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Paul Camion, Bernard Courteau, Philippe Delsarte

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Domaine de Voluceau
Rocquencourt
BP 105

78153 Le Chesnay Cedex
France

Tél. (1) 39 63 55 11

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IN HAMMING SPACES**

**Paul CAMION
Bernard COURTEAU
Philippe DELSARTE**

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ON r -PARTITIONS DESIGNS IN HAMMING SPACES
SUR LES CONFIGURATIONS DE
 r -PARTITIONS DANS LES ESPACES DE HAMMING

Paul CAMION
INRIA
Domaine de Voluceau
78153 - Le Chesnay - France

Bernard COURTEAU *
Dept. de mathematiques et informatique
Universite' de Sherbrooke
Sherbrooke - Quebec -
Canada J1K 2R1

Philippe DELSARTE
Philips Research Laboratory
B-1170 Brussels
Belgium

ABSTRACT

We introduce the combinatorial matrix of a code, the notion of r -partition-design and using these notions we give a characterization of completely regular codes and a combinatorial interpretation to the fact that the distance matrix of a non-linear code contains the least possible number of distinct rows.

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RESUME

Nous introduisons la matrice combinatoire d'un code et la notion de configuration de r -partition. En utilisant ces notions nous donnons une caractérisation des codes totalement réguliers et une interprétation combinatoire du fait que la matrice des distances d'un code non-linéaire comporte le plus petit nombre possible de lignes distinctes.

Keywords : Codes, association schemes, completely regular codes, coherent partitions.

INTRODUCTION

In this paper we introduce the combinatorial matrix of a code, the notion of r -partition-design and we relate these notions to fundamental concepts of coding theory.

Section 1. gives a combinatorial interpretation of the matrix $S = (\alpha_{ij})$ giving the basis $\{P_i(\mathbf{x})\}$ in terms of $\{P_j(\mathbf{x})\}$ the basis of the ring of polynomials over the finite field $F = GF(q)$ formed by the Krawtchouk polynomials $P_j(\mathbf{x})$. The element α_{ij} is the number of paths of length j joining two vectors of the Hamming space \mathbb{F}^n at distance i apart. We then give a recurrence relation and the exponential generating function for these numbers α_{ij} .

In section 2. we introduce the combinatorial matrix $A = (A(\mathbf{x}, j))$ of a code C : $A(\mathbf{x}, j)$ is the number of paths of length j joining $\mathbf{x} \in \mathbb{F}^n$ to the code C . This matrix A is related to the distance matrix $B[2]$ by the relation $A = BS$ and the sequence of columns of A satisfy a recurrence of minimum order $s' + 1$ if and only if s' is the external distance of C . Moreover the characteristic polynomial of this minimum order recurrence admits as zeroes $P_1(l) = n(q-1) - ql$ being the dual distances of C . The preceding are extensions to non-linear codes of notions and results already obtained in [4], [5] and [14].

In section 3. we start the study of r -partition-designs (called coherent partitions by Higman [15]) which are partitions $\Pi = \{C_0, C_1, \dots, C_r\}$ of \mathbb{F}^n into

$r + 1$ classes such that for any $x \in C_u$ the number σ_{uv} of elements of C_v at distance one from x is independent of the choice of x in C_u . A code C is said to admit the partition Π if C is the union of some classes C_u . Perfect, uniformly packed and more generally completely regular codes are then characterized in terms of r -partition-designs. For example we prove the following result : Let C be a code with covering radius ρ . Then C is completely regular if and only if C admits a r -partition-design for $r = \rho$. Moreover $\rho = s'$ the external distance of C and the eigenvalues of the associated matrix $\sigma = (\sigma_{uv})$ are $P_1(l) = n(q-1) - ql$ for $l \in \{0, d'_1, \dots, d'_s\}$ where d'_1, \dots, d'_s are the dual distances of C .

In general, if C admits a r -partition-design, then $r \geq s'$. The case $r = s'$ is characterized as follows : C admits a s' -partition-design if and only if the number of distinct rows in the distance matrix is $s' + 1$. This is an analogue of theorem 6.11 of [1] in the non linear case. On the other hand in the linear case, we may apply theorem 6.10 and 6.11 of [1] to obtain the result : the linear code C admits a s' -partition-design $\Pi = \{C_0, C_1, \dots, C_{s'}\}$ if and only if the partition Π of the quotient group $C' = \mathbb{F}^n / C$ defines an association scheme over C' (called the coset scheme determined by Π) if and only if the restriction to C of the Hamming scheme is a subscheme. The P -matrix of the coset scheme has been determined by A. Montpetit in-terms of the s' -partition-design Π : it is the left eigenmatrix of the matrix σ associated to Π . Finally in section 4. we give numerous examples of codes admitting r -partition-design for $r = s'$ and an example where there doesn't exist such a s' -partition-design.

1. - PATHS IN HAMMING SPACE

Let $\mathbb{F} = GF(q)$ be the field with q elements, q a prime power and $H(n, q)$ the Hamming space of dimension n over F that is the n -dimensional vector space F^n over F equipped with Hamming distance d defined by $d(x, y) =$ number of components in which the n -vectors x and y differ. $H(n, q)$ is a metric association scheme and we refer to [1,2] for all notions and results on association schemes that will be needed in the following.

Definition 1.1 A path of length j joining x to y in F^n is a sequence $x_{(0)} = x, x_{(1)}, \dots, x_{(j)} = y$ of points in F^n such that $d(x_{(k-1)}, x_{(k)}) = 1$ for $k = 1, \dots, j$. The Hamming distance between x and y is the length of the shortest path joining x to y .

The i -th adjacency matrix D_i is the $q^n \times q^n$ matrix with rows and columns indexed by \mathbb{F}^n defined by

$$D_i(x, y) = \begin{cases} 1 & \text{if } d(x, y) = i \\ 0 & \text{otherwise.} \end{cases}$$

Since $\{D_0 = I, D_1, \dots, D_n\}$ is a basis of the Bose-Mesner algebra of $H(n, q)$, we have

$$D_j = \sum_{i=0}^n \alpha_{ij} D_i \quad (1)$$

for uniquely defined complex numbers α_{ij} .

If $x, y \in \mathbb{F}^n$ are such that $d(x, y) = i$, then for $j \geq 0$ we have $D_j^i(x, y) = \alpha_{ij}$. So α_{ij} is the number of paths of length j joining two points x and y at distance i apart and this number does not depend on the particular choice of x and y but only on the distance i between them.

Let $S = (\alpha_{ij})$ be the $n \times \infty$ non-negative integer matrix with α_{ij} in position (i, j) .

Proposition 1.2

- a) If $i > j$, then $\alpha_{ij} = 0$. So S is a $n \times \infty$ upper triangular matrix.
- b) If $i \leq j$, then $i!$ divides α_{ij} .

Proof : a) is evident. To prove b), remark that one passes from one element to the following in a path in \mathbb{F}^n by modifying one and only one component of a n -vector. If $d(x, y) = i$, then there is exactly $i!$ paths of length i joining x to y . So $\alpha_{ii} = i!$. Moreover, any path of length j joining x to y must contain one and only one of these $i!$ minimal paths. Since these last paths play completely symmetric roles, the number of length j paths containing a given minimal path must be a constant α independent of the chosen minimal path. Hence $\alpha_{ij} = \alpha i!$.

Remark 1.3 If $P_i(l)$, $l = 0, \dots, n$ are the eigenvalues of D_i , $i = 0, \dots, n$, then by (1)

$$P_j^i(l) = \sum_{i=0}^n \alpha_{ij} P_i(l) \quad \text{for } j \geq 0. \quad (2)$$

Since the Hamming scheme $H(n, q)$ is P-polynomial with respect to the class of Krawtchouk polynomials we may suppose that in (2) $P_i(l)$ is the evaluation on l of the i -th Krawtchouk polynomial which shall be denoted by the same letter P_i . If P is the Krawtchouk matrix containing in position (l, i) the number $P_i(l)$, we may express equation (2) in matrix form as follows.

Proposition 1.4 $S=q^{-n}PV$ where the matrix V having in position (l, j) the number $P_j^l(l)$ is an infinite Vandermonde matrix.

Remark 1.5 Definition of matrix S , proposition 1.2 a) and equation (2) hold in any metric scheme (X, R) . In this general case the matrix equality in proposition 1.4 should be read $S=|X|^{-1}QV$ with $PQ=QP=|X|I$ where in the eigenmatrix P the Krawtchouk polynomials are replaced by another convenient class of orthogonal polynomials $\Phi_i(x)$. In the case of Hamming scheme, we have $Q=P$. Many of the following results may be extended to arbitrary metric schemes.

Using the order two recurrence satisfied by Krawtchouk polynomials written in the form

$$P_1(l)P_j(l)=(j+1)P_{j+1}(l)+(q-2)jP_j(l)+(q-1)(n-j+1)P_{j-1}(l) \quad (3)$$

we deduce from (2) the linear recurrence

$$\alpha_{i,j+1}=i\alpha_{i-1,j}+i(q-2)\alpha_{ij}+(n-i)(q-1)\alpha_{i+1,j} \quad (4)$$

In matrix form, this gives

$$S_{j+1}=MS_j$$

where $S_j=[\alpha_{0j}, \dots, \alpha_{nj}]^T$ is the j -th column of $S(S_0=[1, 0, \dots, 0]^T)$ and M is the following tridiagonal $(n+1) \times (n+1)$ matrix :

$$M = \begin{bmatrix} 0 & n(q-1) & 0 & 0 & 0 & 0 \\ 1 & q-2 & (n-1)(q-1) & \dots & 0 & \dots \\ 0 & 2 & 2(q-2) & \dots & 0 & \dots \\ \vdots & \vdots & 3 & \dots & 2(q-1) & \dots \\ \vdots & \vdots & \vdots & \dots & (n-1)(q-2) & (q-1) \\ 0 & \vdots & 0 & \dots & n & n(q-2) \end{bmatrix} \quad (5)$$

So $S_j=M^j S_0$ is the first column of M^j and we note that the eigenvalues of

M are $P_1(0)=n(q-1), \dots, P_1(l)=n(q-1)-lq, \dots, P_1(n)$ and the associated eigenvectors are the corresponding columns in the Krawtchouk matrix P because the recurrence (3) may be written in matrix form as $PM=\Delta P$ which gives

$$PMP^{-1}=\Delta$$

where $\Delta = \text{diag}\{P_1(0), \dots, P_1(l), \dots, P_1(n)\}$.

Proposition 1.6 $S=[M^0S_0, MS_0, \dots, M^jS_0, \dots]$ where M the matrix given by (5) have $P_1(l)=n(q-1)-ql, l=0, \dots, n$ as eigenvalues.

Remark 1.7 In the general case of a metric scheme we obtain a similar result by using in place of (3) the order two recurrence satisfied by orthogonal polynomials $\Phi_i(x)$ which are associated to the scheme [1].

We may also look at the numbers α_{ij} in matrix S by means of exponential generating functions. Here is the result.

Proposition 1.8

$$\sum_{j \geq 0} \alpha_{ij} \frac{Z^j}{j!} = q^{-n} \left(e^{(q-1)Z} - e^{-Z} \right)^i \left(e^{(q-1)Z} + (q-1)e^{-Z} \right)^{n-i}, \quad i=0, \dots, n \quad (6)$$

Proof : Let x and y be given in F^n such that $d(x,y)=i$ and let $\gamma = \text{supp}(x-y) = \{k \mid x_k - y_k \neq 0\}$ be the support of $x-y$. Consider the numbers

a_m = number of paths of length m joining two points at distance 1 apart obtained by modifying only the component where they differ (this component being in γ),

b_m = number of cycles of length m starting from a given point and obtained by modifying only one component (exterior to γ),...

$c_m = \alpha_{im}$ = number of paths of length m joining the two points x and y (hence $c_m = \alpha_{im}$) and the associated exponential generating functions

$$a(Z) = \sum_{m \geq 0} a_m \frac{Z^m}{m!}, \quad b(Z) = \sum_{m \geq 0} b_m \frac{Z^m}{m!} \quad \text{and} \quad c(Z) = \sum_{m \geq 0} \alpha_{im} \frac{Z^m}{m!}$$

Interpreting, as usual, the product of two exponential generating

functions [3] as a kind of shuffle product, we may write

$$c(Z) = (a(Z))^i (b(Z))^{n-i} = \sum_{j \geq 0} \alpha_{ij} \frac{Z^j}{j!}$$

and it remains to determine $a(Z)$ and $b(Z)$. Remark that $a_m = (q-2)a_{m-1} + b_{m-1}$ and $b_m = (q-1)a_{m-1}$ with $a_0=0, a_1=1, b_0=1, b_1=0$, so that we have the recurrence $a_m - (q-2)a_{m-1} - (q-1)a_{m-2} = 0, a_0=0, a_1=1$ which gives in terms of generating functions the differential equation

$$a''(Z) - (q-2)a'(Z) - (q-1)a(Z) = 0$$

with initial conditions $a(0)=0, a'(0)=1$. The solution of this problem is

$$a(Z) = \frac{e^{(q-1)Z} - e^{-Z}}{q}$$

By integration, we deduce from $b'(Z) = (q-1)a(Z)$

$$b(Z) = \frac{e^{(q-1)Z} + (q-1)e^{-Z}}{q} \text{ since } b(0)=1.$$

This completes the proof of proposition 1.8.

Remark 1.9 The preceding is a combinatorial proof. We may give a shorter algebraic proof by using the generating function of Krawtchouk polynomials $P_k(i)$ and proposition 1.4.

The generating function of polynomials $P_k(i)$ is

$$(X-Y)^i (X+(q-1)Y)^{n-i} = \sum_{0 \leq k \leq n} P_k(i) X^{n-k} Y^k$$

Setting $X = e^{(q-1)Z}$ and $Y = e^{-Z}$ in this relation gives

$$(e^{(q-1)Z} - e^{-Z})^i (e^{(q-1)Z} + (q-1)e^{-Z})^{n-i} = \sum_{0 \leq k \leq n} P_k(i) e^{(q-1)Z(n-k) - Zk}$$

$$= \sum_{0 \leq k \leq n} P_k(i) e^{ZP_1(k)} \text{ with } P_1(k) = n(q-1) - qk$$

$$\begin{aligned}
 &= \sum_{0 \leq k \leq n} P_k(i) \sum_{j \geq 0} \frac{(ZP_1(k))^j}{j!} = \sum_{j \geq 0} \left[\sum_{0 \leq k \leq n} P_k(i) P_1^j(k) \right] \frac{Z^j}{j!} \\
 &= q^n \sum_{j \geq 0} \alpha_{ij} \frac{Z^j}{j!}
 \end{aligned}$$

by proposition 1.4.

2. - THE COMBINATORIAL MATRIX OF A CODE

Let $C \subset \mathbb{F}^n$ be an unrestricted code of length n over \mathbb{F} .

Definition 2.1 For any $x \in \mathbb{F}^n$, let $A_j(x)$ be the number of paths of length j joining x to an element of C . The **combinatorial matrix of \mathbb{F}^n with respect to C** is then the $q^n \times \infty$ matrix A whose element in position (x, j) is

$$A(x, j) = A_j(x).$$

If $B_i(x)$ is the number of elements of C at distance i apart from x , then the **distance matrix of \mathbb{F}^n with respect to C** [2] is the $(q^n \times (n+1))$ matrix B whose element in position (x, i) is

$$B(x, i) = B_i(x).$$

By the very definition of the numbers in question we have

$$A_j(x) = \sum_{i=0}^n \alpha_{ij} B_i(x) \tag{7}$$

giving in matrix form the following equality.

Proposition 2.2 $A = BS$

As a consequence of proposition 1.8 and 2.2 we also have the following property generalizing theorem 3.3 of [4].

Proposition 2.3

$$q^n \sum_{j \geq 0} A_j(x) \frac{Z^j}{j!} = \sum_{i=0}^n B_i(x) \left[e^{(q-1)Z} - e^{-Z} \right]^i \left[e^{(q-1)Z} + (q-1)e^{-Z} \right]^{n-i} \tag{8}$$

Remark This result combined with proposition 1.4 may be viewed as generalized Pless identities. The classical Pless identities [6] are obtained when the code C is linear and $x=0$ in the formula. This is because, on the one hand

$$q^n A_j(x) = \sum_{i=0}^n \frac{d^j}{dZ^j} \left[(e^{(q-1)Z} - e^{-Z})^i (e^{(q-1)Z} + (q-1)e^{-Z})^{n-i} \right]_{Z=0} B_i(x)$$

and on the other hand, by proposition 2.2 and 1.4,

$$\begin{aligned} q^n A_j(x) &= \sum_{i=0}^n \alpha_{ij} B_i(x) = \sum_{i=0}^n \sum_{l=0}^n P_l(i) P_l(l) B_i(x) \\ &= \sum_{l=0}^n B'_l(x) [n(q-1) - ql]^j \end{aligned}$$

Proposition 2.4 Let C be an unrestricted code in \mathbb{F}^n . Then the following two conditions are equivalent.

(i) s' is the external distance of C

(ii) s' is the minimum of the natural numbers t for which there exists a linear recurrence of order $t+1$

$$\sum_{j=0}^{t+1} c_j A_{j+m}(x) = 0, \quad x \in \mathbb{F}^n$$

where c_0, c_1, \dots, c_{t+1} are integers with $c_{t+1} \neq 0$.

Moreover, the recurrence of minimum order $s'+1$ with $c_{s'+1} = 1$ is unique and the coefficients c_j are determined by

$$\sum_{j=0}^{s'+1} c_j Z^j = \prod_{l \in J} (Z - P_l(l)), \quad J = \{0, d'_1, \dots, d'_s\}$$

where d'_1, \dots, d'_s are the dual distances of C .

Proof : We shall work in the group algebra $\mathcal{C}[\mathbb{F}^n]$ of \mathbb{F}^n over the complex numbers and use the polynomial notation

$$a = \sum_{x \in \mathbb{F}^n} a_x Z^x, \quad a_x \in \mathcal{C}$$

to represent an element $a \in \mathcal{C}[\mathbb{F}^n]$.

Remark that, if $C = \sum_{g \in C} Z^g$ and $Y_1 = \sum_{w(h)=1} Z^h$ then in $\mathcal{C}[F^n]$ we have

$$CY_1^j = \sum_{x \in F^n} A_j(x) Z^x \quad (7)$$

This is because, by definition of convolution product,

$$CY_1^j = \left(\sum_{g \in C} Z^g \right) \left(\sum_{w(h)=1} Z^h \right)^j = \sum_{x \in F^n} \left[\sum_{x=g+h_1+\dots+h_j} 1 \right] Z^x$$

where in the last sum $g \in C$ and $W(h_1)=1, \dots, W(h_j)=1$, and the fact that

$$A_j(x) = \text{card} \{ (g, h_1, \dots, h_j) \mid x = g + h_1 + \dots + h_j, g \in C, W(h_1) = \dots = W(h_j) = 1 \}$$

Now we shall prove that if there exists a linear recurrence of order $t+1$

$$\sum_{j=0}^{t+1} c_j A_{j+m}(x) = 0, \quad x \in F^n, \quad m \geq 0 \quad (8)$$

then

$$\sum_{j=0}^{t+1} c_j [P_1(l)]^j = 0 \quad (9)$$

for $l \in \{0, d'_1, \dots, d'_s, d'_1, \dots, d'_s\}$, being the dual distances of C .

From (8) and by (7) we may write

$$\begin{aligned} \sum_{x \in F^n} \left(\sum_{j=0}^{t+1} c_j A_j(x) \right) Z^x &= 0 \\ \sum_{j=0}^{t+1} c_j (CY_1^j) &= 0 \\ C \left(\sum_{j=0}^{t+1} c_j Y_1^j \right) &= 0 \text{ in } \mathcal{C}[F^n] \end{aligned} \quad (10)$$

Now, by theorem 7 p. 139 of [7], for all l such that $l=0$ or $l=d'_i$, $i=1, \dots, s'$, there exists $u \in F^n$ such that $X_u(C) \neq 0$ and $w(u)=l$ where X_u is the character associated with u . For such an u , we have

$$X_u \left(\sum_{j=0}^{t+1} c_j Y_1^j \right) = 0$$

i.e

$$\sum_{j=0}^{t+1} c_j [X_u(Y_1)]^j = 0.$$

i.e

$$\sum_{j=0}^{t+1} c_j [P_1(l)]^j = 0.$$

Hence the polynomial $c(Z) = \sum_{j=0}^{t+1} c_j Z^j$ is divisible by

$$p(Z) = \prod_{l \in J} (Z - P_1(l)), \quad J = \{0, d'_1, \dots, d'_{s'}\} \text{ and } s' \leq t.$$

Finally we shall exhibit a linear recurrence of order $s'+1$. Take the annihilator polynomial $\beta(Z) = Z \prod_{i=1}^{s'} (Z - d'_i)$ (up to a factor) decompose it in the basis $\{P_1^0(Z), P_1(Z), \dots, P_1^i(Z), \dots\}$

$$\beta(Z) = \sum_{j=0}^{s'+1} c_j P_1^j(Z)$$

and note that in the group algebra $\mathcal{G}[F^n]$

$$C \left[\sum_{j=0}^{s'+1} c_j Y_1^j \right] = 0$$

because for all character X_u either $X_u(C) = 0$ or in case $w(u) = d'_i, i = 1, \dots, s', X_u \left[\sum_{j=0}^{s'+1} c_j Y_1^j \right] = \sum_{j=0}^{s'+1} c_j [X_u(Y_1)]^j = \sum_{j=0}^{s'+1} c_j P_1^j(d'_i) = \beta(d'_i) = 0$

Hence for all $m \geq 0$

$$Y_1^m C \left[\sum_{j=0}^{s'+1} c_j Y_1^j \right] = 0$$

i.e

$$\sum_{j=0}^{s'+1} c_j C Y_1^{m+j} = 0$$

and by (7)

$$\sum_{j=0}^{s'+1} c_j \left[\sum_{x \in \mathbb{F}^n} A_{j+m}(x) Z^x \right] = 0$$

i.e

$$\sum_{x \in \mathbb{F}^n} \left[\sum_{j=0}^{s'+1} c_j A_{j+m}(x) \right] Z^x = 0.$$

This gives the recurrence

$$\sum_{j=0}^{s'+1} c_j A_{j+m}(x) = 0.$$

To show how the preceding are generalization of notions introduced in [4,5], we specialize to the particular case where the code C is linear.

Proposition 2.5 Let $C \subset \mathbb{F}^n$ be a $(n, n-k)$ linear code with parity check matrix H and let $\Omega \subset \mathbb{F}^k$ be the ordered set of columns (supposed distinct) of H .

If $x \in \mathbb{F}^n$ and $h = Hx$ is the syndrome of x , then $A_j(x) = \text{card } \mathcal{E}_j(x)$ where

$$\mathcal{E}_j(x) = \left\{ (h_1, \dots, h_j, \lambda_1, \dots, \lambda_j) \mid h = \lambda_1 h_1 + \dots + \lambda_j h_j, \lambda_i \in \mathbb{F}^n, h_i \in \Omega, i=1, \dots, j \right\}$$

Proof :

Let $\mathcal{D}_j(x) = \left\{ (x = x_{(0)}, x_{(1)}, \dots, x_{(j)}) \mid x_{(j)} \in C, d(x_{(i-1)}, x_{(i)}) = 1, x_{(i)} \in \mathbb{F}^n, i=1, \dots, j \right\}$ be

the set of paths of length j joining x to the code C .

Define $\rho: \mathcal{D}_j(x) \dashrightarrow \mathcal{E}_j(x)$ by

$$\rho \left[(x, x_{(1)}, \dots, x_{(j)}) \right] = (h_1, \dots, h_j, \lambda_1, \dots, \lambda_j)$$

where $h = Hx = H \left[(x - x_{(1)}) + (x_{(1)} - x_{(2)}) + \dots + (x_{(j-1)} - x_{(j)}) \right]$

$$= \lambda_1 h_1 + \lambda_2 h_2 + \dots + \lambda_j h_j$$

This mapping ρ is well defined because $w(x_{(i-1)} - x_{(i)}) = 1$. In fact, the value of the non-zero component of $x_{(i-1)} - x_{(i)}$ gives λ_i and its index gives the index of h_i in Ω .

It is clear that ρ is onto. It is one-to-one because we have supposed the columns of H distinct. The inverse Ψ of ρ is defined by

$$\Psi(h_1, \dots, h_j, \lambda_1, \dots, \lambda_j) = (x, x_{(1)}, \dots, x_{(j)})$$

where $x_{(1)}, \dots, x_{(j)}$ are determined as follows :

$$x_{(1)} = x - \lambda_1 e_{i_1}, \text{ where } i_1 \text{ is the unique index such that } H_{i_1} = h_1.$$

This gives $Hx_{(1)} = \lambda_2 h_2 + \dots + \lambda_j h_j$. We then repeat the argument to obtain $x_{(2)}, \dots, x_{(j)}$. Finally $Hx_{(j)} = 0$ and $x_{(j)} \in C$.

3. - r-PARTITION DESIGNS

Definition 3.1 A **r-partition-design** of the Hamming scheme $H(n, q)$ is a partition of F^n into $r+1$ classes C_0, C_1, \dots, C_r such that for any $x \in C_u$ the number σ_{uv} of elements in C_v at distance one from x is independant of the choice of x in its class C_u .

We shall say that a code C admits the r-partition-design $\{C_0, C_1, \dots, C_r\}$ if $C = \cup\{C_v \mid v \in J\}, J \subseteq \{0, 1, \dots, r\}$. (We shall also say that the partition-design contains the code C).

Remark 3.2 If $\{C_0, C_1, \dots, C_r\}$ is a r-partition design, then for all $u, v \in \{0, \dots, r\}$

$$(\text{card } C_u) \sigma_{uv} = (\text{card } C_v) \sigma_{vu} = \text{card}\{(x, y) \in C_u \times C_v \mid d(x, y) = 1\}.$$

In matrix form this gives,

$$\sigma^T = K \sigma K^{-1} \text{ where } K = \text{diag}\{\text{card } C_0, \dots, \text{card } C_r\}.$$

Remark 3.3 Let C be a $(n, n-k)$ -linear code admitting a r -partition-design C_0, C_1, \dots, C_r with associated matrix $\sigma = (\sigma_{uv})$ such that each C_u is an union of cosets of C . If $\Omega \subseteq F^k$ is the set of columns (supposed distinct) of a parity check matrix H for the code C , define the sets $\Omega_0, \Omega_1, \dots, \Omega_r$ as follows :

$$\Omega_u = \{Hx \mid x \in C_u\}, \quad 0 \leq u \leq r$$

that is Ω_u is the set of syndromes of elements in C_u .

The set $\{\Omega_0, \Omega_1, \dots, \Omega_r\}$ is a partition of F^k because $\text{rank } H = k$ and $\Omega_u \cap \Omega_v \neq \emptyset$ implies $\Omega_u = \Omega_v$ ($h \in \Omega_u \cap \Omega_v \implies h = Hx = Hy$ for $x \in C_u, y \in C_v \implies x \in y + C \implies x \in C_v \implies C_u = C_v$.)

Then we have the following interpretation of the numbers σ_{uv} :

for $u, v \in \{0, 1, \dots, r\}$ and $a \in \Omega_u$

$$\sigma_{uv} = \text{card}\{(b, h) \in \Omega_v \times F^r \mid a = b + h\}. \quad (11)$$

This is because, if $\mathcal{E} = \{y \in C_v \mid d(x, y) = 1\}$ for $x \in C_u$ and $\mathcal{D} = \{(b, h) \in \Omega_v \times F^r \mid a = b + h\}$ for $a \in \Omega_u$, then $\varphi: y \mapsto (Hy, H(x-y))$ is a bijection of \mathcal{E} onto \mathcal{D} .

We note that Ω is the union of some Ω_u because if $\Omega_u \cap \Omega \neq \emptyset$ then $\sigma_{u0} \neq 0$ which implies that $\Omega_u \subset \Omega$.

Conversely, if $\Omega_0, \Omega_1, \dots, \Omega_r$ is a partition of F^k satisfying (11) where $\Omega = \bigcup_{u \in J} \Omega_u$, then we may define the r -partition design $\{C_0, C_1, \dots, C_r\}$ by $C_u = \{x \in F^n \mid Hx \in \Omega_u\}$ where H is the matrix having Ω as columns set. Moreover each C_u is the union of some cosets of the code C having parity check matrix H .

This is a slightly more general definition of r -partition design than the one given in [4] for the case of linear codes. Note also that a 2-partition design is a partial difference set with two parameters [8]. So we may consider r -partition-design as some kind of generalized difference sets.

Remark 3.4 For any $u = 0, \dots, r$, $\sum_{v=0}^r \sigma_{uv} = n(q-1)$. Hence $n(q-1)$ is an eigenvalue of σ .

Remark 3.5 Let C be any code in \mathbb{F}^n . Then C always admits the **trivial** r -partition design C_0, C_1, \dots, C_r where $r=q^n-1$, the classes C_i consisting of only one element. In this case $\sigma=D_1$.

Remark 3.6 Let ρ be the covering radius of C and $\bar{C}_0, \dots, \bar{C}_\rho$ be the classes defined by

$$\bar{C}_i = \{x \in \mathbb{F}^n \mid d(x, C) = i\}, \quad i = 0, \dots, \rho$$

If $\{C_0, C_1, \dots, C_r\}$ is a r -partition design containing C , then

$$\bar{C}_i = \bigcup \{C_u \mid C_u \cap \bar{C}_i \neq \emptyset\}, \quad i = 0, \dots, \rho.$$

This is proved by induction on i . Hence $\rho \leq r$. Naturally the extremal case $r = \rho$ is expected to show interesting combinatorial structures.

Remark 3.7 Consider the partition $\bar{C}_0 = C, \bar{C}_1, \dots, \bar{C}_\rho$ where the code C is error-correcting with covering radius ρ . Let $\sigma(x) = (\sigma_{ij}(x)), x \in \mathbb{F}^n$, be the matrix defined by

$$\sigma_{ij}(x) = \begin{cases} \text{card}\{y \in \bar{C}_j \mid d(x, y) = 1\} & \text{if } x \in \bar{C}_i \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

- 1/ $\sigma_{ij}(x) = 0$ for $|i-j| \geq 2$
- 2/ $\sigma_{i, i-1}(x) = i$ for $x \in \bar{C}_i$ and $1 \leq i \leq e$
- 3/ $\sigma_{i, i}(x) = i(q-2)$ for $x \in \bar{C}_i$ and $0 \leq i \leq e-1$
- 4/ $\sum_{j=0}^{\rho} \sigma_{ij}(x) = n(q-1)$ for $x \in \bar{C}_i$.

Hence for all i, j such that $|i-j| \geq 2$, $0 \leq i \leq e-1$ and for $i=e, j=e-1$ the numbers $\sigma_{ij}(x)$ are independant of the choice of x into the class \bar{C}_i . From this we deduce the following results.

Proposition 3.8 Let C be an e -error-correcting code over \mathbb{F} . Then C is perfect if and only if C admits an e -partition design. Moreover this e -partition design is unique.

Proposition 3.9 Let C be an e -error-correcting quasi-perfect code over \mathbb{F} . Then

C is (λ, μ) -uniformly packed code [9] if and only if C admits an $(e+1)$ -partition design. Moreover this $(e+1)$ -partition design is unique and $\sigma_{ee} = (e+1)\lambda + e(q-2)$, $\sigma_{e+1,e} = (e+1)\mu$.

Example 3.10 If G is a subgroup of the group of Hamming isometries of \mathbb{F}^n and if C_0, C_1, \dots, C_r are its orbits, then $\{C_0, C_1, \dots, C_r\}$ is a r -partition design.

We now give some general results that show the interest of the combinatorial matrix A .

Proposition 3.11 Let $C \subseteq \mathbb{F}^n$ be an unrestricted code of length n over the alphabet \mathbb{F} . If C admits a r -partition design $\{C_0, C_1, \dots, C_r\}$ with associated matrix $\sigma = (\sigma_{uv})$, then

a) for all $x, y \in C_u$, $A_j(x) = A_j(y) = A_j(u)$, $j \geq 0$.

b) the numbers $A_j(u)$, $u \in \{0, \dots, r\}$, $j \geq 0$ satisfy the linear recurrence of order r

$$A_j(u) = \sum_{v=0}^{r-1} \sigma_{uv} A_{j-1}(v), \quad (11)$$

c) the number of distinct rows in the distance matrix B and in the combinatorial matrix A is less than or equal to $r+1$,

d) the external distance s' of C is less than or equal to r .

Proof (by induction on j). Let $C = \bigcup \{C_v \mid v \in J\}$, $J \subseteq \{0, 1, \dots, r\}$. If $x \in C_v$, $v \in J$, then $A_0(x) = 1$ and if $x \in C_v$, $v \notin J$, then $A_0(x) = 0$.

Now suppose that for $m \leq j-1$ the numbers $A_m(y) = A_m(v)$ only depends on

the class C_v to which y belongs and let $x \in C_u$.

For any path γ of length j joining x to C , there exists one and only one v such that γ is obtained by concatenation of a path of length one joining x to $y \in C_v$ and a path of length $j-1$ joining y to C . Since the number σ_{uv} of length 1 paths joining x to C_v does not depend on the chosen x in C_u and that, by induction hypothesis, the number $A_{j-1}(v)$ of paths of length $j-1$ joining y to C does not depend on y , we deduce that $A_j(x)$ does not depend on the chosen x in C_u and moreover that

$$A_j(x) = A_j(u) = \sum_{v=0}^r \sigma_{uv} A_{j-1}(v)$$

This proves a) and b). Finally, condition a) means that the combinatorial matrix A has at most $r+1$ distinct rows. Proposition 2.2 then implies that the distance matrix B has also at most $r+1$ distinct rows proving part c). Part d) of the proposition is then an immediate consequence of theorem 3.1 of [2].

Corollary 3.12 Let C be an unrestricted code in \mathbb{F}^n with external distance s' , d'_1, \dots, d'_s being the dual distances.

If C admits a s' -partition-design with associated matrix σ , then the eigenvalues of σ are $P_1(l)$ for $l \in \{0, d'_1, \dots, d'_s\}$.

Proof : First note that (11) may be written in matrix form

$$A_j = \sigma A_{j-1}, \quad j \geq 1$$

with initial vector A_0 defined by

$$A_0(u) = \begin{cases} 1 & \text{if } C_u \subseteq C \\ 0 & \text{otherwise} \end{cases}$$

This gives $A_j = \sigma^j A_0$ and

$$A = [A_0, \sigma A_0, \dots, \sigma^j A_0, \dots]$$

where A is the restricted combinatorial matrix obtained from A by taking the $s'+1$ rows $A(u)$, $u = 0, \dots, s'$.

Since $A = BS$ by proposition 2.2, the rank of A is equal to the rank of B which is equal to $s'+1$ by [2,th.3.1]. Hence there exists a unique monic

polynomial of degree $s'+1$ $p(Z) = \sum_{j=0}^{s'+1} p_j Z^j$ such that

$$p(\sigma)A_0 = \sum_{j=0}^{s'+1} p_j \sigma^j A_0 = 0.$$

Multiplying by $\sigma^m, m \geq 0$ this yields

$$\sum_{j=0}^{s'+1} p_j \sigma^{j+m} A_0 = 0$$

and

$$\sum_{j=0}^{s'+1} p_j A_{j+m} = 0.$$

The conclusion of corollary is then obtained by applying proposition 2.4.

Remark 3.13 This corollary yields strong necessary conditions for the existence of s' -partition-design containing a code of external distance s' . The characteristic polynomial of σ may replace Lloyd polynomial to obtain non-existence theorem concerning particular classes of codes for example perfect codes, uniformly packed codes etc. [10,11,12]. This is because the parameter e (and λ, μ in case of uniformly packed codes) completely determines the matrix σ for these classes of codes.

Proposition 3.14 Let $C \subseteq \mathbb{F}^m$ be an unrestricted code over \mathbb{F} and s' be the external distance of C . Then the three following conditions are equivalent.

- (i) C admits a s' -partition design
- (ii) The number of distinct rows in the distance matrix B is $s'+1$.
- (iii) The number of distinct rows in the combinatorial matrix A is $s'+1$.

Moreover if it exists the s' -partition design containing C is unique.

Proof Proposition 3.11 and the fact that $\text{rank } B = s'+1$ [2] prove the implication (i) \Rightarrow (ii). Moreover (ii) \Leftrightarrow (iii) by proposition 2.2.

To prove the converse, consider the equivalence relation on F^n defined by

$x \equiv x'$ if and only if the rows $B(x)$ and $B(x')$ of B are equal and denote by C_0, C_1, \dots, C_s the equivalence classes of this relation. By proposition 2.2, we may also say that $x, x' \in C_u$ if and only if $A_j(x) = A_j(x') = A_j(u)$ for all $j \geq 0$.

For $x \in C_u$ and $v \in \{0, 1, \dots, s\}$, set

$$\sigma_{uv}(x) = \text{card}\{y \in C_v \mid d(x, y) = 1\}$$

Then we have that for all $j \geq 1$ and $x \in C_u$

$$A_j(x) = \sum_{v=0}^{s'} \sigma_{uv}(x) A_{j-1}(v).$$

Now, if $x' \in C_u$, this gives

$$A_j(x) = A_j(x') = A_j(u) = \sum_{v=0}^{s'} \sigma_{uv}(x) A_{j-1}(v) = \sum_{v=0}^{s'} \sigma_{uv}(x') A_{j-1}(v)$$

That is

$$\sum_{v=0}^{s'} [\sigma_{uv}(x) - \sigma_{uv}(x')] A_{j-1}(v) = 0, \quad j \geq 1.$$

Since $\text{rank } A = \text{rank } B = s' + 1$, we conclude that

$$\sigma_{uv}(x) - \sigma_{uv}(x') = 0 \quad \text{for all } x, x' \in C_u$$

and $v \in \{0, 1, \dots, s'\}$. Hence $\{C_0, C_1, \dots, C_s\}$ is a s' -partition design. If C is distance-invariant then C will be one of the classes C_u , otherwise it will be the union of some classes C_u . Finally this s' -partition design is unique by proposition 3.11.

Corollary 3.15 Let C be a code with covering radius ρ . Then C is completely regular if and only if C admits a r -partition-design for $r = \rho$. Moreover $\rho = s'$ and the eigenvalues of the associated matrix σ are $P_1(l)$ for $l \in \{0, d'_1, \dots, d'_s\}$ where $P_1(x) = n(q-1) - qx$ is the degree one Krawtchouk polynomial of parameter n and d'_1, \dots, d'_s are the dual distances of C .

In the particular case where the code C is additive we may use theorems

6.10 and 6.11 of [1] to obtain the following results.

Proposition 3.16 Let $C \subset F^n$ be a additive code and s' be the number of non-zero weights of the dual C^\perp of C . Then the following conditions are equivalent.

- (i) C admits a s' -partition-design $\pi = \{C_0, C_1, \dots, C_{s'}\}$.
- (ii) The partition $\pi = \{C_0, C_1, \dots, C_{s'}\}$ of the quotient group $C' = F^n / C$ defines an association scheme over C' .
- (iii) The restriction to C^\perp of the Hamming scheme $H(n, q)$ is a subscheme.

The association scheme (ii) whose relations R'_i are well defined by

$$(x+C)R'_i(y+C)$$

if and only if

$$x-y \in C_i$$

because, by definition, C_i is an union of cosets of C , is called the **coset scheme** determined by the partition π .

The P-matrix of the coset scheme has been determined by A Montpetit [13].

Let P_σ be the **left eigenmatrix** of σ whose row number i is the vector v_i with first component 1 such that

$$v_i \sigma = P_1(d'_i) v_i, \quad 0 \leq i \leq s'.$$

Proposition 3.17 [13]

The P matrix of the coset association scheme is P_σ the left eigenmatrix

of σ .

4. - EXAMPLES

We shall give the matrices σ and P_σ for perfect codes, some uniformly packed codes and some other codes not of these types.

4.1. - Perfect codes

4.1.1. - One-error-correcting perfect codes

If C is a perfect one-error-correcting q -ary code of length n , then by proposition 3.8 there exists a 1-partition-design $\{C_0=C, C_1\}$ in \mathbb{F}_q^n with associated matrix

$$\sigma = \begin{bmatrix} 0 & n(q-1) \\ 1 & n(q-1)-1 \end{bmatrix}$$

By remark 3.2

$$|C_1| = \sigma_{10} |C| = \sigma_{01} |C_0| = n(q-1) |C|$$

where $|X| = \text{card } X$ denotes the cardinality of X . Hence

$$q^n = |C_0| + |C_1| = |C| [1 + n(q-1)]$$

so that $n = (q^m - 1) / (q - 1)$ and $|C| = q^{n-m}$ for some natural number m . The dual distances are by corollary 3.12

$$d'_0 = [n(q-1) - n(q-1)] / q = 0 \quad \text{and} \quad d'_1 = [n(q-1) - (-1)] / q = q^{m-1}$$

because the eigenvalues of σ are $n(q-1)$ and -1 . Thus the matrices σ and P_σ for one-error-correcting codes over \mathbb{F}_q are

$$\sigma = \begin{bmatrix} 0 & q^m - 1 \\ 1 & q^m - 2 \end{bmatrix} \text{ and } P_\sigma = \begin{bmatrix} 1 & q^m - 1 \\ 1 & -1 \end{bmatrix}$$

4.1.2. - Golay code of length $n=11$

The parameters are $e=2, q=3$. So matrices σ and P_σ are

$$\sigma = \begin{bmatrix} 0 & 22 & 0 \\ 1 & 1 & 20 \\ 0 & 2 & 20 \end{bmatrix} \text{ and } P_\sigma = \begin{bmatrix} 1 & 22 & 220 \\ 1 & 4 & -5 \\ 1 & -5 & 4 \end{bmatrix}$$

the eigenvalues of σ being $P_1(0)=n(q-1)=22, P_1(d'_1)=n(q-1)-qd'_1=4$ and $P_1(d'_2)=n(q-1)-qd'_2=-5$ from which we deduce the two non-zero weights of the orthogonal : $d'_1=6$ and $d'_2=9$.

4.1.3. - Golay code of length $n = 23$

The parameters are $e=3, q=2$. So matrices σ and P_σ are

$$\sigma = \begin{bmatrix} 0 & 23 & 0 & 0 \\ 1 & 0 & 22 & 0 \\ 0 & 2 & 0 & 21 \\ 0 & 0 & 3 & 20 \end{bmatrix} \text{ and } P_\sigma = \begin{bmatrix} 1 & 23 & 253 & 1771 \\ 1 & 7 & 13 & -21 \\ 1 & -1 & -11 & 11 \\ 1 & -9 & 29 & -21 \end{bmatrix}$$

the eigenvalues of σ being $P_1(0)=n(q-1)=23, P_1(d'_1)=n(q-1)-qd'_1=7, P_1(d'_2)=-1$ and $P_1(d'_3)=-9$ from which we deduce the three non-zero weights of the orthogonal : $d'_1=8, d'_2=12, d'_3=16$.

4.2 - Some uniformly packed codes

4.2.1. - BCH 2-error-correcting code of length $n = 2^{2m+1} - 1$.

Here we have a (λ, μ) -uniformly packed code with $\lambda = \frac{n-7}{6}$ and $\mu = \lambda + 1 = \frac{n-1}{6}$. The matrices σ and P_σ are

$$\sigma = \begin{bmatrix} 0 & 2^{2m+1}-1 & 0 & 0 \\ 1 & 0 & 2^{2m+1}-2 & 0 \\ 0 & 2 & 2^{2m}-4 & 2^{2m}+1 \\ 0 & 0 & 2^{2m}-1 & 2^{2m} \end{bmatrix}$$

$$\text{and } P_{\sigma} = \begin{bmatrix} 1 & 2^{2m+1}-1 & (2^{2m}-1)(2^{2m+1}-1) & (2^{2m}+1)(2^{2m+1}-1) \\ 1 & 2^{m+1}-1 & (2^m-1)^2 & -(2^{2m}+1) \\ 1 & -1 & -(2^{2m}-1) & 2^{2m}-1 \\ 1 & -(2^{m+1}+1) & (2^m+1)^2 & -(2^{2m}+1) \end{bmatrix}$$

the eigenvalues of σ being $P_1(0)=2^{2m+1}-1, P_1(d'_1)=2^{m+1}-1, P_1(d'_2)=-1$ and $P_1(d'_3)=-(2^{m+1}+1)$ from which we deduce the three non zero weights of the orthogonal : $d'_1=2^{2m}-2^m, d'_2=2^{2m}$ and $d'_3=2^{2m}+2^m$.

4.2.2. - Golay code of length 24

It is the only (λ, μ) -uniformly packed 3-error-correcting code [12]. The parameters are $e=3, q=2, \lambda=0, \mu=6$.

The matrices σ and P_{σ} are

$$\sigma = \begin{bmatrix} 0 & 24 & 0 & 0 & 0 \\ 1 & 0 & 23 & 0 & 0 \\ 0 & 2 & 0 & 22 & 0 \\ 0 & 0 & 3 & 0 & 21 \\ 0 & 0 & 0 & 24 & 0 \end{bmatrix}$$

$$P_{\sigma} = \begin{bmatrix} 1 & 24 & 276 & 2024 & 1771 \\ 1 & 8 & 20 & -8 & -21 \\ 1 & 0 & -12 & 0 & 11 \\ 1 & -8 & 20 & 8 & -21 \\ 1 & -24 & 276 & -2024 & 1771 \end{bmatrix}$$

the eigenvalues being $P_1(0)=24, P_1(d'_1)=8, P_1(d'_2)=0, P_1(d'_3)=-8, P_1(d'_4)=-24$ from which we deduce the four non-zero weights of the orthogonal : $d'_1=8, d'_2=12, d'_3=16, d'_4=24$.

4.2.3. - Preparata codes of length $n=2^{2m}-1, m \geq 2$.

These are binary 2-error-correcting non-linear (λ, μ) -uniformly packed codes with $\lambda = \frac{1}{3}[2^{2m} - 4]$ and $\mu = \frac{1}{3}[2^{2m} - 1]$. The matrices σ and P_σ are

$$\sigma = \begin{bmatrix} 0 & 2^{2m-1} & 0 & 0 \\ 1 & 0 & 2^{2m-2} & 0 \\ 0 & 2 & 2^{2m-4} & 1 \\ 0 & 0 & 2^{2m-1} & 0 \end{bmatrix}$$

$$P_\sigma = \begin{bmatrix} 1 & 2^{2m-1} & (2^{2m-1})(2^{2m-1}-1) & 2^{2m-1}-1 \\ 1 & 2^m-1 & -(2^m-1) & -1 \\ 1 & -1 & -(2^{2m-1}-1) & 2^{2m-1}-1 \\ 1 & -(2^m+1) & 2^m+1 & -1 \end{bmatrix}$$

The eigenvalues of σ are $P_1(0) = 2^{2m}$, $P_1(d'_1) = 2^m - 1$, $P_1(d'_2) = -1$, $P_1(d'_3) = -(2^m + 1)$, so that the dual distances are $d'_1 = 2^{m-1}(2^m - 1)$, $d'_2 = 2^{2m-1}$ and $d'_3 = 2^{m-1}(2^m + 1)$.

4.2.4. - Van Lint code of length $n=11$ [12]

This is a binary non-linear 2-error-correcting (λ, μ) -uniformly packed code with $\lambda=2$ and $\mu=3$. The matrices σ and P_σ are

$$\sigma = \begin{bmatrix} 0 & 11 & 0 & 0 \\ 1 & 0 & 10 & 0 \\ 0 & 2 & 6 & 3 \\ 0 & 0 & 9 & 2 \end{bmatrix}$$

$$P_\sigma = \begin{bmatrix} 1 & 11 & 55 & 55/3 \\ 1 & 3 & -1 & -3 \\ 1 & -1 & -5 & 5 \\ 1 & -5 & 7 & -3 \end{bmatrix}$$

The eigenvalues of σ are $P_1(0) = 11$, $P_1(d'_1) = 3$, $P_1(d'_2) = -1$ and $P_1(d'_3) = -5$. So the non-zero dual distances are $d'_1 = 4$, $d'_2 = 6$, $d'_3 = 8$.

4.3 - Some non-uniformly packed codes admitting s'-partition-design

We shall use remark 3.3 to define a s'-partition design of F^n by means of subset $\Omega_0 = 0, \Omega_1, \Omega_2, \dots, \Omega_s$, of F^n .

4.3.1. -

Set

$$\Omega_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \Omega_1 = \begin{bmatrix} 1100 \\ 0110 \\ 0011 \end{bmatrix}, \quad \Omega_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \Omega_3 = \begin{bmatrix} 11 \\ 01 \\ 11 \end{bmatrix}$$

$$C = \text{Ker } H \text{ where } H = \begin{bmatrix} 11000 \\ 01101 \\ 00110 \end{bmatrix} \text{ i.e. } \Omega = \Omega_1 \cup \Omega_2$$

We have here

$$\sigma = \begin{bmatrix} 0 & 4 & 1 & 0 \\ 1 & 1 & 1 & 2 \\ 1 & 4 & 0 & 0 \\ 0 & 4 & 0 & 1 \end{bmatrix}, \quad P_\sigma = \begin{bmatrix} 1 & 4 & 1 & 2 \\ 1 & 0 & 1 & -2 \\ 1 & 0 & -1 & 0 \\ 1 & -4 & 1 & 2 \end{bmatrix}$$

and the eigenvalues of σ are 5, +1, -1, -3 that is $n(q-1) - qw_i$ for $n=5, q=2, w_i \in \{0, 2, 3, 4\}$, the w_i being the weights of C .

4.3.2. -

Set

$$\Omega_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \Omega_1 = \begin{bmatrix} 100101 \\ 011001 \\ 010110 \\ 010101 \end{bmatrix}, \quad \Omega_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \Omega_3 = \begin{bmatrix} 101010 \\ 100101 \\ 011001 \\ 010110 \end{bmatrix}, \quad \Omega_4 = \begin{bmatrix} 01 \\ 01 \\ 01 \\ 10 \end{bmatrix} \quad C = C_0 = \text{Ker } \Omega_1$$

We have here

$$\sigma = \begin{bmatrix} 0 & 6 & 0 & 0 & 0 \\ 1 & 0 & 1 & 4 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 2 \\ 0 & 0 & 0 & 6 & 0 \end{bmatrix}, P_\sigma = \begin{bmatrix} 1 & 6 & 1 & 6 & 2 \\ 1 & 2 & 1 & -2 & -2 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & -2 & 1 & -2 & 2 \\ 1 & -6 & 1 & 6 & -2 \end{bmatrix}$$

and the eigenvalues of σ are 6, 2, 0, -2, -6 that is $n(q-1)-qw_i$ for $n=6$ and $w_i \in \{0, 2, 3, 4, 6\}$, the w_i being the weights of C .

Note : If we merge C_0 and C_2 because $\sigma_{0v} = \sigma_{2v}$ and $\sigma_{v0} = \sigma_{v2}$ for all $v=0, \dots, 5$, then $C'_0 = C_0 \cup C_2, C_1, C_3, C_4$ form a 3-partition design but it doesn't contain C .

4.3.3. -

Set

$$\Omega_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \Omega_1 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$\Omega_4 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

and let the columns of Ω_3 be the complements of $\Omega_0 \cup \Omega_1 \cup \Omega_4$ in \mathbb{F}_2^6 .

If $C = \text{Ker } \Omega_1$ then

$$\sigma = \begin{bmatrix} 0 & 15 & 0 & 0 \\ 1 & 2 & 12 & 0 \\ 0 & 4 & 10 & 1 \\ 0 & 0 & 15 & 0 \end{bmatrix}, P_\sigma = \begin{bmatrix} 1 & 15 & 45 & 3 \\ 1 & 3 & -3 & -1 \\ 1 & -1 & -3 & 3 \\ 1 & -5 & 5 & -1 \end{bmatrix}$$

and the eigenvalues of σ are 15, 3, -1, -5 that is $n(q-1)-qw_i$ for $n=15, q=2$ and $w_i \in \{0, 6, 8, 10\}$ the w_i being the weights of C .

4.3.4. - Nordstrom-Robinson of length 16 [MacWilliams-Sloane p. 171]

This is a **non-linear** formally self-dual code C with distances $d_1=6, d_2=8, d_3=10, d_4=16$. C admits the 4-partition design C_0, C_1, C_2, C_3, C_4 where $C_i = \{x \in F_2^{16} \mid d(x, C) = i\}$. The matrices σ and P_σ are

$$\sigma = \begin{bmatrix} 0 & 16 & 0 & 0 & 0 \\ 1 & 0 & 15 & 0 & 0 \\ 0 & 2 & 0 & 14 & 0 \\ 0 & 0 & 15 & 0 & 1 \\ 0 & 0 & 0 & 16 & 0 \end{bmatrix}$$

$$\text{and } P_\sigma = \begin{bmatrix} 1 & 16 & 120 & 112 & 7 \\ 1 & 4 & 0 & -4 & -1 \\ 1 & 0 & -8 & 0 & 7 \\ 1 & -4 & 0 & 4 & -1 \\ 1 & -16 & 120 & -112 & 7 \end{bmatrix}$$

The eigenvalues of σ are 16, 4, 0, -4, -16 that is $n(q-1)-qd_i$ with $n=16, q=2$ and $d_i \in \{0, 6, 8, 10, 16\}$ as it should be.

4.3.5. - An exemple [13] where there doesn't exist a s' partition design : First order Reed-Muller code of length 16.

If C denotes the first order Reed-Muller code of length 16, then the covering radius is $\rho=6$ and $s' = 6$ because the extended Hamming code has weights 0, 4, 6, 8, 10, 12, 16. C admits the 7-partition-design $\{C_0, C_1, C_2, \dots, C_7\}$ where the C_i are the equivalence classes for the relation over F_2^{16} $x \equiv y$ if and only if the cosets $x + C$ and $y + C$ have the same weight distributions.

The matrices σ and P_σ are here

$$\sigma = \begin{bmatrix} 0 & 16 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 15 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 14 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 1 & 12 & 0 & 0 \\ 0 & 0 & 0 & 16 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 15 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 \end{bmatrix}$$

$$P_\sigma = \begin{bmatrix} 1 & 16 & 120 & 560 & 35 & 840 & 448 & 28 \\ 1 & 8 & 24 & 24 & 3 & -24 & -32 & -4 \\ 1 & 4 & 0 & -20 & -5 & 0 & 16 & 4 \\ 1 & 0 & -8 & 0 & 0 & 14 & 0 & -7 \\ 1 & 0 & -8 & 0 & 1 & 12 & 0 & -6 \\ 1 & -4 & 0 & 20 & -5 & 0 & -16 & 4 \\ 1 & -8 & 24 & -24 & 3 & -24 & 32 & -4 \\ 1 & -16 & 120 & -560 & 35 & 840 & -448 & 28 \end{bmatrix}$$

The eigenvalues of σ are 16, 8, 4, 0, -4, -8, -16 ; 0 being a double eigenvalue. Note that there doesn't exist a 6-partition-design containing C because B has 8 distinct rows. We may also note that the eigenvalues are $n(q-1)-qw_i$ with w_i the weights of C .

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