Variational approach of serial multi-level production/inventory systems

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Abstract : We discuss stochastic continuous time models of serial multi-level production/inventory systems using quasi-variational inequalities. Optimal centralized and decentralized policies are characterized and first numerical examples, for the deterministic case, are presented.

APPROCHE VARIATIONNELLE D'UNE CHAINE DE PRODUCTION/STOCKAGE

Résumé : On utilise les techniques des inéquations quasi-variationnelles pour étudier des modèles stochastiques à temps continu d'une chaine de production/stockage. On caractérise des politiques optimales centralisées et décentralisées et on présente des premiers exemples numériques du cas déterministe.

VARIATIONAL APPROACH OF SERIAL MULTI-LEVEL PRODUCTION/INVENTORY SYSTEMS

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1 - INTRODUCTION

In broad terms, multi-level production/inventory systems are concerned with production and/or inventory problems involving two or more interrelated activities.

The most common example of a multi-level inventory system is given by a distribution network for a family of products : it involves at the lowest level a number of retail outlets (i.e. stores) in business to satisfy customers demands for goods and which, in turn, act as customers of higher-level wholesale activties (i.e. warehouses). The wholesale activities themselves may be customers of still higher-level wholesale activities (i.e. factories). We can suppose several interconnection schemes for the activities. The simplest one is that in which each node (activity) has at most one predecessor (Fig.1) ; it gives an arborescent system

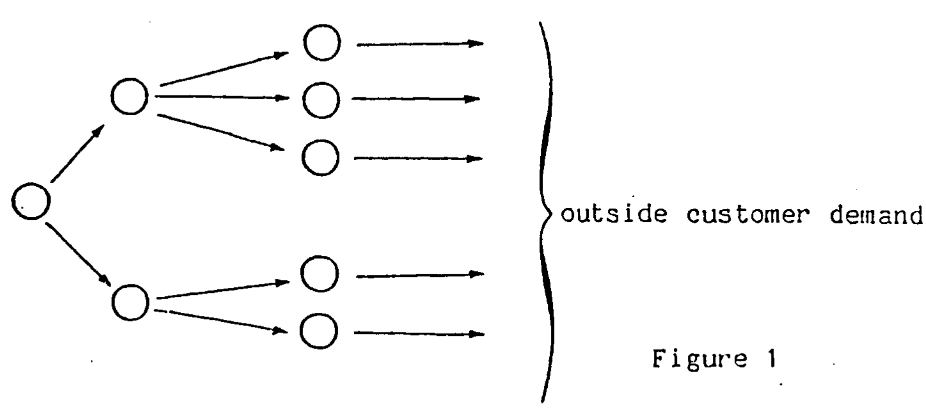


Figure 1

On the other hand, within a manufacturing context, a final product is some times the result of a process which can be decomposed into several levels, broadly corresponding to assembly activities. For simplicity, if we consider that each node has at most one immediate successor, we obtain **the assembly** system illustrated in Figure 2.

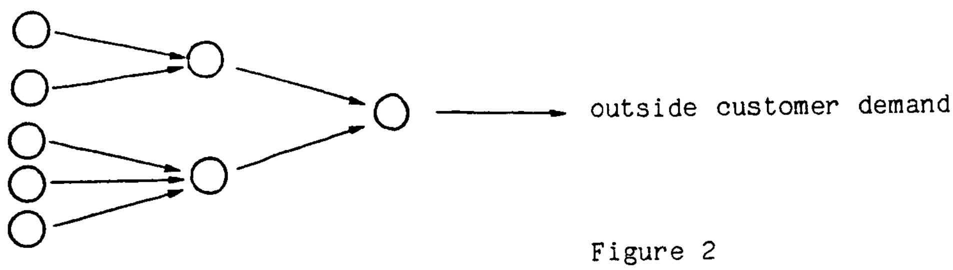


Figure 2

In this paper, we will discuss **serial** systems. In a serial system each node has at most one successor and at most one predecessor (Figure 3). Such a system is both an arborescent and an assembly system.

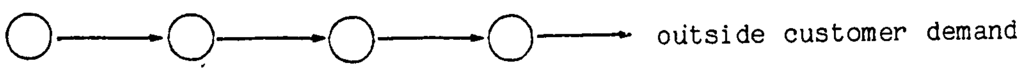


Figure 3

Several inventory or/and production systems can be modelled by a serial system (Clark and Scarf [14] and Zangwill [38]). But, the main reason to discuss such systems is to obtain preliminary results allowing us to attack the study of more complex systems.

Let us remark that an extremely large and rich bibliography exists concerning the theory and the applications of multi-level systems.

As in many cases we will need the same approach used to study single activity models, we will refer here to the surveys of such work given by Scarf [33], Veinnot [37], Aggarwal [1] and the recent book of Bensoussan, Crouhy and Proth [5].

For a general overwiew of contributions to the theory of multi-level production/inventory systems see Clark [13] (covering published results till 1971) and the book edited in 1981 by Schwarz [34] giving recent results and a good picture of what was done in the 1970's.

At this point, let us make two general remarks concerning all the research work attempted in the field of multi-level systems.

The first is that an extremely large quantity of papers concerns only discrete time models (also presented as **periodic review** models). One, but not the only, reason is the well known difficulty in properly discussing continuous time (**continuous review**) models. In fact, under similar assumptions the two models often require quite different methods for analysis. Consider, for example, the time-delay δ for satisfying an order. The case of "no delivery lag" in periodic review models is not equivalent to the case $\delta = 0$ in continuous review ones. As in these last models $\delta = 0$ implies instantaneous delivery, in general we study continuous time models with the more real hypothesis $\delta > 0$; but this makes the discussion difficult. On the other hand, in periodic review models the "no delivery lag" case makes sense and is the easiest one to study.

The second remark concerns the dichotomy between "centralized" and decentralized" solutions.

The first results on the one-item multi-level inventory model involving uncertain demand were introduced in three papers by Clark [12] and Clark-Scarf [14], [15]. In their method the optimality of the system is in some cases achieved by a sequential process of determining optimal policies at each outlet activity (each node). Unfortunately, this nice "decomposition property" of the optimal global policy occurs rarely. Nevertheless, several decentralized procedures can be found in the bibliography. They give **suboptimal** global policies even if they are locally optimal in some suitable sense.

On the other hand, considering centralized solutions introduces a new difficulty. Because of the complexity and dimensionality of some systems, a rigorous recursive computation of such solutions is, in general, completely impractical. In the surveys mentioned above ([13] ; [34]) some efficient algorithms and heuristics are given for particular problems.

The two preceding remarks are closely related with the main lines of our paper.

Let us consider for a single product a serial system with d installations (outlets) :

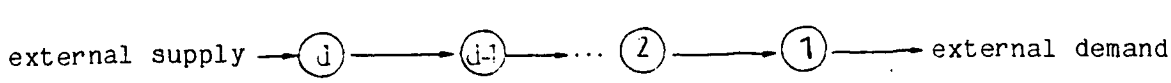


Figure 4

The demand acts in the system at the lowest level (inst. 1) and at no other point in the system. Arrows show the sense of the supply of stock. Obviously orders follow the opposite sense.

Once the parameters and characteristics of the model have been specified (demand distribution, initial stock at each installation, holding and shortage costs, purchase costs, delay for delivering, capacity of each installation, excess demand is backlogged or not, etc) we consider the problem of determining optimal (centralized or decentralized) purchasing policies.

Discrete time models in a dynamic programming appraoch have been introduced to solve this problem by Clark [12] and Clark-Scarf [14], [15]. They assume stochastic external demand, convex (holding and shortage) costs, unlimited capacity of installations. Shipping times and backlogging are allowed. After introducing the essential notion of "echelon" (at each installation an echelon consists of all stocks in the system at that installation and below, including all on-hand and in-transit amounts) they obtain an optimal decentralized policy if purchase costs are assumed linear. Sub-optimal decentralized policies are proposed if fixed reorder costs are allowed.

Centralized solutions are not studied in those papers. The existence of a centralized solution for the discrete time models introduced in [14], [15] is proved in Bensoussan-Crouhy [4].

In Zangwill [38] the theory of concave cost networks is applied to optimize a dynamic economic lot size production system under deterministic demand. A multi-echelon structure models the manufacture of a final product. No backlogging is allowed. Using a discrete dynamic programming approach an algorithm is proposed for calculating the centralized optimal production schedules.

Some other studies (always periodic review is assumed) have followed these major contributions of Clark-Scarf and Zangwill in serial systems. We will only mention an important paper by Bessler and Veinott [9], in which the period-independence notion of Veinott [36] is adapted to the multi-echelon situation. (In their model, at the beginning of a period, each installation may also order stock from an exogenous source).

As is obvious from the title we will consider continuous time models. In the frame of the dynamic programming theory our purpose is to use the quasi-variational technique introduced by A. Bensoussan and J.L. Lions [6] to propose continuous review solutions. Recent numerical and theoretical contributions to the study of quasi-variational inequalities (QVI) will be useful. They will be referred to later.

In §2, we present a non stationary system with stochastic continuous external demand, holding and shortage costs (hypotheses on convexity or concavity are not necessary) while purchase costs contain fixed reorder costs. Backlogging is allowed and fixed shipping times can be considered. The supplies (our controls) are of impulsive type. For this quite general formulation, the existence and properties of an optimal centralized purchase policy are given in Theorems 1 and 2.

In the simplified model of §3, continuous and impulsive supplies are considered. But we will assume linear purchase costs and no delay for delivering. Even if together these two assumptions give a much too unreal model, the decentralized policy obtained for it (Theorems 3 and 4) will be useful for understanding the asymptotic behaviour of more complex models (Theorem 5).

In §5 we take advantage of the results of §2 and §3 for discussing particular models. Numerical methods are proposed. In particular, for centralized policies, we emphazise the interest of using the general procedure introduced by Gonzalez and Rofman [18]. The end of §5 contains features of forthcoming papers.

Finally, in §6, we present some numerical examples of deterministic case. They show that no intuitive solutions can be easily obtained for three level models ; furthermore, it would be possible, thanks to the reduced central memory used, to discuss more complex models having up to 8 warehouses.

More complete numerical results, mainly including the stochastic case (using the methods contained in [20] and [30]) will be included in a forthcoming research report.

2. - CENTRALIZED SOLUTION

2.1. - Notations and assumptions

Suppose d levels of decision in a hierarchic single product system. We denote by $(X_s, s \geq 0)$ the inventory on hand at the time s, precisely $X_s = (X_s^1, \ldots, X_s^d)$ with the following meaning :

> X_s^i : sum from level 1 to i of the inventory on hand and in transit at the time s, i.e. inventory of echelon i at time s.

The demand arises (see fig. 4) at the lowest installation (i=1), each level (i) places order to level (i+1), i=1,2,...,d-1, and the exterior supplies the highest installation (i=d). These purchasing decisions are made at any time and they may modify instantaneously the state of the system. Precisely, we assume that between two consecutive orders, each coordinate $(X_s^i, s \geq 0)$ is a one dimensional jump diffusion process with the same coefficients, i.e. :

$$(2.1) \qquad \begin{cases} dX_s^i = - b(t+s)\, ds + dF_s^t + d\nu_s^i, \ s \geq 0, \\ X_o^i = x_i, \ i = 1,\ldots,d, \end{cases}$$

where t and $x = (x_1,\ldots,x_d)$ are the initial time and state and

$b(s)$: is the mean of the marginal demand,

F_s^t : is the fluctuation of the marginal demand,

ν_s^i : is the cumulative orders from level i.

Note that the process $\nu_s = (\nu_s^1, \ldots, \nu_s^d)$ is our control. We assume that the fluctuation process F_s^t is a stochastic integral in a Wiener-Poisson space $(\Omega, \mathcal{F}, \mathcal{F}^s, P, w_s, \mu_s)$, i.e. :

$$(2.2) \qquad F_s^t = \int_o^s \sigma(t+\lambda)\, dw_\lambda + \iint_{[o,s)\times \mathbb{R}_*} \gamma(t+\lambda,\zeta)\, d\mu_\lambda(\zeta), \quad t,s \geq 0,$$

where $(\Omega, \mathcal{F}, P)$ is a complete probability space, $(\mathcal{F}^s, s \geq 0)$ is an increasing and right continuous family of sub σ-algebras of $\mathcal{F}$, $(w_s, s \geq 0)$ is a standard Wiener process in $\mathbb{R}$ (w.r.t. $(\mathcal{F}^s, s \geq 0)$), $(\mu_s(t), s \geq 0, t \in \mathbb{R}_*)$, $\mathbb{R}_* = \mathbb{R} - \{0\}$, is a martingale random measure constructed from a Poisson random measure $(p(s,.), s \geq 0)$ independent of $(w_s, s \geq 0)$ and with Levy measure Π.

The control process ν_s is defined by means of two sequences, one of random times $(\theta_j, j = 1,2,\ldots)$ and another of random variables $(\xi_j^i, i = 1,\ldots,d, j = 1,2,\ldots)$, i.e. at the random time $t+\theta$ we order from the level i the random quantities ξ_j^i, and the cumulative order on the time interval $(t,t+s]$ is :

$$
(2.3) \qquad \begin{cases} \nu_s^i = \sum_{j=1}^{\infty} \xi_j^i \, \mathbb{1}(\theta_j \leq s), \; i = 1,\ldots,d \\ \xi_j^i \geq 0, \; 0 \leq \theta_j \leq \theta_{j+1}, \; \theta_j \to \infty \text{ as } j \to \infty, \end{cases}
$$

where $\mathbb{1}(\theta_j \leq s)$ denotes the characteristic function of the set $\{\omega : \theta_j(\omega) \leq s\}$. Since the process ν_s is adapted to $(\mathcal{F}^s, s \geq 0)$, the random times θ_j are stopping times and the random variables ξ_j^i are $\mathcal{F}^{\theta_j}$-measurables.

The model has a state-constraint, namely :

$$
(2.4) \qquad x_i + \nu_{s-}^i \geq x_{i-1} + \nu_s^{i-1}, \; \forall s \geq 0, \; \forall i = 2,\ldots,d,
$$

where ν_{s-}^i denotes the limit from the left of ν_s^i.

To any decision $\nu_s = (\nu_s^1,\ldots,\nu_s^d)$ it is associated an expected cost :

$$
(2.5) \qquad \begin{cases} J_{xt}(\nu) = E\{ \int_0^{T-t} f(X_s,t+s) \exp(-\int_0^s \alpha(t+\lambda)d\lambda) \, ds \\ \qquad + \sum_{j=1}^{\infty} k(\xi_j, t+\theta_j) \exp(-\int_0^{\theta_j} \alpha(t+\lambda)d\lambda) \, \mathbb{1}(\theta_j < T-t)\}, \end{cases}
$$

where x,t are the initial conditions, $\xi_j = (\xi_j^1,\dots,\xi_j^d)$ and T is the horizon. Note that hypotheses such as :

(2.6) $$\begin{cases} f(x,t) = f_1(x_1,t) + \dots + f_d(x_d,t), \ \forall x,t \\ k(\xi,t) = k_1(\xi_1,t) + \dots + k_d(\xi_d,t), \ \forall \xi,t \end{cases}$$

are not essential at this moment, because we are now interested in centralized solutions.

The optimal cost function is :

(2.7) $$\hat{u}(x,t) = \inf \{J_{xt}(\nu) : \nu \text{ satisfying (2.3) and (2.4)}\}.$$

Notice that for $s \in (\theta_j,\theta_{j+1})$, not orders are placed from any installations at the time s+t and for $s = \theta_j$ only the installations with $\xi_j^i \neq 0$ places some order for stock. The fact that the fluctuation of the demand is modelled by (2.2) allows us to include a Gaussian and/or Poissonian demand by choosing suitable coefficients $\sigma(s)$ and $\gamma(s,\zeta)$.

Let us precise our technical assumptions on the data. We assume that :

(2.8) either T is finite or $\alpha(t) \geq \alpha_o > 0$, $\forall t \geq 0$;

(2.9) $\begin{cases} b(.),\ \sigma(.),\ \alpha(.) \text{ are uniformly continuous and bounded} \\ \text{functions from } [0,T) \text{ into } \mathbb{R}, \end{cases}$

(2.10) $\begin{cases} \gamma(.,.) \text{ is a Borel measurable function from } [0,T) \times \mathbb{R}_* \\ \text{into } \mathbb{R} \text{ and the function } \gamma_p(t) = \int_{\mathbb{R}_*} |\gamma(t,\zeta)|^p\, \Pi(d\zeta) \text{ is uni-} \\ \text{formly continuous and bounded on } [0,T) \text{ for every } p \geq 1 ; \end{cases}$

(2.11) $\begin{cases} k(.,.) \text{ is a continuous function from } (\mathbb{R}^d-\{0\}) \times [0,T) \text{ into} \\ [k_o,\infty),\ k_o > 0, \text{ uniformly continuous in the first variable} \\ \text{and such that for every } \varepsilon > 0 \text{ there exists } \delta_\varepsilon > 0 \text{ satisfying :} \\ |k(\xi,t) - k(\xi,t')| \leq \varepsilon k(\xi,t) \quad \text{if } |t-t'| \leq \delta_\varepsilon, \end{cases}$

for some number $p \geq 0$ and $\bar{\mathcal{O}} = \{x \in \mathbb{R}^d : x_1 \leq x_2 \leq \dots \leq x_d\}$, (2.12) $\left\{\begin{array}{l} f(.,.) \text{ is a function from } \mathcal{O}\times[0,T) \text{ into } [0,\infty) \text{ satisfying} \\ \text{the following properties : there exists a constant C such} \\ \text{that :} \\ 0 \le f(x,t) \le C(1+|x|)^p,\ \forall x \in \bar{\mathcal{O}},\ t \in [0,T)\ ; \\ \text{for every } \varepsilon > 0 \text{ there exists } \delta_\varepsilon > 0 \text{ such that :} \\ |f(x,t)-f(x',t')| \le \varepsilon(1+|x|+|x'|)^p, \text{ if } |x-x'|\le\delta_\varepsilon,\ |t-t'|\le\delta_\varepsilon. \end{array}\right.$

Denote by $C_p = C_p(\bar{\mathcal{O}}\times[0,T))$ the space of all function satisfying (2.12), regardless of the condition $f \ge 0$.

2.2. - Characterization of the optimal cost function

A formal application of the dynamic programming principle permits us to derive the following conditions :

(2.13) $\left\{\begin{array}{l} Au \le f \text{ and } u \le Mu \text{ in } \bar{\mathcal{O}}\times[0,T), \\ Au = f \text{ if } u < Mu \end{array}\right.$

to be satisfied by the optimal cost function (2.7), where the operators A and M are defined by :

(2.14) $$\left\{\begin{array}{l} Au(x,t) = -\partial_t u(x,t) - \frac{1}{2}\sigma^2(t) \sum_{i,j=1}^{d} \partial^2_{ij} u(x,t) \\ \qquad + b(t) \sum_{i=1}^{d} \partial_i u(x,t) + \alpha(t)\, u(x,t) \\ \qquad + \int_{\mathbb{R}_*} [u(x+\bar{\gamma}(t,\zeta),t)-u(x,t)-\gamma(t,\zeta) \sum_{i=1}^{d} \partial_i u(x,t)]\, \Pi(d\zeta), \end{array}\right.$$

with $\bar{\gamma}(t,z) = (\gamma(t,z),\ldots,\gamma(t,z))$,

(2.15) $$\left\{\begin{array}{l} Mu(x,t) = \inf\{[\,k(\xi,t) + u(x+\xi,t)] : \xi_1,\ldots,\xi_d \ge 0, \\ \qquad \xi \ne 0,\ x_i \ge x_{i-1}+\xi_{i-1},\ i = 2,\ldots,d\}. \end{array}\right.$$

Note that ∂_t, ∂^2_{ij} and ∂_i denote the partial derivatives.

In general, the optimal cost function (2.7) is not a continuously differentiable function then we need to give a meaning to Au when u is not smooth. To this effect, we regard Au as a distribution in the Schwartz' sense, i.e. for any smooth test function φ with compact support in $\mathcal{O} \times (0,T)$:

(2.16)
$$\begin{cases} \langle Au,\varphi\rangle = \int_{\mathcal{O}\times(0,T)} u(x,t)\, A^*\varphi(x,t)\, dx\, dt, \\ A^*\varphi(x,t) = \partial_t \varphi(x,t) - \frac{1}{2}\sigma^2(t) \sum_{i,j=1}^{d} \partial_{ij}^2 \varphi(x,t) \\ \qquad - b(t) \sum_{i=1}^{d} \partial_i \varphi(x,t) + \alpha(t)\varphi(x,t) \\ \qquad + \int_{\mathbb{R}_*} [\varphi(x-\bar{\gamma}(t,\zeta),t) - \varphi(x,t) + \gamma(t,\zeta) \sum_{i=1}^{d} \partial_i \varphi(x,t)]\Pi(d\zeta). \end{cases}$$

Usually, if u,v are two locally integrable function on some open set $Q \subset \mathcal{O} \times (0,t)$ we say :

(2.17)
$$\begin{cases} Au \le v \ (=\text{instead of } \le) \text{ in } \mathcal{D}'(Q) \text{ iff} \\ \int_Q u(x,t)\, A^*\varphi(x,t)\, dx\, dt \le \int_Q v(x,t)\, \varphi(x,t)\, dx\, dt \\ (=\text{instead of } \le) \text{ for every smooth test function } \varphi \text{ with} \\ \text{support on } Q \text{ and } \varphi \ge 0 \ (\varphi \text{ with any sign}). \end{cases}$$

Consider the problem :

(2.18)
$$\begin{cases} \text{find } \hat{u} \text{ in } C_p \ (\text{cfr } (2.12)) \text{ such that :} \\ A\hat{u} \le f \quad \text{in } \mathcal{D}'(\mathcal{O}\times(0,T)), \\ \hat{u} \le M\hat{u} \quad \text{in } \bar{\mathcal{O}} \times [0,T), \\ A\hat{u} = f \quad \text{in } \mathcal{D}'([\hat{u} < M\hat{u}]), \\ \hat{u}(.,T) = 0 \text{ if } T \text{ is finite.} \end{cases}$$

Note that the operator $M\varphi$ is well defined for every φ in C_p, but it may take the value $-\infty$; however the condition $\hat{u} \le M\hat{u}$ implies that $M\hat{u}$ is finite. The expression $[\hat{u} < M\hat{u}]$ denotes the subset points (x,t) in $\mathcal{O} \times (0,T)$ satisfying $\hat{u}(x,t) < M\hat{u}(x,t)$.

THEOREM 1 : Assume the conditions (2.8), ..., (2.12) and

(2.19)
$$\begin{cases} \text{there exists a constant } f_o > 0 \text{ such that} \\ f(x,t) \geq f_o|x^+|^p - \frac{1}{f_o},\ \forall x \in \bar{\mathcal{O}},\ t \in [0,T),\ x^+ = (x_1^+,\dots,x_d^+) \end{cases}$$

hold. Then the optimal cost function $\hat{u}(x,t)$, defined by (2.7), is the unique solution of the problem (2.18). Morever, the function $\hat{u}$ satisfies (2.19) with $\hat{u}.u_o$ instead of $\hat{f}.f_o$ respectively. □

2.3. - Construction of the optimal impulse control policy

We construct an optimal impulse control $\hat{\nu}_s = (\hat{\nu}_s^1,\dots,\hat{\nu}_s^d)$ after introducing the following definitions.

Let $\hat{u}(x,t)$ be either the optimal cost function (2.7) or equivalently the unique solution of the problem (2.18). Since the function $\hat{u}$ is continuous and satisfies (2.19), there exists a Borel measurable function $\hat{\xi}(x,t)$ from $\mathcal{O}\times[0,T)$ into $\mathbb{R}^d-\{0\}$ such that :

(2.20)
$$\begin{cases} M\hat{u}(x,t) = k(\hat{\xi}(x,t),t) + \hat{u}(x+\hat{\xi}(x,t),t)\ , \\ \hat{\xi}_1(x,t),\ \dots,\hat{\xi}_d(x,t) \geq 0, \\ x_{i+1} \geq x_i + \hat{\xi}_i(x,t),\ i = 1,\dots,d-1,\ \forall x \in \bar{\mathcal{O}},\ t \in [0,T). \end{cases}$$

Denote by $q(s,t,\theta)$ the stochastic process :

(2.21)
$$\begin{cases} q(s,t,\theta) = -\int_\theta^s b(t+\lambda)\, d\lambda + \int_\theta^s \sigma(t+\lambda)dw_\lambda \\ \qquad + \iint_{[\theta,s)\times\mathbb{R}_*} \gamma(t+\lambda,\zeta)\, d\mu_\lambda(\zeta),\ s,t \geq 0, \end{cases}$$

and set $\bar{q}(s,t,\theta) = (q(s,t,\theta),\dots,q(s,t,\theta))$ stochastic process with values into $\mathbb{R}^d$.

We define $\hat{\nu}$ by induction for any given initial condition (x,t) in $\bar{\mathcal{O}}\times[0,T)$.

Starting with :

(2.22) $$\begin{cases} y_j(s) = x + \bar{q}(s,t,\theta),\ \forall s \in [0,T-t), \\ \theta_o = 0, \end{cases}$$

and then assuming $(y_j(s),\ 0 \le s \le T-t)$, θ_j to be known,we set :

(2.23) $$\begin{cases} \theta_{j+1} = \inf\{s \in [0,T-t) \ : \ \hat{u}(y_j(s),s) = M\hat{u}(y_j(s),s)\}, \\ \theta_{j+1} = +\infty \text{ if } \hat{u}(y_j(s),s) < M\hat{u}(y_j(s),s),\ \forall s \in [0,T-t), \end{cases}$$

(2.24) $$\begin{cases} y_{j+1}(s) = y_j(s),\ \forall s \in [0,\theta_{j+1}) \\ y_{j+1}(s) = y_j(\theta_{j+1}) + \hat{\xi}(y_j(\theta_{j+1}),\ \theta_{j+1}) + \bar{q}(s,t,\theta_{j+1}), \forall s \in [\theta_{j+1},T-t) \end{cases}$$

Finally :

(2.25) $$\begin{cases} \hat{\nu}_s = (\hat{\nu}_s^1,\dots,\hat{\nu}_s^d),\ s \in [0,T] \\ \hat{\nu}_s^i = \sum_{j=1}^{\infty} \hat{\xi}_j^i\ \mathbb{1}(\hat{\theta}_j \le s),\ i = 1,\dots,d, \\ \hat{\theta}_j = \theta_j,\ \hat{\xi}_j^i = \hat{\xi}_i\ (y_j(\theta_{j+1}),\ \theta_{j+1}). \end{cases}$$

Note that the stochastic process

(2.26) $$\hat{X}_s = \lim_{j\to\infty} y_j(s),\ \forall s \in [0,T-t),$$

is the state of the system under the control (2.25). The construction (2.20),...,(2.25) must be regarded as the impulse control associated with the continuation set $[\hat{u} < M\hat{u}]$.

Theorem 2 : Under the assumptions of Theorem 1, the impulse control policy $\hat{\nu}$ defined by (2.20),..., (2.25) is optimal, i.e. :

(2.27) $$\hat{u}(x,t) = J_{xt}(\hat{\nu}).$$ □

Remark : As a rule, Theorem 2 means that we let the process evolving freely inside the region $[\hat{u} < M\hat{u}]$ and at the precise moment when the process hits the set $[\hat{u} = M\hat{u}]$ we decide to order a quantity $\hat{\xi}$ given by the argument minimum of $M\hat{u}$. □

Results similar to Theorems 1 and 2 can be obtained for a model which includes a lag time, strictly positive, deterministic and unique for each installation. In this case, the operator M takes another form, different from (2.15); essentially we compose the expression (2.15) with :

$$(2.28)\qquad \left\{ \begin{aligned} Qu(x,t) = E\{\int_o^{\delta} f(X_s^o,t+s) \exp\left(-\int_o^s \alpha(t+\lambda)d\lambda\right) ds \\ + u(X_\delta^o,t+\delta) \exp\left(-\int_o^s \alpha(t+\lambda)d\lambda\right)\}, \end{aligned} \right.$$

where X_s^o is the free evolution, i.e. :

$$(2.29)\qquad X_s^o = x + \bar{q}(s,t,o),$$

with $\bar{q}(s,t,o)$ given by (2.21). Clearly δ is the lag time. We refer to Robin [3] and Blankenship and Menaldi [10] for some similar problems in different context.

In a first analysis for the existence and characterization of the optimal value function (2.7), we can roughly consider the impact of a lag time of the form $\delta = \delta(\xi,t)$ in the cost (2.5) as a new ordering cost. Then, if the new $k(\xi,t)$ also includes this contribution, we still are on the model (2.1),...,(2.7).

Notice that a typical example of the cost $k(\xi,t)$ is the function

$$(2.30)\qquad \left\{ \begin{aligned} & k(\xi,t) = \sum_{i=1}^{d} k_i(\xi_i,t) \ , \ \xi = (\xi_1,\ldots,\xi_d) \\ & k_i(\xi_i,t) = \begin{cases} p_i(t)+q_i(t)\ \xi_i & , \text{ if } \xi_i \neq 0 \\ 0 & , \text{ if } \xi_i = 0, \end{cases} \end{aligned} \right.$$

where $p_i(t)$, $q_i(t)$ are positive functions.

3. - A MODEL WITH OPTIMAL DECENTRALIZED POLICY

3.1. - Setting the model

We consider the same serial system of d installations described in § 2 (see fig. 4). As before, X_s^i denotes the inventory of the echelon i at the time s, clearly $X_s = (X_s^1,\dots,X_s^d)$ is the state of our dynamical system.

In this model we assume that the purchasing decisions, i.e. the orders or controls, containt a continuous part and an impulse part. Such a policy is represented by a process $\eta_s = (\eta_s^1,\dots,\eta_s^d)$. The state equation is :

(3.1) $$X_s^i = x_i + \eta_s^i + q(s,t),\ \forall s \in [0,T-t),\ i = 1,\dots,d,$$

where :

(3.2) $$\begin{cases} q(s,t) = -\int_0^s b(t+\lambda)d\lambda + \int_0^s \sigma(t+\lambda)dw_\lambda \\ \qquad + \iint_{[0,s)\times\mathbb{R}_*} \gamma(t+\lambda,\zeta)\ d\mu_\lambda(\zeta), \quad s,t\geq 0. \end{cases}$$

The state-constraint still beeing (2.4), i.e. :

(3.3) $$x_i + \eta_{s-}^i \geq x_{i-1} + \eta_s^{i-1},\ \forall s \geq 0,\ \forall i = 2,\dots,d,$$

and the process η_s satisfies the conditions :

(3.4) $$\begin{cases} \eta_s = (\eta_s^1,\dots,\eta_s^d) \text{ is adapted to } (\mathcal{F}^s,\ s \geq 0), \text{ right-continuous} \\ \text{and limited from the left, } \eta_s^i \geq \eta_{s'}^i \geq 0,\ \forall s \geq s' \geq 0,\ \forall i = 1,\dots,d. \end{cases}$$

Note that the processes X_s^i, η_s^i, i = 1,...,d are both right-continuous having left-hand limits for s in [0,T-t) and we set $X_{s-}^i = x_i$, $\eta_{s-}^i = 0$ at s = 0.

To each policy $\eta = (\eta_s,\ s \geq 0)$ it is associated an expected cost :

(3.5) $$\begin{cases} J_{xt}(\eta) = \sum_{i=1}^d E\{\int_0^{T-t} f_i(X_s^i,t+s)\exp(-\int_0^s \alpha(t+\lambda)d\lambda)\ ds \\ \qquad + C_i(t)\ \eta_0^i + \int_0^{T-t} c_i(t+s)\ \exp(-\int_0^s \alpha(t+\lambda)d\lambda)d\eta_s^i\}, \end{cases}$$

where x,t are the initial conditions and :

$c_i(s)$: is the cost per unit shipped to level i at the time s,

$f_i(z,s)$: is the cost of storage and shortage at level i for a quantity z at the time s,

$\alpha(s)$: is the discount factor at the time s.

As in § 2, T is the horizon (finite or infinite) and η_s^i is the cumulative orders from level i. The last integral in (3.4) is considered in the sense of Stieltjes over the interval (0,T-t), hence :

(3.6) $$\begin{cases} c_i(t)\,\eta_o^i + \int_0^{T-t} c_i(t+s)\exp\left(-\int_0^s \alpha(t+\lambda)d\lambda\right) d\eta_s^i \\ \qquad = \int_{[0,T-t)} c_i(t+s)\exp\left(-\int_0^s \alpha(t+\lambda)d\lambda\right) d\eta_s^i . \end{cases}$$

Note that here we do not include in our model any positive reorder cost. However, we could accept such a cost at the highest level (installation d), but for the sake of simplicity we neglect that possibility.

The optimal cost function is :

(3.7) $$\hat{u}(x,t) = \inf\{J_{xt}(\eta) : \eta \text{ satisfying (3.3) and (3.4)}\}$$

Beside to (2.8),...,(2.10) we also assume that :

(3.8) $c_1(\cdot), \ldots, c_d(\cdot)$ are uniformly continuous and bounded functions from $[0,T)$ into $[0,\infty)$. For each $i = 1,\ldots,d$, either $c_i(\cdot)$ is identically equal to zero or is bounded below by a strictly positive constant,

(3.9) for every $i = 1,\ldots,d$, $f_i = f_i(z,t)$ is a function from $\mathbb{R} \times [0,T)$ into $[0,\infty)$, which is convex in the variable z for every t ; there exist constants $C \geq c > 0$ such that :

$$c(z^+)^p - C \leq f_i(z,t) \leq C(1+|z|^p), \ \forall z \in \mathbb{R},\ t \in [0,T) ;$$

there exists a constant $K > 0$ such that :

$|\partial_z f_i(z,t)| \leq K(1+|z|^{p-1})$, $\forall z \in \mathbb{R}$, $t \in [0,T)$;

for every $\varepsilon > 0$ there is $\delta_\varepsilon > 0$ such that :

$|f_i(z,t) - f_i(z,t')| \leq \varepsilon(1+|z|^p)$, if $|t-t'| \leq \delta_\varepsilon$,

for any z in $\mathbb{R}$, t,t' in $[0,T)$.

The number p is fixed and $p \geq 1$. We denote by V_p the set of all functions $h = h(z,t)$ satisfying (3.9) with f_i replaced by h.

3.2. - Characterization of the optimal cost

We define :

(3.10) $$k(\xi,t) = c_1(t)\xi_1 + \ldots + c_d(t)\xi_d \text{ for } \xi = (\xi_1,\ldots,\xi_d)$$

and we consider the operator A and M given by (2.14) and (2.15) with (3.10).

Using the dynamic programming approach, we can deduce formally that the optimal cost function (3.7) must satisfy :

(3.11) $$\begin{cases} Au \leq f \text{ and } u \leq Mu \text{ in } \bar{\mathcal{O}} \times [0,T), \\ u(\cdot,T) = 0 \text{ if } T \text{ is finite.} \end{cases}$$

Notice that the complementary condition :

(3.12) $$Au = f \text{ if } u < Mu$$

has not a meaning, because for every function u we must have $Mu \leq u$. Theorem 1 does not apply to this situation. However, if we ignore the complementary condition in problem (2.18) then Theorem 1 holds under a weaker hypothesis than (2.11) on the function $k(\cdot,\cdot)$, provide we replace the word "unique" by "maximum" in the statement of the Theorem. The mentioned weaker assumption is just (2.11) with $k_0 = 0$. Clearly, this condition includes the function (3.10). For more details we refer to Menaldi and Robin [24,25] and [28].

As we said in the introduction, our purpose at this point is to show the following decomposition of the optimal cost $\hat{u}(x_1,\ldots,x_d,t)$ defined by (2.7) :

$$(3.13) \qquad \hat{u}(x_1,\ldots,x_d,t) = \hat{u}_1(x_1,t) + \ldots + \hat{u}_d(x_d,t),$$

where $\hat{u}_1,\ldots,\hat{u}_d$ are the solutions of certain one-dimensional state problems to be introduced. A direct consequence of (3.13) is the construction of an optimal decentralized policy.

Consider the one-dimensional stochastic process :

$$(3.14) \qquad Z_s = z + q(s,t) \ , \ s \geq 0,$$

where $q(s,t)$ is given by (3.2) and (z,t) are fixed in $\mathbb{R} \times [0,T)$ and control $(v_s,\ s \geq 0)$ a right-continuous stochastic process, increasing, non negative and adapted.

For given functions $h_1,\ldots,h_d$ satisfying (3.9) we define :

$$(3.15) \qquad \begin{cases} J^i_{zt}(v) = E\{\int_0^{T-t} h_i(Z_s+v_s,t+s) \exp\left(-\int_0^s \alpha(t+\lambda)d\lambda\right) \\ \qquad + c_i(t)\, v_o + \int_o^{T-t} c_i(t+s) \exp\left(-\int_o^s \alpha(t+\lambda)d\lambda\right) dv_s\}, \end{cases}$$

$$(3.16) \qquad \hat{u}_i(z,t) = \inf \{J^i_{zt}(v) \ : \ v \text{ any control}\},$$

for $i = 1,\ldots,d$ and (z,t) in $\mathbb{R} \times [0,T)$.

Recall the definition of the set V_p in (3.9) and consider the problems :

$$(3.17)\quad \begin{cases} \text{find } u_i \text{ in } V_p \text{ such that :} \\ A_z\, u_i \le h_i \text{ in } \mathcal{D}'(\mathbb{R} \times [0,T)), \\ u_i \le M_i\, u_i \text{ in } \mathbb{R} \times [0,T), \\ u_i(.,T) = 0 \text{ if T is finite,} \end{cases}$$

where $\mathcal{D}'$ is the space of distributions,

$$(3.18)\quad \begin{cases} A_z\, \varphi(z,t) = -\partial_t\, \varphi(z,t) - \frac{1}{2}\sigma^2(t)\partial_z^2\, \varphi(z,t) + b(t)\ \partial_z\, \varphi(z,t) \\ \qquad + \alpha(t)\ \varphi(z,t) + \int_{\mathbb{R}_*} [\varphi(z+\gamma(t,\zeta),t) - \varphi(z,t) \\ \qquad - \gamma(t,\zeta)\ \partial_z\, \varphi(z,t)]\ \Pi(d\zeta) \end{cases}$$

$$(3.19)\quad M_i\, \varphi(z,t) = \inf\ \{c_i(t)\zeta + \varphi(z+\zeta,t)\ :\ \zeta \text{ in } [0,\infty)\},$$

for $i = 1,\ldots,d$.

Proposition 1 : Under the assumptions (2.8),...,(2.10), (3.8) and :

$$(3.20)\quad h_1,\ldots,h_d \text{ belong to } V_p,$$

the optimal costs $\hat{u}_1,\ldots,\hat{u}_d$ defined by (3.16) are the maximum solutions of the problems (3.17). Moreover, the functions $\hat{u}_1(z,t), \ldots, \hat{u}_d(z,t)$ are twice continuously differentiable in the variable z and satisfy :

$$(3.21)\quad \begin{cases} A_z\hat{u}_i(z,t) = h_i(z,t), \text{ if } z \ge z_i^*(t),\ 0 \le t < T, \\ \partial_z\hat{u}_i(z,t) = -c_i(t)\ , \text{ if } z \le z_i^*(t),\ 0 \le t < T \end{cases}$$

where

$$(3.22)\quad z_i^*(t) = \inf\ \{z \in \mathbb{R}\ :\ \partial_z\hat{u}_i(z,t) + c_i(t) > 0\},$$

for $i = 1,\ldots,d$, provided the functions (3.22) are continuous and bounded from above, and $\sigma(.)$ never vanishes. □

Now, by induction we define the following functions $h_1,\dots,h_d$:

(3.23) $$h_1 = f_1,\ h_i = f_i + (h_{i-1} - A_z\hat{u}_{i-1}),\ i = 2,\dots,d.$$

Let us suppose that :

(3.24) $$\left\{\begin{array}{l}\text{the functions } z_1^*(.),\dots,z_d^*(.) \text{ defined by (3.22) are uniformly}\\ \text{continuous and bounded from above on } [0,T).\end{array}\right.$$

In order that assumption, (3.24) holds if suffices to know that the function $\sigma(.)$ in (2.9) never vanishes. Then, using (3.24) inductively and based on (3.21), we see that the function $(h_{i-1} - A_z\hat{u}_{i-1})$ is not negative, decreasing and convex in the first variable. Therefore the functions $h_1,\dots,h_d$ defined by (3.23) satisfy the same condition (3.9) of functions $f_1,\dots,f_d$; i.e. (3.20) is valid.

Now, we can state one of our main results of this analysis.

<u>Theorem 3</u> : Let assume that (2.8),...,(2.10), (3.8), (3.9) and (3.24) hold. Then we have the following decomposition :

(3.25) $$\left\{\begin{array}{l}\hat{u}(x_1,\dots,x_d,t) = \hat{u}_1(x_1,t) + \dots + \hat{u}_d(x_d,t),\\ \text{for every } x_1 \le \dots \le x_d,\ 0 \le t < T,\end{array}\right.$$

with the notation (3.5),...,(3.7), (3.15), (3.16) and (3.23). □

3.3. - <u>Decomposition of the optimal centralized policy</u>

A complementary result concerning proposition 1 is the fact that the control $\hat{v}_s^1,\dots,\hat{v}_s^d$ defined by the following expression is optimal:

(3.26) $$\left\{\begin{array}{ll}\hat{v}_s^i = \max\{[\hat{v}_o + q(\lambda,t) + z - z^*(\lambda)]^- : 0 \le \lambda < s\}, & \\ \hat{v}_o^i = (z^*(0) - z)^- & ,\ 0 \le s < T-t,\end{array}\right.$$

for $i = 1,\dots,d$, (z,t) in $\mathbb{R} \times [0,T)$, $q(\lambda,s)$ is given by (3.2) and $(.)^-$ denotes the negative part of a real number. This means that if $\hat{v}^i = (\hat{v}_s^i,\ 0 \le s < T-t)$ is the process (3.26) then :

(3.27) $\hat{u}_i(z,t) = J^i_{zt}(\hat{v}^i)$,

for $i = 1,\ldots,d$.

Now, for given initial conditions $(x_1,\ldots,x_d,t)$ in $\bar{\mathcal{O}} \times [0,T)$ we first construct $\hat{v}^1,\ldots,\hat{v}^d$ by (3.26) with $z = x_i$ as initial state. Next, we define by induction the d-dimensional control $\hat{\eta} = (\hat{\eta}^1,\ldots,\hat{\eta}^d)$ by the formulae :

(3.28)
$$\begin{cases} \hat{\eta}^d_s = \hat{v}^d_s \quad , \forall s \in [0,T-t), \\ \hat{\eta}^{i-1}_s = \begin{cases} \hat{v}^{i-1}_s & \text{if } 0 \le s < \tau, \\ \hat{\eta}^i_s - \hat{\eta}^i_\tau + \hat{v}^{i-1}_\tau & \text{if } \tau \le s < T-t, \end{cases} \\ \text{with } \tau = \inf\{s \ge 0 : z^i + \hat{\eta}^i_s \le z^{i-1} + \hat{v}^{i-1}_s\},\ i = 2,\ldots,d. \end{cases}$$

Theorem 4 : Under the hypotheses of Theorem 3, the control policy $\hat{\eta}$ defined by (3.28) is a decentralized control satisfying (3.3), (3.4) and :

(3.29) $\hat{u}(x,t) = J_{xt}(\hat{\eta})$,

i.e., $\hat{\eta}$ is an optimal control for (3.5), (3.6). □

Remark : Roughly speaking, such an optimal contrôl $\hat{\eta}$ can be described by the following feedback law. At each state of the system $x_1,\ldots,x_d,t$ we proceed as follows; If $x_i > z^*_i(t)$ then not purchase order are made from level $i = 1,\ldots,d$. If $x_i < z^*_i(t)$ then order up to reach the stock $z^*_i(t)$ or x_{i+1}, whatever is smaller, for $i = 1,\ldots,d$ and $x_{d+1} = +\infty$. If $x_i \le z^*_i(t)$ and $x_i \ge z^*_{i+1}(t)$ for some $i = 1,\ldots,d-1$, then adopt the same policy of level i+1. If $x_i = z^*_i(t)$ and $x_{i+1} = z^*_{i+1}(t)$ for some $i = 1,\ldots,d$, $z^*_{d+1}(t) = +\infty$, then order in a continuous fashion just to avoid to exit the admissible states $x_1 \le \ldots \le x_d$. □

Notice that the optimal policy (3.28) expresses exactly what is expected from this model, i.e. with the time, the system becomes completly degenerate in the sense that $\hat{X}^1_s = \ldots = \hat{X}^d_s$ for s sufficiently large depending on the initial state $x_1 \le \ldots \le x_d$.

A natural question to be answered is to see in which measure the model of this section approximates the first model of § 2. To this effect, consider the function $k_\varepsilon(\xi,t)$, $\xi = (\xi_1,\ldots,\xi_d)$, $t \geq 0$, $\varepsilon > 0$:

(3.30) $$k_\varepsilon(\xi_1,\ldots,\xi_d,t) = \varepsilon + c_1(t)\xi_1 + \ldots + c_d(t)\xi_d, \quad \xi \neq 0,$$

and denote by $\hat{u}_\varepsilon(x_1,\ldots,x_d,t)$ the optimal cost (2.7) with $k_\varepsilon(.,.)$ replacing $k(.,.)$ and $f(.,.)$ satisfying the decomposition (2.6).

Theorem 5 : Let the hypotheses (2.8),...,(2.10), (3.8), (3.9), (3.24) and (3.30) hold. Then we have following convergence :

(3.31) $$\hat{u}_\varepsilon(x,t) \to \hat{u}_o(x,t) \text{ as } \varepsilon \to 0,$$

uniformly over bounded subsets of $\bar{\mathcal{O}} \times [0,T)$, and where $\hat{u}_o(x,t)$ denotes the optimal cost (3.7). □

Remark : Estimate (3.31) implies that we can neglect set up or reordering costs when they are sufficiently small. By the way, the analysis of the control (3.28) gives us several possibilities to use the information of the asymptotic model of this section to construct an ε-optimal policy for the model of § 2. In particular, the continuation set of the problems (3.17) may be used as approximation of the true continuation region of problem (2.18). □

4. - PROOF OF RESULTS

4.1. - Some background

Consider the stochastic process :

(4.1) $$\begin{cases} Z_s(x,t) = x - \displaystyle\int_o^s \bar{b}(t+\lambda)d\lambda + \int_o^s \bar{\sigma}(t+\lambda)dw_\lambda \\ \qquad + \displaystyle\iint_{[0,t)\times\mathbb{R}_*} \bar{\gamma}(t+\lambda,\zeta)\, d\mu_\lambda(\zeta), \quad s \geq 0, \end{cases}$$

where $x = (x_1,\ldots,x_d)$, $\bar{b}(.) = \bar{e}b(.)$, $\bar{\sigma}(.) = \bar{e}\sigma(.)$, $\bar{\gamma}(.,.) = \bar{e}\gamma(.,.)$, $\bar{e} = (1,\ldots,1)$ in $\mathbb{R}^d$.

Recall the space $C_p = C_p(\bar{\mathcal{O}} \times [0,T))$, $p \geq 0$ defined in (2.12). We will prove that the condition :

(4.2) $\quad u,f \in C_p,\ Au \leq f$ in $\mathcal{D}'(\mathcal{O} \times (0,T))$

is equivalent to the condition :

(4.3)
$$\begin{cases} \text{for every } (x,t) \text{ in } \bar{\mathcal{O}} \times [0,T) \text{ the process :} \\ \chi_s = u(Z_s,t+s)q_s + \int_0^s f(Z_\lambda,t+\lambda)q_s\, d\lambda,\ 0 \leq s < T-t, \\ \text{is a submartingale, with } Z_s = Z_s(x,t) \text{ and } q_s = \exp(-\int_0^s \alpha(t+\lambda)d\lambda). \end{cases}$$

Indeed, let $\rho(.)$ be a smooth function with compact support on $\mathbb{R}^d$ satisfying :

$$\rho(x) \geq 0,\ \forall x \in \mathbb{R}^d$$
$$\rho(-x) = 0,\ \forall x \in \mathbb{R}^d - \mathcal{O}$$
$$\int_{\mathbb{R}^d} \rho(x)\, dx = 1.$$

Define for every (x,t) in $\bar{\mathcal{O}} \times [0,T)$:

$$u_\varepsilon(x,t) = \int_{\mathbb{R}^d} u(x-\varepsilon y,t)\, \rho(y)\, dy$$

$$f_\varepsilon(x,t) = \int_{\mathbb{R}^d} f(x-\varepsilon y,t)\, \rho(y)\, dy,\ \varepsilon > 0.$$

Note that for every x in $\bar{\mathcal{O}}$ and y in $\mathbb{R}^d$ such that $\rho(y) \neq 0$ we must have $(x-\varepsilon y)$ in $\bar{\mathcal{O}}$, for any $\varepsilon > 0$. Now, if u, f satisfy (4.2) (or (4.3)) then u_ε, f_ε satisfy also (4.2) (or (4.3)) with u,f replaced by $u_\varepsilon, f_\varepsilon$. Thus, using the fact that $u_\varepsilon, f_\varepsilon$ converge to u,f as ε goes to zero, locally uniform in $\bar{\mathcal{O}} \times [0,T)$, we reduce our problem to show the equivalence of (4.2) and (4.3) under the extra hypothesis :

(4.4) $\quad u,\ \partial_i u,\ \partial_{ij} u,\ f,\ \partial_i f,\ \partial_{ij} f$ belong to C_p, $\forall i,j = 1,\ldots,d.$

Next, we define for every (x,t) in $\bar{\mathcal{O}} \times [0,T)$

$$u_\varepsilon(x,t) = \int_{\mathbb{R}} u(x,t-\varepsilon\tau)\ \rho_o(\tau)\ d\tau,$$

$$f_\varepsilon(x,t) = \int_{\mathbb{R}} f(x,t-\varepsilon\tau)\ \rho_o(\tau)\ d\tau,$$

where $\rho_o(.)$ is a smooth function with compact support on $\mathbb{R}$ satisfying :

$$\rho_o(\tau) \geq 0, \ \forall\tau \in \mathbb{R}$$

$$\rho_o(\tau) = 0, \ \forall\tau \leq 0$$

$$\int_{\mathbb{R}} \rho_o(\tau)\ d\tau = 1.$$

Since u is smooth in x, we see that the function :

$$h_\varepsilon(x,t) = Au_\varepsilon(x,t) - \int_{\mathbb{R}} Au(x,t-\varepsilon\tau)\ \rho_o(\tau)\ d\tau$$

converges to zero as ε goes to zero, locally uniform in $\bar{\mathcal{O}} \times [0,T)$. Therefore, if u,f satisfy (4.2) then we have :

$$(4.5) \qquad Au_\varepsilon \leq f_\varepsilon + h_\varepsilon \quad \text{pointwise in } \bar{\mathcal{O}} \times [0,T)$$

for $\varepsilon > 0$ suffiently small. Because u_ε is smooth in x,t, Itô's formula and (4.5) imply (4.3) for $u_\varepsilon, f_\varepsilon + h_\varepsilon$ instead of u,f. Hence, letting ε go to zero we get (4.3).

To prove the converse, let us suppose that u,f satisfy (4.2). Then we have :

$$u(x,t) \leq E\{u(Z_s,t+s)q_s + \int_0^s f(Z_\lambda,t+\lambda)\ q_\lambda\ d\lambda\}$$

for every (x,t) in $\bar{\mathcal{O}} \times [0,T)$, $0 \leq s < T-t$. But, Itô's formula gives :

$$u(Z_s,t+s)q_s = u(x,t+s) - \int_0^s A_x\ u(Z_\lambda,t+s)\ q_\lambda\ d\lambda,$$

where A_x is the integro-differential operator (2. 14) without the derivative ∂_t, i.e. :

$$-\partial_t + A_x = A.$$

Combining the above inequality, we obtain :

$$[u(x,t) - u(x,t+s)] + E\{\int_0^s A_0 u(Z_\lambda,t+s)q_\lambda\, d_\lambda\}$$

$$\leq E\{\int_0^s f(Z_\lambda,t+\lambda)q_\lambda\, d_\lambda\}.$$

Hence, dividing this lost expression by s and letting s go to zero we deduce (4.2). At this point we have established the equivalence between (4.2) and (4.3). □

Let Q be a relatively open subset of $\bar{\mathcal{O}} \times [0,T)$. We will prove that the conditions

(4.6) $$u,f \in C_p,\ Au = f \text{ in } \mathcal{D}'(Q \cap \mathcal{O} \times (0,T))$$

and

(4.7) $$\begin{cases} \text{for every } (x,t) \text{ in } Q \text{ the process} \\ \chi_s = u(Z_{s\wedge\tau},t+s\wedge\tau)q_{s\wedge\tau} + \int_0^{s\wedge\tau} f(Z_\lambda,t+\lambda)q_\lambda\, d\lambda,\ 0 \leq s < T-t \\ \text{is a martingale, with } Z_s = Z_s(x,t), \\ q_s = \exp\left(-\int_0^s \alpha(t+\lambda)d\lambda\right) \\ \text{and :} \\ \tau = \inf\{s \in [0,T-t) : (Z_s,t+s) \notin Q\}. \end{cases}$$

Indeed, we just need to add to previous convolution method, a truncation technique. This means that first we multiply u by a smooth function with compact support in $\bar{Q} - \mathcal{O} \times [0,T)$. Thus, we reduce our problem to the case where $Q = \bar{\mathcal{O}} \times [0,T)$, in which we can use the arguments of (4.2), (4.3). By the way, if τ_n denotes the exit time τ of (4.7) with Q replaced by Q_n, then we have :

(4.8) $$t\wedge\tau_n \to t\wedge\tau, \text{ as } n \to \infty,\ Q_n \subset Q_{n+1},\ \bigcup_{n=1}^{\infty} Q_n = Q.$$ □

With the notation (4.1) we will show that for every number $\alpha_o > 0$ there exists a $\lambda_o > 0$ such that for every (x,t) in $\bar{\mathcal{O}} \times [0,T)$, $0 \le s < T-t$ we have :

$$E\{ (\lambda_o + |Z_s(x,t)|^2)^{p/2} e^{-\alpha_o s} \} \le (\lambda_o + |x|^2)^{p/2} \tag{4.9}$$

Indeed, we consider the function

$$m(x,\lambda) = (\lambda + |x|^2)^{p/2}, \ \forall x \in \mathbb{R}^d, \ \lambda > 0, \tag{4.10}$$

and we denote by A_o the integro-differential operator (2.14) without the terms in ∂_t, $\alpha(t)$. An elementary computation shows that :

$$\begin{cases} A_o m(.,\lambda) \le -\beta(\lambda)\ m(.,\lambda) \text{ in } \mathbb{R}^d \times [0,T), \\ \text{with } \beta(\lambda) > 0, \ \beta(\lambda) \to 0 \text{ as } \lambda \to +\infty, \end{cases} \tag{4.11}$$

after using the fact that the function $b(.)$, $\sigma(.)$, $\gamma(.,.)$ are bounded. Hence, applying Itô's formula to the function :

$$(x,t) \to m(x,\lambda) \exp(-\beta(\lambda)\ t)$$

and the process (4.1), we deduce the estimate (4.9) by means of (4.11). □

Classical estimates on the stochastic integral (4.1) permit us to prove that for every $p \ge 2$, $\varepsilon > 0$ there exist a $\delta > 0$ such that

$$\begin{cases} E\{ \sup\limits_{0 \le \lambda \le s} |Z_s(x,t) - Z_s(x',t')|^p\} \le (s^p + s^{p/2})\varepsilon + |x-x'|^p \\ \forall x,x' \in \mathbb{R}^d, \ \forall s,t,t' \ge 0, \ |t-t'| \le \delta \end{cases} \tag{4.12}$$

Note that if the functions $b(.)$, $\sigma(.)$, $\gamma(.,.)$ in (2.9), (2.10) are Hölder continuous with exponent β, then the ε of (4.12) can be replaced by $C\ |t-t'|^{p\beta}$, i.e. $Z_s(x,t)$ is also Hölder continuous in t with the some exponent β. □

The maximum principle holds for the operator (2.14) in the following sense :

(4.13) $\begin{cases} \text{if } u,v \in C_p, \ Au \leq v \text{ in } \mathcal{D}'(\mathcal{O} \times (0,T)), \ u(.,T) \leq 0 \text{ for} \\ T \text{ finite and } v \leq 0 \text{ whenever } u > 0 \text{ then } u \leq 0. \end{cases}$

Indeed, the equivalence of conditions (4.2) and (4.3) implies that the random process

$$\chi_s = u(Z_s,t+s)q_s + \int_0^s v(Z_\lambda,t+\lambda)q_\lambda \, d\lambda, \quad 0 \leq s \leq T-t$$

is a submartingale, with the notation of (4.2). Since the submartingale is right-continuous having left-hand limits and uniformly integrable, for every stopping time $\tau \leq T-t$ we have

$$u(x,t) \leq E\{u(Z_\tau,t+\tau)q_\tau + \int_0^\tau v(Z_s,t+s)q_s \, ds\}.$$

Thus, choosing

$$\tau = \inf \{s \in [0,T-t] : v(Z_s,t+s) > 0\}$$

and using the fact that $v > 0$ implies $u \leq 0$, we deduce (4.13). □

Remark : Most of the assertions of this section are true under more general assumptions. Actually, they are variants of classic results, at least for $p = 0$ and $\sigma(.)$ strictly positive, cfr. Gihman and Skorohod [17], Stroock and Varadhan [35]. □

4.2. - Centralized model

We are going to prove Theorems 1 and 2 of § 2. Consider the operators :

$$S_\varepsilon : C_p \to C_p, \ \varepsilon > 0$$

$$S_\varepsilon u(x,t) = E\{\int_0^{T-t} [f(Z_s(x,t),t+s)$$

$$+ \frac{1}{\varepsilon} (\psi\wedge u)(Z_s(x,t),t+s)] \exp[-\int_0^s (\alpha(t+\lambda) + \frac{1}{\varepsilon}) d\lambda] ds\},$$

where ψ is a function belonging to the space C_p given by (2.12). We seek a function $u_\varepsilon(x,t)$ such that :

(4.14) $$\begin{cases} u_\varepsilon \in C_p,\ u_\varepsilon(.,T) = 0 \text{ if } T \text{ is finite, } u_\varepsilon \text{ is a fixed point} \\ \text{of } S_\varepsilon, \text{ i.e. } S_\varepsilon u_\varepsilon = u_\varepsilon. \end{cases}$$

We will show that for every $\varepsilon > 0$ and f,ψ in C_p there exists a unique solution u_ε of problem (4.14). Moreover, u_ε is also the unique solution of the following problem :

(4.15) $$\begin{cases} u_\varepsilon \in C_p,\ u_\varepsilon(.,T) = 0 \text{ if } T \text{ is finite,} \\ Au_\varepsilon = f - \frac{1}{\varepsilon}(u_\varepsilon - \psi)^+ \text{ in } \mathcal{D}'(\mathcal{O}\times(0,T)). \end{cases}$$

Indeed, the space C_p with the norm :

(4.16) $$||u||_p = \sup \{|u(x,t)|\ (\lambda + |x|^2)^{-p/2} \exp(-\beta(t-t)) :$$
$$\forall (x,t) \in \bar{\mathcal{O}} \times [0,T)\},$$

for fixed $\lambda \geq 1$, $p \geq 0$, $\beta \geq 0$, $\beta = 0$ if $T = \infty$, becomes a Banach space. Throughout this section,we choose :

(4.16)' $$\begin{cases} \lambda = \lambda_0 \text{ given in (4.9) for } \alpha_0 > 0 \text{ satisfying either} \\ \alpha(.) \geq 2\alpha_0,\ \beta = 0 \text{ if } T = \infty, \text{ or } \alpha(.) \geq 2\alpha_0 - \beta_0,\ \beta = \beta_0 \geq 0 \\ \text{if } T \text{ is finite.} \end{cases}$$

Based on the property (4.9) and the choice (4.16)' we get for any $\varepsilon > 0$:

$$||S_\varepsilon u - S_\varepsilon v||_p \leq (1+\varepsilon\alpha_0)^{-1}\ ||u-v||_p, \forall u,v \in C_p$$

Hence, S_ε is a contraction operator on the Banach space C_p, therefore the problem (4.14) possesses one and only one solution denoted by u_ε. Now, by means of Markov's property we deduce from (4.14) that the process

$$\chi^\varepsilon = u_\varepsilon(Z_s,t+s)q_s^\varepsilon + \int_0^s f_\varepsilon(Z_s,t+s)q_s^\varepsilon \, ds, \quad 0 \le s < T-t,$$

is a martingale, with $Z_s = Z_s(x,t)$,

$$q_s^\varepsilon = \exp\left[-\int_0^s (\alpha(t+\lambda) + \frac{1}{\varepsilon})d\lambda\right] \text{and } f_\varepsilon = f + \frac{1}{\varepsilon}(\psi \wedge u_\varepsilon).$$

Then, by virtue of the equivalence of (4.6) and (4.7), we obtain

$$Au_\varepsilon + \frac{1}{\varepsilon} u_\varepsilon = f + \frac{1}{\varepsilon}(\psi \wedge u_\varepsilon) \text{ in } \mathcal{D}'(\mathcal{O} \times (0,T)),$$

which is equivalent to the last condition of (4.15). □

We will prove, that the unique solution of (4.15) admits the following representation :

$$(4.17) \quad \begin{cases} u_\varepsilon(x,t) = \inf\{J_{xt}^\varepsilon(v) : v\}, \\ J_{xt}^\varepsilon(v) = E\{\int_0^{T-t} (f + \frac{v(s)}{\varepsilon}\psi)(Z_s(x,t),t+s) \\ \qquad \times \exp\left[-\int_0^s (\alpha(t+\lambda) + \frac{v(\lambda)}{\varepsilon})d\lambda\right]ds\}, \end{cases}$$

where the infimum is taken over all adapted processes $v = v(t,\omega)$ whith values in $[0,1]$. Indeed, from (4.15) and by means of Markov's property we can deduce that

$$\frac{1}{\varepsilon} E\{\int_0^{T-t} [(u_\varepsilon-\psi)^+ - v(s)(u_\varepsilon-\psi)] (Z_s(x,t),t+s)$$
$$\times \exp\left[-\int_0^s (\alpha(t+\lambda) + \frac{v(\lambda)}{\varepsilon})d\lambda\right]ds\} = J_{xt}^\varepsilon(v) - u_\varepsilon(x,t),$$

for every control v and any x,t, $\varepsilon > 0$. Because $0 \le v(.) \le 1$, this last equality implies (4.17). □

From the representation (4.17) follows that if u_ε and $\tilde{u}_\varepsilon$ denote either the unique solution of (4.15) or the optimal cost (4.17) corresponding to the data f,ψ and $f,\tilde{\psi}$ then we have :

$$(4.18) \qquad ||u_\varepsilon-\tilde{u}_\varepsilon||_p \leq \alpha_o^{-1} ||f-\tilde{f}||_p + ||\psi-\tilde{\psi}||_p, \forall\varepsilon > 0,$$

under the notation (4.15), (4.16). □

Let $\hat{u}(x,t)$ be the following optimal cost :

$$(4.19) \quad \begin{cases} \hat{u}(x,t) = \inf \{J_{xt}(\theta) \ : \ 0 \leq \theta \leq T-t\}, \\ J_{xt}(\theta) = E\{\int_o^\theta f(Z_s(x,t),t+s) \exp(-\int_o^s \alpha(t+\lambda)d\lambda) \, ds \\ \qquad + \psi(Z_\theta(x,t),t+\theta) \exp(-\int_o^\theta \alpha(t+s)ds) \, \mathbb{1}(\theta < T-t)\}, \end{cases}$$

where the infimum ranges over all stopping time θ satisfying $0 \leq \theta \leq T-t$. Assume that the function ψ is smooth, i.e.

ψ, $A\psi$ belong to C_p and $\psi(.,T) \geq 0$ if T is finite.

Then we will show that for every $\varepsilon > 0$,

$$(4.20) \qquad u_\varepsilon \geq \hat{u}, \ ||u_\varepsilon-\hat{u}||_p \leq \varepsilon||(f-A\psi)^+||_p .$$

Indeed, the definition (4.19) and Markov's property imply that $\hat{u}$ satisfies the condition (4.3) with $\hat{u}$ replacing u. Hence, using (4.2) and (4.15) we obtain :

$$A(\hat{u}-\hat{u}_\varepsilon) \leq \frac{1}{\varepsilon} (u_\varepsilon-\psi)^+ \text{ in } \mathcal{D}'(\mathcal{O}\times (0,T)).$$

Since $\hat{u} \leq \psi$, the maximum principle (4.13) provides the first part of (4.20). Now, for any stopping time θ with values in $[0,T-t]$, we associate a random process $(v_\theta(s), \ 0 \leq s \leq T-t)$ defined by :

$$v_\theta(s) = \begin{cases} 0 & \text{if } s \leq \theta \\ 1 & \text{if } s > \theta \end{cases}$$

Thus, from the definitions (4.17) and (4.19) we have

$$u^{\varepsilon}(x,t) - J_{xt}(\theta) \leq J_{xt}(v_\theta) - J_{xt}(\theta)$$

$$= E\{\int_{\theta}^{T-t} h(Z_s(x,t),t+s)\exp[-\int_{0}^{s} (\alpha(t+\lambda)+\frac{1}{\varepsilon})d\lambda]ds$$

$$- \psi(Z_{T-t}(x,t),T) \exp[-\int_{0}^{T-t} (\alpha(t+s)+\frac{1}{\varepsilon})ds]\},$$

with $h = f - A\psi$. By means of (4.9) and the assumption on ψ, we deduce (4.20). □

Since the optimal cost (4.19) satisfies some estimate similar to (4.18), we deduce from all the above results that the function $\hat{u}$ defined by (4.19) is the maximum solutions of the following problem :

(4.21) $$\begin{cases} \text{find } u \in C_p, u(.,T) = 0 \text{ if } T \text{ is finite,} \\ Au \leq f \text{ in } \mathcal{D}'(\mathcal{O}\times (0,T)),\ u \leq \psi \text{ in } \bar{\mathcal{O}}\times [0,T), \end{cases}$$

i.e. $\hat{u}$ satisfies (4.21) and any other function u with the property (4.21) results $u \leq \hat{u}$. Moreover :

(4.22) $$||u_\varepsilon - \hat{u}||_p \to 0 \text{ as } \varepsilon \to 0.$$

Clearly, all this holds under the assumption :

(4.23) $$\psi \in C_p,\ \psi(.,T) \geq 0 \text{ if } T \text{ is finite.}$$ □

Based on the uniform convergence (4.22), we deduce from (4.15) that

(4.24) $$A\hat{u} = f \text{ in } \mathcal{D}'([\hat{u} < \psi]).$$ □

Now, we will prove that the optimal cost (4.19) is the unique function satisfying (4.21) and (4.24) simultaneously. Moreover, we have

(4.25) $$\hat{u}(x,t) = J_{xt}(\hat{\theta}(x,t)),\ \forall (x,t) \in \bar{\mathcal{O}}\times [0,T-t),$$

where

$$(4.26) \qquad \hat{\theta}(x,t) = \inf \{s \in [0,T-t] : (Z_s(x,t),t+s) \notin [\hat{u} < \psi]\},$$

$\hat{\theta}(x,t) = T-t$ if $\hat{u}(Z_s(x,t),t+s) < \psi(Z_s(x,t),t+s)$ for every s in $[0,T-t)$ and $[\hat{u} < \psi]$ denotes the subset of all points (y,s) in $\bar{\mathcal{O}} \times [0,T)$ such that $\hat{u}(y,s) < \psi(y,s)$. Indeed, let $\hat{u}$ be a function satisfying (4.21) and (4.24).

By means of the equivalence (4.2), (4.3) we get :

$$\hat{u}(x,t) \leq J_{xt}(\theta), \ \forall\theta \text{ valued in } [0,T-t].$$

Next, from (4.6), (4.7) and (4.24) we deduce (4.25). ◻

At this moment, we have on hand all necessary facts to proceed with the proof of Theorems 1 and 2. Let $(u^o,u^1,\ldots)$ be the sequence of functions defined by induction as follows :

$$(4.27) \qquad \begin{cases} u^n \in C_p, \ u^n(.,T) = 0 \text{ if } T \text{ is finite,} \\ Au^n \leq f \text{ in } \mathcal{D}'(\mathcal{O} \times (0,T)), \ u^n \leq Mu^{n-1} \text{ in } \bar{\mathcal{O}} \times [0,T), \\ Au^n = f \text{ in } \mathcal{D}'([u^n < Mu^{n-1}]), \end{cases}$$

for $n = 1,2,\ldots$, and

$$(4.28) \qquad \begin{cases} u^o \in C_p, \ u^o(.,T) = 0 \text{ if } T \text{ is finite,} \\ Au^o = f \text{ in } \mathcal{D}'(\mathcal{O} \times (0,T)), \end{cases}$$

where the operator M is defined by (2.15). From the previous results, we see that the sequence $(u^o,u^1,\ldots)$ is well defined and :

$$(4.29) \qquad 0 \leq u^n \leq u^{n-1}, \ \forall n = 1,2,\ldots$$

Moreover, based on the representation (4.19), the strong Markov's property and the definition of M, we have

$$(4.30) \qquad u^n(x,t) = \inf \{J_{xt}(\nu) : \nu \in \mathcal{V}_n\}, \; n = 1,2,\ldots,$$

where $\mathcal{V}_n$ denotes the set of all impulse control satisfying (2.3) and (2.4) with $\theta_{n+1} = \infty$, i.e. with only n impulses, and $J_{xt}(\nu)$ is the cost (2.5). □

We will show that

$$(4.31) \qquad \begin{cases} u^n \to u^*, \text{ as } n \to \infty, \text{ uniformly on bounded subsets of } \bar{\mathcal{O}} \times [0,T), \\ \text{and } u^* \text{ belongs to } C_p \end{cases}$$

where u^* is the pointwise limit of the sequence $(u^1,u^2,\ldots)$. Indeed, in view of (4.29), we need only to prove that for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for any $n = 1,2,\ldots,$

$$(4.32) \qquad |u^n(x,t) - u^n(x',t)| \le \varepsilon(1+|x|+|x'|)^p, \text{ if } |x-x'| \le \delta_\varepsilon$$

$$(4.33) \qquad |u^n(x,t) - u^n(x,t')| \le \varepsilon(1+|x|)^p \qquad , \text{ if } |t-t'| \le \delta_\varepsilon$$

Now, by means of the hypothesis (2.19) we deduce that there exists a constant $C > 0$ such that for computing the infima (2.7) and (4.30) for $n = 1,2,\ldots,$ we need only to consider all impulse controls ν which satisfy also the condition

$$(4.34) \qquad \begin{cases} E\{\int_0^{T-t} m(\nu_s,\lambda_o) \exp\left(-\int_0^s \alpha(t+\lambda)d\lambda\right) \\ \quad + \sum_{j=1}^{\infty} k(\xi_j,t+\theta_j) \exp\left(-\int_0^{\theta_j} \alpha(t+s)ds\right) \mathbb{1}(\theta_j < T-t)\} \le C\, m(x,\lambda_o) \end{cases}$$

where $m(x,\lambda)$ is given by (4.10) and the constant C is clearly independent of (x,t) in $\bar{\mathcal{O}} \times [0,T)$. At the same time, we obtain another constant $u_o > 0$ such that

$$(4.35) \qquad u^n(x,t) \ge u_o\, |x^+|^p - \frac{1}{u_o}, \; \forall (x,t) \in \bar{\mathcal{O}} \times [0,T),$$

i.e., the same property (2.19) of the function f. Then, let ν be any arbitrary impulse control satisfying (2.3), (2.4) and (4.34) for a fixed (x,t) in $\bar{\mathcal{O}} \times [0,T)$,

$$\nu_s^i = \sum_{j=1}^{\infty} \xi_j^i \mathbb{1}(\theta_j \le s), \; i = 1,\dots,d.$$

We associate to ν another impulse control $\tilde{\nu}$, with

$$\tilde{\theta}_j = \theta_j, \; \tilde{\xi}_{j+1} = \xi_{j+1}, \; \forall j = 1,2,\dots,$$

$$\tilde{\xi}_1^i = \xi_1^i + (x_{i+1}-\tilde{x}_{i+1})^- + (x_i-\tilde{x}_i)^+ + (x_{i-1}-\tilde{x}_{i-1})^- + \dots + (x_1-\tilde{x}_1)^-, \; i = 1,\dots,d$$

where $(.)^-$ denotes the negative part of a real number, $\tilde{x} = (\tilde{x}_1,\dots,\tilde{x}_d)$ is a point in $\bar{\mathcal{O}}$ and $x_{d+1}-\tilde{x}_{d+1} = 0$. We check that the control $\tilde{\nu}$ just defined satisfies the conditions (2.3), (2.4) and

(4.36) $$E\{\int_0^{T-t} m(\tilde{\nu}_s,\lambda_o) \exp(-\int_0^s \alpha(t+\lambda)d\lambda)ds\} \le C(m(x,\lambda_o) + |x-\tilde{x}|^p),$$

where C is some apropriate constant independent of x, $\tilde{x}$ and t. In order to show (4.32) we are going to estimate the difference $(J_{xt}(\nu) - J_{\tilde{x}t}(\tilde{\nu}))$ uniformly in ν. To this purpose, we write

$$|J_{xt}(\nu)-J_{\tilde{x}t}(\tilde{\nu})| \le E\{\int_0^{T-t} \beta(s) \exp(-\int_0^s \alpha(t+\lambda)d\lambda)\, ds$$

$$+ |k(\xi_1,t+\theta_1)-k(\tilde{\xi}_1,t+\theta_1)| \exp(-\int_0^{\theta_1} \alpha(t+s)ds)\},$$

$$\beta(s) = |f(Z_s(x,t) + \nu_s,t+s) - f(Z_s(\tilde{x},t) + \tilde{\nu}_s,t+s)|.$$

Since

$$[Z_s(x,t) + \nu_s] - [Z_s(\tilde{x},t) + \tilde{\nu}_s] = (x-\tilde{x}) + (\xi_1-\tilde{\xi}_1),$$

$$|\xi_1-\tilde{\xi}_1| \le d\, |x-\tilde{x}|,$$

and the fact that f satisfies (2.12), we deduce that for every $\varepsilon > 0$ there exist $\delta_\varepsilon > 0$ such that

$$\beta(s) \le \varepsilon(1+ |Z_s(x,t) + \nu_s| + |Z_s(\tilde{x},t) + \tilde{\nu}_s|)^p$$
$$\text{if } |x-\tilde{x}| \le \delta_\varepsilon .$$

In view of the uniformly continuity of the function $k(.,.)$, the estimates (4.34), (4.36) and the last condition , we can affirm that for every $\varepsilon > 0$ there is a $\delta_\varepsilon > 0$, independent of $\nu, x, \tilde{x}$ and t, such that :

$$|J_{xt}(\nu)-J_{\tilde{x}t}(\tilde{\nu})| \le \varepsilon(1 + |x| + |\tilde{x}|)^p \quad \text{if } |x-\tilde{x}| \le \delta_\varepsilon,$$

which provides (4.32). To show (4.33) we suppose first that $t \le t'$ in $[0,T)$. For any impulse control ν satisfying (2.3), (2.4) and (4.34) relative to (x,t) in $\bar{\mathcal{O}} \times [0,T)$, we have

$$J_{xt'}(\nu)-J_{xt}(\nu) \le E\{\int_0^{T-t'} (\gamma_s(t') - \gamma_s(t))ds + \sum_{j=1}^{\infty} (q_j(t')-q_j(t))\},$$
$$\gamma_s(t) = f(Z_s(x,t) + \nu_s, t+s)\, \exp(-\int_0^s \alpha(t+\lambda))\, d\lambda,$$
$$q_j(t) = k(\xi_j, t+\theta_j) \exp(-\int_0^{\theta_j} \alpha(t+\lambda)d\lambda)\, \mathcal{I}(\theta_j < T-t),$$

because $f \ge 0$ and $T-t \ge T-t'$. Hence, combining (2.11), (2.12), (4.12), (4.34) and the facts that :

$$|1 - \exp(-\int_0^s \alpha(t+\lambda)d\lambda + \int_0^s \alpha(t'+\lambda)d\lambda)| \le C|t-t'| \exp(C|t-t'|),$$
$$\exp(-\int_0^s \alpha(t+\lambda)d\lambda + \int_0^s \alpha(t'+\lambda)d\lambda) \le \exp(C|t-t'|)$$
$$C = 2 \sup \{|\alpha(\lambda)| : 0 \le \lambda < T\},$$
$$|k(\xi,t+\theta) - k(\xi,t'+\theta)| \le \varepsilon\, k(\xi,t+\theta)$$
$$\text{if } |t-t'| < \delta_\varepsilon,$$

we deduce that for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$, independent of ν, x, t and t' such that

(4.37) $$J_{xt'}(\nu) - J_{xt}(\nu) \le \varepsilon(1 + |x|)^p \text{ if } 0 \le t'-t \le \delta_\varepsilon .$$

Next, we assume $t' < t$; then, by virtue of (4.27) and the equivalence of the conditions (4.2), (4.3), we can write

$$u^n(x,t') \le E\{ \int_0^{t-t'} f(Z_s(x,t'),t'+s) \exp(-\int_0^s \alpha(t'+\lambda)d\lambda) + u^n(Z_{t-t'}(x,t'), t) \exp(-\int_0^{t-t'} \alpha(t'+s)ds)\}.$$

Thus, using the fact that u^n satisfies (4.32) and the estimate :

$$E\{|Z_s(x,t) - x|^2\} \le C\, s, \ \forall s > 0$$

for some constant C independent of x,t,s, we deduce that for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$, independent of x,t and t', such that :

$$u^n(x,t') - u^n(x,t) \le \varepsilon(1 + |x|)^p, \text{ if } 0 < t-t' < \delta_\varepsilon.$$

Clearly, this last condition and (4.37) give (4.33). □

Gathering the above results, we can state that the function u^* defined in (4.29),...,(4.31) is the maximum solution of the problem :

$$(4.38) \qquad \begin{cases} \text{find } u \in C_p, \ u(.,T) = 0 \text{ if } T \text{ is finite,} \\ Au \le f \quad \text{in } \mathcal{D}'(\mathcal{O} \times (0,T)), \\ u \le Mu \quad \text{in } \bar{\mathcal{O}} \times [0,T), \end{cases}$$

i.e., û satisfies (4.38) and any other function u with the property (4.38) must verify $u \le u^*$. Moreover, the function u^* satisfies :

$$(4.39) \qquad Au^* = f \text{ in } \mathcal{D}'([u^* < Mu^*]).$$ □

To complete the proof of Theorems 1 and 2, we will show that for any function u^* satisfying (4.38) and (4.39) we have

$$(4.40)\qquad u^*(x,t) = J_{xt}(\hat{\nu}) = \hat{u}(x,t)$$

where $\hat{\nu}$ is the impulse control defined by (2.20),..., (2.25), and $\hat{u}$ is the optimal cost (2.7). Indeed, from the strong Markov's property we get

$$u^*(x,t) \le E\{\int_0^{\theta} f(Z_s(x,t)+\nu_s,t+s)\, \exp(-\int_0^s \alpha(t+\lambda)d\lambda)\, ds$$

$$+ \sum_{j=1}^{\infty} k(\xi_j,t+\theta_j)\, \exp(-\int_0^{\theta_j} \alpha(t+s)ds)\, \mathbb{1}\,(\theta_j < T-t)\}$$

$$+ E\{u^*(Z_\theta(x,t)+\nu_\theta,t+\theta)\, \exp(-\int_0^{\theta} \alpha(t+s)ds)\},$$

for any impulse control ν and $\theta = \theta_n$, $n = 1,2,\ldots$ Since (4.34) implies that

$$\liminf_{s\to\infty} E\{|Z_s(x,t)+\nu_s|^p\, \exp(-\int_0^s \alpha(t+\lambda)d\lambda)\} = 0$$

we deduce from the last inequality that for every ν satisfying (2.3), (2.4) and (4.34) :

$$(4.41)\qquad u^*(x,t) \le J_{xt}(\nu),\ \forall(x,t) \in \bar{\mathcal{O}} \times [0,T).$$

Similarly, for the particular impulse contrôl $\hat{\nu}$ defined by (2.20),..., (2.25) we have

$$u^*(x,t) = E\{\int_0^{\theta_n} f(Z_s(x,t)+\hat{\nu}_s,t+s)\, \exp(-\int_0^s \alpha(t+\lambda)d\lambda)ds$$

$$+ \sum_{j=1}^{n} k(\xi_j,t+\theta_j)\, \exp(-\int_0^{\theta_j} \alpha(t+s)ds)\, \mathbb{1}\,(\theta_j < T-t)\}$$

$$+ E\{u^*(Z_{\theta_n}(x,t)+\nu_{\theta_n},t+\theta_n)\, \exp(-\int_0^{\theta_n} \alpha(t+s)ds)\}.$$

This equality and the growth condition of u^* imply that $\hat{\nu}$ must satisfy (4.34). Hence, as before we obtain :

$$u^*(x,t) = J_{xt}(\nu)\,.$$

Clearly, this last equality and (4.41) provide (4.40). □

Remark : Part of the methods used here are classic, at least when the function f is bounded. We refer in general to Bensoussan and Lions [7, 8], Robin [31] and [21,29]. □

4.3. - Decentralized model

We are going to prove Proposition 1 and Theorems 3, 4 and 5 of § 3.

Recall the definition of convex cone V_p,

(4.42)
$$\begin{cases} h : \mathbb{R} \times [0,T) \to [0,\infty) \text{ belongs to } V_p \text{ iff } h(z,t) \text{ is convex is} \\ \text{the variable } z \text{ for every } t, \text{ there exist constants } C \geq c > 0 \\ \text{such that :} \\ c(z^+)^p - C \leq h(z,t) \leq C(1+|z|^p),\ \forall z \in \mathbb{R},\ t \in [0,T), \\ \text{there exists a constant } K > 0 \text{ such that :} \\ |\partial_z^+ h(z,t)| \leq K(1+|z|^{p-1}),\ \forall z \in \mathbb{R},\ t \in [0,T), \\ \text{and for every } \varepsilon > 0 \text{ there is } \delta_\varepsilon > 0 \text{ such that :} \\ |h(z,t)-h(z,t')| \leq \varepsilon(1+|z|^p), \text{ if } |t-t'| \leq \delta_\varepsilon, \\ \text{for any } z \text{ in } \mathbb{R},\ t,t' \text{ in } [0,T), \end{cases}$$

where $\partial_z^+ h$ denotes the partial derivative of h with respect to z from the right. Denote by $u_\varepsilon(z,t)$ the following optimal cost :

(4.43)
$$\begin{cases} u_\varepsilon(z,t) = \inf\{J_{zt}(v) : v \in \mathcal{V}_\varepsilon\},\ \varepsilon > 0, \\ J_{zt}(v) = E\{\int_0^{T-t} [h(z+q(s,t)+v_s,t+s) + c(t+s)\dot{v}_s] \\ \qquad\qquad \times \exp(-\int_0^s \alpha(t+\lambda)d\lambda)\, ds\}, \\ \mathcal{V}_\varepsilon = \{v \in \mathcal{V} : v_s = \int_0^s \dot{v}_\lambda\, d\lambda, 0 \leq \varepsilon\dot{v}_s \leq 1,\ \forall s\}, \end{cases}$$

where $q(s,t)$ is given by (3.2), $\mathcal{V}$ is the set of all right-continuous stochastic processes, increasing, non negative and adapted, and h,c are given functions, h in V_p and $c(.)$ with the property (3.8). We will prove that $(u_\varepsilon, \varepsilon > 0)$ belongs to V_p uniformly in $\varepsilon > 0$, i.e. u_ε satisfies (4.42) uniformly in $\varepsilon > 0$. Indeed, to see that $u_\varepsilon(z,t)$ is convex in the variable in the variable z we notice that :

$$(z,v) \to J_{zt}(v), \text{ t fixed in } [0,T)$$

is a convex function from $\mathbb{R} \times \mathcal{V}_\varepsilon$ into $\mathbb{R}$. Next, the growth condition on h and c allows us to consider only controls v in $\mathcal{V}_\varepsilon$ satisfying also :

$$\text{(4.44)} \qquad E\{\int_0^{T-t} [m(v_s,\lambda_0) + c_0 \dot{v}_s] \exp(-\int_0^s \alpha(t+\lambda)d\lambda)\, ds \leq C\, m(z,\lambda_0),$$

for some appropriate constants C, c_0 ($c_0 > 0$ if $c(.) \neq 0$) independent of z,t,v, and where $m(z,\lambda_0)$ is the function (4.9), (4.10). At the same time, we get two other constants $C \geq c > 0$ such that :

$$c(z^+)^p - C \leq u_\varepsilon(z,t) \leq C(1+|z|^p), \ \forall z \in \mathbb{R},\ t \in [0,T),\ \varepsilon > 0.$$

Let us estimate the difference :

$$|u_\varepsilon(z,t)-u_\varepsilon(z',t)| \leq \sup\{|J_{zt}(v)-J_{z't}(v)| : v \in \mathcal{V}_\varepsilon,$$
$$v \text{ satisfying (4.44)}\}.$$

We have :

$$|J_{zt}(v)-J_{z't}(v)| \leq K\,|z-z'|$$
$$\times E\{\int_0^{T-t} (1+|z+q(s,t)+v_s|+|z'+q(s,t)+v_s|)^{p-1}$$
$$\exp(-\int_0^s \alpha\,(t+\lambda)d\lambda)ds\}.$$

Hence, by using Hölder inequality and (4.44) we obtain :

$$(4.45)\quad \begin{cases} |u_\varepsilon(z,t)-u_\varepsilon(z',t)| \le K\ |z-z'|(1+|z|+|z'|)^{p-1} \\ \forall z,z' \in \mathbb{R},\ t \in [0,T),\ \varepsilon > 0, \end{cases}$$

for some constant K independent of z,z',t and ε. Now, we suppose $0 \le t \le t' < T$, z in $\mathbb{R}$, v in $\mathcal{V}_\varepsilon$ satisfying (4.44). We have :

$$J_{zt'}(v)-J_{xt}(v) \le E\{\int_0^{T-t'} (\gamma_s(t')-\gamma_s(t))ds\},$$

$$\gamma_s(t) = [h(z+q(s,t)+v_s,t+s) + c(t+s)\dot{v}_s] \exp(-\int_0^s \alpha(t+\lambda)d\lambda),$$

because $h \ge 0$, $c(.) \ge 0$, $\dot{v}_s \ge 0$ and $T-t' \le T-t$. Since h belongs to V_p and v satisfies (4.44), we can deduce that for every $\varepsilon' > 0$ there exists $\delta_{\varepsilon'} > 0$ such that :

$$u_\varepsilon(z,t') - u_\varepsilon(z,t) \le \varepsilon'(1+|z|^p),\ \text{if } 0 \le t'-t \le \delta_{\varepsilon'},$$

for every z in $\mathbb{R}$, $\varepsilon > 0$. On the other hand, from the definition (4.43) and the Markov's property, we get for $0 \le t' < t < T$:

$$u_\varepsilon(z,t') \le E\{\int_0^{t-t'} h(z+q(s,t'),t'+s)\ \exp(-\int_0^s \alpha(t'+\lambda)d\lambda)\ ds$$

$$+ u_\varepsilon(z+q(t-t',t),t)\ \exp(-\int_0^{t-t'} \alpha(t'+s)ds)\},$$

which together which (4.45) show that for every $\varepsilon' > 0$ there exists $\delta_{\varepsilon'} > 0$ such that :

$$u_\varepsilon(z,t')-u_\varepsilon(z,t) \le \varepsilon'(1+|z|^p),\ \text{if } 0 \le t-t' \le \delta_{\varepsilon'}.$$

Hence, the functions $(u_\varepsilon,\ \varepsilon > 0)$ satisfy (4.42) uniformly in $\varepsilon > 0$. □

Approximating $\sigma(.)$ by a sequence of $\sigma_\eta(.)$ satisfying :

$$\sigma_\eta^2(t) \ge \eta > 0,\ \forall t \in [0,T)$$

we can use classic results about stochastic control problems (cfr. Fleming and Rishel [16]). Since the a priori estimates on $(u_\varepsilon, \varepsilon > 0)$ are uniformly in $\eta > 0$, we deduce that u_ε satisfies the Hamilton-Jacobi-Bellman equation associate with (4.43), namely :

(4.46) $$\begin{cases} u_\varepsilon \in V_p,\ u_\varepsilon(.,T) = 0 \text{ if } T \text{ is finite,} \\ A_z u_\varepsilon + \frac{1}{\varepsilon} (Bu_\varepsilon)^+ = h \text{ in } \mathcal{D}' \ (\mathbb{R} \times (0,T)), \end{cases}$$

where A_z is the operator (3.18) and :

(4.47) $$Bu(z,t) = -c(t)\ \partial_z^+ u(z,t).$$ □

Let $\hat{u}$ be the following optimal cost :

(4.48) $$\hat{u}(z,t) = \inf \{J_{zt}(v) : v \in \mathcal{V}\},$$

where $J_{zt}(v)$ and $\mathcal{V}$ are defined in (4.43). We will prove that :

(4.49) $$\begin{cases} u_\varepsilon \to \hat{u} \text{ as } \varepsilon \to 0, \text{ uniformly over bounded subsets of } \mathbb{R} \times [0,T), \\ \text{and } \hat{u} \text{ belongs to } V_p. \end{cases}$$

Indeed, in view of the a priori estimates on $(u_\varepsilon, \varepsilon > 0)$, we need only to show that u_ε converges to $\hat{u}$ pointwise in $\mathbb{R} \times [0,T)$. To this effect, we notice that the set

$$\mathcal{V}_o = \cup \{\mathcal{V}_\varepsilon : \varepsilon > 0\}$$

is dense in $\mathcal{V}$ as subsets of the Lebesgue's space $L^p((0,T) \times \Omega)$. Next, the fact that

$$v \to J_{xt}(v)$$

is a lower semicontinuous functional allows us to obtain our claim. □

Let M be the operator

$$(4.50)\qquad Mu(z,t) = \inf \{c(t)\zeta + u(z+\zeta,t) \;:\; \forall\zeta \geq 0\}$$

We will show that the optimal cost $\hat{u}$ defined by (4.48) is the maximum solution of the following problem :

$$(4.51)\qquad \begin{cases} \text{find } u \in V_p,\ u(\cdot,T) = 0 \text{ if } T \text{ is finite,} \\ Au \leq h \text{ in } \mathcal{D}'(\mathbb{R} \times (0,T)), \\ u \leq Mu \text{ in } \mathbb{R} \times [0,T), \end{cases}$$

i.e., $\hat{u}$ satisfies (4.51) and any other function u enjoying the property (4.51) must verify $u \leq \hat{u}$. Indeed, notice that for functions u in V_p we have

$$u \leq Mu \text{ equivalent to } Bu \leq 0.$$

Let v be any control in $\mathcal{V}_\varepsilon$, $\varepsilon > 0$. Then, applying Itô's formula to a sequence of smooth functions approximating u and the process

$$s \to (z+q(s,t) + v_s,\ t+s)$$

we obtain

$$u(z,t) \leq J_{zt}(v),\ \forall v \in \mathcal{V}_\varepsilon,\ \varepsilon > 0,$$

for any function u satisfying (4.51). These facts, together with the properties (4.46) and (4.49) permit us to prove that the optimal cost (4.49) is the maximum solution of (4.51). Notice that, we have

$$(4.52)\qquad A\hat{u} = h \quad \text{in } \mathcal{D}'([B\hat{u} > 0]),$$

provided $[B\hat{u} > 0]$ is an open set in $\mathbb{R} \times [0,T)$. □

Define

$$(4.53)\qquad z^*(t) = \inf\{z \in \mathbb{R} : \partial_z^+ \hat{u}(z,t) + c(t) > 0\},\ t \in [0,T).$$

Assume that the function $z^*(.)$ is uniformly continuous and bounded from above. Then, from (4.52) and convexity of the function $\hat{u}(.,t)$ we deduce :

$$(4.54)\qquad \begin{cases} A\hat{u}(z,t) = h(z,t) \text{ if } z \geq z^*(t),\ t \in [0,T), \\ \partial^+\hat{u}(z,t) = -c(t) \text{ if } z \leq z^*(t),\ t \in [0,T), \end{cases}$$

as soon as $A\hat{u}(z,t)$ is pointwise meaningfull. Clearly, this is the case when $\sigma(.)$ never vanishes. □

In order to get an optimal policy, we need a stochastic process $(\hat{v}_s, s \geq 0)$ in $\mathcal{V}$ with the following properties :

$$(4.55)\qquad \begin{cases} z + q(s,t) + \hat{v}_s \geq z^*(s),\ \forall s \in [0,T-t), \\ (\hat{v}_s,\ s \geq 0) \text{ belongs to } \mathcal{V},\ \hat{v}_o = (z(o)-z)^+ \text{ and :} \\ \chi_\varphi(s) = \varphi(z+q(s,t)+\hat{v}_s,t+s) - \varphi(z+\hat{v}_o,tT) + \int_o^s A_o\varphi(z+q(\lambda,t)+\hat{v}_\lambda,t+\lambda)d\lambda \\ \qquad - \int_o^s \partial_z \varphi(z^*(\lambda),t+\lambda)d\hat{v}_\lambda,\ s \in [0,T-t), \\ \text{is a martingale for any smooth function } \varphi, \end{cases}$$

where A_o denotes the operator A with $\alpha(.) = 0$. Indeed, let us assume the existence of such a process $\hat{v}_s$. Then, applying (4.54) to a sequence of smooth functions converging to $\hat{u}$, we obtain

$$\hat{u}(z+\hat{v}_o,t) = E\{\int_{\hat{o}}^{t'} h(z+q(s,t)+\hat{v}_s,t+s) \exp(-\int_o^s \alpha(t+\lambda)d\lambda)\, ds$$
$$+ \int_o^{t'} c(t+s)d\hat{v}_s\} + E\{\varphi(z+q(t',t)+\hat{v}_{t'},t+t')$$
$$\exp(-\int_o^{t'} \alpha(t+s)ds)\},$$

and by letting t' go to T-t we get

$$\hat{u}(z,t) = J_{zt}(\hat{v}),$$

i.e. $\hat{v}$ is an optimal control. The formulae

$$(4.56) \qquad \begin{cases} \hat{v}_s = \max\{[\hat{v}_o + q(\lambda,t) + z - z^*(\lambda)]^- : 0 \le \lambda \le s\}, \\ \hat{v}_o = (z - z^*(o))^-, \end{cases}$$

provides the process $(\hat{v}_s,\ s \ge 0)$ satisfying (4.55). When no integral term is involved, i.e. $\gamma(.,.) = 0$ this fact corresponds to the existence of a reflected diffusion process. If $\sigma(.)$ and $b(.)$ are constants, this is a reflected Brownian motion. For the case of an integral term $\gamma(.,.) \neq 0$, we obtain the process $\hat{v}$ as a particular case of the results in [26] and the reference therein. □

At this point, we have proved Proposition 1 and its complement (3.26), (3.27). To show Theorems 3 and 4, we need only to prove :

$$(4.57) \qquad \hat{u}(x_1,\dots,x_d,t) \ge \hat{u}_1(x_1,t) + \dots + \hat{u}_d(x_d,t),$$

$$(4.58) \qquad \hat{u}_1(x_1,t) + \dots + \hat{u}_d(x_d,t) = J_{xt}(\hat{\eta}),$$

with the notation of § 3.2 and § 3.3. Indeed, let η be an impulse control satisfying (3.3) and (3.4). By definition of $\hat{u}_d(x_d,t)$ we have

$$\begin{aligned} \hat{u}_d(x_d,t) &\le J^d_{x_d t}(\eta^d) \\ &= E\{\int_o^T f_d(x_d+q(s,t)+\eta^d_s,t+s)\ \exp(-\int_o^s \alpha(t+\lambda)d\lambda)ds \\ &\quad + c_d(t)\eta^d_o + \int_o^T c_d(t+s)\ \exp(-\int_o^s \alpha(t+\lambda)d\lambda)\ d\eta^d_s\} + I, \end{aligned}$$

with

$$I = E\{\int_o^T (h_{d-1} - A_z\hat{u}_{d-1})(x_d+q(s,t)+\eta^d_s,t+s)\ \exp(-\int_o^s \alpha(t+\lambda)d\lambda)\ ds\}.$$

Since the function

$$z \to h_{d-1}(z) - A_z\hat{u}_{d-1}(z)$$

is decreasing and $\eta = (\eta^1,\ldots,\eta^d)$ satisfies (3.3), we obtain

$$I \le E\{\int_0^T (h_{d-1}-A_z\hat{u}_{d-1})(x_{d-1}+q(s,t)+\eta_s^{d-1},t+s)\ \exp(-\int_0^s \alpha(t+\lambda)d\lambda)\ ds$$

$$= E\{\int_0^T h_{d-1}(x_{d-1}+q(s,t)+\eta_s^{d-1},t+s)\ \exp(-\int_0^s \alpha(t+\lambda)d\lambda)\ ds$$

$$+ \int_0^T c_d(t+s)\ \exp(-\int_0^s \alpha(t+\lambda)d\lambda)d\eta_s^{d-1}\} - E\{\hat{u}_{d-1}(x_{d-1}+\eta_o^{d-1},t)\}.$$

Because

$$E\{\hat{u}_{d-1}(x_{d-1}+\eta_o^{d-1},t)\} \ge \hat{u}_{d-1}(x_{d-1}) - E\{c_{d-1}(t)\eta_o^{d-1}\}$$

we can write

$$u_{d-1}(x_{d-1},t)+u_d(x_d,t) \le \sum_{i=d-1}^{d} E\{\int_0^T f_i(x_i+q(s,t)+\eta_s^i,t+s)$$

$$\times \exp(-\int_0^s \alpha(t+\lambda)d\lambda)ds + c_i(t)\eta_o^i$$

$$+ \int_0^T c_i(t+s)\ \exp(-\int_0^s \alpha(t+\lambda)d\lambda)ds\}$$

$$+ E\{\int_0^T (h_{d-2}-A_z\hat{u}_{d-2})(x_{d-1}+q(s,t)+\eta_s^{d-1},t+s)$$

$$\times \exp(-\int_0^s \alpha(t+\lambda)d\lambda)ds\}.$$

Hence, by induction we deduce

$$u_1(x_1,t) + \ldots + u_d(x_d,t) \le J_{xt}(\eta),\ \forall\eta\ ,$$

which implies (4.57). To get (4.58) we notice that we have all equalities in the above induction when $\eta = \hat{\eta}$ given by (3.28). □

The last part is to prove Theorem 5. To this effect,we proceed as in § 4.2. Estimate of the form (4.32), (4.33) are valid for the functions involved in the convergence (3.31), i.e., the optimal costs $\hat{u}_\varepsilon$ associated to k_ε belong to the space C_p, uniformly in $\varepsilon > 0$. From this fact we conclude. □

Remarks : More general version of these problem are treated in Chow and al. [11], Menaldi and Robin [24, 25]. Results involving Theorem 5 for bounded data have been studied in Menaldi, Quadrat and Rofman [23] and [28] under different context. Most of the results presented in this paper, have been anounced in [29]. Particular situations, i.e. time-independent case without Poisson fluctuation were considered in [27]. □

5. OTHER MODELS AND NUMERICAL TECHNIQUES

The variational approach used in §2 for the characterization of the optimal value function can be applied in more complex models. For example, if a purchasing policy includes continuous and impulsive orders, with different unit costs and reorder cost for the impulsive part, we can formulate the QVI relative to these new assumptions.

It is also possible to consider installations with finite maximum capacities. In fact, after introducing stopping time controls the problem can be modelized by means of suitable QVI taking into account this capacity upper bound .

A final obvious remark is concerning deterministic demand. In this case we will find first order differential operators in the definitions (2.14), (2.15) allowing a simplified discussion of the problem. We will obtain the same essential resuts of §2.

Let us give some information about numerical solutions. Centralized solutions for deterministic models can be obtained using the procedure recently presented in Gonzalez and Rofman [18]. The method lean upon the characterization of the optimal value function $\hat{u}(x,t)$ as the maximum solution of a deterministic QVI similar to (2.18). Applications to energy production systems with state x belonging to $\mathbb{R}^1$, $\mathbb{R}^2$ and $\mathbb{R}^3$ can be found in Bancora-Imbert et al. [2,3], [32] and Gonzalez and Rofman [19]. First numerical examples of deterministic hierarchical inventory systems can be seen in [36].

On the other hand we know that decentralized solutions of the problem presented in §3 are easily computed. If we consider this model giving the asymptotic behaviour of those described in §2 (as it was shown in Theorem 5) its solution may give us useful information to obtain approximate solutions of short run models of type (2.1) - (2.5).

In a forthcoming paper we will discuss short run arborescent models. We intend divide them in groups of activities for which centralized policies may be computed.

6. NUMERICAL EXAMPLES

This paragraph proposes some examples of the model developed in this paper. In fact we consider the deterministic stationnary case, and apply the algorithm developed in [18], to solve the sequence of non linear fixed point problems related to the optimization of serial two-level (respectively three-level) inventory systems. As a consequence of the results : short computation time and reduced memory place, we intend to solve more complex systems.

In the following we will suppose the order cost k linear.

Case 1 : Two-Level System

u(x) is the maximum element of the set W with :

$$W = \{w : \theta \to \mathbb{R} \ / \ (1.1),\ (1.2)\}$$
$$\theta = \{(X^1,X^2) \in \mathbb{R}^2,\ X^1 \leq X^2\}.$$

(1.1) $\quad Au(X) \leq f(X) \qquad X = (X^1,X^2)$

(1.2) $\quad u(X) \leq M_0u(X)$

with

$$Au(X) = \alpha u(X) + b\left(\frac{\partial u}{\partial X^1} + \frac{\partial u}{\partial X^2}\right)$$

$$M_0u(X) = \inf_{X^2+\xi^1 \leq X^2} \left[u(X^1+\xi^1, X^2+\xi^2) + C_1^1\, 1_{\xi^1>0} + C_1^2\, \xi^1 + C_2^1\, 1_{\xi^2>0} + C_2^2\, \xi^2\right]$$

$C_i^1 + C_i^2\xi^i$: comand-cost of quantity ξ^i by level i $\quad (i = 1,2)$.

In fact, we can show that, without loss in generality we can consider the constant cost only. That, we have to solve the system :

(1.3) $$\alpha u(X) + b\left(\frac{\partial u}{\partial X^1} + \frac{\partial u}{\partial X^2}\right) \leq f(X)$$

(1.4) $$u(X) \leq \inf_{\substack{X^1+\xi^1 \leq X^2 \\ \xi^i \geq 0 \\ (\xi^1,\xi^2)\neq(0,0)}} [C_1^1 1_{\xi^1\geq 0} + C_2^1 1_{\xi^2 \geq 0} + u(X^1 + \xi^1, X^2 + \xi^2)]$$

Discretized problem

a - The set Ω is approximated with a triangulation Ω^h, union of simplices of vertices (x_p^1, x_q^2), $p = 0 \to N_{X^1}$ $q = 0 \to N_{X^2}$.

b - Let Z be the set of admissible, impulse controls, and Z^h the discrete approximation :

$$Z^h = \{(n_1h_1, n_2h_2) \in Z, (n_1,n_2) \in N^2\}.$$

c - In the set of linear finite elements W^h defined in Ω^h we consider the set W^h :

$$W^h = \{w^h : \Omega^h \to \mathbb{R} / (1.5), (1.6)\}$$

(1.5) $$\frac{\partial w^h}{\partial x_b} (S_i^h).\vec{b} + f(S_i^h) - \alpha w^h(S_i^h) \geq 0$$

where $\frac{\partial w^h}{\partial x_b}(S_i^h).\vec{b}$ is the derivative of w^h in direction of the vector $\vec{b} = (-b, -b)$

$$\frac{\partial w^h}{\partial x_b}(S_i^h).\vec{b} = \frac{w^h(a_i^h) - w^h(S_i^h)}{\|a_i^h - S_i^h\|} \|\vec{b}\|$$

where

$$S_i^h(i_1h_1, i_1h_1 + i_2h_2) \qquad i_1 \in Z$$

$$a_i^h((i_1-1)h_1, (i_1-1)h_2 + i_2h_2) \qquad i_2 \in \mathbb{N}$$

$$\|a_i^h - S_i^h\| = \sqrt{2}\, h_1.$$

(1.5) becomes :

$$(1.5)' \qquad \frac{w^h(a_i^h) - w^h(S_i^h)}{h_1} \,.\, b + f(S_i^h) - \alpha w^h(S_i^h) \geqq 0$$

$$(1.6) \qquad w^h(S_i^h) \leqq \inf_{(n_1,n_2)\in D_i^h} \{C_1^1\, 1_{n_1>0} + C_2^1\, 1_{n_2>0} + w^h(Y_i^h)\}$$

for each S_i^h in Ω^h.

Where :

$$D_i^h = \{(n_1,n_2) \in N^2,\ (n_1,n_2) \neq (0,0) \text{ and } /\ n_1h_1 \leqq i_2h_2\}$$

Y_i^h : stock's level after an order $(n_1h_1,\ n_2h_2)$

and if we note

$$S_{i_1}^h = ((i_1+n_1)h_1,\ (i_1+n_1)h_1 + (i_2+n_2)h_2 - h_2\ E(n_1\frac{h_1}{h_2}))$$

$$S_{i_2}^h = ((i_1+n_1)h_1,\ (i_1+n_1)h_1 + (i_2+n_2)h_2 - h_2[1 + E(n_1\ \frac{h_1}{h_2})])$$

two vertices of Ω^h , E(X) being the integer part of X, and if we consider

$$\lambda_1 = 1 - (n_1\ \frac{h_1}{h_2} - E(n_1\ \frac{h_1}{h_2}))$$

$$\lambda_2 = n_1\ \frac{h_1}{h_2} - E(n_1\ \frac{h_1}{h_2}).$$

We can write :

$$(1.7) \qquad Y_i^h = \lambda_1\ S_{i_1}^h + \lambda_2\ S_{i_2}^h.$$

Finally, using (1.7), we rewrite (1.5)', (1.6) :

$$(1.8) \qquad w^h(S_i^h) \leqq \frac{1}{b+\alpha h_1}\ [w^h(a_i^h).b + h_1\ f(S_i^h)]$$

$$(1.9) \qquad w^h(S_i^h) \leqq \min_{(n_1,n_2)\in D_i^h} [C_1^1\, 1_{n_1\geqq 0} + C_2^1\, 1_{n_2\geqq 0} + \lambda_1\ w^h(S_{i_1}^h) + \lambda_2\ w^h(S_{i_2}^h)]$$

In the case of $h_1 = h_2 = h$, we have :

$$(1.10) \qquad w^h(i_1h,\ i_2h) \leq \frac{b\, w^h((i_1-1)h,\ (i_2-1)h) + hf(i_1h,\ i_2h)}{\alpha h + b}$$

$$(1.11) \qquad w^h(i_1h,\ i_2h) \leq \min_{\substack{(n_1,n_2)\in \mathbb{N}^{2*} \\ (i_1+n_1 \leq i_2)}} [C_1^1\, 1_{n_1\geq 0} + C_2^1\, 1_{n_2\geq 0} + w^h(i_1+n_1)h,(i_2+n_2)h)],$$

The discretized problem is to find the maximum element w^h of the set W^h.

On the other hand, if we define the operator M by :

$$\frac{1}{b+\alpha h_1}(w(i_1-1,i_2).b + h_1 f(i_1,i_2)) \ ;$$

$$Mw(i_1,i_2) = \inf_{(n_1,n_2)\in D_i^h} (C_1^1\, 1_{n_1>0} + C_2^1\, 1_{n_2>0} + \lambda_1 w(i_1+n_1,i_2+n_2-E(n_1\frac{h_1}{h_2}) + \lambda_2 w(i_1+n_1,i_2+n_2-E(n_1\frac{h_1}{h_2})-1))$$

It is shown [18] that w^h is the solution of the non linear fixed point problem :

$$Mw = w.$$

The resolve of the model gives us, for each stock's level (X^1,X^2), the optimal cost $w(X^1,X^2)$ and the level's orders (n_1,n_2).

These results give the optimal policy of stocks management. This is meaning that, at each stock's level, $X = (X^1,X^2)$, we know the optimal decision to take : to let the system freely run, or order at some level.

<u>Example 1</u>

The stocking cost (x_1,x_2) is :

a - $x_1 \geq 3\ h$

$$f(x_1,x_2) = 2x_1 + 3x_2 - 20\ h$$

$x_2 \geq y_1$

b - $x_1 \geq 3h$

$$f(x_1,x_2) = 4.57x_2 + .25x_2 - 9\ h$$

$x_2 \leq y_1$

c - $0 \leqq x_1 \leqq 3h$

$$f(x_1,x_2) = -2.5x_1 + 3x_2 - 6{,}5\ h$$

$x_2 \geqq y_1$

d - $0 \leqq x_1 \leqq 3\ h$

$$f(x_1,x_2) = .25(x_1+x_2) + 4{,}5\ h$$

$x_2 \leqq y_1$

e - $x_1 \leqq 0$

$$f(x_1,x_2) = -\ 8x_1 + 3x_2 - 6{,}5\ h$$

$x_2 \geqq y_1$

f - $x_1 \leqq 0$

$$f(x_1,x_2) = -\ 5.25x_1 + .25x_2 + 4.5\ h$$

$x_2 \leqq y_1$

where : $y_1 = x_1 + 4\ h.$
$h = 100$
$b = 10$
$C_1^1 = 3$
$C_2^1 = 6.$

For the level 1 we consider 15 discretization points, (9 points are in rupture region), while the level 2 is divided in 8 intervals.

The time of computation for the whole optimization was 14" on DPS 8/70 Multics computer.

Figure 2 shows a simulation of system's optimal policy, with initial stock : $(X_0^1,\ X_0^2) = (200,\ 800)$. We deduce from results that level 2 is not necessary and level 1 could order from exterior directly.

Example 2

In the following examples, we will illustrate the system's sensibility to rate $C_1^1\ /\ C_2^1$.

a - Stocking cost is :

. $x_1 \geqq 0$ $(x_1,x_2) = .5(x_1+x_2)$

.. $x_1 \leqq 0$ $f(x_1,x_2) - 25x_1 + .5x_2$

. $h = 100$

. $b = 1$

. $C_1^1 = 100$

. $C_2^1 = 200$

Figure 3 shows a simulation of systems optimal policy, with initial stock : $(X_0^1, X_0^2) = (0,600)$.

b - Data are the same that is in (a), except for C_1^1, C_2^1 :

. $C_1^1 = 100$

.. $C_2^1 = 300$

Figure 4 shows a simulation of systems optimal policy, with initial stock : $(X_0^1, X_0^2) = (0,600)$.

We deduce that as long as C_2^1/C_1^1 is relatively small, it is intersting for level 1 to order directly from exterior, and if C_2^1/C_1^1 increases, intermediary stock becomes necessary.

See [Figure 3, Figure 4].

<u>Case 2 : Three level-system</u>

We introduce in the inventory system of previous case a third level. This time, the set Ω is aproximated with union of tetrahedrons $\Omega^{h'}$ (= Ω). The discretized system is :

$$w^h(S_i^h) \leq \frac{h\ f(S_i^h) + b\ w^h(a_i^h)}{\alpha h + b}$$

$$w^h(S_i^h) \leq C_1^1\ 1_{n_1 \geq 0} + C_2^1\ 1_{n_2 \geq 0} + C_3^1{}_{n_3 \geq 0} + w^h((S_i^h)')$$

with :

$$a_i^h = ((i_1-1)h,\ ((i_1-1)+i_2)h,\ (i_1+i_2+i_3-1)h)$$

$$(S_i^h)' = ((i_1+n_1)h,\ (i_1+i_2+n_2)h,\ (i_1+i_2+i_3+n_3)h).$$

$(S_i^h)'$ is the level of stocks after an order.

In fact :

$$(S_i^h)' = (i_1+n_1)h,\ (i_1+n_2)h + (i_2+n_2-n_1)h,\ (i_1+i_2+n_2)h + (i_3-n_3)h)$$

is also a vertex of Ω^h because :

$$n_1 \leqq i_2$$
$$n_3 \leqq i_3.$$

<u>Example 1</u>

Stocking cost is :

. $x_1 \geqq 1$ $\qquad f(x) = .5(x_1+x_2+x_3)$

.. $x_1 \leqq 1$ $\qquad f(x) = -25x_1 + .5x_2 + .5x_3 + 25.5$

. $h = 100$

. $b = 1$

. $C_1^1 = 1$

. $C_2^1 = 2$

. $C_3^1 = 3$

For level 1 we consider 10 discretization points (4 in the rupture region), while the level 2 was divided in 9 intervals and level 3 in 19 intervals.

The results go out after 19 iterations. We deduce from them that intermediate levels are not necessary and level 1 could order to exterior directly. Figure 5 shows simulation of system's optimal policy, with initial state $(X_0^1, X_0^2, X_0^3) = (0,0,100)$.

Example 2

We keep the same data except for :

$C_1^1 = 100$

$C_2^1 = 200$

$C_3^1 = 300$

In this example level 3 is not necessary while level 1 orders two times faster than level 2.

The result go out after 21 iterations. Figure 6 shows or simulation of system's optimal policy, with initial state : $(X_0^1, X_0^2, X_0^3) = (0,0,0)$.

Example 3

We increase C_3^1 : $C_3^1 = 600$. In this case level 2 is not necessary (figure 7).

Example 4

With the same data the previous example except for the stocking cost :

. $x_1 \geqq 1$ $\quad f(x_1,x_2,x_3) = .1x_1 + .07x_2 + .06x_3 + 9$

. $x_1 \leqq 1$ $\quad f(x_1,x_2,x_3) = - 24.94x_1 + .07x_2 + .06x_3 + 34.5$

In this case, the time of computation for the whole optimization was 14'.

Simulation of the system shows that level 3 orders 2 times less than level 2, which orders two times less than level 1. Figure 8 shows a simulation of the system.

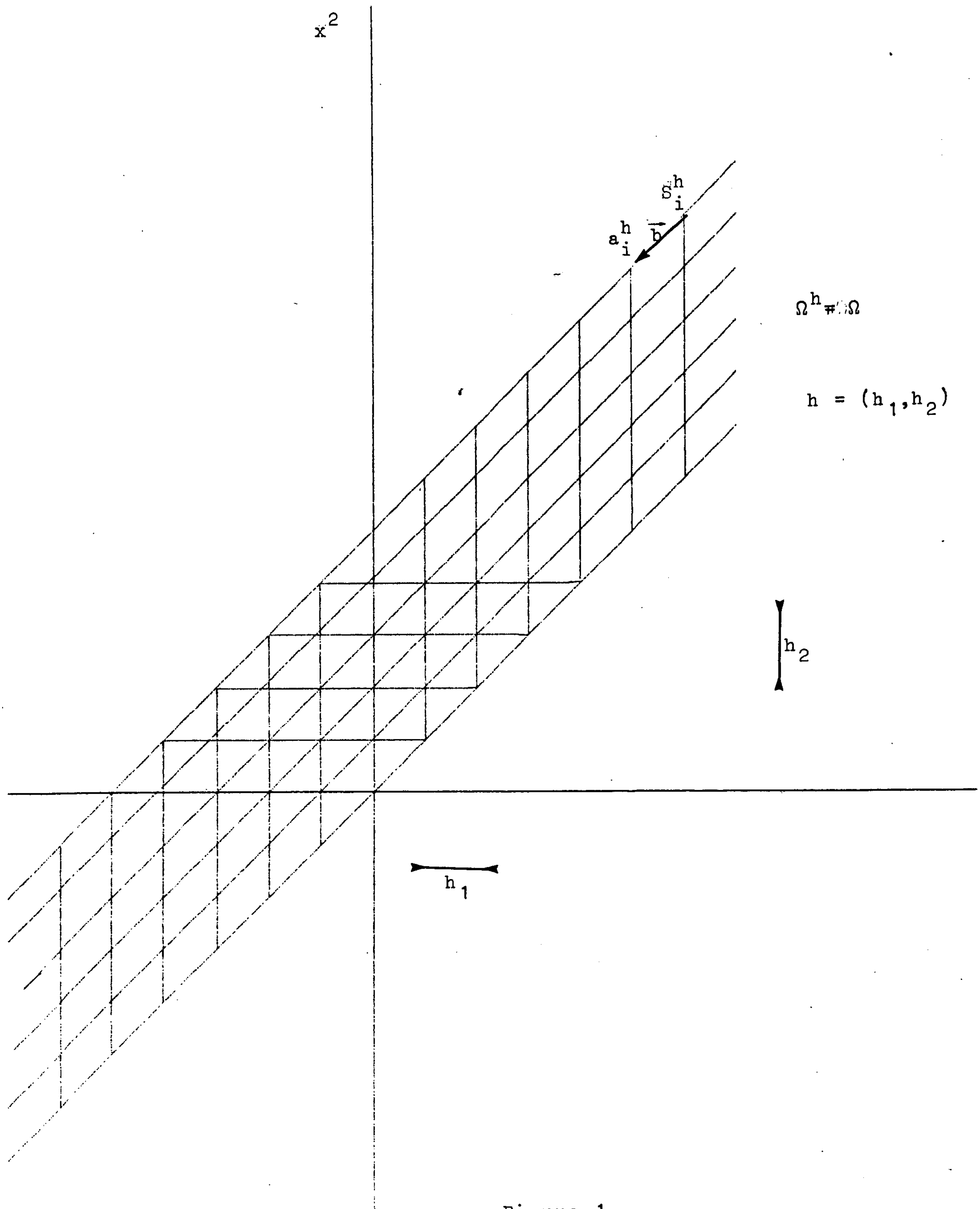


Figure 1

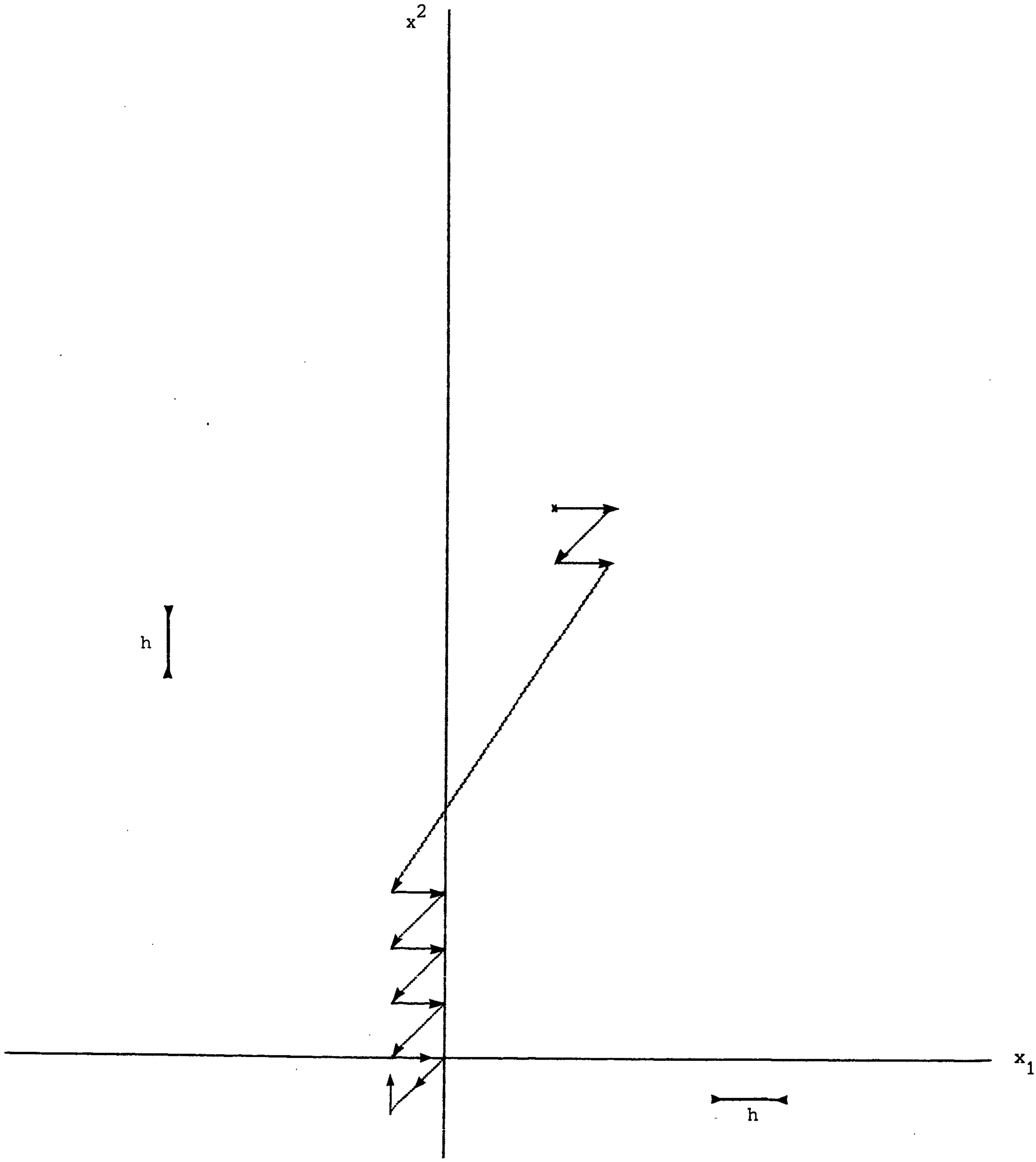


Figure 2.

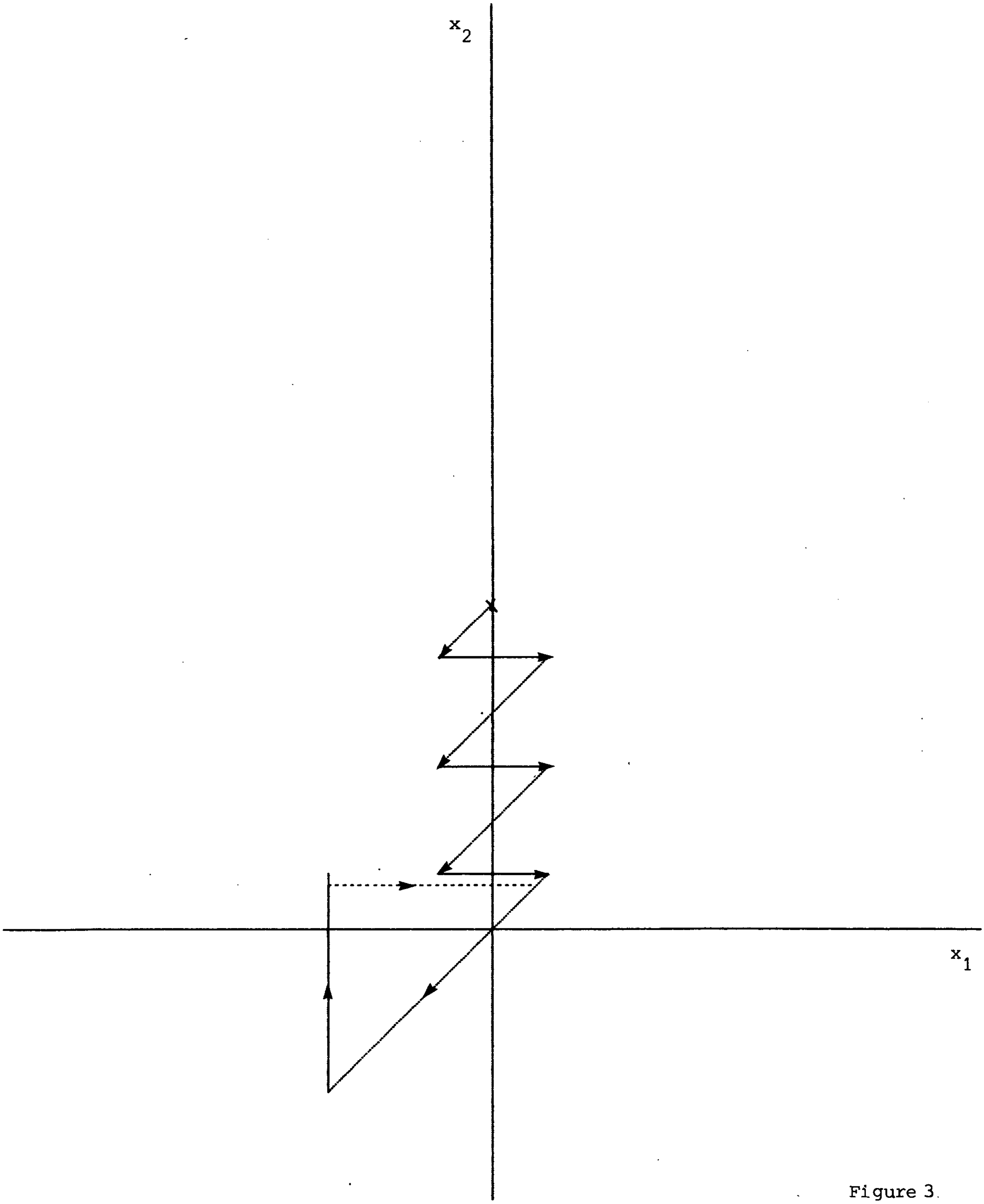


Figure 3

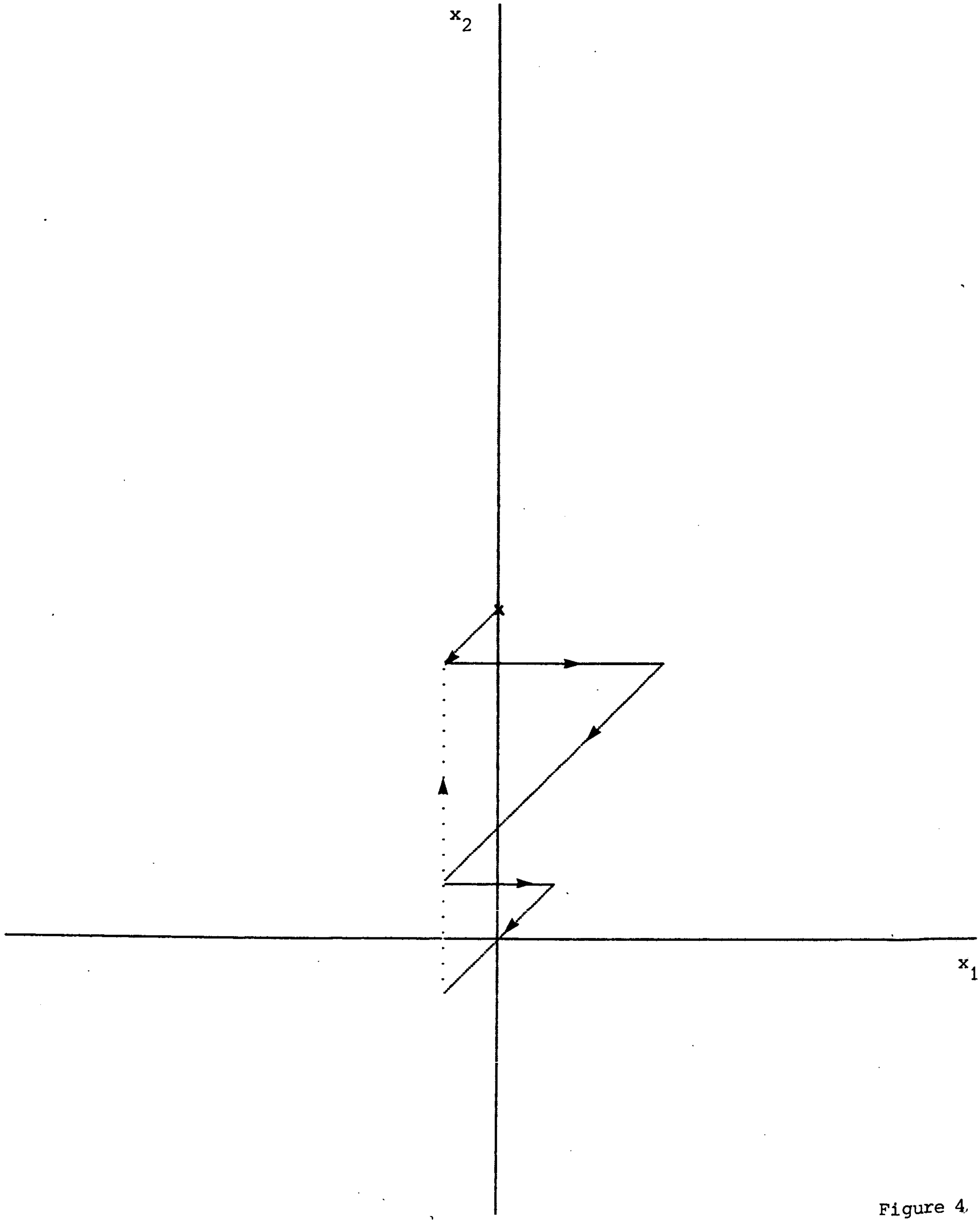


Figure 4

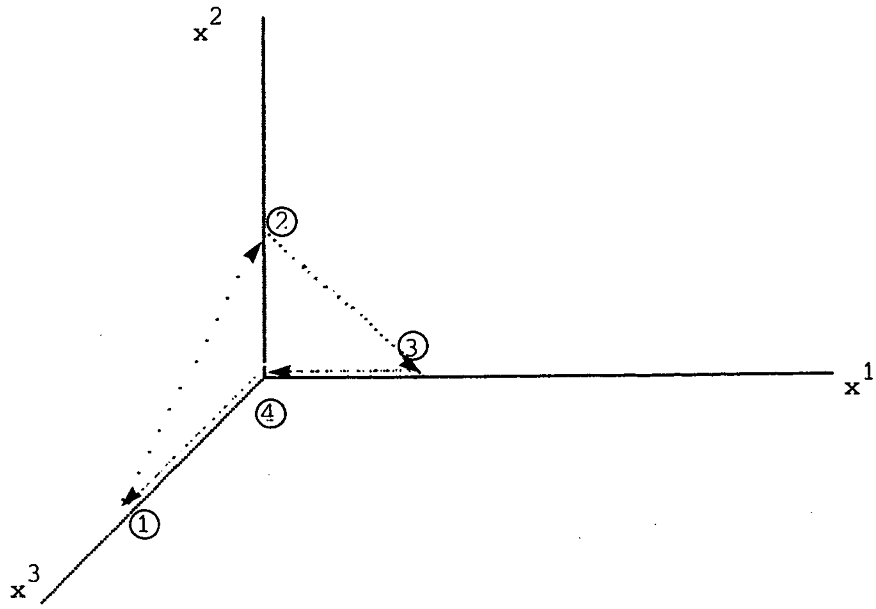


Figure 5

1 (0,0,0)

2 (0,0,300)

3 (0,300,0)

4 (200,100,0)

5 (100,100,0)

6 (0,100,0)

7 (100,0,0)

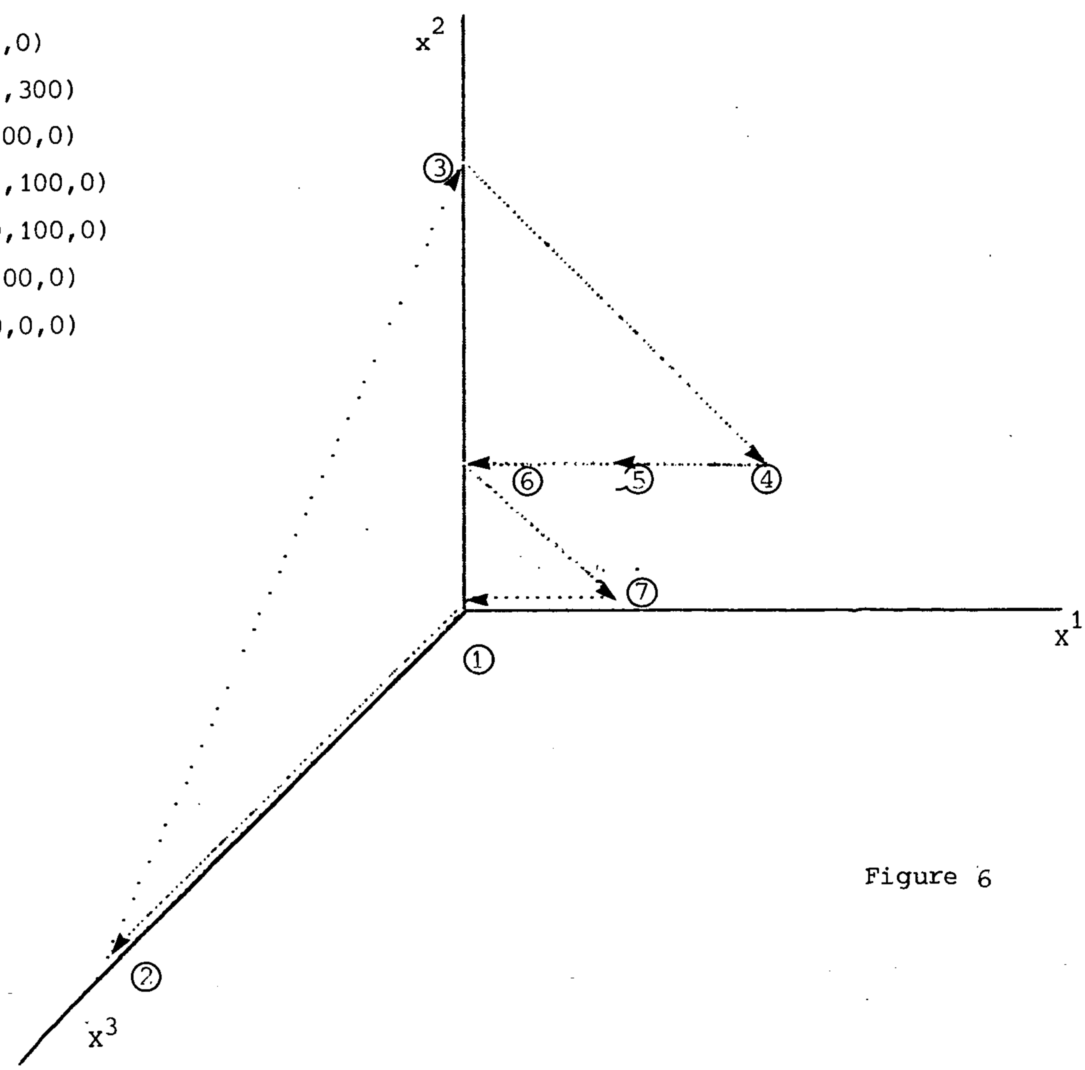


Figure 6

1 (0,0,0)
2 (0,0,400)
3 (0,200,200)
4 (200,0,200)
5 (100,0,200)
6 (0,0,200)
7 (0,200,0)
8 (200,0,0,)
9 (100,0,0)

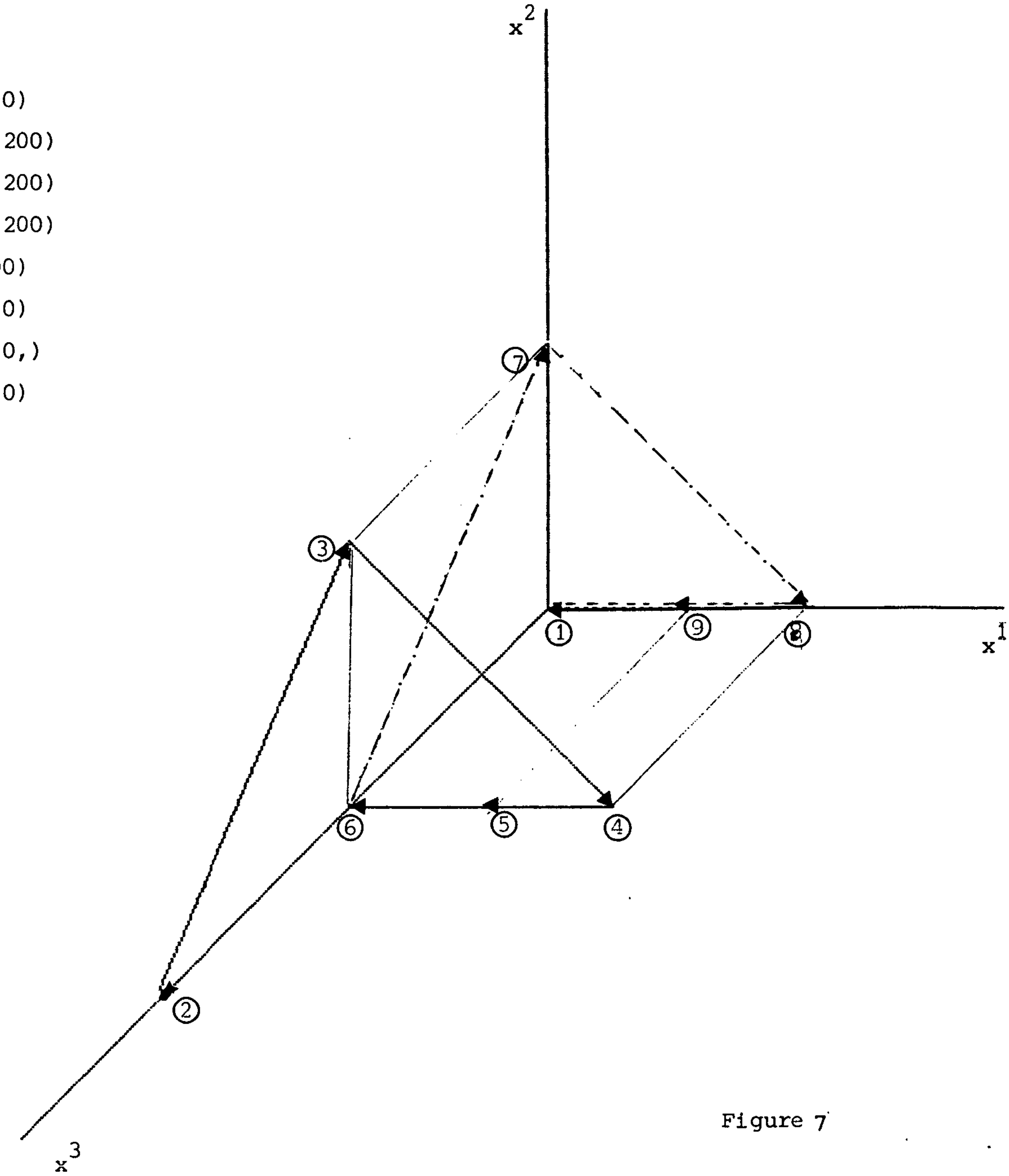


Figure 7

1	(0,0,0)
2	(0,0,700)
3	(0,500,200)
4	(300,200,200)
5	(200,200,200)
6	(100,200,200)
7	(0,200,200)
8	(200,0,200)
9	(100,0,200)
10	(0,0,200)
11	(0,200,0)
12	(200,0,0)
13	(100,0,0)

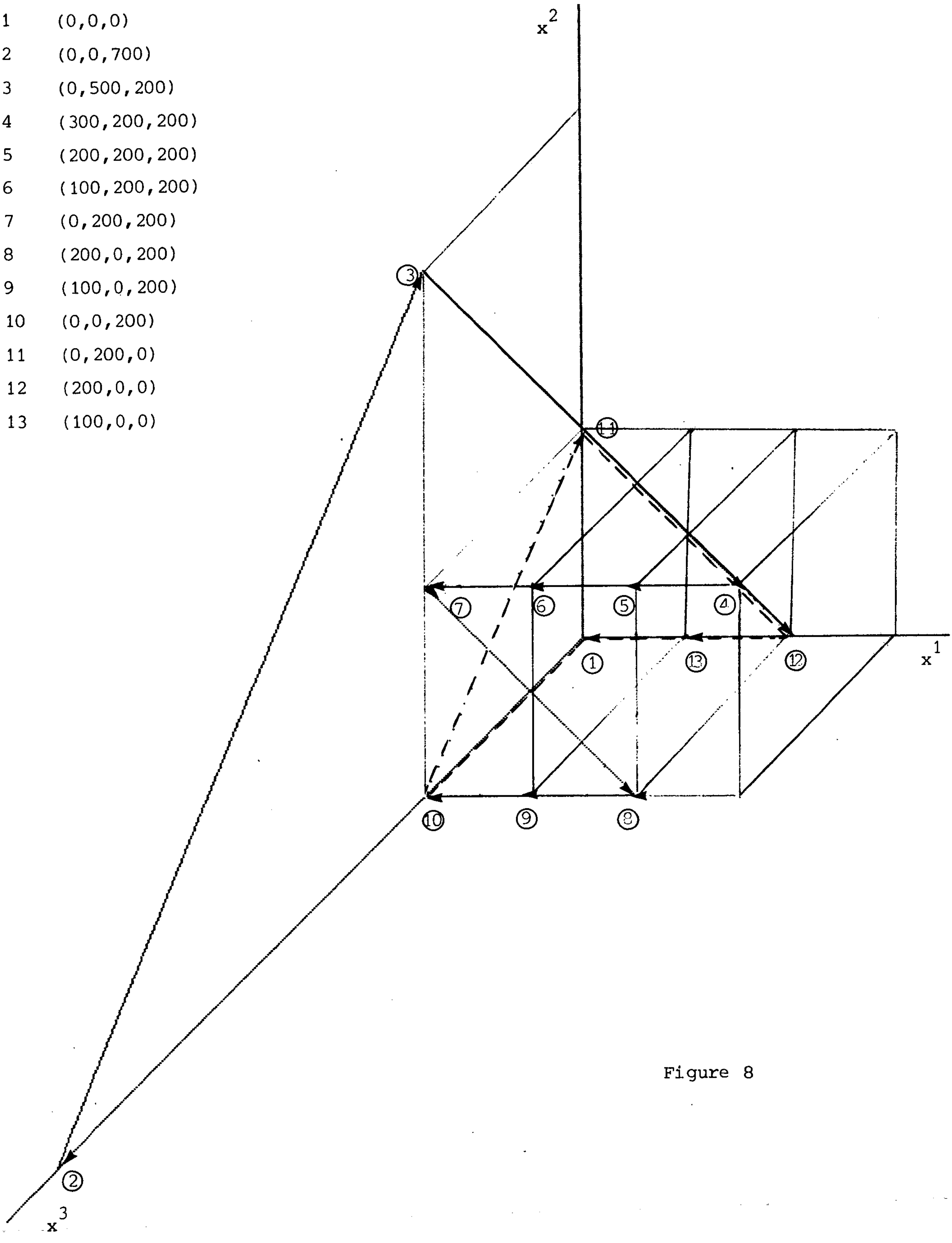
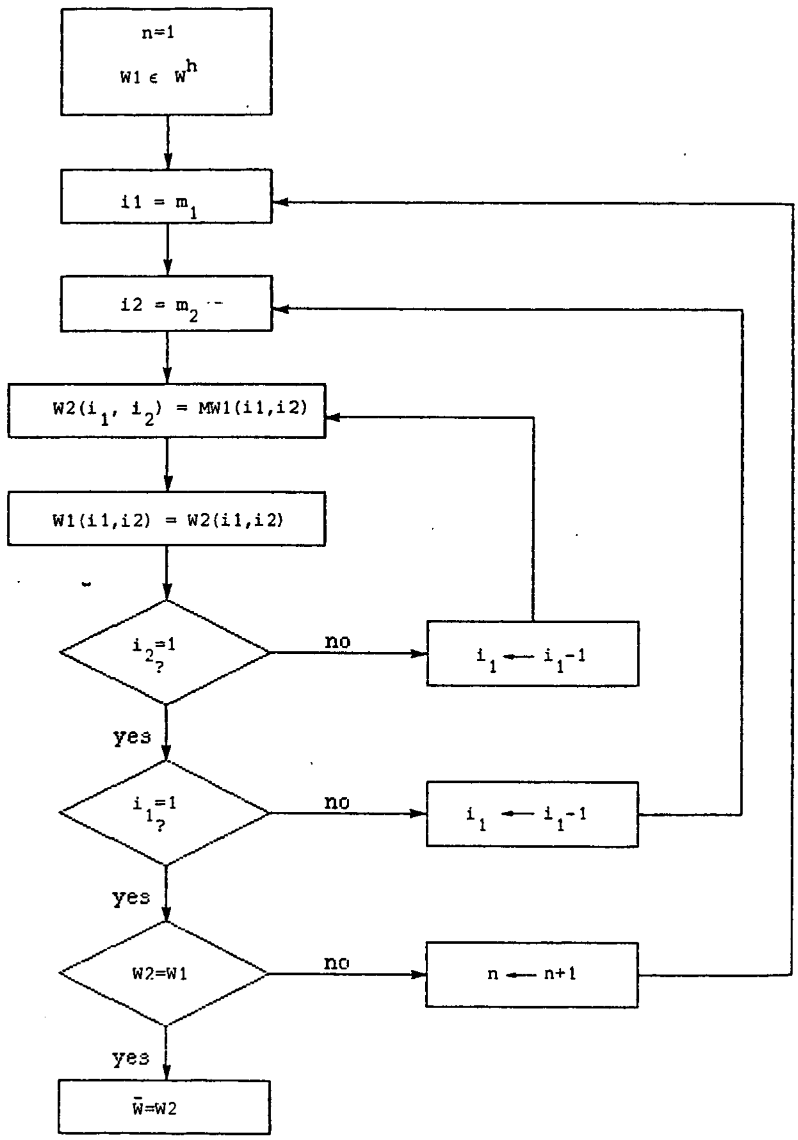


Figure 8

Algorithm



Remark : The algorithm needs only the values of the functions w_1^h at the points a_i^h and Y_i^h to compute w_2^h (S_i^h). Then, it's only necessary to have in the central memory of the computer the values of w_1^h (S_i^h) for vertices in a neighbourhood of S_i^h. This property allows the application of this algorithm in computers with small central memories. Otherwise, with non negligible discretized points (on the average, we take 10 discretized points for each level), we can study systems having upto 8 serial level on DP58/70 Multics.

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