi^-_{cn}(Solution)$

($c = 0.5,\ n = 20$)

TRACE DE COURBES

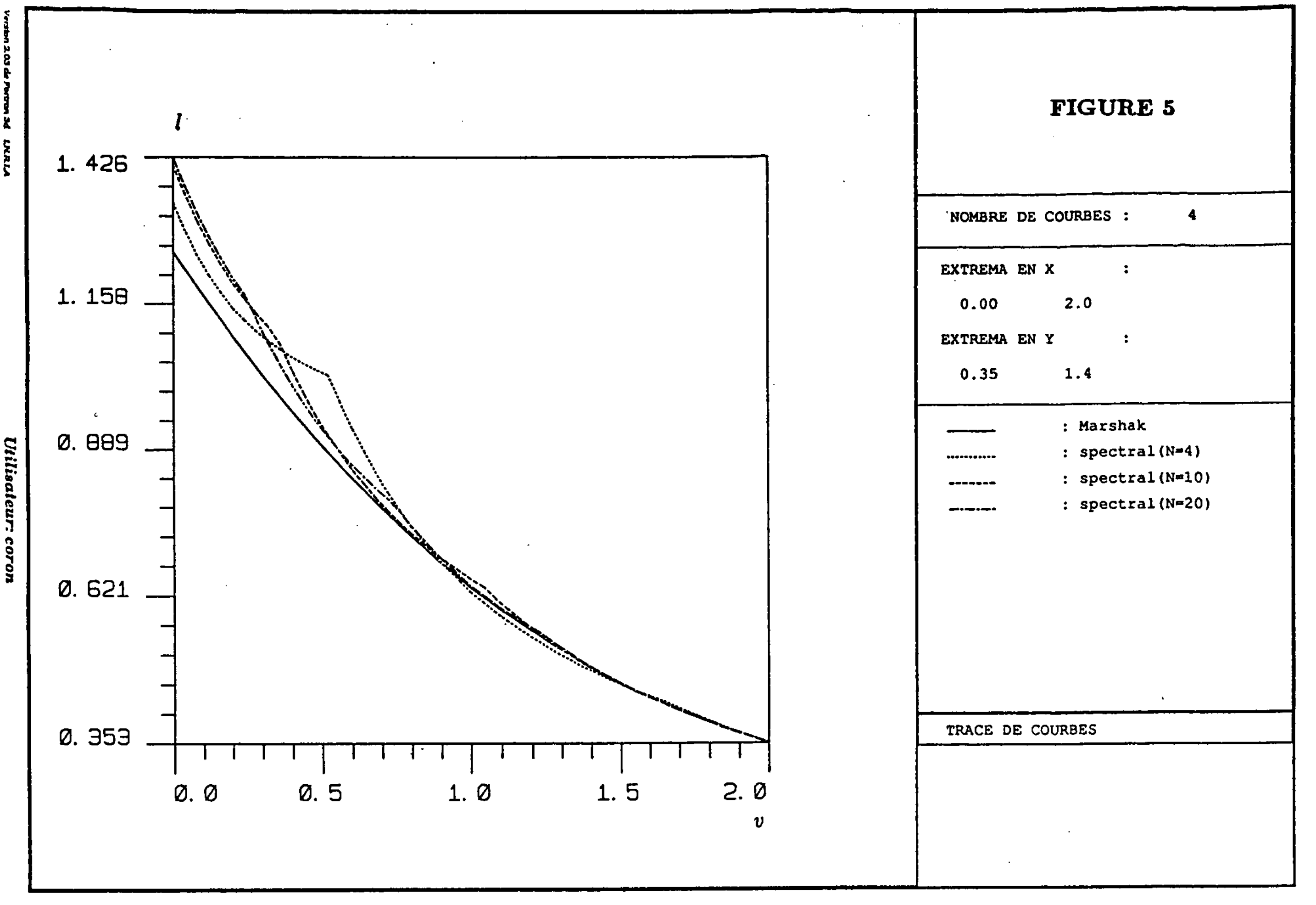

FIGURE 5

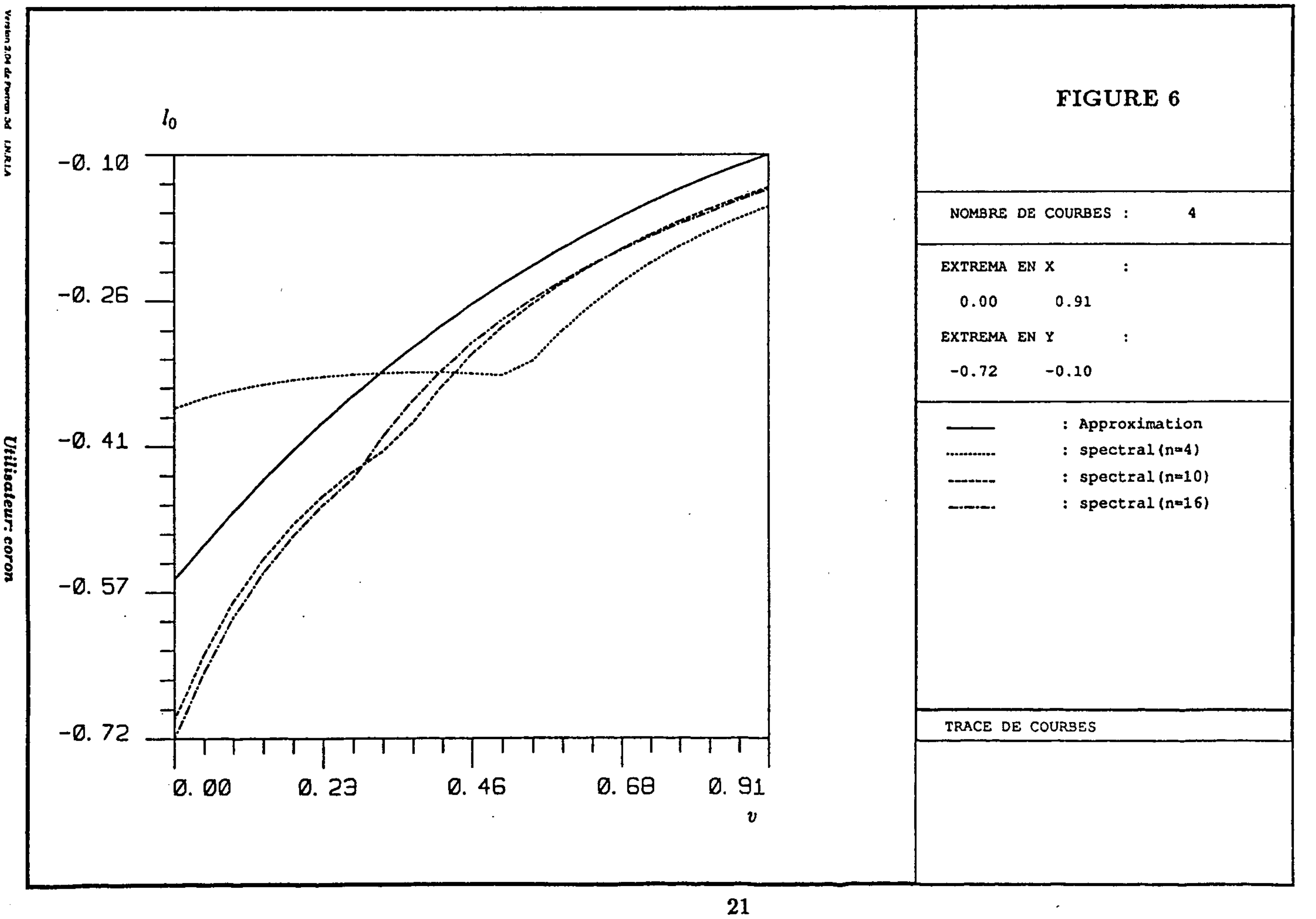

FIGURE 6

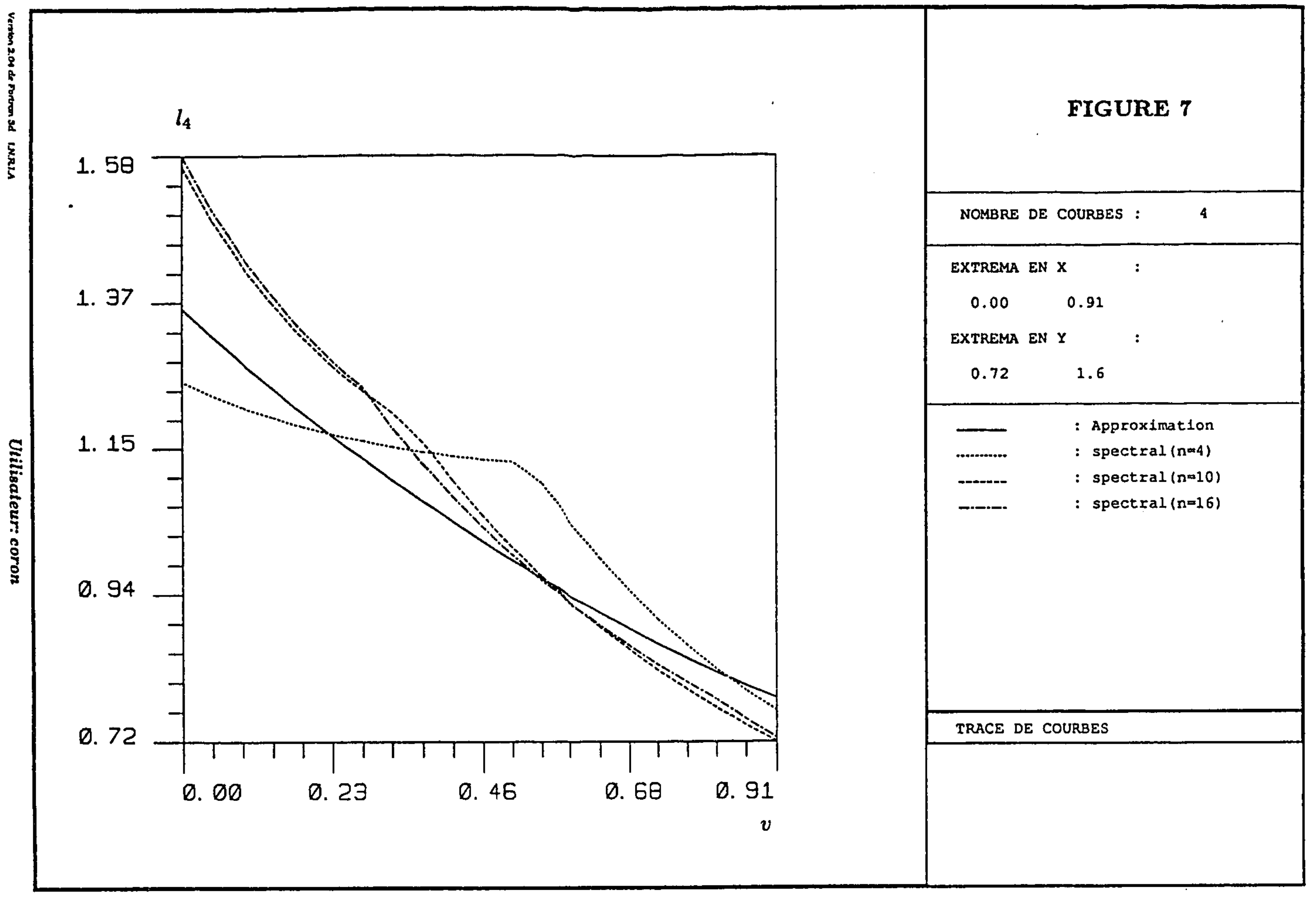

FIGURE 7

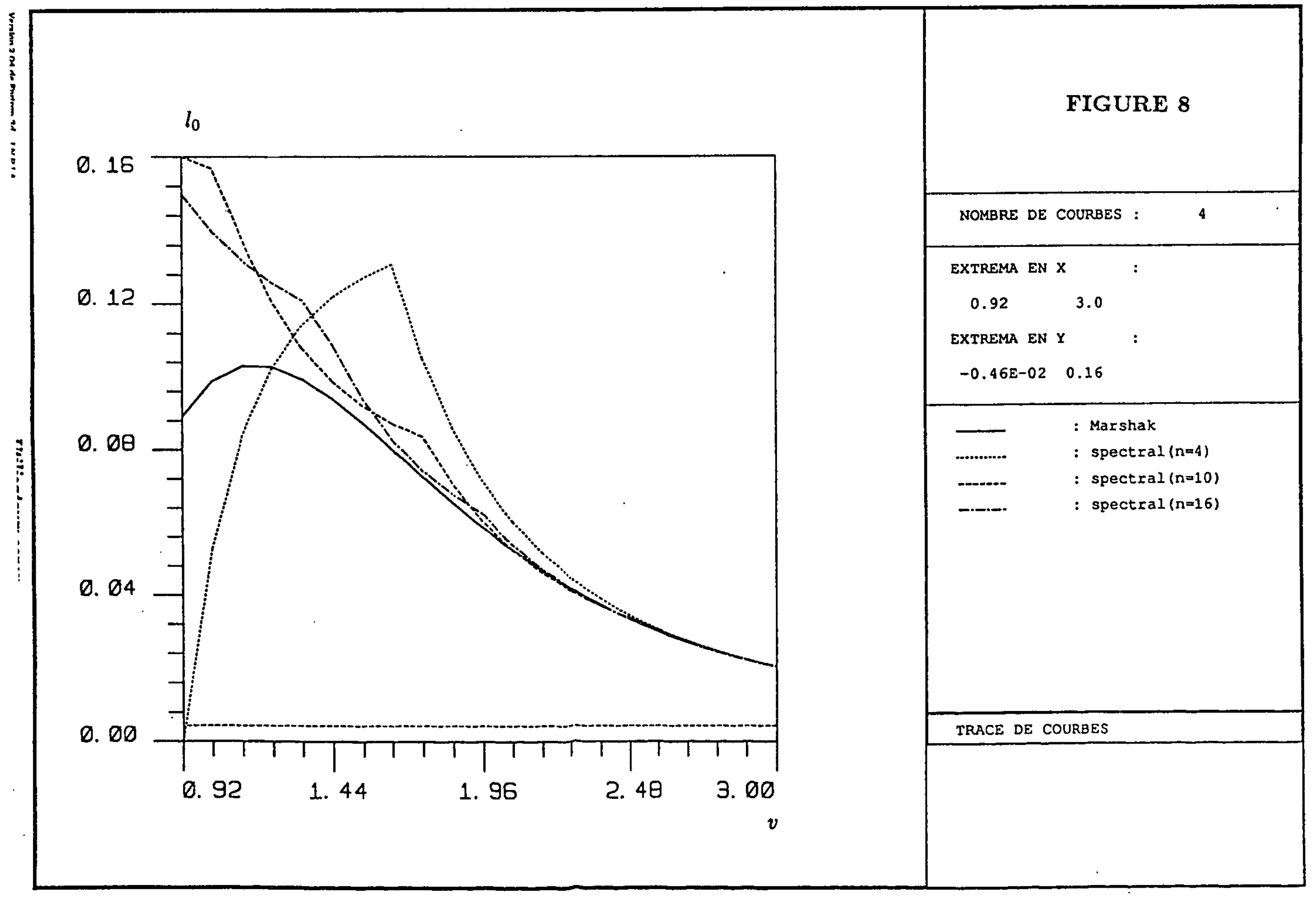

FIGURE 8

Utilisateur: coron

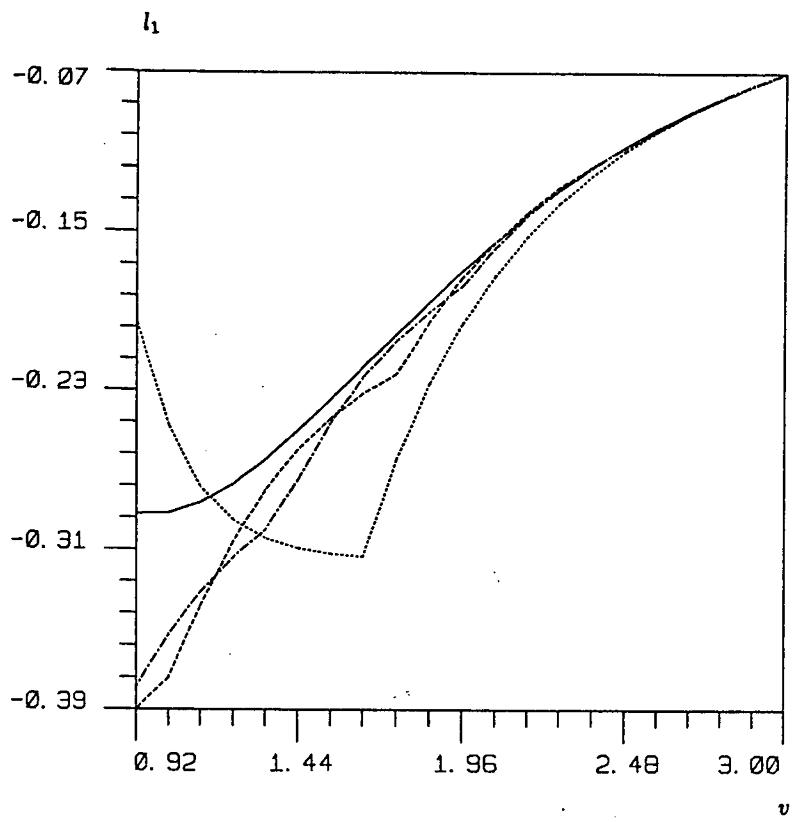

FIGURE 9

NOMBRE DE COURBES : 4

EXTREMA EN X :

0.92 3.0

EXTREMA EN Y :

-0.39 -0.72E-01

- ——— : Marshak
- ········ : spectral(n=4)
- – – – : spectral(n=10)
- –·–·– : spectral(n=16)

TRACE DE COURBES

Utilisateur: coron

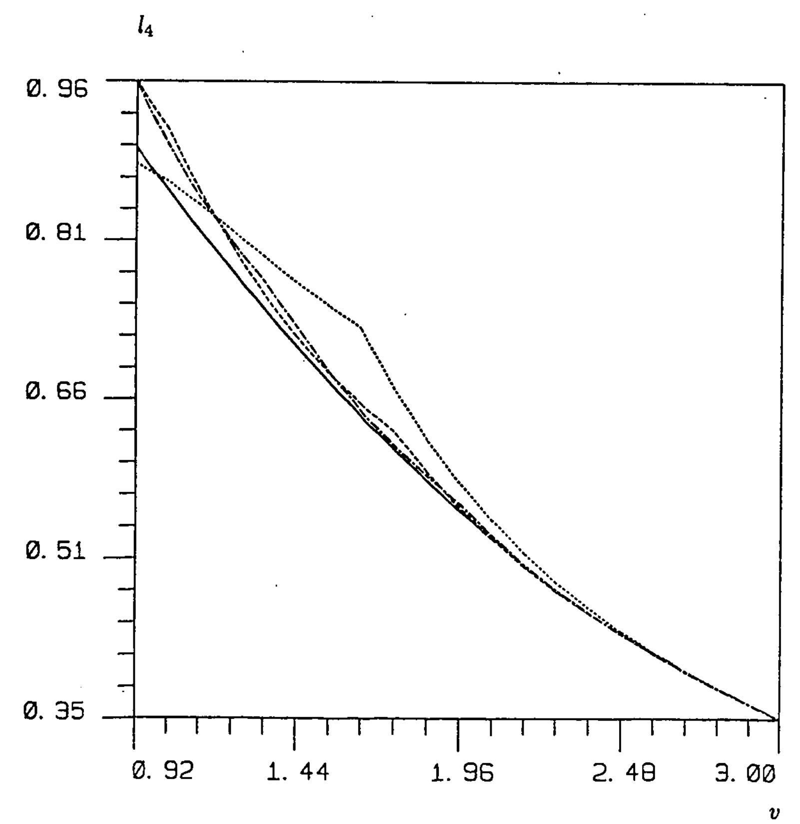

FIGURE 10

NOMBRE DE COURBES : 4

EXTREMA EN X :

0.92 3.0

EXTREMA EN Y :

0.35 0.96

- ——— : Marshak
- ········ : spectral(n=4)
- – – – : spectral(n=10)
- –·–·– : spectral(n=16)

TRACE DE COURBES